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SOME APPLICATIONS OF LYAPUNOV REGULARITY

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Abstract. For a non-autonomous dynamics with discrete time defined by a tempered sequence of upper-triangular matrices, we obtain lower and upper bounds for the Grobman regularity coefficients. We also give two applications of these results: we obtain an upper bound for the Grobman coefficients of the exterior powers of a tempered sequence, and we give a simple proof of Oseledets' multiplicative ergodic theorem for cocycles over a measure-preserving transformation without using Kingman's subadditive ergodic theorem.

1. Introduction

Our main objective is to discuss various properties and several applications of regularity coefficients, and specifically of the Grobman coefficient. More precisely, we consider a non-autonomous dynamics with discrete time defined by a tempered sequence of upper-triangular matrices and we obtain lower and upper bounds for their Grobman coefficients. The applications of these bounds include a sharp upper bound for the Grobman coefficients of the exterior powers of a tempered sequence of upper-triangular matrices as well as a simple proof of Oseledets' multiplicative ergodic theorem in [8] for cocycles over a measure-preserving transformation.

1.1. **Lyapunov regularity.** We start with a brief discussion about the relevance and in fact the importance of Lyapunov regularity. The notion was introduced by Lyapunov in [6] (see [7] for an English translation) and plays an important role in the Lyapunov stability theory. It turns out that it allows one to study the persistence of the stability and of the conditional stability of a linear dynamics under sufficiently small nonlinear perturbations (see for example [2]). This is particularly effective in the context of ergodic theory since for a smooth dynamics preserving a finite measure satisfying a mild integrability assumption, the linearization along almost all trajectories is Lyapunov regular. This is a consequence of Oseledets' multiplicative ergodic theorem (further details are given below).

We illustrate the role of Lyapunov regularity with a simple example in the case of continuous time. Consider a linear equation

$$v' = A(t)v \tag{1.1}$$

on $\mathbb{R}^q$ for some $q \times q$ matrices $A(t)$ whose entries vary continuously with $t \in \mathbb{R}$. We assume that $\sup_{t\in\mathbb{R}} \|A(t)\| < +\infty$. Its Lyapunov exponent is the function

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$\lambda\colon \mathbb{R}^q \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\lambda(v_0) = \limsup_{t\to\infty} \frac{1}{t} \log \|v(t)\|, \tag{1.2}$$

where $v(t)$ denotes the solution of (1.1) with $v(0) = v_0$ (note that all solutions are global), with the convention that $\log 0 = -\infty$. The *Lyapunov coefficient* of equation (1.1) is defined by

$$\sigma(A) = \min \sum_{i=1}^{q} \lambda(v_i) - \liminf_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} A(\tau)\, d\tau,$$

where the minimum is taken over all bases $v_1, \dots, v_q$ for $\mathbb{R}^q$ (see [7]). One can easily verify that $\sigma(A) \ge 0$. The equation is said to be *Lyapunov regular* or simply *regular* if $\sigma(A) = 0$ or, equivalently, if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} A(\tau)\, d\tau = \sum_{i=1}^{q} \lambda(v_i)$$

for some basis $v_1, \dots, v_q$ for $\mathbb{R}^q$. If the Lyapunov exponent λ takes only negative values, then equation (1.1) is asymptotically stable: any solution $v(t)$ of the equation tends to zero when $t \to +\infty$. However, the equation need not be *uniformly* asymptotically stable: in order that a given solution is contained in an ε-neighborhood of the origin, one may need to take the initial condition sufficiently small depending on the initial time. A consequence is that the type of stability may change under arbitrarily small nonlinear perturbations. Indeed, it was shown by Perron in [10] that there are examples of asymptotically stable equations $v' = A(t)v$ for which some perturbation

$$u' = A(t)u + f(u)$$

has a solution $u(t)$ with

$$\limsup_{t\to\infty} \frac{1}{t} \log \|u(t)\| > 0.$$

On the other hand, Lyapunov [7] showed that for a *regular* equation the asymptotic stability persists under sufficiently small nonlinear perturbations, such as for example when

$$\|f(u) - f(v)\| \le c\|u - v\|(\|u\|^r + \|v\|^r)$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{R}^q$, for some constants $c, r > 0$ (see [2] for details).

1.2. **Context of ergodic theory.** As noted above, the notion of Lyapunov regularity is particularly effective in the study of the persistence of stability and conditional stability under sufficiently small nonlinear perturbations in the context of ergodic theory.

More precisely, Lyapunov regularity turns out to be typical under fairly general assumptions in ergodic theory. To formulate a precise statement, recall that a measure μ on $\mathbb{R}^q$ is *invariant* under a flow $(\phi_t)_{t\in\mathbb{R}}$ if

$$\mu(\phi_t(A)) = \mu(A) \text{ for every measurable set } A \subset \mathbb{R}^q \text{ and } t \in \mathbb{R}.$$

In particular, any flow defined by a differentiable vector field with zero divergence, such as any Hamiltonian vector field, preserves the Lebesgue measure. Now consider a differential equation $x' = F(x)$ on $\mathbb{R}^q$ for some C^1 vector field F. We assume that the equation generates a flow $(\phi_t)_{t\in\mathbb{R}}$ preserving a finite measure μ with compact support on $\mathbb{R}^n$. Then for μ-almost every $x \in \mathbb{R}^q$ the linear variational equation

$$v' = A_x(t)v \quad \text{with } A_x(t) = d_{\phi_t(x)}F \tag{1.3}$$

is Lyapunov regular. This statement is a special case of Oseledets' multiplicative ergodic theorem in [8].

On the other hand, since the general solution of equation (1.3) is $v(t) = (d_x\phi_t)v_0$, with $t \in \mathbb{R}$ and $v_0 \in \mathbb{R}^q$, the Lyapunov exponent λ in (1.2) becomes

$$\lambda(v_0) = \limsup_{t\to+\infty} \frac{1}{t} \log \|(d_x\phi_t)v_0\|$$

for a given $x \in \mathbb{R}^q$. Therefore, the values of the Lyapunov exponent are naturally related to the exponential growth rates of contraction and expansion of the matrices $d_x\phi_t$ and so to the hyperbolicity of the linear variational equation in (1.3). Indeed, as noted above, if the Lyapunov exponent λ takes only negative values, then the equation is asymptotically stable (although not necessarily uniformly asymptotically stable).

This leads naturally to the notion of nonuniform hyperbolicity. The classical notion of uniform hyperbolicity, essentially introduced by Perron in [11], plays an important role in a large part of the theory of differential equations and dynamical systems. Some of its consequences are the existence of topological conjugacies and of stable and unstable invariant manifolds under sufficiently small nonlinear perturbations. On the other hand, the existence of uniform hyperbolicity is a rather stringent condition. The notion of nonuniform hyperbolicity introduced by Pesin in [12] is a much weaker requirement and so it is also much more common. Among the most important properties due to nonuniform hyperbolicity are the existence of stable and unstable manifolds and their absolute continuity, the ergodic properties of dynamical systems preserving a finite measure that is absolutely continuous with respect to the volume, a formula for the Kolmogorov-Sinai entropy in terms of the Lyapunov exponents, a rich orbit structure, an exponential growth rate for the number of periodic points, and an approximation by uniformly hyperbolic horseshoes. We refer the reader to the book [3] for references and for a detailed presentation of a large part of the theory.

1.3. **Regularity coefficients.** There are several regularity coefficients that can be used to define Lyapunov regularity, such as those introduced by Perron (see [9, 11]) and Grobman (see [5]). Some of them are more amenable than others in specific situations, although all of them are related. More precisely, not only all of them vanish if and only if the dynamics is regular, but also the ratio of any two of them (over all nonregular dynamics) is bounded and bounded away from zero. From this point of view, it is irrelevant to obtain lower and upper bounds for any specific regularity coefficient since then these can be transferred to any other regularity coefficient. Here we use the regularity coefficient due to Grobman. We refer the reader to the books [1, 5] for detailed accounts of the theory.

Also as noted above, the type of stability persists when the Lyapunov coefficient is sufficiently small when compared to the Lyapunov exponents, but not necessarily zero. Since the Lyapunov coefficient may be hard to compute, it is convenient to have sharp bounds (the best would be to have in terms of the Lyapunov exponents but these are also hard to compute). This justifies the interest of the present work.

Now we formulate briefly our results that give lower and upper bounds for the Grobman coefficients. Consider a sequence of invertible upper-triangular $q \times q$ matrices $(A_m)_{m\in\mathbb{N}}$ and let $e_1, \dots, e_q$ be the canonical basis for $\mathbb{R}^q$. Since the matrices A_m are upper-triangular, the space $E_k \subset \mathbb{R}^q$ generated by the first k vectors $e_1, \dots, e_k$, for $k = 1, \dots, q$, is invariant under all the matrices. Moreover, E_k is isomorphic to $\mathbb{R}^k$ and the restriction of each matrix A_m to E_k can be identified with a $k \times k$ upper-triangular matrix $A_{m,k}$. We denote by γ_k the Grobman coefficient of the pair of Lyapunov exponents associated with the sequences of matrices $(A_{m,k})_{m\in\mathbb{N}}$ and $(B_{m,k})_{m\in\mathbb{N}}$, where $B_{m,k} = (A^*_{m,k})^{-1}$ (see Section 2 for the definition of the Grobman coefficient). The sequence $(A_m)_{m\in\mathbb{N}}$ is said to be *tempered* if

$$\limsup_{m\to+\infty} \frac{1}{m} \log^+ \|A_m\| = 0, \tag{1.4}$$

where $\log^+ x = \max\{0, \log x\}$. Moreover, denoting by a^m_{ij} the entries of A_m, for each $i = 1, \dots, q$ let

$$\underline{\alpha}_i = \liminf_{m\to+\infty} \frac{1}{m} \log \prod_{l=1}^{m} |a^l_{ii}| \quad \text{and} \quad \overline{\alpha}_i = \limsup_{m\to+\infty} \frac{1}{m} \log \prod_{l=1}^{m} |a^l_{ii}|.$$

The following result is a consequence of Theorems 3.1 and 4.7 below.

Theorem 1.1. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of invertible $q \times q$ matrices whose Lyapunov exponent λ takes only finite values on $\mathbb{R}^q \setminus \{0\}$. If A_m is upper-triangular for every $m \in \mathbb{N}$, then*

$$\max_{k=1,\dots,q} \gamma_k \geq \frac{1}{q^2} \sum_{i=1}^{q} (\overline{\alpha}_i - \underline{\alpha}_i) \quad \text{and} \quad \gamma_q \leq \sum_{i=1}^{q} (\overline{\alpha}_i - \underline{\alpha}_i). \tag{1.5}$$

We emphasize that there is no loss of generality in considering only upper-triangular (or lower-triangular) matrices. Indeed, given a sequence of invertible $q\times q$ matrices $(A_m)_{m\in\mathbb{N}}$ there exists a sequence of orthogonal $q \times q$ matrices $(U_m)_{m\in\mathbb{N}}$ such that $C_m = U^*_{m+1} A_m U_m$ is upper-triangular for each $m \in \mathbb{N}$ (see [1, Theorem 3.2.1]). Note that since the matrices U_m are orthogonal, the values of the Lyapunov exponent of the sequence $(C_m)_{m\in\mathbb{N}}$ are the same as those of the Lyapunov exponent of the sequence $(A_m)_{m\in\mathbb{N}}$, together with their multiplicities. To the best of our knowledge, the lower bound in (1.5) is new —a different lower bound, although not expressed in terms of the numbers $\underline{\alpha}_i$ and $\overline{\alpha}_i$, was given in [4]. The upper bound in (1.5) appeared before, although here we correct a mistake in the original proof (namely, it is tacitly assumed in [4, Lemma A.1] that the limit of the left-hand side of equation (A.6) is zero).

The methods of proof to obtain the lower and upper bounds in (1.5) are different. The lower bound is obtained through estimates for the numbers $\underline{\alpha}_i$ and $\overline{\alpha}_i$ in terms of the exponential growth rate of volumes for the restrictions of the dynamics to the subspaces E_k. On the other hand, the upper bound is obtained by first constructing dual bases with specific estimates for their Lyapunov exponents that then are used to obtain the upper bound for the Grobman coefficient. More precisely, we show that there exist dual bases $v_1 \dots, v_q$ and $w_1 \dots, w_q$ for $\mathbb{R}^q$ such that

$$\lambda(v_j) \leq \overline{\alpha}_j + \sum_{k=1}^{j-1} (\overline{\alpha}_k - \underline{\alpha}_k),$$

$$\mu(w_j) \le -\underline{\alpha}_j + \sum_{k=j+1}^{q} (\overline{\alpha}_k - \underline{\alpha}_k)$$

for $j = 1, \dots, q$ (see Theorems 4.1 and 4.4). These estimates are then used to obtain the sharp upper bound for the Grobman coefficient in (1.5).

The lower and upper bounds in (1.5) allow one to deduce the following characterization of Lyapunov regularity.

Theorem 1.2. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of invertible $q \times q$ upper-triangular matrices whose Lyapunov exponent λ takes only finite values on $\mathbb{R}^q \setminus \{0\}$. Then the sequence is Lyapunov regular if and only if $\underline{\alpha}_i = \overline{\alpha}_i$ for $i = 1, \dots, q$.*

Theorem 1.2 is proved in Section 5. Since all norms on the space $\mathbb{R}^q$ are equivalent, all the results in the paper are independent of the particular choice of norm. So we can fix any norm on $\mathbb{R}^q$ from the beginning.

1.4. **Applications of the results.** We give two applications of the former results: we obtain an upper bound for the Grobman coefficient of the exterior powers of a tempered sequence of upper-triangular matrices and we give a simple proof of Oseledets' multiplicative ergodic theorem for cocycles over a measure-preserving transformation.

The exterior powers of a sequence of matrices (see Section 6 for the definition) relate well to the k-volumes in $\mathbb{R}^q$ with $k \le q$. Indeed, the k-volume determined by the vectors $v_1, \dots, v_k$ is given by

$$vol(v_1, \dots, v_k) = \|v_1 \wedge \dots \wedge v_k\|.$$

In view of the importance of the exponential growth rate of volumes in ergodic theory (see also [13, 14] for the relation with the entropy conjecture), it is quite relevant a corresponding study to that in the former section for exterior powers. In Section 6 we show that the Grobman coefficient $\gamma(\lambda_k, \mu_k)$ of the sequence of matrices $(\Lambda^k(A_m))_{m\in\mathbb{N}}$ satisfies

$$\gamma(\lambda_k, \mu_k) \le \binom{q-1}{k-1} \sum_{i=1}^{q} (\overline{\alpha}_i - \underline{\alpha}_i)$$

for each integer $k \in [1, q]$ (see Theorem 6.1). Together with Theorem 1.2, this implies that if the sequence of matrices $(A_m)_{m\in\mathbb{N}}$ is regular, then so is the sequence $(\Lambda^k(A_m))_{m\in\mathbb{N}}$ for $k = 1, \dots, q$.

Finally, in Section 7 we give a proof of Oseledets' multiplicative ergodic theorem for cocycles over a measure-preserving transformation, which says that under a certain integrability assumption the set of all regular points has full measure. The proof is based on the upper bound for the Grobman coefficient after making an appropriate upper-triangular reduction of the cocycle. Although the upper-triangular reduction is reminiscent of the original proof by Oseledets, our argument does not require Kingman's subadditive ergodic theorem and is streamlined after having the upper bound. We also provide full technical details for this reduction, which is obtained through a lift of the cocycle to a linear extension on the product of the original space and the orthogonal group.

2. Lyapunov regularity

In this section we introduce the notion of Lyapunov regularity for the nonautonomous dynamics defined by a sequence of matrices and we recall one of its main characterizations. This will be used later on in the paper.

Consider a sequence of invertible $q \times q$ matrices $(A_m)_{m\in\mathbb{N}}$. We shall always assume that

$$\limsup_{m\to+\infty} \frac{1}{m} \log \|\mathcal{A}_m^{\pm 1}\| < +\infty, \tag{2.1}$$

where

$$\mathcal{A}_m = \begin{cases} A_{m-1} \cdots A_1 & \text{if } m > 1, \\ \mathrm{Id} & \text{if } m = 1. \end{cases}$$

This happens for example if the sequences $(A_m)_{m\in\mathbb{N}}$ and $(A_m^{-1})_{m\in\mathbb{N}}$ are bounded. The Lyapunov exponent $\lambda\colon \mathbb{R}^q \to \mathbb{R}\cup\{-\infty\}$ associated with $(A_m)_{m\in\mathbb{N}}$ is defined by

$$\lambda(v) = \limsup_{m\to+\infty} \frac{1}{m} \log \|\mathcal{A}_m v\|.$$

Condition (2.1) implies that λ does not take infinite values on $\mathbb{R}^q\setminus\{0\}$. The sequence of matrices $(A_m)_{m\in\mathbb{N}}$ is said to be *(Lyapunov) regular* if

$$\lim_{m\to+\infty} \frac{1}{m} \log |\det \mathcal{A}_m| = \min \sum_{i=1}^{q} \lambda(v_i) \tag{2.2}$$

with the minimum taken over all bases $v_1, \dots, v_q$ of $\mathbb{R}^q$. We emphasize that the notion of regularity includes the requirement that the limit in the left-hand side of (2.2) exists.

We also consider the sequence of matrices $B_m = (A_m^*)^{-1}$ for $m \in \mathbb{N}$, where A^* denotes the adjoint of A, and we let $\mathcal{B}_m = (\mathcal{A}_m^*)^{-1}$. The Lyapunov exponent $\mu\colon \mathbb{R}^q \to \mathbb{R} \cup \{-\infty\}$ associated with $(B_m)_{m\in\mathbb{N}}$ is defined by

$$\mu(w) = \limsup_{m\to+\infty} \frac{1}{m} \log \|\mathcal{B}_m w\|$$

Condition (2.1) implies that μ does not take infinite values on $\mathbb{R}^q \setminus \{0\}$. We define the *Grobman coefficient* of λ and μ by

$$\gamma(\lambda, \mu) = \min \max\bigl\{\lambda(v_i) + \mu(w_i) : 1 \le i \le q\bigr\}$$

with the minimum taken over all dual bases $v_1, \dots, v_q$ and $w_1, \dots, w_q$ of $\mathbb{R}^q$, that is, such that $\langle v_i, w_j\rangle = \delta_{ij}$ for all i and j, where δ_{ij} is the Kronecker symbol. Of course, condition (2.1) ensures that the Grobman coefficient is well defined.

We recall that a sequence of matrices $(A_m)_{m\in\mathbb{N}}$ is said to be *tempered* if (1.4) holds. For example, any bounded sequence is tempered. It can be shown that a tempered sequence of matrices $(A_m)_{m\in\mathbb{N}}$ is regular if and only if $\gamma(\lambda, \mu) = 0$ (see for example [4]). We shall use this characterization of regularity together with an upper bound for the Grobman coefficient (see Theorem 4.7) to give a new streamlined proof of Oseledets' multiplicative ergodic theorem in [8] (see Theorem 7.1).

3. Lower bound for the Grobman coefficients

In this section we establish a lower bound for the Grobman coefficients of a sequence of upper-triangular matrices $(A_m)_{m\in\mathbb{N}}$ expressed in terms of the entries in the main diagonal.

For each $k = 1, \dots, q$, let $A_{m,k}$ be the $k \times k$ upper-triangular matrix obtained from A_m considering only the first k columns of its first k rows. We denote by $\gamma_k = \gamma_k(\lambda, \mu)$ the Grobman coefficient of the pair of Lyapunov exponents associated with the sequences of matrices $(A_{m,k})_{m\in\mathbb{N}}$ and $(B_{m,k})_{m\in\mathbb{N}}$, where $B_{m,k} = (A^*_{m,k})^{-1}$. Moreover, denoting by a^m_{ij} the entries of A_m, for each $i = 1, \dots, q$ let

$$\underline{\alpha}_i = \liminf_{m\to+\infty} \frac{1}{m} \log \prod_{l=1}^m |a^l_{ii}| \quad \text{and} \quad \overline{\alpha}_i = \limsup_{m\to+\infty} \frac{1}{m} \log \prod_{l=1}^m |a^l_{ii}|. \tag{3.1}$$

Theorem 3.1. *Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $q \times q$ matrices satisfying* (2.1)*. If A_m is upper-triangular for every $m \in \mathbb{N}$, then*

$$\max_{i=k,\dots,q} \gamma_k(\lambda, \mu) \geq \frac{1}{q^2} \sum_{i=1}^q (\overline{\alpha}_i - \underline{\alpha}_i).$$

Proof. Given a basis $v_1, \dots, v_q$ for $\mathbb{R}^q$ such that $v_1, \dots, v_k$ is a basis for E_k (the space generated by the vectors $e_1, \dots, e_k$) for each $k = 1, \dots, q$, one can always assume that

$$\sum_{i=1}^q \lambda(v'_i) \geq \sum_{i=1}^q \lambda(v_i) \tag{3.2}$$

for all bases $v'_1, \dots, v'_q$ for $\mathbb{R}^q$. Moreover, let $w_1, \dots, w_q$ be a basis for $\mathbb{R}^q$ such that

$$\sum_{i=1}^q \mu(w'_i) \geq \sum_{i=1}^q \mu(w_i)$$

for all bases $w'_1, \dots, w'_q$ for $\mathbb{R}^q$ (note that $v_1, \dots, v_q$ and $w_1, \dots, w_q$ need not be dual bases). By Hadamard's inequality, we have

$$|\det \mathcal{B}_m| \leq |\det W|^{-1} \prod_{i=1}^q \|\mathcal{B}_m w_i\|,$$

where W is the matrix with columns $w_1, \dots, w_q$, and so

$$\begin{aligned}
\liminf_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_m| &= -\limsup_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{B}_m| \geq -\sum_{i=1}^q \mu(w_i) \\
&= \sum_{i=1}^q \lambda(v_i) - \sum_{i=1}^q (\lambda(v_i) + \mu(w_i)).
\end{aligned}$$

Now let

$$\lambda'_1 \leq \dots \leq \lambda'_q \quad \text{and} \quad \mu'_1 \geq \dots \geq \mu'_q$$

be the ordered values of the numbers $\lambda(v_1), \dots, \lambda(v_q)$ and $\mu(w_1), \dots, \mu(w_q)$, respectively, counted with their multiplicities. The number

$$\pi(\lambda, \mu) = \max\{\lambda'_i + \mu'_i : 1 \leq i \leq q\}$$

satisfies $\pi(\lambda,\mu) \le \gamma(\lambda,\mu)$ (see [1, Theorem 2.4.2]). Therefore,

$$\sum_{i=1}^q (\lambda(v_i)+\mu(w_i)) = \sum_{i=1}^q (\lambda_i'+\mu_i') \le q\pi(\lambda,\mu) \le q\gamma(\lambda,\mu)$$

and so

$$\liminf_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_m| \ge \sum_{i=1}^q \lambda(v_i) - q\gamma(\lambda,\mu).$$

Letting

$$\mathcal{A}_{m,k} = \begin{cases} A_{m-1,k}\cdots A_{1,k} & \text{if } m>1, \\ \mathrm{Id} & \text{if } m=1, \end{cases}$$

one can show in a similar manner that

$$\liminf_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_{m,k}| \ge \sum_{i=1}^k \lambda(v_i) - k\gamma_k$$

for $k=1,\dots,q$. On the other hand, since

$$\det \mathcal{A}_{m,k} = (\det \mathcal{A}_{m,k-1}) \prod_{l=1}^{m-1} a_{kk}^l, \tag{3.3}$$

we obtain

$$\begin{aligned}
\overline{\alpha}_k &\le \limsup_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_{m,k}| - \liminf_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_{m,k-1}| \\
&\le \sum_{i=1}^k \lambda(v_i) - \sum_{i=1}^{k-1} \lambda(v_i) + (k-1)\gamma_{k-1} \\
&= \lambda(v_k) + (k-1)\gamma_{k-1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\underline{\alpha}_k &\ge \liminf_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_{m,k}| - \limsup_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_{m,k-1}| \\
&\ge \sum_{i=1}^k \lambda(v_i) - k\gamma_k - \sum_{i=1}^{k-1} \lambda(v_i) = \lambda(v_k) - k\gamma_k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{\alpha}_k - \underline{\alpha}_k &\le \lambda(v_k) + (k-1)\gamma_{k-1} - \lambda(v_k) + k\gamma_k \\
&= (k-1)\gamma_{k-1} + k\gamma_k \\
&\le (2k-1)\max_k \gamma_k,
\end{aligned}$$

which yields the inequality

$$\sum_{k=1}^q (\overline{\alpha}_k - \underline{\alpha}_k) \le \sum_{k=1}^q (2k-1)\max_k \gamma_k = q^2 \max_k \gamma_k.$$

This completes the proof of the theorem. □

4. Upper bound for the Grobman coefficients

In this section we establish a corresponding upper bound for the Grobman coefficient of a tempered sequence of upper-triangular matrices in terms of the entries in the main diagonal. We start by constructing certain special bases that are then used to establish an upper bound for $\gamma(\lambda,\mu)$ in terms of the numbers in (3.1).

Theorem 4.1. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of invertible $q\times q$ matrices satisfying* (2.1)*. If A_m is upper-triangular for every $m\in\mathbb{N}$, then there exists a basis $v_1\dots,v_q$ for $\mathbb{R}^q$ such that*

$$\lambda(v_j)\le\overline{\alpha}_j+\sum_{k=1}^{j-1}(\overline{\alpha}_k-\underline{\alpha}_k)\quad\text{for } j=1,\dots,q.$$

Proof. For each $i,j=1,\dots,q$ with $i\le j$, let

$$\beta_{ij}=\overline{\alpha}_j-\underline{\alpha}_i+\sum_{k=i+1}^{j-1}(\overline{\alpha}_k-\underline{\alpha}_k),\tag{4.1}$$

$$\kappa_{ij}=\beta_{ij}+\max_{i\le l\le j}(\overline{\alpha}_l-\underline{\alpha}_l).\tag{4.2}$$

In what follows we make the convention that $\prod_{l=p}^{q}c_l=1$ when $q<p$. For each $m\in\mathbb{N}$ we consider the matrix $Z(m)$ with entries z_{ij}^m defined recursively as follows:

(1) for $i>j$ let $z_{ij}^m=0$;
(2) for $i=j$ let $z_{ii}^m=\prod_{l=1}^{m-1}a_{ii}^l$;
(3) for $i<j$ with $\kappa_{ij}\ge 0$ let $z_{ij}^1=0$ and for $m\ge 2$ let

$$z_{ij}^m=\sum_{p=0}^{m-2}\sum_{l=i+1}^{j}a_{il}^{m-p-1}z_{lj}^{m-p-1}\prod_{r=m-p}^{m-1}a_{ii}^r;\tag{4.3}$$

(4) for $i<j$ with $\kappa_{ij}<0$ let

$$z_{ij}^m=-\sum_{p=1}^{+\infty}\sum_{l=i+1}^{j}a_{il}^{m+p-1}z_{lj}^{m+p-1}\prod_{r=m}^{m+p-1}(a_{ii}^r)^{-1}.\tag{4.4}$$

Lemma 4.2. *Assuming that the series in* (4.4) *converge whenever $\kappa_{ij}<0$, we have $Z(m+1)=A_mZ(m)$ for all $m\in\mathbb{N}$.*

Proof. We first assume that $\kappa_{ij}\ge 0$. Then

$$\begin{aligned}z_{ij}^{m+1}&=\sum_{p=0}^{m-1}\sum_{l=i+1}^{j}a_{il}^{m-p}z_{lj}^{m-p}\prod_{r=m-p+1}^{m}a_{ii}^r\\&=\sum_{l=i+1}^{j}a_{il}^m z_{lj}^m+\sum_{p=1}^{m-1}\sum_{l=i+1}^{j}a_{il}^{m-p}z_{lj}^{m-p}\prod_{r=m-p+1}^{m}a_{ii}^r.\end{aligned}$$

We have

$$\sum_{p=1}^{m-1}\sum_{l=i+1}^{j}a_{il}^{m-p}z_{lj}^{m-p}\prod_{r=m-p+1}^{m}a_{ii}^r=a_{ii}^m\sum_{p=1}^{m-1}\sum_{l=i+1}^{j}a_{il}^{m-p}z_{lj}^{m-p}\prod_{r=m-p+1}^{m-1}a_{ii}^r=a_{ii}^m z_{ij}^m$$

since $\kappa_{ij} \geq 0$ and so

$$z_{ij}^{m+1} = \sum_{l=i+1}^{j} a_{il}^m z_{lj}^m + a_{ii}^m z_{ij}^m = \sum_{l=i}^{j} a_{il}^m z_{lj}^m.$$

Now we assume that $\kappa_{ij} < 0$. Then

$$\begin{aligned} z_{ij}^{m+1}(a_{ii}^m)^{-1} &= -\sum_{p=1}^{+\infty} \sum_{l=i+1}^{j} a_{il}^{m+p} z_{lj}^{m+p} \prod_{r=m}^{m+p} (a_{ii}^r)^{-1} \\ &= -\sum_{p=0}^{+\infty} \sum_{l=i+1}^{j} a_{il}^{m+p} z_{lj}^{m+p} \prod_{r=m}^{m+p} (a_{ii}^r)^{-1} + \sum_{l=i+1}^{j} a_{il}^m z_{lj}^m (a_{ii}^m)^{-1}. \end{aligned}$$

Since $\kappa_{ij} < 0$, we have

$$-\sum_{p=0}^{+\infty} \sum_{l=i+1}^{j} a_{il}^{m+p} z_{lj}^{m+p} \prod_{r=m}^{m+p} (a_{ii}^r)^{-1} = z_{ij}^m$$

and so

$$z_{ij}^{m+1}(a_{ii}^m)^{-1} = z_{ij}^m + \sum_{l=i+1}^{j} a_{il}^m z_{lj}^m (a_{ii}^m)^{-1},$$

which gives

$$z_{ij}^{m+1} = a_{ii}^m z_{ij}^m + \sum_{l=i+1}^{j} a_{il}^m z_{lj}^m = \sum_{l=i}^{j} a_{il}^m z_{lj}^m.$$

This completes the proof. $\square$

Now let

$$\lambda_{ij} = \limsup_{m \to +\infty} \frac{1}{m} \log |z_{ij}^m|.$$

Lemma 4.3. *For $i, j = 1, \dots, q$, we have $\lambda_{ii} = \overline{\alpha}_i$ and for each $i < j$ the entry z_{ij}^m is well defined for all $m \in \mathbb{N}$ and satisfies*

$$\lambda_{ij} \leq \overline{\alpha}_j + \sum_{k=i}^{j-1} (\overline{\alpha}_k - \underline{\alpha}_k).$$

Proof. Clearly, $\lambda_{ii} = \overline{\alpha}_i$ for $i = 1, \dots, q$. For the entries with $i < j$ we proceed by backwards induction on i. Given $i < q$, assume that for $i + 1 \leq l \leq j$ the entry z_{lj}^m is well defined for all $m \in \mathbb{N}$ and satisfies

$$\lambda_{lj} \leq \overline{\alpha}_j + \sum_{k=l}^{j-1} (\overline{\alpha}_k - \underline{\alpha}_k).$$

We show that z_{ij}^m is well defined for all $m \in \mathbb{N}$ (which is immediate when $\kappa_{ij} \geq 0$ because it is given by a finite sum) and

$$\lambda_{ij} \leq \overline{\alpha}_j + \sum_{k=i}^{j-1} (\overline{\alpha}_k - \underline{\alpha}_k). \tag{4.5}$$

Assume that $\kappa_{ij} < 0$. By (1.4) and the induction hypothesis, for each $\varepsilon > 0$ there exists $D > 0$ such that $|a_{il}^n| \le De^{\varepsilon n}$,

$$D^{-1}e^{(\underline{\alpha}_i-\varepsilon)n} \le \prod_{l=1}^{n} |a_{ii}^l| \le De^{(\overline{\alpha}_i+\varepsilon)n},$$

$$|z_{lj}^n| \le De^{[\overline{\alpha}_j+\sum_{k=l}^{j-1}(\overline{\alpha}_k-\underline{\alpha}_k)+\varepsilon]n}$$

for all $n \in \mathbb{N}$ and $i+1 \le l \le j$. Therefore,

$$\begin{aligned}
A &:= \sum_{p=1}^{+\infty}\sum_{l=i+1}^{j} |a_{il}^{m+p-1} z_{lj}^{m+p-1}| \prod_{r=m}^{m+p-1} |a_{ii}^r|^{-1} \\
&\le \sum_{p=1}^{+\infty}\sum_{l=i+1}^{j} De^{\varepsilon(m+p-1)} De^{[\overline{\alpha}_j+\sum_{k=l}^{j-1}(\overline{\alpha}_k-\underline{\alpha}_k)+\varepsilon](m+p-1)} |a_{ii}^r|^{-1} \prod_{r=1}^{m-1} |a_{ii}^r| \\
&\le D' \sum_{p=1}^{+\infty}\sum_{l=i+1}^{j} e^{[\overline{\alpha}_j-\underline{\alpha}_i+\sum_{k=l}^{j-1}(\overline{\alpha}_k-\underline{\alpha}_k)+3\varepsilon](m+p)} \prod_{r=1}^{m-1} |a_{ii}^r|
\end{aligned}$$

for some constant $D' > 0$. Since $\kappa_{ij} < 0$, we have $\beta_{ij} < 0$. Now take $\varepsilon > 0$ such that $\beta_{ij} + 3\varepsilon < 0$. Then

$$A \le D'q \sum_{p=1}^{+\infty} e^{(\beta_{ij}+3\varepsilon)(m+p)} \prod_{r=1}^{m-1} |a_{ii}^r| < \frac{D'q}{1-e^{\beta_{ij}+3\varepsilon}} \prod_{r=1}^{m-1} |a_{ii}^r| < +\infty. \tag{4.6}$$

This shows that z_{ij}^m is also well defined when $\kappa_{ij} < 0$.

Finally, we establish the estimate in (4.5). First assume that $\kappa_{ij} \ge 0$. It follows from (4.3) that

$$\begin{aligned}
\lambda_{ij} &\le \limsup_{m\to+\infty} \frac{1}{m} \log \sum_{p=0}^{m-2}\sum_{l=i+1}^{j} |a_{il}^{m-p-1} z_{lj}^{m-p-1}| \prod_{r=m-p}^{m-1} |a_{ii}^r| \\
&\le \limsup_{m\to+\infty} \frac{1}{m} \log \sum_{p=0}^{m-2}\sum_{l=i+1}^{j} D^2 e^{\varepsilon(m-p-1)+[\overline{\alpha}_j+\sum_{k=l}^{j-1}(\overline{\alpha}_k-\underline{\alpha}_k)+\varepsilon](m-p-1)} \\
&\quad \times \prod_{r=1}^{m-1} |a_{ii}^r| De^{(-\underline{\alpha}_i+\varepsilon)(m-p-1)} \\
&\le \overline{\alpha}_i + \limsup_{m\to+\infty} \frac{1}{m} \log\Big(D'e^{3\varepsilon m} \sum_{p=0}^{m-2} e^{\beta_{ij}(m-p)}\Big) \\
&\le \overline{\alpha}_i + \limsup_{m\to+\infty} \frac{1}{m} \log(me^{\beta_{ij}m}) + 3\varepsilon = \overline{\alpha}_j + \sum_{k=i}^{j-1}(\overline{\alpha}_k - \underline{\alpha}_k) + 3\varepsilon
\end{aligned}$$

for some constant $D' > 0$. Since ε is arbitrary, this yields inequality (4.5). Now assume that $\kappa_{ij} < 0$, which implies that $\beta_{ij} < 0$. Take $\varepsilon > 0$ such that $\beta_{ij} + 3\varepsilon < 0$. By (4.6) we have

$$\lambda_{ij} \le \limsup_{m\to+\infty} \frac{1}{m} \log\Big(D'q \frac{e^{(\beta_{ij}+3\varepsilon)(m+1)}}{1-e^{\beta_{ij}+3\varepsilon}} \prod_{r=1}^{m-1} |a_{ii}^r|\Big)$$

$$\begin{aligned}&= \beta_{ij} + 3\varepsilon + \limsup_{m\to+\infty} \frac{1}{m} \log(De^{(\overline{\alpha}_i+\varepsilon)m})\\ &= \overline{\alpha}_j + \sum_{k=i}^{j-1} (\overline{\alpha}_k - \underline{\alpha}_k) + 3\varepsilon.\end{aligned}$$

Again, since ε is arbitrary, this yields inequality (4.5). $\square$

Now consider the basis formed by the columns of $Z(1)$, that is, the vectors

$$v_j = Z(1)e_j \quad \text{for } j = 1, \dots, q.$$

By Lemma 4.3 we obtain

$$\lambda(v_j) = \max\big\{\lambda_{ij} : 1 \le i \le q\big\} \le \overline{\alpha}_j + \sum_{k=1}^{j-1} (\overline{\alpha}_k - \underline{\alpha}_k)$$

for $j = 1, \dots, q$, which completes the proof of the theorem. $\square$

Now we consider the matrices B_m.

Theorem 4.4. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of invertible $q \times q$ matrices satisfying* (2.1)*. If A_m is upper-triangular for every $m \in \mathbb{N}$, then there exists a basis $w_1 \dots, w_q$ for $\mathbb{R}^q$ such that*

$$\mu(w_j) \le -\underline{\alpha}_j + \sum_{k=j+1}^{q} (\overline{\alpha}_k - \underline{\alpha}_k) \quad \text{for } j = 1, \dots, q.$$

Proof. We continue to define β_{ij} and κ_{ij} by (4.1) and (4.2). For each $m \in \mathbb{N}$ we consider the matrix $W(m)$ with entries w^m_{ij} defined recursively by:

1. for $i < j$ let $w^m_{ij} = 0$;
2. for $i = j$ let $w^m_{ii} = \prod_{l=1}^{m-1} (a^l_{ii})^{-1}$;
3. for $i > j$ with $\kappa_{ji} \ge 0$ let $w^1_{ij} = 0$ and for $m \ge 2$ let
$$w^m_{ij} = -\sum_{p=0}^{m-2} \sum_{l=j}^{i-1} a^{m-p-1}_{li} w^{m-p}_{lj} \prod_{r=m-p-1}^{m-1} (a^r_{ii})^{-1}; \tag{4.7}$$
4. for $i > j$ with $\kappa_{ji} < 0$ let
$$w^m_{ij} = \sum_{p=1}^{+\infty} \sum_{l=j}^{i-1} a^{m+p-1}_{li} w^{m+p}_{lj} \prod_{r=m}^{m+p-2} a^r_{ii}. \tag{4.8}$$

Lemma 4.5. *Assuming that the series in* (4.8) *converge whenever $\kappa_{ji} < 0$, we have $W(m+1) = B_m W(m)$ for all $m \in \mathbb{N}$.*

Proof. In view of the definition of B_m, the statement is equivalent to show that

$$W(m) = B_m^{-1} W(m+1) = A^*_m W(m+1).$$

We first assume that $\kappa_{ji} \ge 0$. Then

$$\begin{aligned}w^{m+1}_{ij} &= -\sum_{p=0}^{m-1} \sum_{l=j}^{i-1} a^{m-p}_{li} w^{m+1-p}_{lj} \prod_{r=m-p}^{m} (a^r_{ii})^{-1}\\ &= -\sum_{l=j}^{i-1} a^m_{li} w^{m+1}_{lj} (a^m_{ii})^{-1} - \sum_{p=1}^{m-1} \sum_{l=j}^{i-1} a^{m-p}_{li} w^{m+1-p}_{lj} \prod_{r=m-p}^{m} (a^r_{ii})^{-1}.\end{aligned}$$

We have

$$\begin{aligned} &-\sum_{p=1}^{m-1}\sum_{l=j}^{i-1} a_{li}^{m-p} w_{lj}^{m+1-p} \prod_{r=m-p}^{m} (a_{ii}^r)^{-1} \\ &= -(a_{ii}^m)^{-1} \sum_{p=1}^{m-1}\sum_{l=j}^{i-1} a_{li}^{m-p} w_{lj}^{m+1-p} \prod_{r=m-p}^{m-1} (a_{ii}^r)^{-1} = (a_{ii}^m)^{-1} w_{ij}^m \end{aligned}$$

since $\kappa_{ji} \geq 0$ and so

$$w_{ij}^{m+1} = -\sum_{l=j}^{i-1} a_{li}^m w_{lj}^{m+1} (a_{ii}^m)^{-1} + (a_{ii}^m)^{-1} w_{ij}^m,$$

which yields

$$w_{ij}^m = a_{ii}^m w_{ij}^{m+1} + \sum_{l=j}^{i-1} a_{li}^m w_{lj}^{m+1} = \sum_{l=j}^{i} a_{li}^m w_{lj}^{m+1}.$$

Now we assume that $\kappa_{ji} < 0$. Then

$$\begin{aligned} a_{ii}^m w_{ij}^{m+1} &= \sum_{p=1}^{+\infty}\sum_{l=j}^{i-1} a_{li}^{m+p} w_{lj}^{m+p+1} \prod_{r=m}^{m+p-1} a_{ii}^r \\ &= \sum_{p=0}^{+\infty}\sum_{l=j}^{i-1} a_{li}^{m+p} w_{lj}^{m+p+1} \prod_{r=m}^{m+p-1} a_{ii}^r - \sum_{l=j}^{i-1} a_{li}^m w_{lj}^{m+1}. \end{aligned}$$

Since $\kappa_{ji} < 0$, we have

$$\sum_{p=0}^{+\infty}\sum_{l=j}^{i-1} a_{li}^{m+p} w_{lj}^{m+p+1} \prod_{r=m}^{m+p-1} a_{ii}^r = w_{ij}^m$$

and so

$$w_{ij}^{m+1} a_{ii}^m = w_{ij}^m - \sum_{l=j}^{i-1} a_{li}^m w_{lj}^{m+1},$$

which gives

$$w_{ij}^m = a_{ii}^m w_{ij}^{m+1} + \sum_{l=j}^{i-1} a_{li}^m w_{lj}^{m+1} = \sum_{l=j}^{i} a_{li}^m w_{lj}^{m+1}.$$

This completes the proof. □

Now let

$$\mu_{ij} = \limsup_{m\to+\infty} \frac{1}{m} \log |w_{ij}^m|.$$

Lemma 4.6. *For $i, j = 1, \dots, q$, we have $\mu_{ii} = -\underline{\alpha}_i$ and for each $i > j$ the entry w_{ij}^m is well defined for all $m \in \mathbb{N}$ and*

$$\mu_{ij} \leq -\underline{\alpha}_j + \sum_{k=j+1}^{i} (\overline{\alpha}_k - \underline{\alpha}_k).$$

Proof. Clearly, $\mu_{ii} = -\underline{\alpha}_i$ for $i = 1, \dots, q$. For the entries with $i > j$ we proceed by induction on i. Given $i < q$, assume that for $j \le l \le i-1$ the entry w_{lj}^m is well defined for all $m \in \mathbb{N}$ and satisfies

$$\mu_{lj} \le -\underline{\alpha}_j + \sum_{k=j+1}^{l} (\overline{\alpha}_k - \underline{\alpha}_k).$$

We show that w_{ij}^m is well defined for all $m \in \mathbb{N}$ (which is immediate when $\kappa_{ji} \ge 0$ because it is given by a finite sum) and

$$\mu_{ij} \le -\underline{\alpha}_j + \sum_{k=j+1}^{i} (\overline{\alpha}_k - \underline{\alpha}_k). \tag{4.9}$$

Assume that $\kappa_{ji} < 0$. By (1.4) and the induction hypothesis, for each $\varepsilon > 0$ there exists $D > 0$ such that $|a_{li}^n| \le De^{\varepsilon n}$,

$$D^{-1} e^{(\underline{\alpha}_i - \varepsilon)n} \le \prod_{l=1}^{n} |a_{ii}^l| \le De^{(\overline{\alpha}_i + \varepsilon)n},$$
$$|w_{lj}^n| \le De^{[-\underline{\alpha}_j + \sum_{k=j+1}^{l} (\overline{\alpha}_k - \underline{\alpha}_k)]n}$$

for all $n \in \mathbb{N}$ and $j \le l \le i-1$. Therefore,

$$\begin{aligned}
B :&= \sum_{p=1}^{+\infty} \sum_{l=j}^{i-1} |a_{li}^{m+p-1} w_{lj}^{m+p}| \prod_{r=m}^{m+p-2} |a_{ii}^r| \\
&\le \sum_{p=1}^{+\infty} \sum_{l=j}^{i-1} De^{\varepsilon(m+p-1)} De^{[-\underline{\alpha}_j + \sum_{k=j+1}^{l} (\overline{\alpha}_k - \underline{\alpha}_k)](m+p)} \prod_{r=1}^{m+p-2} |a_{ii}^r| \prod_{r=1}^{m-1} |a_{ii}^r|^{-1} \\
&\le D' \sum_{p=1}^{+\infty} \sum_{l=j}^{i-1} e^{[\overline{\alpha}_i - \underline{\alpha}_j + \sum_{k=j+1}^{l} (\overline{\alpha}_k - \underline{\alpha}_k) + 3\varepsilon](m+p)} | \prod_{r=1}^{m-1} |a_{ii}^r|^{-1}
\end{aligned}$$

for some constant $D' > 0$. Since $\kappa_{ji} < 0$, we have $\beta_{ji} < 0$. Now take $\varepsilon > 0$ such that $\beta_{ij} + 3\varepsilon < 0$. Then

$$B \le D'q \sum_{p=1}^{+\infty} e^{(\beta_{ji} + 3\varepsilon)(m+p)} \prod_{r=1}^{m-1} |a_{ii}^r|^{-1} < \frac{D'q}{1 - e^{\beta_{ji} + 3\varepsilon}} \prod_{r=1}^{m-1} |a_{ii}^r|^{-1} < +\infty. \tag{4.10}$$

This shows that w_{ij}^m is also well defined when $\kappa_{ji} < 0$.

Finally, we establish the estimate in (4.9). First assume that $\kappa_{ji} \ge 0$. It follows from (4.7) that

$$\begin{aligned}
\mu_{ij} &\le \limsup_{m \to +\infty} \frac{1}{m} \log \sum_{p=0}^{m-2} \sum_{l=j}^{i-1} |a_{li}^{m-p-1} w_{lj}^{m-p}| \prod_{r=m-p-1}^{m-1} |a_{ii}^r|^{-1} \\
&\le \limsup_{m \to +\infty} \frac{1}{m} \log \sum_{p=0}^{m-2} \sum_{l=j}^{i-1} D^2 e^{\varepsilon(m-p-1) + [-\underline{\alpha}_j + \sum_{k=j+1}^{l} (\overline{\alpha}_k - \underline{\alpha}_k) + \varepsilon](m-p)} \\
&\quad \times \prod_{r=1}^{m-1} |a_{ii}^r|^{-1} De^{(\overline{\alpha}_i + \varepsilon)(m-p-2)}
\end{aligned}$$

$$
\begin{aligned}
&\le -\underline{\alpha}_i + \limsup_{m\to+\infty} \frac{1}{m} \log\Big(D' e^{3\varepsilon m} \sum_{p=0}^{m-2} e^{\beta_{ji}(m-p)}\Big) \\
&\le -\underline{\alpha}_i + \limsup_{m\to+\infty} \frac{1}{m} \log(m e^{\beta_{ji} m}) + 3\varepsilon \le -\underline{\alpha}_j + \sum_{k=j+1}^{i} (\overline{\alpha}_k - \underline{\alpha}_k) + 3\varepsilon
\end{aligned}
$$

for some constant $D' > 0$. Since ε is arbitrary, this yields inequality (4.9). Now assume that $\kappa_{ji} < 0$, which implies tat $\beta_{ij} < 0$. Take $\varepsilon > 0$ such that $\beta_{ji} + 3\varepsilon < 0$. By (4.10) we have

$$
\begin{aligned}
\mu_{ij} &\le \limsup_{m\to+\infty} \frac{1}{m} \log\Big(D' q \frac{e^{(\beta_{ji}+3\varepsilon)(m+1)}}{1 - e^{\beta_{ji}+3\varepsilon}} \prod_{r=1}^{m-1} |a_{ii}^r|^{-1}\Big) \\
&= \beta_{ji} + 3\varepsilon + \limsup_{m\to+\infty} \frac{1}{m} \log(D e^{(-\underline{\alpha}_i+\varepsilon)m}) \\
&= -\underline{\alpha}_j + \sum_{k=j+1}^{i} (\overline{\alpha}_k - \underline{\alpha}_k) + 3\varepsilon.
\end{aligned}
$$

Again, since ε is arbitrary, this yields inequality (4.9). ☐

Now consider the basis formed by the columns of $W(1)$, that is, the vectors

$$
w_j = W(1) e_j \quad \text{for } j = 1, \dots, q.
$$

By Lemma 4.6 we obtain

$$
\mu(w_j) = \max\big\{\mu_{ij} : 1 \le i \le q\big\} \le -\underline{\alpha}_j + \sum_{k=j+1}^{q} (\overline{\alpha}_k - \underline{\alpha}_k)
$$

for $j = 1, \dots, q$, which completes the proof of the theorem. ☐

Finally, we establish the upper bound for $\gamma(\lambda, \mu)$ using the bases $v_1, \dots, v_q$ and $w_1, \dots, w_q$ constructed in Theorems 4.1 and 4.4.

Theorem 4.7. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of invertible $q \times q$ matrices satisfying* (2.1). *If A_m is upper-triangular for every $m \in \mathbb{N}$, then*

$$
\gamma(\lambda, \mu) \le \sum_{i=1}^{q} (\overline{\alpha}_i - \underline{\alpha}_i). \tag{4.11}
$$

Proof. Let $Z(m)$ and $W(m)$ be the matrices constructed in the proofs of Theorems 4.1 and 4.4 (whose columns are the vectors, respectively, $\mathcal{A}_m v_i$ and $\mathcal{B}_m w_i$ for $i = 1, \dots, q$). By Lemmas 4.2 and 4.5, for each $m \in \mathbb{N}$ we have

$$
\begin{aligned}
W(m+1)^* Z(m+1) &= (B_m W(m))^* A_m Z(m) \\
&= W(m)^* A_m^{-1} A_m Z(m) \\
&= W(m)^* Z(m)
\end{aligned}
$$

and it follows by induction that

$$
W(m)^* Z(m) = W(1)^* Z(1). \tag{4.12}
$$

Since $z_{ij}^1 = 0$ for $i > j$ and $w_{ij}^1 = 0$ for $i < j$, we have

$$
(W(m)^* Z(m))_{ij} = 0 \quad \text{for } i > j,
$$

$$(W(m)^*Z(m))_{ij} = \sum_{l=1}^{q} w_{li}^m z_{lj}^m = \sum_{l=i}^{j} w_{li}^m z_{lj}^m$$

for $i \le j$. In particular, since $z_{ii}^1 = w_{ii}^1 = 1$ for $i = 1, \dots, q$, we have

$$(W(m)^*Z(m))_{ii} = (W(1)^*Z(1))_{ii} = w_{ii}^1 z_{ii}^1 = 1$$

for $i = 1, \dots, q$.

Now take $i < j$. When $\kappa_{ij} \ge 0$, since $\beta_{ij} = \beta_{il} + \beta_{lj}$, we have

$$\begin{aligned}
0 \le \kappa_{ij} &= \beta_{ij} + \max_{i \le k \le j} (\overline{\alpha}_k - \underline{\alpha}_k) \\
&= \beta_{il} + \beta_{lj} + \max_{i \le k \le j} (\overline{\alpha}_k - \underline{\alpha}_k) \\
&\le \beta_{il} + \max_{i \le k \le l} (\overline{\alpha}_k - \underline{\alpha}_k) + \beta_{lj} + \max_{l \le k \le j} (\overline{\alpha}_k - \underline{\alpha}_k) \\
&= \kappa_{il} + \kappa_{lj}.
\end{aligned}$$

Hence, for each l either $\kappa_{il} \ge 0$ or $\kappa_{lj} \ge 0$, and so either $w_{li}^1 = 0$ or $z_{lj}^1 = 0$. Therefore,

$$(W(1)^*Z(1))_{ij} = \sum_{l=i}^{j} w_{li}^1 z_{lj}^1 = 0$$

because each term in the sum vanishes. Finally, when $\kappa_{ij} < 0$, by Lemmas 4.3 and 4.6 we have

$$\begin{aligned}
&\limsup_{m \to \infty} \frac{1}{m} \log |(W(m)^*Z(m))_{ij}| \\
&= \limsup_{m \to \infty} \frac{1}{m} \log \sum_{l=i}^{j} |w_{li}^m z_{lj}^m| \\
&\le \max_{i \le l \le j} \limsup_{m \to \infty} \frac{1}{m} \log |w_{li}^m z_{lj}^m| \le \max_{j \le l \le i} (\mu_{li} + \lambda_{lj}) \\
&\le \max_{i \le l \le j} \Big(-\underline{\alpha}_i + \sum_{k=i+1}^{l} (\overline{\alpha}_k - \underline{\alpha}_k) + \overline{\alpha}_j + \sum_{k=l}^{j-1} (\overline{\alpha}_k - \underline{\alpha}_k) \Big) \\
&= \overline{\alpha}_j - \underline{\alpha}_i + \sum_{k=i+1}^{j-1} (\overline{\alpha}_k - \underline{\alpha}_k) + \max_{i \le l \le j} (\overline{\alpha}_l - \underline{\alpha}_l) \\
&= \beta_{ij} + \max_{i \le l \le j} (\overline{\alpha}_l - \underline{\alpha}_l) = \kappa_{ij} < 0
\end{aligned}$$

and so it follows from (4.12) that

$$(W(1)^*Z(1))_{ij} = \lim_{m \to +\infty} (W(m)^*Z(m))_{ij} = 0.$$

Summing up, $W(1)^*Z(1)$ is the identity matrix and so

$$\langle v_i, w_j \rangle = \langle Z(1)e_i, W(1)e_j \rangle = e_j^* W(1)^* Z(1) e_i = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. In other words, the bases $v_1, \dots, v_q$ and $w_1, \dots, w_q$ are dual. On the other hand, by Lemmas 4.3 and 4.6 we have

$$\lambda(v_j) \le \overline{\alpha}_j + \sum_{k=1}^{j-1} (\overline{\alpha}_k - \underline{\alpha}_k),$$

$$\mu(w_j) \le -\underline{\alpha}_j + \sum_{k=j+1}^{q} (\overline{\alpha}_k - \underline{\alpha}_k).$$

Hence, for $j = 1, \dots, q$ we obtain

$$\lambda(v_j) + \mu(w_j) \le \sum_{k=1}^{q} (\overline{\alpha}_k - \underline{\alpha}_k)$$

and since the bases are dual, inequality (4.11) follows readily from the definition of the Grobman coefficient. □

5. Proof of Theorem 1.2

Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of invertible $q\times q$ upper-triangular matrices whose Lyapunov exponent λ takes only finite values on $\mathbb{R}^q \setminus \{0\}$.

Proof of Theorem 1.2. First assume that $\underline{\alpha}_i = \overline{\alpha}_i$ for $i = 1, \dots, q$. Then it follows readily from Theorem 4.7 that $\gamma(\lambda, \mu) = 0$ and so the sequence of matrices $(A_m)_{m\in\mathbb{N}}$ is regular.

Now assume that the sequence $(A_m)_{m\in\mathbb{N}}$ is regular. As in the proof of Theorem 3.1, let $v_1, \dots, v_q$ be a basis for $\mathbb{R}^q$ such that $v_1, \dots, v_k$ is a basis for E_k (the space generated by the vectors $e_1, \dots, e_k$) for each $k = 1, \dots, q$ that satisfies (3.2) for all bases $v'_1, \dots, v'_q$ for $\mathbb{R}^q$. Then

$$\lim_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_m| = \sum_{i=1}^{q} \lambda(v_i). \tag{5.1}$$

Let $A_{m,k}$ be the $k \times k$ upper-triangular matrix obtained from A_m considering only the first k columns of its first k rows. Taking $k = q - 1$ in (3.3) we obtain

$$\det \mathcal{A}_{m,q-1} = (\det \mathcal{A}_m) \prod_{l=1}^{m-1} (a^l_{qq})^{-1}$$

and so

$$\begin{aligned}
&\liminf_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_{m,q-1}| \\
&= \lim_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_m| + \liminf_{m\to+\infty} \frac{1}{m} \log \prod_{l=1}^{m-1} |a^l_{qq}|^{-1} \\
&= \lim_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_m| - \overline{\alpha}_q.
\end{aligned} \tag{5.2}$$

Since $v_1, \dots, v_{q-1}$ is a basis for E_{q-1}, the vector v_q has a nonzero component along e_q and so the component of $\mathcal{A}_m v_q$ along e_q is equal to $c\prod_{l=1}^{m-1} a^l_{qq}$ for some constant $c \ne 0$ independent of m. This readily implies that $\lambda(v_q) \ge \overline{\alpha}_q$. Combining (5.1) and (5.2), we conclude that

$$\liminf_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_{m,q-1}| = \sum_{i=1}^{q} \lambda(v_i) - \overline{\alpha}_q \ge \sum_{i=1}^{q-1} \lambda(v_i).$$

On other hand, we also have

$$\limsup_{m\to+\infty} \frac{1}{m} \log|\det \mathcal{A}_{m,q-1}| \le \sum_{i=1}^{q-1} \lambda(v_i)$$

and so

$$\lim_{m\to+\infty}\frac{1}{m}\log|\det\mathcal{A}_{m,q-1}|=\sum_{i=1}^{q-1}\lambda(v_i).$$

This shows that the sequence $(A_{m,q-1})_{m\in\mathbb{N}}$ is regular. One can proceed inductively to show that the same happens to each sequence of matrices $(A_{m,k})_{m\in\mathbb{N}}$ and so $\gamma_k(\lambda,\mu)=0$ for $k=1,\dots,q$. Hence, it follows readily from Theorem 3.1 that $\underline{\alpha}_i=\overline{\alpha}_i$ for $i=1,\dots,q$. $\square$

6. Regularity of exterior powers

In this section we consider the non-autonomous dynamics defined by the exterior powers of a tempered sequence of upper-triangular matrices and we obtain a sharp upper bound for its Grobman coefficient.

We first recall some basic notions. For each integer $k\in[1,q]$, let $\Lambda^k(\mathbb{R}^q)$ be the kth exterior power of $\mathbb{R}^q$, that is, the vector space of dimension $\binom{q}{k}$ spanned by the k-vectors $v_1\wedge\cdots\wedge v_k$ with $v_1,\dots,v_k\in\mathbb{R}^q$. We define a scalar product on $\Lambda^k(\mathbb{R}^q)$ by requiring that

$$\langle v_1\wedge\cdots\wedge v_k, w_1\wedge\cdots\wedge w_k\rangle=\det J,$$

where J is the $k\times k$ matrix with entries $b_{ij}=\langle v_i,w_j\rangle$ for $i,j=1,\dots,k$. Any $q\times q$ matrix A induces a linear map $\Lambda^k(A)$ on $\Lambda^k(\mathbb{R}^q)$ by requiring that

$$\Lambda^k(A)(v_1\wedge\cdots\wedge v_k)=Av_1\wedge\cdots\wedge Av_k \tag{6.1}$$

for all $v_1,\dots,v_k\in\mathbb{R}^q$.

Now consider a sequence of invertible $q\times q$ matrices $(A_m)_{m\in\mathbb{N}}$ satisfying property (2.1). The Lyapunov exponent $\lambda_k\colon\Lambda^k(\mathbb{R}^q)\to\mathbb{R}\cup\{-\infty\}$ associated with the sequence $(\Lambda^k(A_m))_{m\in\mathbb{N}}$ is given by

$$\lambda_k(v)=\limsup_{m\to+\infty}\frac{1}{m}\log\|\Lambda^k(\mathcal{A}_m)v\|.$$

Indeed, it follows readily from (6.1) that

$$\Lambda^k(\mathcal{A}_m)=\begin{cases}\Lambda^k(A_{m-1})\cdots\Lambda^k(A_1) & \text{if } m>1,\\ \mathrm{Id} & \text{if } m=1.\end{cases}$$

Moreover, since $\|\Lambda^k(\mathcal{A}_m)\|\le\|\mathcal{A}_m\|^k$, it follows from condition (2.1) that the Lyapunov exponent λ_k does not take infinite values on $\mathbb{R}^q\setminus\{0\}$.

We also consider the sequence of matrices $B_m=(A_m^*)^{-1}$ for $m\in\mathbb{N}$. The Lyapunov exponent $\mu_k\colon\Lambda^k(\mathbb{R}^q)\to\mathbb{R}\cup\{-\infty\}$ associated with the sequence $(\Lambda^k(B_m))_{m\in\mathbb{N}}$ is given by

$$\mu_k(w)=\limsup_{m\to+\infty}\frac{1}{m}\log\|\Lambda^k(\mathcal{B}_m)w\|,$$

where $\mathcal{B}_m=(\mathcal{A}_m^*)^{-1}$ (and condition (2.1) ensures that μ_k does not take infinite values on $\mathbb{R}^q\setminus\{0\}$). The Grobman coefficient of λ_k and μ_k is given by

$$\gamma(\lambda_k,\mu_k)=\min\max\big\{\lambda_k(v_i)+\mu_k(w_i): 1\le i\le q\big\}$$

with the minimum taken over all dual bases $v_1,\dots,v_q$ and $w_1,\dots,w_q$ of $\Lambda^k(\mathbb{R}^q)$.

In the following result we establish an upper bound for $\gamma(\lambda_k,\mu_k)$ when the matrices are upper-triangular. We continue to denote by $\underline{\alpha}_i$ and $\overline{\alpha}_i$ the numbers defined by (3.1) for $i=1,\dots,q$.

Theorem 6.1. *Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of invertible $q\times q$ matrices satisfying* (2.1). *If A_m is upper-triangular for every $m\in\mathbb{N}$, then for every integer $k\in[1,q]$ we have*

$$\gamma(\lambda_k,\mu_k)\le\binom{q-1}{k-1}\sum_{i=1}^{q}(\overline{\alpha}_i-\underline{\alpha}_i). \tag{6.2}$$

Proof. Consider the canonical basis $e_1,\dots,e_q$ for $\mathbb{R}^q$. Then the k-vectors

$$e_{i_1\cdots i_k}=e_{i_1}\wedge\cdots\wedge e_{i_k}\quad\text{with } i_1<\cdots<i_k$$

form a basis for $\Lambda^k(\mathbb{R}^q)$. Note that a matrix A is upper-triangular if and only if Ae_i is a linear combination of the vectors $e_1,\dots,e_i$ for $i=1,\dots,q$. Hence, since A_m is upper-triangular, the k-vector $A_me_{i_1}\wedge\cdots\wedge A_me_{i_k}$ is a linear combination of the k-vectors $e_{j_1\cdots j_k}$ such that

$$j_1<\cdots<j_k\quad\text{and}\quad j_l\le i_l \text{ for } l=1,\dots,k. \tag{6.3}$$

In other words, the linear map $\Lambda^k(A_m)$ is upper-triangular with respect to the basis $e_{i_1\cdots i_k}$ for $\Lambda^k(\mathbb{R}^q)$ ordered so that a k-vector $e_{j_1\cdots j_k}$ comes before another k-vector $e_{i_1\cdots i_k}$ if and only if condition (6.3) holds.

In particular, this implies that the elements on the main diagonal of the linear representation of $\Lambda^k(\mathcal{A}_m)$ with respect to the basis $e_{i_1\cdots i_k}$ are given by

$$\prod_{l=1}^{m-1}a_{i_1i_1}(l)e_{i_1}\wedge\cdots\wedge\prod_{l=1}^{m-1}a_{i_ki_k}(l)e_{i_k}=\prod_{p=1}^{k}\prod_{l=1}^{m-1}a_{i_pi_p}(l)e_{i_1}\wedge\cdots\wedge e_{i_k}.$$

This allows one to apply Theorem 4.7 to deduce that

$$\gamma(\lambda_k,\mu_k)\le\sum_{i_1<\cdots<i_k}(\overline{\alpha}_{i_1\cdots i_k}-\underline{\alpha}_{i_1\cdots i_k}), \tag{6.4}$$

where

$$\overline{\alpha}_{i_1\cdots i_k}=\limsup_{m\to+\infty}\frac{1}{m}\log\prod_{p=1}^{k}\prod_{l=1}^{m-1}a_{i_pi_p}(l),$$
$$\underline{\alpha}_{i_1\cdots i_k}=\liminf_{m\to+\infty}\frac{1}{m}\log\prod_{p=1}^{k}\prod_{l=1}^{m-1}a_{i_pi_p}(l).$$

Observe that

$$\overline{\alpha}_{i_1\cdots i_k}\le\sum_{p=1}^{k}\overline{\alpha}_{i_p}\quad\text{and}\quad\underline{\alpha}_{i_1\cdots i_k}\ge\sum_{p=1}^{k}\underline{\alpha}_{i_p}.$$

Therefore, it follows from (6.4) that

$$\gamma(\lambda_k,\mu_k)\le\sum_{i_1<\cdots<i_k}\Big(\sum_{p=1}^{k}\overline{\alpha}_{i_p}-\sum_{p=1}^{k}\underline{\alpha}_{i_p}\Big)=\sum_{i_1<\cdots<i_k}\sum_{p=1}^{k}(\overline{\alpha}_{i_p}-\underline{\alpha}_{i_p}).$$

The number of summands in the right-hand side is equal to $\binom{q}{k}k=q\binom{q-1}{k-1}$, with each term $\overline{\alpha}_i-\underline{\alpha}_i$ repeated the same number of times for $i=1,\dots,q$. Hence,

$$\sum_{i_1<\cdots<i_k}\sum_{p=1}^{k}(\overline{\alpha}_{i_p}-\underline{\alpha}_{i_p})=\binom{q-1}{k-1}\sum_{i=1}^{q}(\overline{\alpha}_i-\underline{\alpha}_i),$$

which yields inequality (6.2). $\square$

Let $(A_m)_{m\in\mathbb{N}}$ be a tempered sequence of invertible $q\times q$ upper-triangular matrices. It follows from Theorems 1.2 and 6.1 that if the sequence of matrices $(A_m)_{m\in\mathbb{N}}$ is regular than so is the sequence $(\Lambda^k(A_m))_{m\in\mathbb{N}}$ for $k=1,\dots,q$.

7. Regularity of cocycles

In this section we give a simple proof of Oseledets' multiplicative ergodic theorem for cocycles over a measure-preserving transformation. It says that under a certain integrability assumption the set of all regular points for such a cocycle has full measure. The proof is based on the upper bound for the Grobman coefficient established in Theorem 4.7 after making an appropriate upper-triangular reduction of the cocycle.

Let (X,μ) be a Lebesgue probability space, that is, a probability space isomorphic to an interval with the Lebesgue measure together with a finite or countable number of atoms. Moreover, let $f\colon X\to X$ be a measurable transformation preserving μ. This means that

$$\mu(f^{-1}A)=\mu(A)$$

for any measurable set $A\subset X$. We recall that X is metrizable, with the corresponding topology generating the original σ-algebra, and that for each $\varepsilon>0$ there exists a compact set $K\subset X$ with $\mu(K)\ge 1-\varepsilon$. By considering the transformation induced by f on K (which in view of Poincaré's recurrence theorem is well defined), we may assume from the beginning without loss of generality that X is compact.

We denote by GL_q the set of all invertible $q\times q$ matrices with real entries. A measurable function $\mathcal{A}\colon X\times\mathbb{N}\to GL_q$ is called a *cocycle over* f if for every $x\in X$ and $n,m\in\mathbb{N}$ we have

$$\mathcal{A}(x,0)=\mathrm{Id}\quad\text{and}\quad \mathcal{A}(x,n+m)=\mathcal{A}(f^n(x),m)\mathcal{A}(x,n).$$

Given a measurable function $A\colon X\to GL_q$, we define a cocycle over f by

$$\mathcal{A}(x,m)=\begin{cases}A(f^{m-1}(x))\cdots A(x) & \text{if } m>0,\\ \mathrm{Id} & \text{if } m=0.\end{cases}\tag{7.1}$$

Conversely, any cocycle $\mathcal{A}$ can be obtained as in (7.1) taking $A(x)=\mathcal{A}(x,1)$. The function A is called the *generator* of the cocycle $\mathcal{A}$.

Now let $\mathcal{A}\colon X\times\mathbb{N}\to GL_q$ be a cocycle over f. The *Lyapunov exponent* of a pair $(x,v)\in X\times\mathbb{R}^q$ is defined by

$$\lambda(x,v)=\limsup_{m\to+\infty}\frac{1}{m}\log\|\mathcal{A}(x,m)v\|.$$

Consider a point $x\in X$ such that $\lambda(x,v)$ is finite for all $v\ne 0$. We say that x is *(Lyapunov) regular* for $\mathcal{A}$ if the sequence of matrices $(A_m(x))_{m\in\mathbb{N}}$ given by $A_m(x)=A(f^m(x))$ is regular, that is, if and only if (see (2.2))

$$\lim_{m\to+\infty}\frac{1}{m}\log|\det\mathcal{A}(x,m)|=\min\sum_{i=1}^{q}\lambda(x,v_i)$$

with the minimum taken over all bases $v_1,\dots,v_q$ of $\mathbb{R}^q$.

Finally, we give a simple proof of Oseledets' multiplicative ergodic theorem for cocycles over a measure-preserving transformation.

Theorem 7.1 (see [8]). *Let $\mathcal{A}$ be a cocycle over a measure-preserving transformation f. If its generator satisfies $\log^+\|A^{\pm 1}\| \in L^1(X,\mu)$, then the set of all regular points for $\mathcal{A}$ has full μ-measure.*

Proof. Consider the μ-integrable functions

$$\phi^\pm(x) = \log^+\|A(x)^{\pm 1}\|.$$

It follows from Birkhoff's ergodic theorem that the limit

$$\psi^\pm(x) = \lim_{m\to+\infty}\frac{1}{m}\sum_{k=0}^{m-1}\phi^\pm(f^k(x))$$

exists for μ-almost every $x \in X$. Moreover, each $\psi^\pm$ is μ-integrable and

$$\int_X \psi^\pm\,d\mu = \int_X \phi^\pm\,d\mu.$$

In particular, $\psi^\pm$ is finite μ-almost everywhere. Note also that

$$\|v\| \le \|\mathcal{A}(x,m)^{-1}\|\cdot\|\mathcal{A}(x,m)v\|.$$

Therefore,

$$\lambda(x,v) \le \limsup_{m\to+\infty}\frac{1}{m}\log^+\|\mathcal{A}(x,m)\| \le \psi^+(x) < +\infty$$

and

$$\begin{aligned}
\lambda(x,v) &\ge \limsup_{m\to+\infty}\frac{1}{m}\log(\|\mathcal{A}(x,m)^{-1}\|^{-1}) \\
&\ge -\limsup_{m\to+\infty}\frac{1}{m}\log\|\mathcal{A}(x,m)^{-1}\| \\
&\ge -\psi^-(x) > -\infty
\end{aligned}$$

for μ-almost every $x \in X$ and any $v \ne 0$. This allows one to compute the corresponding Grobman coefficient. In addition, we have

$$\frac{1}{m}\sum_{k=0}^{m-1}\phi^\pm(f^k(x)) = \frac{m+1}{m}\cdot\frac{1}{m+1}\sum_{k=0}^{m}\phi^\pm(f^k(x)) - \frac{1}{m}\phi^\pm(f^m(x))$$

and again it follows from Birkhoff's ergodic theorem that

$$\lim_{m\to+\infty}\frac{1}{m}\phi^\pm(f^m(x)) = 0$$

for μ-almost every $x \in X$. In particular, the sequence $(A_m(x))_{m\in\mathbb{N}}$ is tempered for μ-almost every $x \in X$. Therefore, one would be able to apply Theorem 4.7 provided that the matrices $A_m(x)$ were upper-triangular.

Now we make an upper-triangular reduction of the matrices. Let SO_q be the set of orthogonal $q\times q$ matrices and let $Y = X\times SO_q$. Given $(x,U)\in Y$, we apply the Gram–Schmidt process to the columns of $A(x)U$ to obtain

$$A(x)U = V(x,U)B(x,U), \tag{7.2}$$

where $V(x,U)$ is orthogonal and $B(x,U)$ is upper-triangular with positive entries on the main diagonal. Note that these matrices are uniquely defined. Moreover, it follows readily from the steps in the Gram–Schmidt process that both are measurable in (x, U). We consider the measurable linear extension $F\colon Y \to Y$ of the map f defined by

$$F(x, U) = (f(x), V(x, U)).$$

For the projection $\pi\colon (x, U) \mapsto U$, it follows from (7.2) that

$$B(x, U) = \pi(F(x, U))^{-1} A(x) \pi(x, U). \tag{7.3}$$

Let $\bar{\mathcal{A}}$ and $\mathcal{B}$ be the cocycles over F with generators, respectively, $\bar{A}(x, U) = A(x)$ and $B(x, U)$. We have

$$\bar{\mathcal{A}}((x, U), m) = \mathcal{A}(x, m)$$

and it follows from (7.3) that

$$\mathcal{B}((x, U), m) = \pi(F^{m-1}(x, U))^{-1} \mathcal{A}(x, m) \pi(x, U). \tag{7.4}$$

In particular, $\det \mathcal{B}((x, U), m) = \det \mathcal{A}(x, m)$. Moreover, since $\|U\| = 1$ for $U \in SO_q$, it follows from (7.4) that

$$\limsup_{m \to +\infty} \frac{1}{m} \log \|\mathcal{B}((x, U), m) v\| = \lambda(x, v)$$

for all $(x, U) \in Y$ and $v \in \mathbb{R}^q$. This readily implies that given $x \in X$, the sequence $(A_m(x))_{m \in \mathbb{N}}$ is regular if and only if the sequence of upper-triangular matrices

$$B_m(x, U) = B(F^{m-1}(x, U)) \quad \text{for } m \in \mathbb{N} \tag{7.5}$$

is regular for some $U \in SO_q$ (and so for all $U \in SO_q$). By (7.3) we have

$$B_m(x, U) = \pi(F^m(x, U))^{-1} A_m(x) \pi(x, U)$$

and since $(A_m(x))_{m \in \mathbb{N}}$ is tempered for μ-almost every point $x \in X$, the sequence $(B_m(x, U))_{m \in \mathbb{N}}$ is tempered for μ-almost every $x \in X$ and every $U \in SO_q$.

Now let ν be a probability measure on Y such that

$$\nu(B \times SO_q) = \mu(B)$$

for any measurable set $B \subset X$ and define measures

$$\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu \circ F^{-k}$$

for each $n \in \mathbb{N}$. Then

$$\begin{aligned} \nu_n(B \times SO_q) &= \frac{1}{n} \sum_{k=0}^{n-1} \nu(F^{-k}(B \times SO_q)) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \nu(f^{-k} B \times SO_q) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \mu(f^{-k} B) = \mu(B) \end{aligned} \tag{7.6}$$

since the measure μ is f-invariant. Let $\bar{\nu}$ be the weak* limit of some sequence of measures ν_{k_n}. In order to show that $\bar{\nu}$ is F-invariant we proceed as follows. First observe that for any continuous function $\psi\colon Y \to \mathbb{R}$ we have

$$\int_Y \psi \, d\nu_{k_n} \to \int_Y \psi \, d\bar{\nu} \tag{7.7}$$

when $n \to +\infty$ (recall that we are assuming that X is compact and so Y is also compact). Moreover,

$$\int_Y (\psi \circ F)\, d\nu_{k_n} = \int_Y \psi\, d(\nu_{k_n} \circ F^{-1}) = \int_Y \psi\, d\Big(\nu_{k_n} + \frac{\nu \circ F^{-k_n} - \nu}{k_n}\Big)$$

and so

$$\int_Y (\psi \circ F)\, d\nu_{k_n} - \int_Y \psi\, d\nu_{k_n} \to 0 \tag{7.8}$$

when $n \to +\infty$. We will show that

$$\int_Y (\psi \circ F)\, d\nu_{k_n} \to \int_Y (\psi \circ F)\, d\bar\nu \tag{7.9}$$

when $n \to +\infty$. Together with (7.7) and (7.8) this implies that

$$\int_Y (\psi \circ F)\, d\bar\nu = \int_Y \psi\, d\bar\nu$$

for any continuous function $\psi \colon Y \to \mathbb{R}$ and so $\bar\nu$ is F-invariant. Moreover, it follows from (7.6) that

$$\bar\nu(B \times SO_q) = \mu(B) \tag{7.10}$$

for any measurable set $B \subset X$.

To establish (7.9), for each $m \in \mathbb{N}$ consider a compact set $K_m \subset X$ with measure $\mu(K_m) \ge 1 - 1/m$ such that both f and the generator A of the cocycle are continuous on K_m (which by Lusin's theorem is always possible). Then F is continuous on $K'_m := K_m \times SO_q$ and

$$\nu_n(K'_m) = \mu(K_m) \ge 1 - 1/m \tag{7.11}$$

for all $n \in \mathbb{N}$. Let $\psi \colon Y \to \mathbb{R}$ be a continuous function. Then $\kappa = \psi \circ F$ is continuous on K'_m and by Tietze's extension theorem it has a continuous extension $\bar\kappa$ to the whole Y. For each $l \in \mathbb{N}$ we consider the continuous function $\kappa_l \colon Y \to \mathbb{R}$ defined by

$$\kappa_l(x) = \max\big\{\bar\kappa(x) - l \operatorname{dist}(x, K'_m), 0\big\}.$$

Given $\varepsilon > 0$, there exists $l \in \mathbb{N}$ such that

$$\int_Y \kappa_l\, d\bar\nu < \int_{K'_m} \kappa\, d\bar\nu + \varepsilon.$$

Then

$$\limsup_{n\to\infty} \int_{K'_m} \kappa\, d\nu_{k_n} \le \lim_{n\to\infty} \int_Y \kappa_l\, d\nu_{k_n} = \int_Y \kappa_l\, d\bar\nu < \int_{K'_m} \kappa\, d\bar\nu + \varepsilon$$

and it follows from the arbitrariness of ε that

$$\limsup_{n\to\infty} \int_{K'_m} \kappa\, d\nu_{k_n} \le \int_{K'_m} \kappa\, d\bar\nu.$$

Replacing ψ by $-\psi$, we also obtain

$$\liminf_{n\to\infty} \int_{K'_m} \kappa\, d\nu_{k_n} \ge \int_{K'_m} \kappa\, d\bar\nu$$

and so

$$\lim_{n\to\infty} \int_{K'_m} \kappa\, d\nu_{k_n} = \int_{K'_m} \kappa\, d\bar\nu. \tag{7.12}$$

One can now establish (7.9). It follows from (7.11) that

$$\begin{aligned}\Big|\int_Y \kappa\,d\nu_{k_n} - \int_Y \kappa\,d\bar\nu\Big| &\le \Big|\int_Y \kappa\,d\nu_{k_n} - \int_{K'_m} \kappa\,d\nu_{k_n}\Big| + \Big|\int_{K'_m} \kappa\,d\nu_{k_n} - \int_{K'_m} \kappa\,d\bar\nu\Big| \\ &\quad + \Big|\int_{K'_m} \kappa\,d\bar\nu - \int_Y \kappa\,d\bar\nu\Big| \\ &\le \frac{1}{m}\max|\psi| + \Big|\int_{K'_m} \kappa\,d\nu_{k_n} - \int_{K'_m} \kappa\,d\bar\nu\Big| + \frac{1}{m}\max|\psi|.\end{aligned}$$

Hence, by (7.12) we obtain

$$\limsup_{n\to+\infty}\Big|\int_Y \kappa\,d\nu_{k_n} - \int_Y \kappa\,d\bar\nu\Big| \le \frac{2}{m}\max|\psi|$$

and letting $m\to+\infty$ yields property (7.9) holds.

Finally, we establish the regularity of the sequence of matrices $B_m(x,U)$ in (7.5) for μ-almost every $x\in X$ and all $U\in SO_q$. Let $b_{ij}(x,U)$ be the entries of $B(x,U)$. Moreover, for each $i=1,\dots,q$ let

$$\begin{gathered}\underline{\beta}_i(x,U) = \liminf_{m\to+\infty}\frac{1}{m}\log\prod_{l=0}^{m-1} b_{ii}(F^l(x,U)),\\ \overline{\beta}_i(x,U) = \limsup_{m\to+\infty}\frac{1}{m}\log\prod_{l=0}^{m-1} b_{ii}(F^l(x,U)).\end{gathered}$$

Since $\log^+\|A^{\pm1}\|\in L^1(X,\mu)$ and $B(x,U)$ is upper-triangular with positive entries on the main diagonal, the functions $\log b_{ii}$ are $\bar\nu$-integrable and so it follows from Birkhoff's ergodic theorem that

$$\underline{\beta}_i(x,U) = \overline{\beta}_i(x,U) = \lim_{m\to+\infty}\frac{1}{m}\log\prod_{l=0}^{m-1} b_{ii}(F^l(x,U))$$

for $\bar\nu$-almost every $(x,U)\in Y$. By Theorem 4.7, this implies that the tempered sequence of upper-triangular matrices $B_m(x,U)$ is regular for $\bar\nu$-almost every $(x,U)\in Y$. On the other hand, we showed before that $(A_m(x))_{m\in\mathbb{N}}$ is regular if and only if the sequence $B_m(x,U)$ is regular for some $U\in SO_q$. The desired statement follows now readily from (7.10). $\square$

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