REFLECTIVE JOURNALING AS A TOOL TO SUPPORT LEARNING MATHEMATICAL PROOF

by

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DEDICATION

To my parents, who showed me that math is not something to be afraid of, and to my brother Michael, who motivated me to stay interested in math and learn as much as I can. I would also like to dedicate this to my Opa, who saw in me a potential to earn a PhD from an early age.

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DEFINITION OF TERMS

Some of the terms used may have several interpretations and require clarification. Definitions of those terms are provided here to give clarity to the reader.

Journal. For this study, students are asked to write unstructured reflections on their perceptions of the proofs course. Students are also asked to respond weekly to structured prompts about proof and the process of proving. Both the unstructured and structured responses constitute the students' journals, but the two types of journal activities will be classified as separate data types for coding purposes. This will be done because one of the research questions looks at what students choose to write about in their journals, so including the journals where students were told what to write about in the analysis for this question would skew the responses.

Proof. This study will be primarily focused on the process of proving and not the formal proof itself. Therefore, the term proof will be used consistent with Harel and Sowder's (2007) definition of proof as "the process by which an individual (or community) removes doubts about the truth of an assertion" (p. 60). A proof is thus an argument that removes doubts about the truth of an assertion.

Learning. This study will take a constructivist approach to learning. For this study, learning will be defined as the incorporation of knowledge and reorganization of one's cognitive schemes based on experiences. Learning is achieved by students actively constructing knowledge by relating previous and new knowledge and experiences (Emig, 1977).

Views on the functions of proof. This term will be used to represent the students' views on what purpose(s) proof serves and why mathematicians engage in proving. De Villiers' (1990) identified five functions that proof serves in mathematics: verification, explanation, systematization, discovery, and communication.

ABSTRACT

This study investigates how reflective writing supported students' learning to prove in an Introduction to Advanced Mathematics course. The students submitted weekly journal entries that were composed of unstructured prompts and structured, proof-related prompts. Students' reported benefits from the journals were coded according to Borasi and Rose's (1989) framework for student benefits from journaling in mathematics, and students' journals about their proof writing process were coded according to Raman's (2003) ideas about proof writing. In the unstructured journals, students demonstrated primarily therapeutic, problem solving, and content benefits. However, students reported experiencing mostly problem solving and content benefits, as well benefits related to dialoguing with the instructor. A positive and significant correlation was found between the number of journals completed and course grade, which suggests a relationship present between the two. Over half of the students felt the journals influenced their learning to prove by helping them pin down their understandings and write about proof ideas in their own words, which they then connected to the more formal writing in their proofs. There did not seem to be a relationship between the journals and students' views about mathematics, likely because students rarely wrote about their views related to the nature of mathematics or proving

I. INTRODUCTION

Reading and writing proofs are key activities in mathematics, and mathematicians and mathematics educators agree that proof should play a key role in mathematics classes. In fact, the National Council of Teachers of Mathematics[NCTM] (2000) and National Governors Association Center for Best Practices' Common Core State Standards [CCSS] (2010) urge that proof should be part of mathematics curricula beginning in prekindergarten. However, proof is also seen as a challenging activity for students to learn, with evidence that students at all levels have difficulties with proofs (Moore, 1994; Weber, 200; Hoyles & Healy, 1999; Harel & Rabin, 2010; Salazar, 2012).

Although the NCTM Standards (2000) and CCSS (2010) call for proof to be a central aspect of mathematics classrooms, students often have little experience with writing proofs when they enter a university. In high school, students may have experienced proofs in their geometry classes, but generally formal proofs are presented by the teacher and the students often accept the proofs without understanding them. This encourages the students' belief that the teacher is the sole arbiter of correctness (Harel & Rabin, 2010), and discourages students from attempting to write proofs of their own. Students may have written two column proofs or other similarly structured proofs in their geometry class, but these are often to prove theorems that students already believe to be true, thereby promoting the students' view of proof for verification and fail to help students develop a view of proof for discovery, explanation, systematization, and communication (De Villiers, 1990).

At the undergraduate level, researchers have observed that transitioning from school mathematics to proof-based mathematics courses represents a significant challenge for students (Moore, 1994; Tall, 1992; Dreyfus, 1999). Dreyfus (1999) notes that transitioning from Calculus courses to proofs-based courses requires students to transition from thinking computationally about mathematics to thinking about mathematics as a set of intricately related structures. Tall (1992) describes the transition as "from a position where concepts have an intuitive basis founded on experience, to one where they are specified by formal definitions and their properties reconstructed through logical deductions" (p.495). This shift in thinking is not easy for students, who are often answer-centered and view mathematics as a set of rules for calculations. Students exhibit frustration at being asked to prove theorems that seem obvious to them or theorems with proofs provided in their textbook. (Salazar, 2012). At this stage, the students do not see the purpose of proving "already known" facts such as the sum of two even numbers is even. Further, students often fail to connect their personal understanding of a theorem with knowledge of how to construct a formal proof of that theorem (Raman, 2003) and fail to possess strategic knowledge needed to write proofs (Weber, 2001). Mathematics educators also find that students find proofs confusing and unnecessary and often cannot identify correct proofs (Selden & Selden, 2003; Weber, 2010; Salazar, 2012).

As a result of their early interactions with proofs, college students often hold different views of the definition and functions of proof than their instructors and other mathematicians generally do (Weber, 2014; Moore, 1994; Salazar, 2012; Raman, 2001). In traditional proof instruction, students are presented complete proofs, then must learn and memorize the proofs and reproduce them on the exams. However, researchers (Lai & Weber, 2013) found that mathematics professors often value presenting proofs to students using diagrams and featuring main ideas, but this does is necessarily enacted in students' practice. For example, some professors may talk students through a proof, only verbally explaining key ideas or what they consider small details without writing them down; however, when students turn in proofs, the professors expect those ideas to be explicitly written into the proof. Lai and Weber note that this results in students' confusion about what constitutes complete proofs.

As a result of students' struggles learning to proof, many institutions of higher education now offer an undergraduate introduction to advanced mathematics or introduction to mathematical proof course. In this course, students are taught basic logic and proof techniques in the hopes of preparing them for proof writing in upper-level mathematics courses. Though there is considerable evidence about various difficulties students have with reading and writing proofs, there is a lack of research on pedagogical tools to help students learn proofs. Researchers call for a shift away from traditional instruction of proofs where professor presents formal proofs to the students to having students participate in the proving process (Yoo, 2008; Jones, 2000; Blanton, Stylianou, & David, 2009; Dreyfus, 1999).

Somewhere along their educational path, mathematics students, those who continue on to earn graduate degrees and become practitioners in the field, develop proving strategies and appreciation for varying functions of proof. Just where and how this happens remains elusive to researchers. Furthermore, secondary teachers generally do not earn graduate degrees in mathematics, and therefore do not get as many opportunities to develop their beliefs about proof as mathematicians do, which impacts their inclusion of proof in their teaching (Knuth, 2002). Thus, research is needed to investigate how students develop their ideas about proof and the functions that proofs serve in order to provide students with opportunities to develop sophisticated notions of proof and its functions.

My interest in this topic grew out of my own experiences as an undergraduate mathematics major learning to prove. Before my introduction to advanced mathematics course, I had no previous experience with proofs. I had always excelled at mathematics, but struggled when I had to start writing proofs. For me, transitioning from Calculus, which focused on procedures and calculations, to a proofs course, where I was suddenly expected to write correct proofs, proved to be a great challenge. I struggled to make sense of proofs and why they were necessary. I also had trouble using mathematical language and definitions precisely. It wasn't until two years later in my junior year, in a real analysis course, that I started to make progress at proving. My real analysis course was taught using a modified Moore method, and my instructor did an excellent job of helping me to take an active role in my learning of proof. This allowed me to reevaluate my beliefs regarding the nature of proofs and the functions of proofs. Additionally, the course helped me learn to be aware of and communicate my thinking and understanding of a theorem in order to write the proof.

Not only was I getting better at proving, but I was also starting to love proofs. I began seeing the exploration, discovery, and communication functions of proof. Once my view of proofs began changing, I realized how important communication really is in mathematics and how critical it is for students to develop their own personal understandings and views of mathematics and proof.

Later, in my Mathematics Instruction and Assessment course in my doctoral program, I read "Assessing Students' Thinking through Writing" by Jennifer Mayer and

Susan Hillman (1996) about how journaling can be used as an assessment tool in mathematics classes. I was intrigued by Mayer and Hillman's use of journals to create discourse with their students. I have enjoyed keeping journals on a regular basis, but had previously not considered the benefits of journaling in math courses. Further research led me to articles (Borasi & Rose, 1989; Clarke, Waywood, & Stephens, 1993) about how journaling promotes students' learning of mathematics concepts and the development of sophisticated views of mathematics.

In light of these and other articles supporting the use of journaling in mathematics courses, I was surprised to find very little research on how journaling could be used to help students learn to prove. I hypothesize that my initial confusion with learning proofs and trying to understand what they represent and how they are used could have been lessened through journaling. As a result, I decided to conduct my dissertation research on the topic of journaling in an introductory proofs course.

Focus of this Study

A review of the relevant literature revealed students taught proofs using traditional methods often fail to possess a solid understanding of proofs (Harel & Rabin, 2010; Selden & Selden, 2003; Weber, 2010; Raman, 2003; Knuth, 2002; Jones, 2000). Further, there is a gap in the research on students' thinking as they learn to read and write mathematical proofs. That is, the research studies that have been done provide snapshots into students' understandings related to certain topics or aspects of proof, but generally do not provide a longitudinal glimpse of students' development throughout the semester. The focus of this study was to investigate how regular reflective journaling supports students' learning of proofs. The study took place in two sections of an undergraduate introduction to advanced mathematics course at a major public four-year university in Texas.

In this course, students are introduced to advanced mathematical thinking and proving through logic, set theory, number theory, properties of real numbers, and functions. Students also learn proof techniques in this course. Aside from high school geometry experiences in two-column proofs, this course is often students' first formal introduction to mathematical proofs. The prerequisite mathematics courses at the university are Calculus I and Calculus II, neither of which is generally taught emphasizing proofs. The Introduction to Advanced Mathematics course is composed of students majoring in STEM fields – primarily mathematics and applied mathematics. Also in the course are those students majoring in mathematics with a mathematics teaching certification for grades 7-12. This course is a prerequisite for Analysis I, Modern Algebra, and Topology, all of which are required for mathematics majors and require proof writing proficiency.

The study focuses on students' thoughts and experiences as they learned to prove in the course. The study investigated how the students' interpretations of the course and content in the course develop throughout the semester. Additionally, the study focused on the ideas students have about approaching proving and the functions that proofs and proving serve.

Purpose of this Study

The purpose of this study is to examine how the incorporation of regular journaling tasks in an introduction to proofs course enhances the students' learning of proofs. This study investigates how the journaling tasks affected the students' perceptions of proofs and mathematics and their performance in their proof-writing course. Additionally, the students' journals were analyzed to explore the content of the journals and how the students used them. Incorporating journaling into the regular course curriculum has the potential to help students transition from computation-based mathematics to higher-level proofs-based mathematics.

Research Questions

The research questions for this study are as follows:

1. What beneficial themes arise in undergraduate students' journals in an introduction to proofs course? How do students use the journals?
2. How does journaling in an introduction to proofs course affect students' ideas about proof writing?
3. How does journaling in an introduction to proofs course affect students' views on the functions of proof in mathematics?
4. How does journaling in an introduction to proofs course affect students' learning of proof, as demonstrated in their proof writing performance?

Significance of this Study

This study is significant in the fields of mathematics and mathematics education. The journals shed light on the ideas students have regarding proof and their perceptions of their learning of proof. This information will add a new dimension to the literature on how students learn to prove and the thought processes they have during their learning of proof. The journaling tasks used in this survey are easy and inexpensive to implement in the classroom, and may be implemented in classrooms at all levels.

As well as being easy to implement, the journals can provide the instructor with a unique glimpse into their students' thinking about the course and proving, and may help instructors adjust classroom practices according to the needs of the students. The journals also provided an opportunity for future middle and high school mathematics teachers to reflect on their learning of proof, particularly the different functions that proof may serve, something they may otherwise not think about. This may impact their future teaching and help them implement the NCTM and CCSS mathematical practices for K-12, as research (Knuth, 2002) indicates that teachers' beliefs about proof influence their teaching practices associated with proof.

II. LITERATURE REVIEW

The purpose of the study was to investigate the effects of journaling in an introduction to proofs course on students' learning of proofs and their beliefs about mathematics and proofs. In this chapter, existing research related to this study will be presented to form a background for this study. The first section establishes/ highlights the importance of proof writing in mathematical practice and summarizes the research on the teaching and learning of proof at the undergraduate level. The second section will look at research on *Writing to Learn*, particularly journaling. This section will discuss general research on journaling, and will also present research on journaling and reflection in mathematics courses.

Proof in Mathematics

In this section on mathematical proof, I provide a brief historical context regarding mathematical proof – its definition and importance to the field. Next, I discuss research on proof instruction, demonstrating a need for research on innovative pedagogical approaches to teaching proof. Finally, I discuss students' experiences with learning proof, specifically their proof writing performance and views about proof. This section will attempt to synthesize research on the development of proof writing ability, and show that there is still much to learn on that topic.

Prior to the 20th century, most mathematicians held fixed, absolute views of mathematics (Yoo, 2008; Harel and Sowder, 2007; Weber, 2014). Mathematics was seen as a "static but coherent body of absolute truths" (Yoo, 2008, p. 8). Yoo explains that, with this viewpoint, a proof was seen as a method to verify the truth of a statement within a formal system, with emphasis placed on formal proof, the use of the axiomatic method, and precision with definitions. In this ideology, a proof is a formal object (Weber, 2014). Mathematics was seen as independent of human construction, and therefore proofs were independent of human or social context.

However, mathematicians and mathematics educators have increasingly argued against this static view of mathematics, claiming that mathematics is indeed social, and that proofs are dependent on a community of practitioners (Weber, 2014; Wheeler, 1990; Raman, 2003; Harel & Sowder, 2007). Mathematicians began viewing mathematics as a human activity driven by the needs of everyday life and science, with mathematical results up for revision as mathematics develops (Yoo, 2008). Further, Weber (2014) argues that viewing proof as a formal object, while useful in identifying proofs, is inconsistent with practices of mathematicians when writing proofs. In an attempt to classify proofs consistently with the mathematical practice, Harel and Sowder (2007) distinguish between ascertaining (proving something to oneself) and persuading (proving something to someone else). Harel and Sowder define a proof as an argument that convinces the truthfulness of a statement to a person or community. The rise of deductive logic in proof, Harel and Sowder contend, arose as a consequence of the changing views of mathematics and the property that proofs must convince the mathematical community.

Though there is still no consensus among mathematicians as to what exactly constitutes a proof, there is increasingly agreement that proofs and proving serve numerous purposes, not just verifying the truthfulness of an assertion (De Villiers, 1990; Hanna, 2000). Proofs can be used for verification, explanation, systematization, discovery, communication, construction, exploration, and incorporation (Hanna, 2000).

This fluid, broad view of proof allows for proof to be incorporated into mathematics classrooms at all levels in different forms.

Proof Instruction

Stylianides (2007) discusses the state of proof in the elementary through secondary grades, claiming that although standards documents call for proof and justification throughout mathematics curricula, there is not comprehensive research on conceptual or instructional issues of proof in K-12 mathematics. Stylianides explains that teachers' conceptions of proof have been shown to affect their opinions about the standards documents promoting proof and their teaching approaches to proof. Further, there is not a unified conception of the notion of proof for school mathematics. To remedy this, Stylianides provides a framework for conceptualizing proof in school mathematics. Proof, as defined by Stylianides, is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the characteristics:

> 1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;
> 2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of the classroom community

This definition, Stylianides claims, seeks to balance the considerations of mathematics as

a discipline and students as mathematical learners. Stylianides' goal is for this framework to serve as a fluid conceptualization of proof that can be adjusted based on the classroom community and students' levels of mathematical learning.

Though Stylanides provides a framework for proof in elementary through secondary mathematics education, proof remains a secondary concern for teachers in those grades (Knuth, 2002). When proof is present in secondary education, it is generally in the form of two-column proofs in high-school geometry courses. Thus, many students begin university studies with little experience in proving.

In traditional undergraduate proof instruction, students are presented formal proofs by the instructor and then must re-create the proofs on exams. Students may also be expected to write new proofs – proofs not presented to them by the instructor - on exams. Researchers (Lai & Weber, 2013; Yoo, 2008; Jones, 2000; Blanton, Stylianou, & David, 2009) have noted numerous issues with this type of instruction.

Lai and Weber (2013) conducted a study in which they observed 10 university mathematics professors constructing and revising pedagogical proofs, proofs intended to be used as a teaching tool, and subsequently interviewed them about the proofs. The interviews revealed that most professors believed that pedagogical proofs depend on the audience, and that eight of the ten professors described ways in which they would modify the proof or emphasize certain points depending on the ability level of the audience. Five of the professors mentioned that the level of rigor of the proof they presented would depend on the audience, and that they would leave out some logical details for advanced audiences. The professors also mentioned that they would use diagrams and stress main ideas of the proofs in their instruction. However, Lai and Weber observed discrepancies between the pedagogical considerations that the professors mentioned in their interviews and the professors' proof write-ups. Instead, most of the written proofs produced were formal proofs lacking notes on how they would be implemented for different audiences. Lai and Weber conclude that some mathematics professors often value pedagogical proofs with diagrams and featuring main ideas, but this does not necessarily translate into the way they teach proofs. It is important to note, however, that the professors in the study were not observed teaching the proofs, so Lai and Weber's conclusion is based on speculation from the written pedagogical proofs.

However, Lai and Weber's (2013) findings are consistent with the results of Alcock's (2010) study of five university mathematics professors' views about the teaching and learning of proof. Based on in-depth interviews with each professor, Alcock identified four modes of thinking of successful provers identified by the professors: instantiation, which aims "to meaningfully understand a mathematical statement by thinking about the objects to which it applies" (p. 78); structural thinking, which aims "to generate a proof for a statement by using its formal structure" (p. 78); creative thinking, which involves examining "instantiations of mathematical objects in order to identify a property or set of manipulations that can form the crux of a proof" (p. 78); and critical thinking, which aims "to check the correctness of assertions" (p. 78). However, when discussing their teaching strategies, the professors described few teaching strategies directly relating to instantiation or creative thinking. Instead, most of the strategies described by the professors were aimed at structural thinking or critical thinking. This suggests a disconnect between what professors claim to value in regards to proving and the strategies they use to teach proving.

As a result of students' poor performance with proofs and issues identified by empirical research, researchers increasingly call for a shift away from traditional instruction of proofs where students are presented formal proofs to having students participate in the proving process (Yoo, 2008; Jones, 2000; Blanton, Stylianou, & David, 2009)

Blanton, Stylianou, and David (2009) discuss classroom discourse as a method for teaching students to prove. The researchers provide a framework for assessing classroom discourse and how student and teacher utterances scaffold student learning of proof. The framework was developed based on a 1-year teaching experiment in a university discrete mathematics course. During the course, the teacher/researcher actively created a classroom culture supporting daily discourse and small group construction of proofs. Based on classroom observations, the researchers categorized the different types of utterances of the teacher and students and concluded that the classroom discourse in proof writing served as a powerful tool for helping students learn to prove. However, they did not measure students' performance, so their conclusion is based on observations of students discussing and presenting proofs.

Stylanides and Stylanides (2009) provide another example of an innovative instructional sequence to help students learn about proofs, specifically aimed at helping students realize a need for proofs. Stylianides and Stylanides begin by lamenting the fact that students often do not see the need for proof or recognize that empirical arguments are not proofs. To address this, the researchers developed an instructional sequence during a four-year design experiment in an undergraduate mathematics course that serves as a prerequisite course for the elementary teacher certification program. The experiment consisted of 5 experiment cycles of enactment, analysis, and refinement that included eight enactments of the course. In the instructional sequence, problems were carefully chosen with examples and counterexamples that would challenge the students' unsophisticated notions of examples and counterexamples in proving, leading the students to re-conceptualize their notions of proving. Rather than explaining the solutions to the problems to the students and giving them a proof of the solution, the instructors had the students work on the problems together and discuss their results. Stylanides and Stylanides claim that this instructional sequence can be modified for other populations of students with similar positive learning outcomes.

Blanton, Stylianou, and David's (2009) and Stylanides and Stylanides' (2009) examples of innovative proof instructional techniques show how research on students' learning of proof can be incorporated into proof pedagogy. In both examples, students were active learners of proof, and the instructor supported their learning through their choice of problems and through the use of classroom discourse. However, more research is needed to explore these and other proof instructional methods that support students' development of comprehensive proof understandings.

Learning of Proof

Numerous researchers (Weber, 2001; Raman, 2003; Harel & Sowder, 2007) have shown that developing proof competence presents a great challenge to students. Much of the research discusses student deficits in proof competence; for example, students have trouble following or constructing logical, deductive arguments, differentiating deductive arguments from empirical evidence or examples, and using deductive arguments to derive additional results (Healy & Hoyles, 2000; Weber, 2001). Researchers often note that these deficiencies stem from the students failing to have a comprehensive understanding of what it means to prove in mathematics or what a mathematical proof is (Weber, 2001; Moore, 1994). However, there is less research on what students understand and do well with respect to learning proof, and how students develop comprehensive proof understandings. Dreyfus (1999) notes that transitioning from Calculus courses to proofs-based courses requires students to transition from thinking computationally about mathematics to thinking about mathematics as a set of intricately related structures, but exactly how this done has not yet been extensively explored by researchers.

In high school, students often develop the authoritative proof scheme (Harel & Sowder, 2007; Martin & McCrone, 2004) in which students accept proofs as true based on the authority of the presenter rather than the content or form of the proof. In 2010, Harel and Rabin observed two high school algebra classes in order to study how the teachers presented proofs to the class to explore how teachers inadvertently support students' development of authoritative proof schemes. Harel and Rabin noted the high occurrences of the initiate, respond, evaluate [IRE] model of instruction in the classes, and observed that this reinforced the students' belief that the teacher is the sole arbiter of correctness. If students believe that it is the teacher who decides the correctness of a proof, then the students are lacking a comprehensive understanding of what it means to prove in mathematics or what a mathematical proof is.

This incorrect notion of proof appears to stay with students as they transition into undergraduate mathematics courses. Weber (2001) explored undergraduate students' conceptions of proof and how these might affect the students' proof writing. Weber begins by citing literature on challenges faced by students in regard to proof, and describes two main areas of difficulty: first, students do not have an accurate conception of mathematical proof, and second, students may lack understandings about a theorem or concept and in turn misapply it. However, Weber notes that there is a large class of failed proof attempts that were not caused by one of the two difficulties identified by the research.

To investigate whether a student who knew what proof was, could reason logically, and apply the facts and concepts could correctly construct a proof, Weber (2001) observed 4 undergraduate and 4 advanced doctoral students proving statements about abstract algebra. The researcher used verbal protocol analysis to observe the participants 'thinking aloud' as they attempted to prove five theorems. Weber also assessed the participants' factual knowledge by providing them with a list of group theory statements to categorize as true or false. Doctoral students proved 90% of the propositions whereas undergraduates proved only 30%. Weber determined that in 57% of the undergraduates' failed proof attempts, they failed to construct their proof because they did not apply their syntactic knowledge. The results of the study indicate that an understanding of mathematical proof and syntactic knowledge were are not sufficient to competently write proofs. Interpreting these results, Weber identified four types of strategic knowledge that doctoral students appeared to possess and the undergraduates appeared to lack: knowledge of the domain's proof techniques, knowledge of which theorems are important, when the important theorems will be useful, and knowledge of when and when not to use "syntactic" strategies. Future research with more participants and within other mathematical domains is needed on strategic knowledge to investigate how students develop efficiency.

Hoyles and Healy (2000) surveyed students' proof conceptions in algebra to see what the students viewed as correct proofs. The researchers quantitatively surveyed high-achieving 14- and 15-year olds in algebra, asking them to score given proofs, identify proofs that best represent their method for arguing a conclusion, as well as identify proofs that were most likely to get a high score by the teacher. In addition, students were asked to construct proofs for the researchers. Descriptive statistics based on frequency tables, simple correlations, and tests of significance, as well as multilevel modelling were employed by the researchers to analyze the results of the surveys. Hoyles and Healy observed that students overwhelmingly differentiated between the proof that represented their method for arguing a task and the proof that would be given high marks by the teacher. The results also suggest that students are better at identifying correct proofs than constructing them. In their proofs, only 40% of students used some deductive reasoning when attempting to prove a familiar theorem (22% completely correct and 18% partial proofs), and only 3% were able to construct a complete proof of an unfamiliar theorem. In addition, students predominantly provided empirical evidence in in their proofs.

The difficulties with validating proofs do not only occur with 14- and 15- year olds, but extend to undergraduate students as well. Selden and Selden (2010) interviewed eight undergraduate mathematics majors and asked them to read and judge the correctness of four proofs twice, and then asked them 8 general questions about how the students read, understand, and validate proofs. Overall, the students had trouble validating a proof's correctness. Upon first reading, 46% of the 8 participants' validations were correct. After they were encouraged to reflect further on their validations and the proofs, the percentage of correct validations rose to 81%. This suggests the large role that reflection has in learning proofs even though the authors note that students are often not directly encouraged to reflect. This oversight could be hurting students' performance.

In addition to struggling to identify correct proofs, undergraduate students often fail to connect their personal understanding of a theorem with knowledge of how to construct a formal proof of that theorem (Raman, 2003). Raman interviewed a purposive sample of 11 undergraduate mathematics students, 4 graduate mathematics students, and 5 university mathematics professors to investigate how they approached proof writing. In the interviews, the participants described their thinking as they wrote a proof that the derivative of an even function is odd and then were asked to evaluate 5 proofs of the statement for their clarity and completeness. Upon analyzing the interview responses for the types of approaches to proof-writing used, Raman (2001) found that university professors and mathematics students often think about the public and private aspects of proof differently. For the university professors and mathematicians, private aspects of proving (ascertaining) are deeply linked with public aspects (persuading). For them, the act of ascertaining often provides the basis for them to persuade. However, students tend to view the two as separate. In her study, Raman found that the students believed proofs were created from nothing. In other words, students did not see the ascertaining, private side of creating a proof. They focused solely on trying to create a mathematically correct proof and failed to see a connection between their private understandings of the idea and what they viewed as a formal proof.

In order to help students learn to prove, research is needed that focuses not only on what students cannot do when learning to prove, but how students' thinking about proof develops. Moore (1994) conducted a study that consisted of nonparticipant observations, interviews with the professor and students, and tutorial sessions with the 16 students in an introduction to advanced mathematics course to see what difficulties arose in the students' learning to prove. In the study, Moore attempted to shed light on students' difficulties with the transition to learning to prove. Moore noted that may of the students' difficulties learning proof were cognitive, and he found seven major sources of difficulties in doing proofs:

> DI. The students did not know the definitions. That is, they were unable to state the definitions.
>
> D2. The students had little intuitive understanding of the concepts.
>
> D3. The students' concept images were inadequate for doing the proofs.
>
> D4. The students were unable, or unwilling, to generate and use their own examples.
>
> D5. The students did not know how to use definitions to obtain the overall structure of proofs.
>
> D6. The students were unable to understand and use mathematical language and notation.
>
> D7. The students did not know how to begin proofs.

Moore also noted that students' perceptions of mathematics and proof influenced the students' proof writing and sometimes hindered the students' success. Moore distinguishes between a concept image (Vinner, 1983), concept definition, concept usage, and concept-understanding scheme to clarify how one must understand mathematical concepts in order to use them in proofs. Students had difficulties with definitions partly because they did not sense the importance of precise definitions in mathematics and proof. The research revealed that both a concept image and concept definition were required for students to write correct proofs.

Moore also noted the importance of language in mathematics, and the difficulty that arises because of it, observing that language is fundamental to introducing students to new ideas and definitions, yet mathematical language is often an obstacle to students' understanding.

Students' Views on Proof

Researchers have observed that, in addition to struggling to learn to read and write correct proofs, college students often hold different views of the definition and functions of proof than their instructors, mathematicians, do (Weber, 2014; Moore, 1994).

Mejia –Ramos and Weber (2014) quantitatively surveyed 118 practicing mathematicians at universities in the USA about why and how they read published proofs, and found that mathematicians do not just read to check the correctness of the proof. Instead, mathematicians read proofs to gain insight in proof methods that may be applicable to their own work. Second, most (91%) of the participants said they try and understand the proof in terms of its main ideas and overall methods (77%), and in fact many mathematicians (73%) claimed that understanding the methods is sometimes sufficient to judge the correctness of the proof. Also, participants said they often try to understand the main idea of the proof before line-by-line reading for errors. It is also noteworthy that 56 % of participants said that empirical evidence is sometimes enough to accept a claim as true, highlighting the importance of examples for understanding and validating claims and illustrating why professors often show examples to students when presenting proofs. This induces confusion, however, as many students struggle when validating proofs (Selden and Selden, 2003) and confuse examples with proofs (Healy and Hoyles, 2000).

Instead of viewing proofs as a way to gain insight or to understand main ideas like professors do, many undergraduate students see proofs as static, tricky, and confusing (Raman, 2003; Salazar, 2012; Moore, 1994). In her 2001 and 2003 works, Raman interviewed university mathematics students and had them "think aloud" as they attempted to write proofs. She observed that many students described proving as formal, not relying on intuition and informal understandings, and even thought that proving involves creating something out of nothing. The students in Raman's studies did not see a connection between their privately held understandings of how to prove the theorems and how to construct the written proof.

Weber (2014) notes the disconnect between students' and instructors' views on the functions of proof. He categorizes proofs into four types, based on the function they represent: proofs that provide knowledge about mathematical truth, proofs that justify the use of terminology, proofs that illustrate technique, and proofs that justify the use of a definition or axiomatic structure. Weber claims that instructors of advanced mathematics courses often treat all proofs the same, and that they should instead be explicitly aware of the purpose each proof they present to their students serves. Weber concludes that by being aware of and explicit with students about the purpose of proofs presented can help students recognize and appreciate the broader purposes of proof besides just convincing.

In addition to undergraduate students holding views about proof inconsistent with those of practicing mathematicians, it appears that secondary mathematics teachers' views of proof are also inconsistent with those of mathematicians. Eric Knuth (2000) conducted semi-structured and task-based interviews with sixteen in-service secondary mathematics teachers with varying teaching experience in the USA in order to clarify the teachers' conceptions of proof. The interviews were coded using an analytical-inductive method. In contrast to mathematicians' views, none of the teachers interviewed by Knuth viewed the role of proof as promoting understanding, and only half of the teachers viewed proof as a means of creating knowledge or systematization. Further, in the task where the teachers validated whether statements were proofs or nonproofs, every teacher rated at least one of the nonproofs as a proof, with 11 teachers rating multiple nonproofs as proofs. Five teachers rated the empirically based argument as a proof, and the teachers overall "focused on the correctness of the manipulations performed in the argument as opposed to the nature of the argument itself" (p. 394). Knuth's research suggests that secondary teachers often have narrower, and sometimes incorrect, views on the functions and the definition of proof than mathematicians.

It is not surprising, then, that secondary students and, later, undergraduate students struggle with developing sophisticated notions of proof. In fact, the difficulties they face when learning to prove often cause undergraduate mathematics students to express a dislike of proofs and feel anxiety at the thought of writing proofs (Salazar, 2012; Raman, 2003).

In this section of the literature review on proof, I have attempted to demonstrate that, although a critical mathematical practice, proof is difficult to learn; traditional methods of proof instruction are ineffective at helping students develop advanced conceptions of proof; and research on innovative proof instruction techniques, as well as research on the processes by which students develop sophisticated conceptions of proof, is needed to fill holes in the literature on proof in mathematics education. I will next discuss the w*riting to learn* movement, how it has been incorporated into mathematics education, and how it may be a promising tool for the teaching and learning of proof.

Writing to Learn

In this section, I provide a historic overview of the development of the *writing to learn* movement and its theoretical components. I then look at what research has been done on w*riting to learn* in mathematics. Finally, because of a lack of research available on *writing to learn* in mathematical proof, I explore the literature on student reflection in the learning of mathematical proof and demonstrate that reflective practices have been shown to improve students' learning of proof.

History of Writing to Learn

Starting in the 1970s, educational researchers are increasingly exploring how writing could be used as a pedagogical tool for supporting learning (Bazerman, Little, Bethel, Chavkin, Fouqette, & Garufis, 2005; Loud, 1999), citing the theories of Vygotsky and Emig as motivation for their work. The movement emphasizes that *writing to learn* is more than *learning to write*; *writing to learn* is not about learning the mechanics of writing or how to write in a certain domain, and instead focuses on the importance of language in knowledge acquisition (Loud, 1999). The movement is based on the idea that students' thinking and understandings can develop and achieve clarity through the process of writing their thoughts and observations. Proponents of the *writing to learn* movement recognized that writing is present in all domains; writing is a method of learning, not just a mode of communication; and writing is a complex process that needs to be developed and practiced. As such, writing can provide a unique opportunity for learning (Emig, 1977).

James Britton and Janet Emig are the two pioneering educators of the pedagogical approach to w*riting to learn* (Bazerman et al., 2005). Britton was an educator in Britain in the 1960s and developed the British model of language instruction, which was distinct from the American model that emphasized rigor, standard curricula, and standardized, objective assessment (Bazerman et al., 2005). Britton et al. (1975) analyzed the writing of over 2,000 British secondary students aged 11-18 to illuminate the kinds of writing students were being asked to do in British secondary schools and identified three functional modes of writing: transactional, poetic, and expressive. In transactional writing, Britton and his colleagues explained, one is writing to communicate information, to inform, or to persuade, whereas poetic writing is creative writing intended to create beautiful objects such as stories. The third type of writing, expressive writing, is for exploring and reflecting one's ideas and is intended for one's own use; this is the type of writing that *writing to learn* proponents focus on. In Britton et al.'s study, they found that only 4% of the writing assigned was expressive, which showed that writing was used primarily to demonstrate learning and not to support learning. Janet Emig (1977) expanded on Britton's theory, noting that because writing is connective, active, and requires other neurological processes than those used in talking, writing represents its own unique mode for learning. Britton et al. (1975) and their counterparts (Emig, 1977) likened this expressive writing to Vygotsky's notion of inner speak, and claimed it could be useful in every developmental stage of learning.

The *writing to learn* movement continued to gain research traction in the United States, with studies exploring how writing appeared in US classrooms. Similar to Britton et al.'s (1975) claim that the vast majority of the writing done in British school classrooms was transactional, Applebee (1992) found that much of the writing being done in US school classrooms was writing to demonstrate understanding, not writing to gain understanding.

Given these results, researchers began investigating the different ways that writing can be used in the classroom to promote learning. One strand of research focused on note taking (Bazerman et al., 2005), with numerous researchers (Di Vesta & Gray, 1972, as cited in Bazerman et al., 2005; Kulhavy, Dyer, & Silver, 1975) finding note taking to be a more effective study strategy than only reading or listening. Another strand of research investigated student-written summaries of reading passages, and found that students who created their own summaries or analogies learned significantly more of the textual material than did the students who only read the text (Loud, 1999). Researchers also investigated writing to learn at the undergraduate level, and found that the results of student-written summaries versus only reading the text carried over to college level students (Wittrock & Alseandrini, 1990, as cited in Loud, 1999).

One criticism of research in *writing to learn*, however, is that researchers were using widely varying writing tasks and measures of learning (Penrose, 1992). To explore the effects of the writing task and learning measure on the research results, Penrose conducted a study in a freshman English class. The 40 students were split into two groups: one group was assigned to "study for a test" over the reading and the other group was told to "write a report" about the reading (a scientific passage about hurricanes); students in both groups were asked to think-aloud as they completed their writing tasks. It is important to note that the "write a report" group was not told they were going to be tested over the reading. The students then took a comprehension test over the reading they completed. The think-aloud transcripts were analyzed using a coding scheme of eight cognitive operation categories derived from the data, and the comprehension tests were scored based on the sorting of each question based on the nature of the question. The experiment was then repeated with the same groups of students, but the students switched assignments, this time reading a passage about philosophy. Altogether, each student read two passages and completed the "write a report" task and the "study for a test" task. Each student took two comprehension tests, and completed two think-aloud activities.

Data analysis revealed that the report-writing group scored lower than the study group on only two of the four comprehension measures. This may be explained by the fact that the report-writing group did not know they were going to be tested and the study group did. Also, Penrose noted that the "cognitive operations students engaged in were determined more by students' interpretation of the tasks than by the tasks themselves" (p. 489). Further, Penrose found that the within task differences in cognitive operations were as large as the differences between tasks. These results are significant because they illuminate that writing is a complex, individual process, and that people have different experiences and benefits from writing to learn depending on not only the type of task, but the individual's interpretation of the task. Despite this challenge, researchers continued to explore and promote *writing to learn* because of its ability to help students discover and discuss connections and relationships during their learning process, even if individual students experienced writing to learn differently.

Writing to Learn in Mathematics

As a result of the promising results *of writing to learn* in other disciplines, primarily English and reading, researchers began to wonder how writing to learn may be used in other fields. In mathematics, *writing to learn* grew alongside reform movements led by National Council of Teachers of Mathematics in the 1980s (Loud, 1999). In their 1989 document, *Curriculum and Evaluation Standards for School Mathematics*, the NCTM called for more emphasis on conceptual understanding, problem solving, and constructivist-based theories on how children learn. As mathematics educators considered how to meet the NCTM's call, they began incorporating *writing to learn* into mathematics curricula. This inclusion of writing in mathematics classes overwhelmingly took the form of journal writing (Clarke, Waywood, & Stephens, 1993; Santos & Semana, 2014; Powell, 1997; Borasi & Rose, 1989; Hari, 2002), with researchers reporting positive results in students' performance, attitudes, and beliefs about mathematics.

At the secondary mathematics level, Clarke, Waywood, and Stephens (1993) stress the importance of communication in mathematics and discuss the need for a language for internalization of mathematics in order for learners to make meaning. To investigate how the development of this language for internalization can be facilitated, Clarke et al. studied a Catholic secondary girls' school of 500 students in which journaling was implemented as a regular activity in all mathematics classrooms. Starting in grade 7, the students were given journals with the sections what we did, what we learned, and examples and questions, and were expected to write every day after class.

Clarke et al. developed three questionnaires and administered them to all the students at the school: the "Mathematics" questionnaire attempted to measure students' perceptions of school mathematics and the field of mathematics, the "Journals- Part A" and "Journals- Part B" surveys focused on the students' perceptions of the use, purposes, difficulties, values, and teacher actions related to journaling. The surveys were analyzed to reveal that the majority (54 percent) of students reported writing after every lesson and seventy-five percent wrote at least twice per week. The texts of the journals were also analyzed and revealed three categories (in order of sophistication) of journal use: Recount, in which students list events from class; Summary, in which students discuss and summarize content topics; and Dialogue, in which students examine their own perceptions and beliefs regarding mathematics and learning. Clarke et al. did notice, however, that less than half of the students predominantly employed one category of writing; instead, the students employed multiple modes of journal use. However, regression analysis revealed that the more sophisticated the mode of journal use by a student, the more likely he or she was to use the journal, find journaling easy and enjoyable, and express higher appreciation for journal completion.

In 1989, Borasi and Rose investigated the educational value of journaling in an undergraduate algebra course and found that the journals provided the students an outlet to express and reflect on their feelings about mathematics and the course and their knowledge of mathematical processes and the curricular content of the course. The 29 student participants wrote at least three open-ended journal entries per week, and the professor read and responded to the students' entries every weekend.

Borasi and Rose (1989) qualitatively analyzed the journals to see what the students were writing about and how they were using the journals. Borasi and Rose noted four benefits to the journaling that arose from themes in students' use of the journals: a therapeutic effect as students wrote about their feelings related towards mathematics and the course; increased content knowledge as students wrote about the course material; improved problem-solving and learning skills as students reflected on their own process of learning and doing mathematics; and shifts towards a more “appropriate view” (p. 352) of mathematics as students became aware of and evaluated their beliefs. The researchers state that students were not likely to write about course content or their views of mathematics unless prompted, and suggested that teachers give students prompts to write about these topics. This study and that by Clarke et al. (1993) are both limited, however, by the fact that the researchers did not measure achievement gains of the students as a result of the journals.

To investigate how journaling in an undergraduate mathematics course affects performance, Loud (1999) studied an applications-based college mathematics course. In the comparison study, students in the comparison group completed weekly structured journal writing assignments while students in the control group did not. A beliefs survey was administered to both groups, and a common final exam was used to measure performance in the course. The students in the experimental group performed significantly higher on the final exam than the students in the control group did. Additionally, beliefs and attitudes about mathematics improved significantly for students in the experimental group, but not students in the control group. Loud also found that the more complex the writing tasks, in which the students explained mathematical concepts and documented solutions to multiple step problems, the more detailed explanations that they were able to give. This is consistent with Clarke et al.'s (1993) finding that the more sophisticated students' writing, the larger the effect of the performance gains.

In addition to research studies highlighting the benefits of journaling in mathematics courses as a form of *writing to learn*, there are numerous practitioners' articles about journaling as a tool for learning mathematics (Vacaretu, 2008; Miller, 1992; Mayer & Hillman, 1996; Casler- Failing, 2013). These experienced classroom teachers give suggestions for practice based on anecdotal evidence from years of classroom implementation. The suggestions include having students write and critique journal entries about their problem solving processes (Vacaretu, 2008), beginning each class meeting with a five-minute writing assignment (Miller, 1992), having students turn in structured, written reports after group problem solving sessions (Mayer & Hillman, 1996), and having students submit weekly, unstructured journal assignments (Casler-Failing, 2013).

Along with studies focusing on the use of journals, research has also been done on *writing to learn* in mathematics using other methodologies of writing. Lesnak (1989) conducted a comparative study between four sections of a remedial algebra course in a liberal arts college in the US. Two sections were randomly assigned to the control, and the other two were the experimental group. The only difference in instruction was that the experimental group students were asked to write step-by-step explanations of how to solve problems. The experimental group had significantly higher test averages than the control group, and all 52 students in the experimental group gave positive reviews of the *writing to learn* algebra course, stating that it increased their confidence and gave them more positive attitudes towards algebra.

In 2014, Santos and Semana studied students' expository writings, which were similar to a written report of a problem solving session. In their qualitative study, Santos and Semana observed four eighth grade mathematics students in Portugal as they developed three expository writing assignments, analyzed their expository writings, and conducted and analyzed interviews with the students. Santos and Semana observed improvements in justifications and explanations between each submitted writing assignments, and noticed a decrease in the use of vague statements, rules, or procedural descriptions. The authors conclude that the expository writing helped the students develop more sophisticated justifications and mathematical language.

Proponents of *writing to learn* in mathematics argue that there is a difference between *writing to learn* and writing to demonstrate learning (Champman, 2002). In *writing to learn*, writing is used as a tool to develop understanding and support learning, whereas in writing to demonstrate learning, writing takes place after the learning has occurred. Another component of *writing to learn* in mathematics is that writing about mathematics helps students learn mathematical vocabulary and language (Thompson & Rubenstein, 2000). The main theoretical underpinnings of *writing to learn* in mathematics will be discussed in depth in the theoretical framework section of this proposal.

Reflection as a Tool for Learning Mathematical Proof

Though *writing to learn* has become increasingly popular in mathematics education, there is a lack of research studies done on *writing to learn* in advanced mathematics and the learning of proof. However, there have been studies done on reflection in various forms as a tool for learning proof (Blanton & Stylianou, 2014; Hodds, Alcock, & Inglis, 2014). These will be discussed in this section of the literature review to show that providing students opportunities for reflection in proofs-based mathematics courses is a promising instructional tool.

Hodds, Alcock, and Ingils (2014) explored the idea of using self-reflection (in the form of self-explanation training) as a tool for improving students' proof comprehension in a quasi-experimental study. The authors summarize the research on students' proof comprehension, noting that students' failures are often not from an inability to reason, but a failure in execution. In the self-explanation intervention group, students were shown a brief PowerPoint about self-explanation and its key principles: identifying key ideas in each line of the proof and explaining each line of the proof in terms of the ideas presented in the previous lines. After the self-explanation training (or brief PowerPoint about the history of mathematics for the control group), the students were presented a proof on the computer screen and asked to study it silently. Then, they were presented the proof again line by line and asked to explain their thinking when reading and understanding each line of the proof. The students then completed a proof comprehension task designed by the researchers and based on the work of Mejia-Ramos (2012).

The students' responses from the line-by-line reading of the proof were coded, and the results indicated that the self-explanation group students were more insightful in their explanations, making significantly more observations based upon theorems, ideas, and definitions not explicitly stated in the proof, more observations related to the structure of the proof, and more observations related to ideas used earlier than the proof than the control group did. Conversely, the control group gave significantly more explanations that were false, paraphrased the line or part of the line, or issued monitoring statements such as "I understand this" or "I don't understand this" than the self-explanation group did. Further, the self-explanation group performed significantly better that the control group in the proof comprehension test with a large effect size of d = 0.950.

Hodds, Alcock, and Ingils (2014) suggests the large impact that opportunities for reflection on proof-reading strategies can have on students' abilities to comprehend proofs. While this study did not have the students engage in journaling, self-explanation is a form of real-time reflecting on the proof that the students read. It is verbal, not written, but still a form of reflecting on one's understanding. In Hodds, Alcock, and Ingils' study, the students were trained in self-explanation, but only once and via a short (written) PowerPoint, yet the researchers reported a large effect size, suggesting the power that reflection may have in helping students learn about proof.

Another form of reflection that has been shown to improve proof writing performance is class discourse (Blanton & Stylianou, 2014; Blanton, Stylianou, & David, 2009). Blanton and Stylianou discuss the framework for assessing class discourse developed in Blanton, Stylianou, and David (2009), and how it can be used to evaluate the types of statements made by the teacher in response to students' claims. In their classroom observations, the authors found that transactive prompts, or "prompts that urge students to challenge their own and their peers' reasoning" (p. 143) to reason transactively, encourage students to engage in metacognitive activity. In doing so, the students continually evaluate their reasoning and that of their peers, leading them to develop more complete notions of proof and the process of constructing arguments used in proving. Although class discourse is not a written form of reflection, Stylianou and Blanton's observations provide another example of the importance of reflection of one's thinking in the learning of proof.

While there is considerable evidence that reflective writing increases performance in mathematics courses, there is a gap of research on how reflective writing may be used to aid students in their learning of proof writing. This study will attempt to begin filling this gap in research.

Theoretical Framework

This study is built primarily on Emig's (1977) theory on writing as a mode of learning, and will use Raman's (2003) framework for ideas about proof. Emig demonstrates that writing serves as a unique mode of learning separate from talking, and Borasi and Rose (1989) provide a framework for how reflective writing may be used as a tool for learning mathematics. Further, De Villiers (1990) enumerates the different functions that proofs serve, and Raman characterizes the different ideas people have regarding proofs that this study will relate to reflective journaling. The figure below provides an illustration of the theoretical framework for this study. This section of the dissertation will describe each component of the framework in detail.

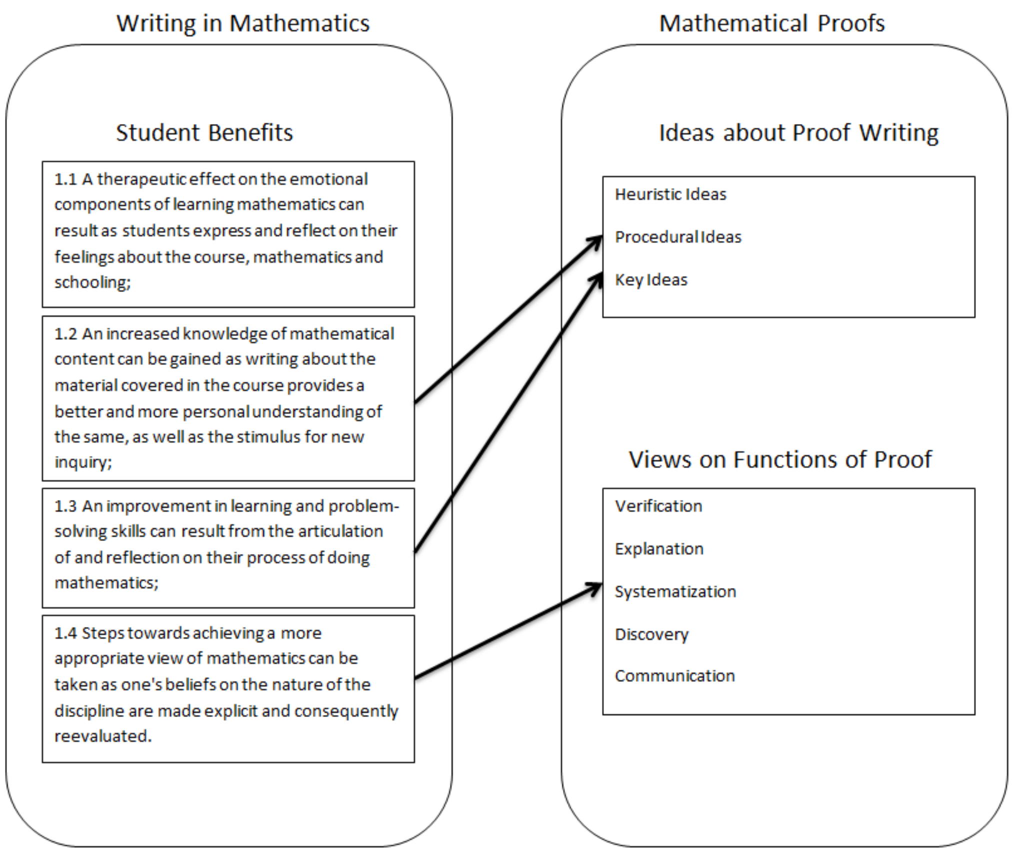

Figure 1. Theoretical Framework Overview.

Writing to Learn in Mathematics Courses

Many mathematics educators (NCTM, 2000; CCSS, 2010; Emig, 1977; Clarke, Waywood, & Stephens, 1993; Moore, 1994) cite communication as a crucial aspect of mathematics because it is through communication in some form – written, verbal, or nonverbal – that students demonstrate their knowledge and teachers assess their students. Writing in mathematics courses has gained popularity in recent years (Borasi & Rose, 1989; Loud, 1999; Santos & Semana, 2014) with researchers citing numerous benefits to students' learning as a result of *writing to learn.* Emig (1977) discusses the differences between talking and writing. Talking is not pre-writing, as writing uses different cognitive structures than talking. Emig provides the following list of ways in which writing is different than talking:

- writing is learned, talking is natural
- writing is a technological device and produces a visual graphic product
- writing is generally slower than talking
- writing is "stark, naked, even barren" while talking is "rich, luxuriant, inherently redundant" (Emig, 1977)
- writing creates context, talking is situated in an environment
- writing the audience is absent, talking present
- writing tends to be more responsible and committed than talking
- writing shows our representation of the world, embodying both a process and a product, and claims therefore writing more valuable for learning

Since writing is a unique activity, separate from talking, it is important to consider how writing may impact learning. First, however, it is crucial to discuss what it means to learn and how we learn. Although theories about what learning is differ (Bruner, 1971; Vygotsky, 1962), some commonalities include: they cite the importance of re-enforcement and feedback and claim that learning is connective and selective, active, and personal. In other words, learning can be defined as reorganizing or confirming a cognitive theme as the result of an experience.

Bruner (1971) gives three types of learning - enactive (by doing), iconic (by seeing), and representational or symbolic (by restating that which we have learned into words). Emig argues that writing allows one to enact all three of these learning experiences and also creates a multirepresentational tool for learning. Another important component of writing as a form of learning is the unique demand of connecting and interpreting "inner speak" that writing puts on the writer. In this process, the writer must deliberately and consciously reflect on their thought schemes. Writing can also promote learning because writing follows the pace of the writer. In other words, writing is self - rhythmed and allows the writer to reflect on past and present experiences to make meaning.

Emig's theory on writing as a mode for learning can be summarized in the following figure:

Unique Cluster of Correspondences between Certain Learning Strategies and Certain Attributes of Writing

Selected Characteristics of Successful Learning Strategies	Selected Attributes of Writing, Process and Product
(1) Profits from multi-representational and integrative re-inforcement	(1) Represents process uniquely multi-representational and integrative
(2) Seeks self-provided feedback:	(2) Represents powerful instance of self-provided feedback:
(a) immediate	(a) provides product uniquely available for *immediate* feedback (review and re-evaluation)
(b) long-term	(b) provides record of evolution of thought since writing is epigenetic as process-and-product
(3) Is connective:	(3) Provides connections:
(a) makes generative conceptual groupings, synthetic and analytic	(a) establishes explicit and systematic conceptual groupings through lexical, syntactic, and rhetorical devices
(b) proceeds from propositions, hypotheses, and other elegant summarizers	(b) represents most available means (verbal language) for economic recording of abstract formulations
(4) Is active, engaged, personal—notably, self-rhythmed	(4) Is active, engaged, personal—notably, self-rhythmed

Figure 2. Emig's 1977 Framework for Writing as a Mode of Learning.

Accepting that writing represents a unique mode of learning, researchers began to explore how writing affects the learning of mathematics. One strand of research (Borasi

& Rose, 1989; Clarke, Waywood, & Stephens, 1993) is focused on how reflective journaling can be used to help students make sense of the mathematics they are learning. In their 1989 study, Borasi and Rose investigated how writing in the form of reflecting journaling could be used to help students learn mathematics. Students in a freshman algebra course were required to keep weekly journals, and the researchers performed a content analysis of the journal entries and the instructors' written responses. Borasi and Rose's analysis revealed the following taxonomy of benefits to the students' mathematics learning as a result of the journals:

A taxonomy of potential benefits of journal writing

<u>Potential benefits as the students write their journal</u>:

1.1 A therapeutic effect on the emotional components of learning mathematics can result as students express and reflect on their feelings about the course, mathematics and schooling;
1.2 An increased knowledge of mathematical content can be gained as writing about the material covered in the course provides a better and more personal understanding of the same, as well as the stimulus for new inquiry;
1.3 An improvement in learning and problem-solving skills can result from the articulation of and reflection on their process of doing mathematics;
1.4 Steps towards achieving a more appropriate view of mathematics can be taken as one's beliefs on the nature of the discipline are made explicit and consequently reevaluated.

<u>Potential benefits as the teacher reads the students' journals:</u>

2.1 More appropriate evaluation and remediation of individual students can result from the increased individual knowledge of each student gained through the journals;
2.2 Immediate changes and improvements in the course itself can be made in response to students' feedback on the course.
2.3 Long-term improvements in teaching approach and methodologies may be induced in response to the new insights gained about students, learning and teaching;

<u>Potential benefits as students and teacher dialogue in the journals:</u>

> 3.1 More individualized teaching can be achieved as the teacher directly responds to questions, problems and suggestions expressed by students in their journals;
> 3.2 A more caring and non-adversarial classroom atmosphere, conducive to students' taking learning risks and supporting the teacher's commitment to continuous improvement, may be created by the mutual trust built through the journal exchange.
>
> (Borasi & Rose, 1989)

Although the framework provided by Borasi and Rose was developed in an algebra course and the current study took place in a mathematical proofs course, nonetheless since proofs fall under the domain of mathematics, the taxonomy can still be applied. Further, this study focused solely on the benefits to the students and not the two other categories of benefits provided by Borasi and Rose. Future studies can investigate the benefits to the teacher and student-teacher dialogue from journaling in relation to mathematical proofs.

Mathematical Proofs

Few mathematicians and mathematics education researchers would argue the fundamental nature of proofs in mathematics. Proofs represent the written mathematical language and are the convention by which mathematicians communicate with each other. However, despite being a key aspect of mathematics, there is not complete consensus among mathematicians or mathematics education researchers about proof (Harel & Sowder, 2007) – what it is, why it is important, and how it is learned. There is agreement among some researchers and mathematicians that proofs serve primarily to promote understanding and that rigor is secondary (Hanna, 2000; Healy & Hoyles, 2000).

Researchers (Harel and Sowder, 2007; De Villiers, 1990; Raman, 2003; Kidron & Dreyfus, 2014) have increasingly strayed away from viewing proof as merely a formal set of logically sequenced statements to verify a mathematical assertion. Instead, a proof is defined as what establishes truth for a person or community (Harel & Sowder, 2007). Using this definition, proof is an activity that can be prominent in mathematics curricula of all levels. This definition does not discount the formal aspect of proving; rather, the definition allows for the observation that proof has both private and public components.

Increasingly, researchers note the distinction between proof as a public activity and proof as a private discourse. Harel and Sowder (2007), for example, differentiate between ascertaining and convincing oneself of the truthfulness of an assertion, versus persuading and convincing others of the truthfulness of an assertion. In addition to proof being multi-dimensional in terms of proving for oneself versus proving for others, proof also requires multiple types of knowledge. As Weber (2001) and Raman (2003) point out, there is more to proof writing than merely possessing a conceptual understanding of the theorem to be proved and knowledge of proof methods. The prover must also somehow connect those two related, but distinct types of knowledge. Journaling, which is a private activity, may help students bridge the private and public aspects of proof.

In 2001, Raman conducted her dissertation research under Alan Schoenfeld at the University of California, Berkley to explore how university mathematics professors, graduate mathematics students, and undergraduate mathematics students view the public and private aspects of proof. A purposive sample of 11 undergraduate mathematics students, 4 graduate mathematics students, and 5 university mathematics professors were selected by Raman to be interviewed. In the interviews, the participants described their thinking as they wrote a proof that the derivative of an even function is odd and then were asked to evaluate five proofs of the statement for their clarity and completeness.

Upon analyzing the interview responses for the types of approaches to proof-writing used, Raman (2001) found that university professors and mathematics students often think about the public and private aspects of proof differently. For the university professors and mathematicians, private aspects of proving (ascertaining) are deeply linked with public aspects (persuading). For them, the act of ascertaining often provides the basis for them to persuade. However, students tend to view the two as separate. In her study, Raman found that the students believed proofs were created from nothing. In other words, the students did not see the conceptual connection to the proof, and rather thought the proof was generated procedurally with no consideration for the underlying meaning. Thus, the students did not recognize the ascertaining, private side of creating a proof. They focused solely on trying to create a mathematically correct proof and failed to see a connection between their private understandings of the idea and what they viewed as a formal proof. Based on her 2001 research, Raman identified three types of ideas people have about proof writing:

Heuristic idea: Heuristic ideas are based on informal understandings that a theorem or assertion ought to be true, but with little or no ideas about how to turn the argument into a formal proof. This is related to Harel & Sowder's (2007) idea of ascertaining. Here, the focus is on understanding for oneself and convincing oneself that a theorem or assertion is true, without consideration of convincing others. In addition, heuristic ideas are related to Tall and Vinner's (1981) definition of concept image, or the cognitive structure in an individual's mind about a concept, in this case a theorem to be proven. The heuristic idea encompasses the private domain of proving, with no consideration for the public domain.

Procedural idea: Procedural ideas are based on general known proof strategies, such as starting by writing definitions of all terms involved and trying to connect the definitions to form a proof. These ideas are based on logic and formal manipulation and can lead to a proof that lacks a link to informal understandings. We can compare procedural ideas to persuading (Harel & Sowder, 2007). In persuading, one attempts to convince others the truthfulness of a statement. Procedural ideas can also be likened to Tall and Vinner's (1981) formal concept definitions, which are the verbal expressions of a concept or idea that would be "accepted by the mathematical community at large" (p. 2). If one is able to create a proof that indeed does persuade, but without really understanding the theorems and concepts behind the proof, then one has used purely procedural ideas.

Key idea: A key idea links together the private and public domains of proof, and represents an idea that conveys understanding and knowledge of why a certain claim is true that can be translated into a formal proof. With key ideas, one is able to not only understand for oneself why a theorem or assertion is true, but also build from that understanding to an idea of how to demonstrate to others that the assertion is true. Mathematics faculty often possess key ideas, whereas many of their students do not. Key ideas represent the bridge between private and public domains of proving.

This framework can be summarized in the following figure:

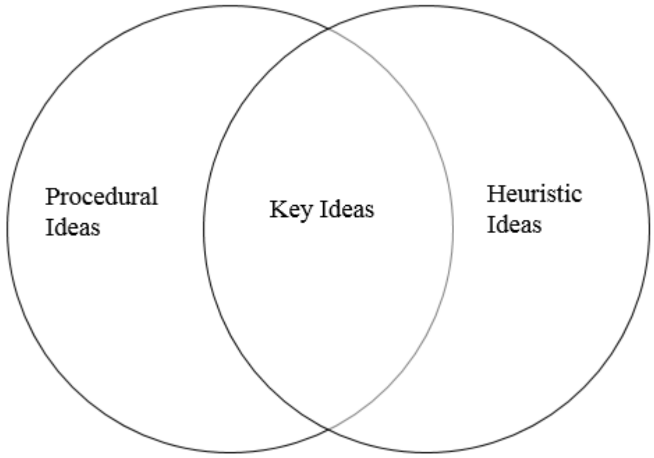

Figure 3. Raman's 2003 Framework for Proof Ideas.

In addition to having ideas about the process constructing proofs, researchers, students, and mathematicians all have views about the functions mathematical proofs serve. Though there is no consensus on exactly what the functions of proof are, researchers agree that proofs serve more functions than just verifying the truthfulness of a statement. In fact, De Villiers (1990) identified six unique functions of proof in mathematics:

- Verification (concerned with the truth of a statement)
- Explanation (providing insight into why it is true)
- Systematization (the organization of various results into a deductive system of axioms, major concepts and theorems)
- Discovery (the discovery or invention of new results)
- Communication (the transmission of mathematical knowledge)

(De Villiers, 1990)

In 2000, Hanna expanded this list, adding

- construction (of an empirical theory)
- exploration (of the meaning of a definition or the consequences of an assumption)

- incorporation (of a well-known fact into a new framework and thus viewing it from a fresh perspective)

(Hanna, 2000)

Students often do not see all of these functions of proof, and researchers (De Villiers, 1990; Hoyles & Healy, 1999) note that students' views of proofs and the functions of proofs often differ from those of their instructors. One of the goals of mathematics education is to produce future mathematicians by teaching students to think like mathematicians (Schoenfeld, 1992). In order to do so, students must participate in authentic activities that allow them to reflect on their views and reevaluate them in light of the activities they engaged in.

Writing as a Mode of Learning Mathematical Proof

Assigning a reflective journal in which students contemplate aspects of mathematical proof represents an authentic activity to help students develop their proof ideas and views on the functions of proof. Writing provides a unique opportunity for reflection and learning apart from verbal communication (Emig, 1977), and this opportunity extends to the domain of mathematics (Borasi & Rose, 1989). By allowing students to explore their personal understandings and beliefs, reflective journaling may allow the student to build connections between their heuristic and procedural proof ideas, thus helping them develop key proof ideas. In addition, giving the students prompts that force them to reevaluate the "proof is only for validation" viewpoint may encourage students to consider other functions for proof.

In their framework for how journaling affects mathematics learning, Borasi and Rose (1989) identify two benefits to the student that can be related to the development of key ideas: Benefit 1.2: An increased knowledge of mathematical content can be gained as writing about the material covered in the course provides a better and more personal understanding of the same, as well as the stimulus for new inquiry, and Benefit 1.3: An improvement in learning and problem-solving skills can result from the articulation of and reflection on their process of doing mathematics. In the context of learning proofs, students may gain increased knowledge of proof writing methods and strategies, and articulation and reflection of their proving process may help students to link procedural and heuristic proof ideas.

Additionally, Benefit 1.4: Steps towards achieving a more appropriate view of mathematics can be taken as one's beliefs on the nature of the discipline are made explicit and consequently reevaluated, can be linked with the development and refinement of students’ views on the functions of proof.

For an overview of theoretical framework, see Figure 1 on page 36.

III. METHODOLOGY

While there are large fields of research on students' understandings and misunderstandings of proof and research on how reflective writing about proofs and proving enhances the learning of mathematics, there is a lack of research on how reflective writing may be used to enhance students' learning of proofs. Research is needed to investigate how students' thinking develops as they learn to prove. Reflective journals provide a promising medium through which to catch a glimpse of students' thoughts throughout a semester of an introduction to proofs course. The study investigates the following questions:

1. What beneficial themes arise in undergraduate students' journals in an introduction to proofs course?
2. How does journaling in an introduction to proofs course affect students' ideas about proof writing?
3. How does journaling in an introduction to proofs course affect students' views on the functions of proof in mathematics?
4. How does journaling in an introduction to proofs course affect students' learning of proof, as demonstrated in their proof writing performance?

In this study I employed a pragmatist paradigm view: the focus is on the consequences of the research and the research questions, which allows for multiple methods of data collection to address a problem (Creswell & Clark, 2011; Patton, 2002). In this mixed methods study, quantitative data were collected consisting of survey instruments, number of journals completed, and course grades, and were combined with qualitative data in the form of students' journal entries, longitudinal in-depth interviews, and survey instruments to investigate the effects of reflective journaling on students' learning of proof. By collecting numerous data types and sources, I gained insight into how the students used the journals, how the students felt about journaling and its effects on their learning of proof, and how the students developed an understanding and appreciation of proof.

Pilot Study

In the fall of 2014, I joined a research team consisting of two professors (one mathematics and the other mathematics education) at my university to investigate the use reflective journaling in an Honor's Number Theory course. The study became a pilot study for my dissertation, and helped me get an idea of the types of journal entries students produced in an advanced mathematics course. This also helped me develop a methodology and instruments to use in my dissertation study. The pilot study was conducted in an honors number theory course in which students learned to prove mathematical theorems and statements in number theory. Honors college students may use this course as a substitute for College Algebra or Introduction to Advanced Mathematics, often resulting in students from a wide range of majors and varying levels of previous mathematical experiences taking mathematics in the course. The syllabus for the course is attached in Appendix A.

On the first day of class, the researchers introduced themselves to the class, explained the nature of the research, and distributed consent forms that were collected by the instructor and given to the researchers after the next class meeting. The students were also asked to take the Initial Survey online through the class website. The Initial Survey consisted of 2 parts: Part I consists of demographic questions about the students' gender, classification, major, previous mathematics courses, beliefs about mathematics and mathematical proofs, and previous experience with reflective journaling. In addition, the graded students' responses to the first problem set were collected to help determine the students' incoming mathematics performance. Both Part I and Part II of the Initial Survey can be found in Appendix B.

The students' responses to the Initial Survey were collected and put into a spreadsheet. To protect the students' privacy, the participants were assigned ID numbers in the following manner: For each participant, a random number between 1 and 18 was generated. If the number had not already been used, then the ID number was assigned to that participant. If the number had already been generated, a new random number between 1 and 18 was generated and the process was repeated until an open number was generated.

Once a week, the 17 students in the course wrote reflective journal entries that accounted for 10% of their final grade. The prompt given to them each week was: "Please describe the class with comments especially about the pace, difficulty, and any problems you may be having. What do you like the best, and what do you like the least? Suggestions?" This prompt is intentionally broad, allowing the students to write about what they chose as long as it was related to the course. Twice during the semester, more specific, proof-related prompts were given to the students to get them to consider different aspects of mathematical proof. These prompts were designed to address areas for improvement in students' learning of proof identified in the research literature. These included:

Table 1
Structured prompts and their relationship to the theoretical framework.

Weekly Journal Number	Prompt	Relationship to Research Framework
4	1. When assigned to prove a theorem, what is your proving strategy?	• Raman (2003) ideas about proof writing • Borasi and Rose (1989) Benefit 1.3 (process of learning)
4	2. Pick a proof or problem that you recently completed and copy this into your journal. What did you think about and what was your process for solving that problem?	• Borasi and Rose (1989) Benefit 1.2 (content) & 1.3 (process of learning)
4	3. Why do you think mathematicians place so much emphasis on the importance of being precise with language?	• Borasi and Rose (1989) Benefit 1.4 (views)
6	1. Discuss the role that definitions play in writing proofs. How are definitions important? How do you use definitions when writing proofs?	• Borasi and Rose (1989) Benefit 1.3 (process of learning) & 1.4 (views)
6	2. Although an example is not a proof, many mathematicians use them to help with proof writing. What are your thoughts or experiences on how examples can be used to aid proof writing?	• Borasi and Rose (1989) Benefit 1.2 (content), 1.3 (process of learning), & 1.4 (views)

Additionally, an interview protocol (Appendix D) was developed and five students were selected to be interviewed. The students were selected based on the lengths of their journal entries, whether they consistently turned in journal entries, whether they showed insight in their journal entries in the form of an awareness of their own learning, their major, and their grade on the midterm exam. Three students were chosen who wrote consistently, and wrote long and insightful journal entries. Of those students, one did poorly on the exam and is a non-math major, one did well and is a math major, and the other did well and is not a math or science major. Two inconsistent writers were also chosen, one who did well on the exam and one who did not. Both were math majors.

At the end of the semester, a qualitative Post – Survey (See Appendix C) was administered to the class with many of the same questions from the Initial Survey (minus the demographic questions). Questions were also included on the survey asking about the journals: how they affected the students' views about proof, how they affected the students' learning of proof, and what benefits, if any, the students experienced as a result of keeping the journals.

The analysis of the journals indicates that students exhibited numerous benefits from the journaling identified by Borasi and Rose (1989), and the topics of discussion in the journal entries evolved over the course of the semester. The students predominantly used the unstructured journals for Benefits 1.1 (therapeutic effect) and 1.3 (process of learning), and Benefits 1.2 (content) and 1.4 (views) were more prominent in the structured assignments. Additionally, the structured journal assignments provided interesting insights into the students' thinking about proof. For instance, in Journal 4, question 1 (When assigned to prove a theorem, what is your proving strategy?), few (3) students mentioned starting with definitions. Further, only five students mentioned trying to start with something they understand. Rather, most students claimed they list all theorems that might be relevant, work backwards from the conclusion, or try to pick a theorem that will lead to the conclusion. This shows that the students tend to be procedurally oriented, rather than having heuristic or key ideas about proving.

In addition to the journals providing a window into students' thinking, the students persevered with the journals, with most students consistently turning journal entries in. Further, many students who missed a journal assignment turned the journal in late with an apology, even though it may not have been counted for a grade since it was late. This suggests that the students were not just keeping the journals because they were required for a grade, and rather that the students saw some other benefit from the journals besides maintaining a grade in the course. In the interviews, all four interviewees answered that they would recommend keeping a journal to a friend about to enroll in an introduction to proofs course. The interviewees mentioned specific benefits from the journal including it helps “identify areas that I needed to work on” (Benefit 1.3), “it keeps making me remember what I did the past few days” (Benefit 1.2), and “writing proofs, it’s a lot of writing, and if you keep a constant journal you get used to the writing…and so then you can write a proof” (Benefit 1.3).

The pilot study showed that the journal tasks provided students an opportunity to explore their understanding and feelings about the course. The study also showed that the journals provided the instructor and researchers insight into how the students think about proving, which would not have been apparent in the students’ turned-in proof assignments alone. Additionally, the pilot study helped the researcher to create and modify the Pre- Proof Views Survey, Post- Proof Views Survey, Proof Journal Interview Protocol, and structured journal assignments, as well as develop the data analysis plan.

Design and Conceptual Model

First, I will briefly describe my conceptual model, and afterwards I will discuss the research design and interpretive framework of this study. My study investigates numerous relationships regarding the effects of journaling in an introduction to advanced mathematics course. In order to do so, the relationships and constructs of interest must be specified. This conceptual model is guided by the theoretical framework I discussed above, and was used to explain how the different pieces of the study were analyzed and connected. The journals provide the medium to explore two constructs: the depth of writing in mathematics and the opportunity to *write to learn* in mathematics.

The opportunity to *write to learn* in mathematics represents the students being provided this experience of keeping a weekly journal in their mathematics class. My hypothesis is that the *writing to learn*, in the form of the weekly reflective journals, would affect students' performance on learning to write proofs. This relationship represents research question 4, and was measured quantitatively using pre- post comparisons of students' proof writing homework sets, as well as qualitatively through students' responses to open- ended survey questions. However, the students' performance in proof writing was affected not only by their opportunity to *write to learn* in mathematics, but the depth of their writings. Researchers have demonstrated (Clarke et. al, 1993; Borasi & Rose, 1989) that the more depth to students' writings, the larger performance gains as a result of the writing.

This depth of writing construct is difficult to measure and quantify. For this study, the depth of writing was demonstrated quantitatively by each student's number of writing assignments submitted and number of benefits (therapeutic, content, views, and problem solving) the student demonstrated in their journals and reported in their surveys. The depth of students' writings also affected their views about the functions of proof (research question 3), ideas about proof (research question 2), and the benefits they received from the journal (research question 1). The relationships between the depth of the students' writings and their views, ideas, and benefits were explored qualitatively through interviews, open-ended surveys, and the journal entries themselves. Changes in the students' views on the functions of proof as a result of the depth of their writing were also measured quantitatively. Examining all of these relationships allowed me to gain insight and add to the research literature on how journaling may affect students' learning of mathematical proof. My conceptual model is summarized in the figure below:

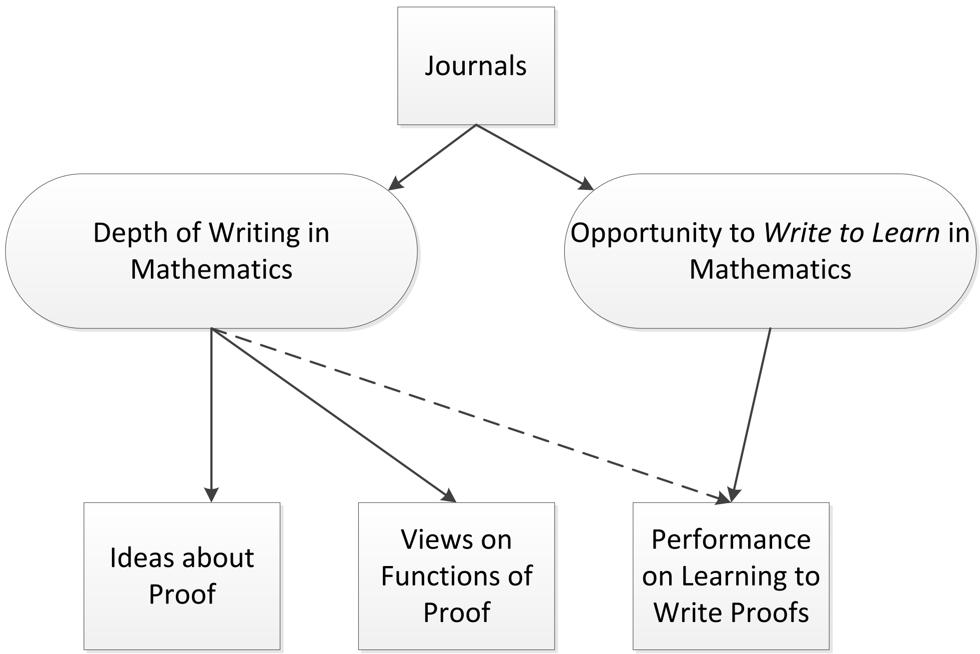

Figure 4. Conceptual Model.

Because this study attempts to examine a phenomenon, the students' use of journals to learn proof, in-depth, a mixed methods approach was used along with the interpretive framework of pragmatism (Creswell & Clark, 2011). With this framework, knowledge is gained through action and reflection with the goal of solving a problem. This focus on the research problem permits me to use multiple sources of data to answer the research questions (Creswell & Clark, 2011). Pragmatism supports the use of a mixed methods design by allowing the researcher to use multiple data sources to deeply understand the research problem.

The journals were viewed through two lenses: 1) focusing on how the journals affect the students' proof writing performance, their ideas about proof, and views on the functions of mathematical proof and 2) looking at how the journals provide insight into students' thinking as they progress through their introduction to advanced mathematics course. As such, the study consists of two parts that, together, provide an overview of journaling in an introductory proofs course.

Overall, the study had a concurrent triangulation design, in which quantitative and qualitative data were both collected concurrently and then analyzed and interpreted after data collection (Tashakkori & Teddlie, 2003). This design allowed me to explore research questions that are not dependent on each other's results (Creswell & Clark, 2011), and is advantageous because it allows for "well-validated and substantiated findings" (Tashakkori & Teddlie, 2003, 229). The design allowed me to study the content of the journals, as well as the effects of journaling on the students' learning. In the concurrent triangulation design, I used two different methods to "confirm, cross-validate, or corroborate findings within a single study" (Tashakkori & Teddlie, 2003, p. 229). Pretest-posttest investigations examining the explanatory effects of journal writing, measured with the explanatory variable level of journal implementation, on performance and views about proof were employed to help answer the research questions. In addition to the quantitative regression analysis, I collected qualitative surveys and conducted interviews to triangulate the quantitative results. Put together, the data shed light on journaling as a tool for helping students learn about proof.

Setting

The setting of the study was an undergraduate introduction to proofs course, which is generally taught by a Ph.D. mathematics faculty member. The population of the study is the set of students enrolled in the introduction to advanced mathematics course taught at a large public university in Texas. At the institution of the study, students must have earned a C or better in Calculus I and Calculus II to enroll in the course. In these two prerequisites classes, students may have been exposed to proofs by the instructor, but often do not have experience writing proofs of their own. Students may have previous experience with proofs in their high school geometry courses, but are unlikely to have had proof writing experiences at the university prior to enrolling in the course. However, that was not the case in this study. In the Pre-Proof Views survey at the beginning of the semester, 21 of 31 participating students self-reported that they had previous experience with proof-based courses, particularly discrete mathematics.

This course is designed to introduce students to the methods of proof and some of the central concepts needed for higher level mathematics courses. Topics of study generally include techniques of proof, set theory, relations, functions, and the algebraic structure and topological properties of Euclidean Space. For students, changing from the Calculus setting, in which their assignments consist of solving problems such as evaluating integrals, calculating areas and volumes, administering series convergence tests, or constructing Taylor series, to the introduction to advanced mathematics course, where the assignments are centered on writing proofs, represents a great shift in thinking and mode of communication. Students are expected to understand theorems and definitions and apply them to construct correct proofs. A major component of the introduction to advanced mathematics course is writing, specifically writing mathematical proofs. A copy of the departmental syllabus for the introduction to advanced mathematics course is provided in Appendix H.

Sample

The study took place in two sections of an introduction to advanced mathematics course in which students are taught mathematical proof techniques. Sixty students enrolled in the course, 27 in Section 1 and 33 in Section 2. After 17 students dropped the course before and throughout the semester, 43 students remained registered at the end of the semester. Thirty-one students filled out the Pre-Proof Survey (18 females and 15 males). Students' majors are given in the table below; 3 students were double majors.

Table 2
Academic major breakdown of participants.

Major	Frequency
Applied Mathematics	3
Mathematics	18
Mathematics with a Teaching Certificate	2
Computer Science	6
Physics	1
Exercise and Sports Science with a Secondary Teaching Certificate	1
Exploratory	1
Biochemistry	1
French	1

There were no freshmen in the course. The breakdown of students' classifications is given in the table below:

Table 3
Classification of participants.

Reported Classification	Freshmen	Sophomores	Juniors	Seniors
Count	0	8	10	13

The two sections were taught by different tenured faculty members. The sample was composed of the students in both sections given their informed consent to participate.

Journal Component of the Course

For the study, both participating instructors agreed to include a weekly journal as part of the course requirements (5% of the total course grade). The journals were assigned and submitted online using the university's online collaborative learning environment. Each instructor created a course page, and the weekly journals were posted as assignments. Students were able to type their journal responses into the online window, upload a word document, or upload a PDF. I acted as a teaching assistant or grader for the class, and read the journals each week. The students were given a completion grade (all or nothing points) based on whether they completed the journal or not. In addition to reading and grading the journal responses, I typed written feedback into the assignment window online, so that the students could see the feedback comments when they checked their grade. The feedback was generally 1-3 sentences, and ranged from a short response that just affirmed to the student that their journal had been read to a longer response that addressed a question or concern a student brought up, or a mistake in the student's proof. Below I give two examples of journal entries and the associated feedback I gave the students:

Table 4
Examples of journal entries and my feedback.

Student	Journal Entry	Feedback
Lauren	"Sets and Relations: What I think is difficult about these sections is the grammar/reading the symbols properly. I might read the problem and then go about solving it a completely different from what it's asking for."	"That's a great point! There can be small differences in symbols/notation that make a huge difference in the problem! It's great that you are aware of that and watching out for it."
Sheila	"This class is going well, the material keeps me challenged but it's not so hard that it discourages me. I am doing well with the pace of the class and do not have any questions right now."	"Great! Thanks for the update!"

Although my study focuses on the benefits of the journals to the students and not the instructors, it was important for the instructors to be aware of the students' comments so that they could make instructional adjustments based on the journals and the needs of the students. I felt that the students would receive and notice more benefits from the journals if the instructors were aware of the journal contents than if the instructors never saw the journals. I also wanted the instructors to get a chance to hear what the students were saying without necessarily having to read each journal entry. I submitted weekly highlights to the instructors, which included examples of responses the students submitted, particularly questions, concerns, or feedback (both positive and negative) to help the instructor gauge where the class was at and that could be used to implement instructional changes. I chose students' responses to include that represented a majority of opinions being stated, were unusual or particularly insightful, contained a question for the instructor, or discussed a concern or praise that the instructor could take into consideration.

Below, I give two examples of the weekly journal highlights I sent to the instructors. Although the instructors did not read the journals, numerous students described noticing positive instructional changes and decisions that the students felt resulted from their journal comments. I include these examples below so that the reader can get an idea of the implementation of the journals and what information the instructors had about the journals, which allowed them to adjust instruction and classroom practices to the students' benefit.

For example, the week 5 (2/16 /2015 – 2/20/2015) prompt was "Please reflect on the class and course content with comments especially about the pace, difficulty, and any observations or questions you may be having." I submitted the following highlights to the instructor:

> Good morning,
> Some highlights from this week's journals:
>
> "The class started out pretty easy and has progressively gotten harder and the pace of the class is really good because many of the material has based off one another and i continue to understand. As we continue to keep going it has helped to go over the easy questions and have open discussion about thoughts of what is being taught and the right answers."
> **- Joe**
>
> "The course is well paced, however I don't learn very well by reading, so I wish we could cover more of the topic before jumping right into practice." – **Mike**
>
> "I get super, easily confused. When I think I have the right answer, it ends up being wrong, in a way, where the correct answer is the opposite of what I think it is. I also have no idea how to even start writing a proof even though apparently our whole class has the basic tools needed to do this. The pace is fine, but I guess I don't truly understand the concepts being learned. One of the reasons why I am probably doing so poorly is because I haven't been studying as much as I need to"
> **– Diana**
>
> "I find the material in this class interesting, but when I do struggle with something I have a hard time coming up with a question to ask. I like that we practice nonstop during class. I find it helpful. " - **Travis**

" I am happy with the class so far. As far as pace, I think we are slightly behind what our professor expected to be, but we have covered all of the material for the test in plenty of time to study. This class has not been as difficult as it's name suggested and I am grateful for that. " **- Ryan**

For week 4 (2/9/2015 – 2/27/2015), I asked the students to

> Choose a definition that you have recently been using in class - it can be one that you understand well or one that you are struggling with. Write the definition using formal terminology, and then write in words how you interpret that definition. How would you describe it to a friend? Please also describe any other questions or comments you have for the course that you think are important.

The highlight email that I sent to the professor was:

> Good Morning,
> This week, I asked the students to choose a definition that they have been using and first write it out in formal notation, and then explain how they understand the definition and how they would describe it to a friend.
> Some highlights:
>
> "Definition 19 states if a and b are integers, a divides b if there exists an integer k so that b=ak. I interpret this as a actually divides b , like you have 49/7. so i was a little confuse. But then i asked Professor Bates to go over it with me and she helped me realize hat the definition meant. So with definitions you really have to take them literal so if it says a divides b for some integer k where b=ak that what it means. so 49=7k if a is 7 and b is 49." -**Mary**
> "Definition 19- If a and b are integers, a divides b if there exists an integer k so that b=ak.
> Interpretation- If some integer b can be rewritten as the product of two integers, a and k, then the number b is a multiple of value a.
> Described to a friend- If some integer value, say b, is divisible by a number, a, and the result is an integer k, then a is said to divide b. For example 2|8 because 8 can be written as 2*4. 2 does not divide 9 though, as 9 cannot be written as 2*integer." - **Sam**
> "Definition:
> If a and b are integers, a divides b if there exists an integer k so that b=ak.
> How I see the definition:
> If there is an integer k, and b=ak, then a is able to divide b. I think its easier for me to understand when I write it in this way. In order for a to divide b, we must see that some other integer, k, is multiplied by a so that they equal b. and if a divides b then b is a multiple of a because we have some integer, k, being multiplied by a.
> Describe to friend:
> In order for us to be able to say that a divides b, we must include another integer, k, as well in order for this to be true and provable. Divides is different than the

word divided by because we are not actually doing the dividing, we are stating that a in fact does divide b. We are saying here that in order for a to divide b, b must be some multiple of a, because we use b=ak in order to say that it is true that a divides b." – **Pam**

"Definition 3: An integer is a positive or negative counting numbers or zero, as in ...,-2,-1,0,1,2,... My definition of an integer would just be any whole number, including 0.
I chose this definition because I believe it to be one of the most important definitions in writing proofs. Stating a variable is an integer restricts it to only being a counting number. As of now, the term integer has been very important in various proofs. " – **Bill**

"Definition 19: If a and b are integers, a divides b if there exists an integer so that b = ak.
If a and b are both real numbers and are whole.
So if a divides b, then a times k would also divide b. (by closure?)
Sometimes I'm still kind of iffy about what by closure actually means. I guess it's more like, understood? But that's really the only problem I have right now in the course " – **Hannah**
"The word "proof" or "prove" or their converse "disprove" are very interesting words that have quite different formal and informal definitions and usage.
Formal definition: A rigorous mathematical argument which unequivocally demonstrates the truth of a given proposition. A mathematical statement that has been proven is called a theorem. (http://mathworld.wolfram.com/Proof.html)
Informal definition: evidence sufficient to establish a thing as true, or to produce belief in its truth; the establishment of the truth of anything; demonstration. (http://dictionary.reference.com/browse/proof)
On a day to day basis most of us use the word "proof" but we don't mean it like a mathematician or a physicist do. Although I cannot prove to you with 100% certainty anything (except perhaps that I shall die someday), and I cannot really prove to you that I wrote this, nor can a reader prove to me that they have actually read it: at a certain point anything requires a bit of believing. The question becomes then, what are you willing to believe? This is where being skeptical and critical are important, because the easiest one to fool is oneself. We are capable of feeling confident about things not happening; that is, disproving things. For instance, that the earth is not flat because you will see different stars if you travel north or south, or that there exists an even prime other than 2 because every even is divisible by two. So I would describe to a friend that for something to be proven I need a good idea about why it has not been disproven. Of course disproving something is not a 100% guarantee that it has not nor will never happen ever, but I am not losing sleep by thinking about the shape of the earth. Alas, this difference in use of "proof" between math and science is very interesting: in math the truth value of a statement is evaluated logically, and in science the truth value of a hypothesis is evaluated against the evidence of reality

> which brings forth my favorite part of science in that it helps us see how nature often defies our sense of what is logical. " –**Neil**

After the semester, I met with both professors to see how they used the summaries. Both instructors said they would read them and take the students' comments into consideration for their instruction the following week.

Instrumentation

To investigate the effects of journal writing on students' learning of proof, numerous instruments were used to collect the data for this study.

Surveys

Surveys were designed to collect both quantitative and qualitative data.

Pre- Proof Views: Part I

In order to answer the research questions about how students' views about proof and develop throughout the semester, it is important to see what views students bring with them into the course. The Pre-Proof Views Survey consists of two parts: Part I is qualitative and Part II is quantitative. A qualitative Pre-Proof Views Survey was developed during the pilot study in fall 2014 to gain insight into students' previous experiences with mathematics, proof, and journaling. The open-ended survey also contains demographic information asking students about their previous mathematics courses taken, major, classification, and gender. Two questions regarding students' motivation for taking the course and their expectations for the course are also included in the survey. This survey helped me to gauge where students are at the beginning of the semester and was useful when triangulating the results from the quantitative instruments.

In the pilot study survey, students were asked to finish the thought, "A mathematical proof to me is_________" in order to elucidate the students' incoming beliefs about the nature and definitions of proof. Pilot data indicated that some students, rather than writing their definitions of proof, wrote feelings that they associated with the topics. For instance, a student wrote: “A mathematical proof to me is confusing.” To attempt to elicit more rich descriptions, the prompt was edited to: “To me, the definition of a mathematical proof is____________” and “In mathematics, people write proofs to___________.” In addition to demographic questions, prompts on the survey include the questions in the table below:

Table 5
Survey questions and their purposes.

Question	Purpose – Research Question
In mathematics, people write proofs to…	Establish a baseline for research question 3 (views on the functions of proof)
To me, the definition of a mathematical proof is…	Inform research question 2 (ideas about proof)
My favorite part of math is…	Background information on views on mathematics
My least favorite part of math is…	Background information on views on mathematics
Math to me is…	Background information on views on mathematics

See Appendix F for a copy of the modified Pre-Proof Views: Part I.

Post - Proof Views: Part I

At the end of the semester, a Post-Proof Views Survey was administered to investigate the students’ views about the nature and definitions of proof after the course, their thoughts on if and how their views changed during the semester, and their perceptions of the journal tasks and the influence of journaling on their learning. The survey consisted of two parts: Part I was qualitative and Part II was quantitative. In Part I, the short, open-ended prompts on the survey consist of the prompts from the Pre-Proof Views Survey (minus the demographic and motivations for the course questions).

Additionally, questions asking about how students' views changed throughout the course and questions about the journals were included in the survey. These include:

Table 6

Survey questions about journals and students' views, and the questions' research purposes.

Question	Purpose – Research Question
How has the course affected your beliefs about proof?	Help answer research question 3 (views on the functions of proof) by distinguishing between general changes from the course and changes from the journal
Table 6 Continued.	
What is your favorite part of proofs?	Answer research question 2 (ideas about proof)
What is your least favorite part of proofs?	Answer research question 2
What purpose or purposes do you think that proof serves in mathematics? In other words, why are proofs important?	Answer research question 3
How do you feel about keeping a weekly math journal this semester?	Answer research question 1 (benefits of journaling)
How did the journaling tasks affect your learning in the course?	Answer research question 4 (performance effects from journaling)
How did the journaling tasks influence your opinions about proof in mathematics?	Answer research question 3
What are the benefits of journal writing in a proof-based mathematics course?	Answer research question 1
How could journal writing be changed to be more effective?	Answer research question 4
Would you recommend keeping a math journal to a friend about to start a proofs course in the future? Why or why not?	Answer research question 1

Questions about students' use of the journals were included in the survey to help determine each student's degree of journal implementation. These questions asked students about how often they wrote a journal, how long they spent writing, and how much thought they put into the journals. See Appendix F for a copy of Part I.

Pre- Proof Views: Part II and Post-Proof Views: Part II

To understand how students' views on the functions of proof changed over the course of the semester of proof instruction, I developed a quantitative survey aligned with DeVillier's (1990) categorization of the functions of mathematical proof as: verification, discovery, systematization, communication, and exploration. The survey measures which function(s) of proof students view as important and the relative importance of each function. The survey consists of twenty statements, with four statements representing each of the five functions of proof. (See Appendix I for a copy of the survey) The students were given 20 points to assign to the 20 statements however they chose to weight the statements in terms of how they viewed the purpose of proofs in mathematics. A statement that a student felt more accurately described the function of mathematical proof should receive more points than a statement that did not align with their beliefs about the functions of proof. The participants were asked to use all 20 points and add them up at the end to make sure all had been distributed. The quantitative survey was included as Part II of the Pre-Proof Views and Post-Proof Views surveys to measure the change in students' views that occurs between the start and end of the semester. Before the survey was administered to the participants, 3 independent researchers who are also mathematics professors at the university looked at the survey and made suggestions, which I incorporated into the finalized version of the instrument.

Journals

Based on Borasi and Rose's (1989) assertion that students should have autonomy over what to write about and that students used journaling for various purposes, the students kept weekly journals about their thoughts and experiences in the course.

However, Borasi and Rose noted that students often did not write about the content of the course or recognize journaling as a way to help them learn without being prompted, thereby lessening the benefits to the students from journaling. Further, research indicates that the more students write about content and reflect on their understandings of the content, the more appreciation they have for the journals and the better performance improvements they exhibit (Clarke et. al, 1993; Borasi & Rose, 1989). Students were thus given a structured prompt roughly every other week (See Table 7 below) about the nature of proving, strategies for writing proofs, the functions of proof, and other proof writing and learning- related topics. In the pilot study, students predominantly used the unstructured journals as a therapeutic outlet and to discuss their learning, which accounted for two of the four benefits identified by Borasi and Rose (1998). Giving the students structured prompts roughly every other week helped them to achieve the other two benefits of journaling while still allowing them opportunities to write about what they wanted in their unstructured journal entries.

Table 7
Journal prompts and benefits by week.

Week	Prompt	Benefit to Student (Borasi & Rose, 1989)
1	Whether you've experienced proving before or whether this is brand new to you, what are some questions you may have or concerns? Is there anything you want me to know that could help teach you? What are your goals for the course? Please respond to the above prompt in a few sentences. Feel free to write more if you want to!	Therapeutic Problem Solving
2	Discuss the role that definitions play in mathematics and writing proofs. How are definitions important? How might you use definitions when writing proofs? Please also describe any other questions or comments you have for the course that you think are important.	Therapeutic Problem Solving Views

Table 7 Continued

3	Please reflect on the class and course content with comments especially about the pace, difficulty, and any observations or questions you may be having.	Therapeutic Content
4	Choose a definition that you have recently been using in class - it can be one that you understand well or one that you are struggling with. Write the definition using formal terminology, and then write in words how you interpret that definition. How would you describe it to a friend? Please also describe any other questions or comments you have for the course that you think are important.	Therapeutic Content Problem Solving
5	Please reflect on the class and course content with comments especially about the pace, difficulty, and any observations or questions you may be having.	Therapeutic Content
6	When given a theorem to prove, what is your proving strategy? How do you judge the completeness of a proof?	Problem Solving
7	Pick a proof or problem that you recently completed and copy this into your journal. What did you think about and what was your process for solving that problem? Please also describe any other questions or comments you have for the course that you think are important.	Therapeutic Content Problem Solving
8	As we near the halfway point of the semester, please reflect on your progress in the course so far. What are two topics you feel very comfortable with and why? What are two topics you feel less comfortable with and how you might go about improving your understanding of those topics	Therapeutic Content Problem Solving
9	Please pick a proof that you recently completed and copy this into your journal. What did you think about and what was your process for solving that problem.	Content Problems Solving
10	Please reflect on the class and course content with comments especially about the pace, difficulty, and any observations or questions you may be having.	Therapeutic Content
11	So far in the course, what is your favorite proof technique? Why? What is your least favorite proof technique? Why?	Therapeutic Content Views
12	Please reflect on the class and course content with comments especially about the pace, difficulty, and any observations or questions you may be having.	Therapeutic Content
13	For this final journal, please reflect on the course and your progress throughout the course this semester. What advice would you give to a student about to take an introduction to proofs course? How, if at all, did your ideas about mathematics and proofs change during the semester? Please also mention any other comments or questions you have.	Therapeutic Problem Solving Views

Though there is evidence (Borasi and Rose, 1989; Clarke et al, 1993) that the instructor feedback on the journal entries and the dialogue between the instructor and students through the journals represent an important aspect of the journaling interaction, that interaction is beyond the scope of this study. Further, one hesitation that many instructors have about implementing journaling is the time requirement for them to read and respond to the journals each week. However, many instructors have access to graders and/or teaching assistants that can help with reading and responding to the journals. Therefore, I simulated a teaching assistant or grader, and read the journals and left feedback each week, and then submitted a weekly report to the instructor describing the content of the journals, with excerpts and summaries of the questions and comments students left. This way, students still received feedback, and the instructor was made aware of what students were saying and was able to address the comments in class.

Three structured prompts specifically asked students about their proving strategy (week 7) and to pick a proof they recently completed, copy it into their journal, and explain their thinking about how they figured out the proof (weeks 6 and 9). The students' responses to these prompts helped the researcher categorize the types of ideas students have about writing proofs as identified by Raman (2003).

Additionally, as part of the structured journal prompts, the students were periodically asked to write about their views on mathematics and/or proving. The students' responses to these prompts were also used as data to triangulate the students' responses on the surveys.

Interviews

At the end of the semester, 5 students were selected to participate in voluntary, task-based interviews. The semi-structured interviews consisted of two parts: open-ended interview questions and a mathematical task (see Appendix D). In Part I of the interview, students were asked questions about their perceptions of the course in general, questions about proof, and questions about the math journals. The questions about the course in general were included to see if any students volunteer opinions about the journals without being prompted. The questions about proof asked how the course affected the students' views about proof, aspects of proof they like and dislike, and their views on the functions of proof in mathematics. Finally, the questions about the journals investigated how students felt about keeping their journals, how the journals affected their learning and views about proof, the benefits they saw in keeping a journal, and suggestions they had for improving math journals. These interviews helped me gain details about how the students used the journals.

In Part II of the interview, the participants were given a theorem to prove or disprove and modify and asked to think aloud as they determined the validity of the theorem and wrote their proof or refutation. The task-based portion of the interview served to identify what type of idea, as defined in Raman (2003), the students have about proof writing, not whether they are able to write a correct proof or not. Thus, it was important to choose a task that the students possessed the knowledge to complete, or at least produce a solid attempt. The chosen theorem was as follows: "Consider the statement: For real numbers $a, b,$ and c, if $a < b$, then $ca < cb$.

- If the statement is true, construct a proof.

- If the statement is not true, please modify the statement to describe the correct relationship between ca and cb and then construct a proof of the modified statement."

I chose this prompt because students completed units on number theory in both sections of the course, and because, though it is not a true statement, it can be modified in numerous ways.

Course Averages

To help look at the students' performance in the class, course averages were used in a correlation calculation. Since proof writing is the main component of the assessments in both sections, using the course grade gives an appropriate score to begin investigation the relationship between journal completion and proof writing performance.

Data Collection

At the beginning of the semester, I introduced myself to the two classes and explained the research goals and rationale, as well as handed out and collected signed consent forms from the students. The course instructor explained how the journals would be incorporated into the participation grade for the class (5% of the course grade in both sections). The students were told that, although the professor may look at the journal entries, I would definitely be reading and responding to them every week and submitting a summary to the course instructor. The students submitted their journals electronically through the course website every week. When students logged on to the page, they would see the prompt – whether structured or unstructured - for the week and a link to upload their document or copy and paste the text into the submission box. The website allows the instructor to attach their feedback comments to the submission, making them visible only to the student.

The first survey was administered online through the course website. However, there were issues with the survey not working on certain internet browsers, so some students filled out the survey and emailed their responses to the instructor. The post-surveys were administered face-to-face in a group setting during regular class time to the present and willing students of one section, and the students had the choice taking home survey or filling it out in person after the final exam to the other section. Extra credit was offered to students as an incentive for completing the surveys. Any student who chooses not to participate were offered an alternative extra credit assignment to complete. The interviews were semi-structured, which allowed the researcher to explore ideas brought up by the participant during the interview. Also, during the task-based portion of the interview, the researcher offered clarifications and small suggestions if the student was stuck to the point of giving up and not being able to continue, but in general the researcher refrained from offering help to the student.

Students were selected for the interviews based on their journaling frequency and their homework average. The selection was made based on two factors: the student' degrees of journal implementation and their homework averages. The interviews took place outside of class, so the participants were given an Amazon gift card in appreciation for their time. The selection categories are summarized in the following table:

Table 8

Interview participant selection.

	Low Journal Implementation	High Journal Implementation
Low Homework Average	none	1 student from each section
High Homework Average	1 student from Section 2	1 student from each section

Having students with different performance and journal implementation levels helped the researcher gain multiple perspectives to better understand the interaction of the journal tasks and student learning. The interviews were recorded using both audio and video recording equipment, allowing me to see gestures and other non-spoken cues.

Below, I give a figure to summarize the data collection, where time passing during the semester is represented by looking from left to right on the graphic.

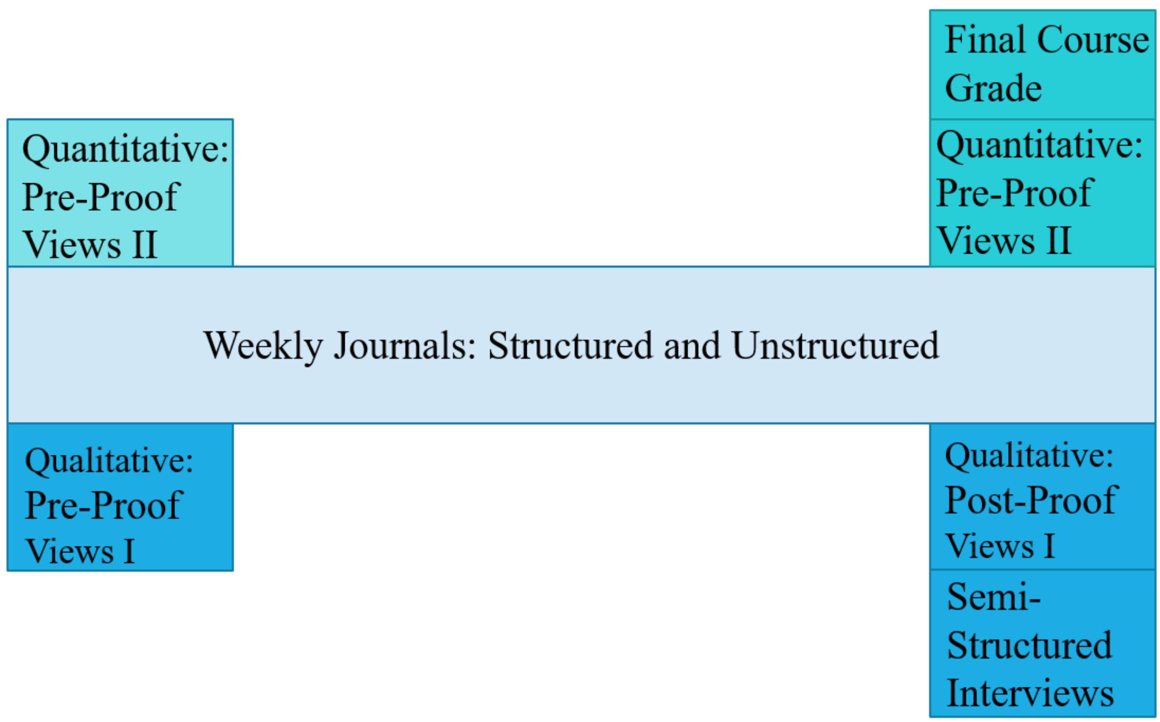

Figure 5. Data Collection Timeline.

Data Analysis

Quantitative Analysis

Though the journal entries themselves are qualitative in nature, one of the research questions of this study aimed to investigate the predictive power of the journals on the students' proof-writing performance. Therefore, the number of journal assignments completed by each student was used as an independent variable. Using this input variable, along with each student's course average, correlation coefficients were calculated for the entire set of participants, as well as various subsets of the data that were determined by students' survey responses.

Additionally, the quantitative Pre- and Post- Proof Views Part II instruments contained 20 items grouped into five categories using De Villier's (1990) framework for the five functions of proof: verification, explanation, systematization, discovery, and communication. For each participant, the total points assigned to each function of proof were calculated. I then also calculated the average points assigned to each function of proof by each section of students and the overall average points assigned to each function of proof by all participants. Since I wanted to compare individual students' pre and post views of the functions of proof, I created a table with all 15 participants who completed both the Pre- and Post- surveys. Because there is not an existing framework or agreement within the mathematics community about the relative worth of each function, I decided to focus on how balanced the functions' weights were. To determine how balanced students' views were, I calculated the maximum difference in points assigned to any two functions by each student in the Pre- and Post- survey. I then classified each student, section, and the overall set of all participants as balanced, somewhat balanced, or unbalanced at both the beginning and end of the semester, depending on the maximum difference of points in the Pre- and Post- surveys. I sorted the 15 participants into groups depending on whether their degree of balance increased, decreased, or stayed the same during the semester. Once I determined the groups, I was able to begin my mixed methods analysis by examining the Part I survey and interview data for members in each group as a sub-case to determine what similarities the group members shared.

Qualitative Analysis

The unstructured journal entries were analyzed to reveal how the students use the journals and what they choose to write about. The journals were coded twice; first, a content analysis in which codes were developed inductively was used, and then the journals were also coded using Borasi and Rose's (1998) categorization of benefits from journaling. A content analysis of the journals was conducted, allowing me to "examine meanings, themes and patterns that may be manifest or latent in a particular text" (Zhang & Wildemuth, 2009, p. 308). Grounded theory was employed in the content analysis of the journals. With this approach, I developed codes inductively from the data instead of from previous research or frameworks (Hsieh and Shannon, 2005). This allowed me to investigate what the students in the course wrote about without any preconceived notions of what they should write.

The content analysis coding process was conducted as follows: the journal assignment entries were first read once to gain an overall feel for what the students were writing. I then read the entries a second time, writing down emergent codes of topics that appeared. After identifying some themes, I refined the coding scheme and examined the existing codes to determine if any could be combined.

Once a coding scheme was developed, successive journal assignments were analyzed using the same coding scheme. Pilot data suggested that themes might emerge during the semester that were not present in the first journal entries. Because I was interested in exploring how the students' thinking, as seen in their journal entries, developed over the semester, I added codes as needed during the semester and recorded when they emerged. In the axial coding phase, the results of the journal coding were organized into a coding matrix and were used to build a more in-depth understanding of each theme and the relationships between them.

After the content analysis was performed, I coded the unstructured journals a second time to see which benefits, as described in Borasi and Rose (1989), were present. Once this secondary coding was done, the benefits were compared with the themes from the content analysis to see in what ways the benefits were present in the journals.

The structured journal entries were analyzed to explore how the students' thinking about proof-related ideas developed and were revealed by the journals. Because the students had various topics and prompts, each structured entry was considered a separate data source and was coded independently from the other structured journal entries. The structured entries were developed from identifying proof topics from the literature review, such as the use of examples or definitions in proof writing, that research had determined students may improve upon. Thus, the structured writing assignments were coded using directed content analysis: the journal entries were first coded deductively according to the framework related to that research topic and then were inductively examined to identify any remaining themes that emerge (Zhang & Wildemuth, 2009).

The semi-structured interviews consisted of two parts, and each part was coded separately. Similar to the unstructured journal entries, Part I of the interviews was coded inductively using grounded theory. This allowed for continual comparison between the participants. Part II of the interviews, the task-based portion, was coded deductively using Raman's (2003) framework for proof ideas. The transcripts of the students' attempts, along with their written work, were analyzed to determine whether the students approached proof writing with heuristic, procedural, or key ideas. Once both parts of the interviews were coded, the results were compared to see if any similarities existed between students in terms of the themes present in Part I and the type of proof idea used in the Part II. Of particular interest was if there were any similarities between the students' responses to the questions about keeping the journal and their proof ideas in the task-based portion.

In the coding for all of the qualitative data, the unit of analysis had to be determined, as Zhang and Wildermuth (2009) note that the unit of analysis has implications for coding decisions. Rather than fixing the unit of data to a particular text length (such as one word or one sentence, etc.), the unit of analysis was a theme that may by represented by an unspecified length of text. For example, in the coding process during the pilot study, the code "confidence" was assigned to "I get a really satisfying feeling when getting the answer", "I just get frustrated and stressed out because I do not always follow exactly what is happening. I know I should ask more questions but my struggle is I don't even know where to begin", and "I'm scared". Even though these segments of text were all varying in length, they all represented the idea that the writer was expressing confidence or a lack of confidence. In coding with this type of unit of analysis, I chose to focus on the expression of an idea (Zhang & Wildermuth, 2009), which was more relevant to my research questions than narrowing my focus to a specific unit of text.

To analyze the Pre- and Post- Proof Views surveys, I began by open coding the responses. For each question, I listed the various responses, then began sorting them into groups to form codes. For instance, in the question asking the participants their favorite aspect of mathematics, I grouped “there’s always a concrete answer”, “getting the right answer”, “finishing a long problem and getting the right answer”, “solving a long problem”, and answers like this into the code Procedural Focus. Similarly, I grouped “understanding how something works”, “connecting multiple solution methods”, and “new ways of thinking about problems” into the code Conceptual Focus. Some responses included statements from multiple codes. For these cases, I coded the response using all of the codes present instead of deciding on one code. For example, when asked why they are taking the course, one participant replied, “The class is required to take, but I would also like to think like a mathematician since I would like to be one, one day.” This was coded as both “required” and “interest”, since both sentiments were present.

Once I had the codes established, I re-read the responses, using highlighters to code the responses and count the occurrences of each code. However, I did not just want discrete tallies of responses. Instead, I wanted to be able to compare how answers to multiple questions related to each other. So, I created a spreadsheet with the participants forming the rows and the survey questions from both the pre and post surveys forming the columns. For each participant, I entered the code for each of their responses into the spreadsheet. I was then able to sort the spreadsheet by multiple columns (looking at responses to multiple questions) to explore connections between questions.

Mixed Methods

Consistent with a concurrent triangulation design, the qualitative and quantitative data were collected and analyzed separately and then compared and interpreted to see how well the results supported each other or explained any divergence that occurs

(Tashakkori & Teddlie, 2003). The results from the qualitative surveys and the journal themes were compared to see how well the students' journals matched their self-reported reflection on how they used the journals. The results of the correlation analysis on students' performance were compared to the qualitative survey responses asking about the impact of the journal tasks on the students' learning. Additionally, the results of the analysis on students' views on the functions of proof were compared with the qualitative survey responses asking about the impact of the journal tasks on the students' views on proof.

Any similarities or inconsistencies in the results of the quantitative analysis and students' self-reported reflections on the journaling were noted. After the comparisons of the quantitative and qualitative data were complete, the data were compiled into a whole unit to provide an overview of the results. Put together, these analyses shed light on how the journals impacted the students' learning of proof throughout the semester.

Trustworthiness

This study employed qualitative and quantitative measures to ensure the validity of the data, results, interpretation. In the development and implementation of the quantitative surveys, measures of reliability and validity were calculated to certify the integrity of the results. The qualitative analysis employed the techniques of triangulation, member checking, and peer checking of coding described by Creswell and Clark (2007) to ensure the validity and reliability of the analysis. The triangulation of data occurred between the students' journals, interviews, and their survey responses. By employing these methods, I was able to draw valid and reliable conclusions to answer my research questions and expand the research body on reflective journaling to aid the learning of proof.

IV. RESULTS

In this chapter I provide the results of the analysis and examples of how the data were analyzed to investigate the effects of reflective writing on students' learning of mathematical proof writing. I begin by describing the course and both sections of the course and then discuss the students' perceptions of the course and journals. Next, I describe the interview cases in detail to give the reader a deeper idea of what the students in the class were like and their perceptions of how they experienced the course and journals. I then discuss the other data sources, including the structured and unstructured journal entries, Pre- and Post-Proof Views surveys, and course performance data. The data are organized by research question instead of by data source to allow the multiple data sources to be presented together and create a more holistic understanding of each research question. Within the section for each research question, I discuss how the different data sources help create the narrative of the effects of the journals. When I describe the participating students and professors, I am using different names to protect their confidentiality.

The research questions in this study are:

1. What beneficial themes arise in undergraduate students' journals in an introduction to proofs course? How do students use the journals?
2. How does journaling in an introduction to proofs course affect students' ideas about proof writing?
3. How does journaling in an introduction to proofs course affect students' views on the functions of proof in mathematics?

4. How does journaling in an introduction to proofs course affect students' learning of proof, as demonstrated in their proof writing performance?

Instruction by Section

To gain a better understanding of how a typical class day in each section was structured, I observed both sections twice, keeping field notes of the instruction techniques and student behaviors and questions. This also helped me interpret any differences that appeared in the analysis. Below, I describe a typical class for each section.

Section 1

Dr. Austin structures the class with a mixture of lecture and practicing exercises from the textbook. They begin by handing back papers from previous classes and asking if students have any questions, then discussing the agenda for the day. Dr. Austin used two textbooks, *Proof: Introduction to Higher Mathematics* by Esty & Esty (2014) and *Book of Proof* (2nd Ed.) by Hammack (2013). Students were required to bring the Hammack book to every class.

Dr. Austin's lectures follow closely with the textbook sequencing for the chapters. Dr. Austin projected the textbook onto the board and used the adjacent, empty board space to elaborate on definitions and conjectures and to provide examples. During the practice exercises, Dr. Austin called on students by name to give the next step for the exercise, sometimes calling on students alphabetically to ensure everyone answers a question. Dr. Austin generated proofs for the class using a mixture of lecturing on the structure of the proof and notation, and asking students in the class for ideas about how to proceed. When students would give incorrect ideas as to how to proceed in a proof, Dr. Austin used it as an opportunity to discuss the nature of doing mathematics and making mistakes, saying “Sometimes you start down a line reasoning that doesn’t work out, and that’s ok.” Further, students would sometimes offer suggestions that skipped steps, and Dr. Austin would make sure to describe the missing steps and write them in detail, explaining reasoning as they went. Students in this section were given 14 homework assignments and once a week to turn in, as well as 14 quizzes throughout the semester.

Section 2

A typical class period in Section 2 began with Dr. Bates asking students to identify where the class left off the previous day. Dr. Bates discussed the topic for the day (equivalence relations) briefly. Dr. Bates did not use a textbook, and instead used a deliberately sequenced list of definitions and theorems, organized into chapters and sections, and structured the course to be inquiry-based. Dr. Bates began with the problem the students brought up from the previous class (an example of a relation and the students were to determine whether was an equivalence relation or not.) Dr. Bates would often pose a question to the students about whether they felt a statement or theorem was true and wait for multiple students to voice their reasoning.

Throughout the class, Dr. Bates was not the primary person talking, and instead the students were prompted to explain reasoning and ideas. Further, when Dr. Bates did write proofs on the board, Dr. Bates wrote what the students told them to write (even if it was incorrect) and constantly asked the students to evaluate the statements and respond. Dr. Bates did occasionally make evaluative statements, for example, “Ok, you think it is transitive by example, which is a good starting point but not enough for a proof.” Dr. Bates also asked the students lots of clarifying questions, and revoiced their explanations, asking for confirmation that Dr. Bates interpreted the comment correctly.

In addition to writing student-guided proofs on the board, Dr. Bates required students to present proofs (20% of their course grade) on the board and “defend them”. Students were expected to write proofs at home or sometimes in class, and then present explanations, answering questions and clarifying their reasoning to their fellow classmates. In Dr. Bates’ class, the students shared the responsibility for determining the correctness and validity of proofs. In addition to the presentations, the students were given six homework assignments to turn in for a grade throughout the semester.

Students’ Initial Views of Proof and the Course

The students’ goals for the course were predominantly focused with learning to prove (18), and developing their thinking (10), earning a certain grade (8), and having fun (4). Nine students listed multiple goals, generally describing a grade goal and then another type as well. The participants overwhelmingly found high school math to be easy (22), and only 2 found it challenging. Five students were taking the course because they were interested in the material, 25 because it was required, and 5 as an elective. Further, 21 students had taken proof-based classes before, while only 10 had not.

When asked to give their definition of a mathematical proof, 4 students gave definitions of proof as a process, 8 gave product-focused definitions of proof, 8 left the questions blank, and two said they did not know. Participants mentioned all 5 functions of proof, though not equally: convince (13), explain (13), discover (1), systematize (1), and communicate (7). Also, 5 had previous negative experiences with journals (ex: forgetting them for a class so their grade was lowered or parents reading their personal journal). Fifteen students had no previous experience with journaling, 5 had some experience, 4 keep a journal regularly, and 4 discussed keeping a math journal as a list of problems or notes from class.

End-of-Semester Course Impressions

At the end of the semester, 28 students completed the Post-Proof Survey instrument. Overall, the students from both sections enjoyed the course, with 17 giving positive impressions of the course, 7 negative, and 4 giving neutral responses in the Post-Proof Views Survey. Sixteen participants felt the course positively influenced their beliefs about proof, showing them the importance of proof (8), increasing their confidence (4), and showing them more diverse proof methods (4). Seven students felt the course negatively influenced their beliefs about proof, claiming the course made them realize how much of a hassle proofs are to write (4) and how "picky" proofs are (3); and 4 students felt their views about proof did not change during the course.

Interview Cases

Here, I describe the 5 interview cases to help the reader get an in-depth idea of the students in the course and how their thinking progressed throughout the semester, according to their survey and interview responses. Before describing the interviews, I give a description of each student at the beginning of the semester based on their responses to the Pre-Proof Views Part I survey. I then discuss each participant's responses to the questions in Part I of the interview, the semi-structured portion, and their progress in Part II, the task portion. By having examples of students' detailed reflections about their progress in the course and impressions of the journals, I was able to get a more detailed idea of exactly what it was about the journals that the students liked and did not like. The interviews were conducted during the last 2 weeks of the semester. The task portion of the interview asked the students:

> Consider the statement: For real numbers a, b, and c, if $a < b$, then $ca < cb$.
> - If the statement is true, construct a proof.
> - If the statement is not true, please modify the statement to describe the correct relationship between ca and cb and then construct a proof of the modified statement.

Later in this chapter, I will reference aspects of the interviews in concurrence with the discussions of each research question.

Bill

Bill was a sophomore mathematics major. He experienced proving before the introduction to advanced mathematics course in his discrete mathematics courses. He was taking the course because it was required, he was interested in the subject, and he hoped the course would show him a new way to think about mathematics. Bill found high school mathematics hard, and his favorite part of mathematics was understanding conceptual ideas. He believed a proof is a product, and that people write proofs to validate statements. He had no previous experience journaling. Bill completed 7 journals and received an 83 in the course.

Part I: Impressions of the Course and Journals

Bill found the course interesting and fun, but not advanced enough for him. He said he was not learning new math, but a new way of thinking. However, when asked how the course affected his beliefs about mathematics and proof, he said it hadn't. His definition of a proof was "a statement that holds true for any, uh, possible case". He was the only participant whose definition of a proof was product-oriented, not process-oriented. However, his favorite aspect of proof, "using proofs to prove new results", represents the process of discovery during proof writing. His least favorite aspect of proof was typing, since he found it tedious. Bill's overall impressions of the journals were that they were unnecessary for him, and that he forgot about them most of the time. When asked if the journals supported his learning to prove, Bill said "it just keeps you, uh, keeps you up to date of ... It just, uh, you know, just see ... It makes you see how, how you're doing, if you have any problems". Here, Bill seems to be recognizing the improvements in learning and problem solving benefit of the journals. Bill did not feel the journals influenced his opinion about proofs, and noted that he did not experience any benefits from the journals, saying "I guess the journal would help people who are, who are more like ... Who, who are kind of, who are kind of a little behind in the class maybe". Bill also noted that he did not ever check the feedback and that he would not recommend keeping a journal to a friend about to take the course. Overall, Bill felt that the class was beneath his ability level, and he was able to be successful without much effort. So for him, the journals were unnecessary and he found them to be just one more thing to remember to do each week.

Part II: Task

Bill was focused on the written proof and wording of the statements. Bill's product-focused definition of a proof was also visible in the task portion of his interview. He was the only participant whose definition of a proof was product-oriented, not process-oriented. He incorrectly referred to the statement of the problem as a proof, noticing immediately that, since the statement did not hold true for negative values of c, "the proof is incorrect". He modified the statement to require a, b, and c to be natural numbers instead of just requiring c to be positive. Once he modified the statement, I asked him to create a proof. He wrote "We know $a < b$, therefore…" and quickly said, "It just seems really like…I don't like…I don't know how to word this. [Wrote ca < cb] This is by, by the definition?" although ca<cb is not by definition, and is in fact the conclusion to be proved. After a few minutes it was clear that Bill did not know how to continue, so I gave him the hint that for real numbers m and n, m <n is equivalent to saying n = m + k, where k is a positive real number. Once he had this, he quickly wrote the rest of his proof, using correct mathematical notation ($k \epsilon \mathbb{N}$, $\Rightarrow$ for an implication), language ("by substitution,…", "therefore the statement holds"), and showed a general grasp of proof construction. However, he incorrectly assumed the conclusion and used $ca < cb$ in his proof. Bill's focus on a proof as a finished product and his procedural knowledge of proof writing were not enough to help him correctly finish the proof. He was hindered by the fact that he didn't consider carefully how each line he wrote compared to the statement he was trying to prove. I coded Bill's proof attempt as demonstrating procedural ideas about proof writing.

Caleb

Caleb was a mathematics major. His goal for the course was to learn to read and write proofs. Although he was interested in the subject, Caleb primarily enrolled in the introduction to advanced mathematics course because it was required for his major. Caleb had no previous proving experience, and mentioned that he would always get confused when professors in courses like Calculus presented proofs. Caleb completed 11 journals and received a 72 in the course.

Part I: Impressions of the Course and Journals

Caleb thoroughly enjoyed the course, saying he liked the professor and thought the course was "definitely a vital course for any math major. And it's useful for anyone who's going into a field of like logic, like computer science, and stuff like that as well." The course affected Caleb's beliefs about mathematics by showing him that "math isn't all about numbers and formulas. It's, it goes much, much deeper into that". Caleb's definition of a proof is a series of steps to prove an idea, a view of proof as a process rather than a finished product. The course affected Caleb's beliefs about proof by showing him that proofs are about the process of how to get an "answer and how it works". Caleb noted that, until the course, all proofs he had seen were presented as complete, finished products, rather than ideas about why things work. Caleb's favorite aspect of proving was the puzzle, and his least favorite aspect was trying to prove things that seem obvious but aren't. Caleb admitted that if he would have kept up more with the journals, they would have been more beneficial. When asked if the journals supported his learning to prove, Caleb said they helped jog his memory of what he learned and needed to work on (improvements in learning benefit) and make mental notes of steps (content benefit). He also felt the journals affected his opinions about proof because they made him "humble myself a little bit, because there's going- there's times where you would just, you know, obsess over a problem and it's, not that it's so difficult, it's just trying to find the right steps, and then, um, sometimes you just need to ask for help." Caleb also listed memory retention (content benefit) as a benefit of the journals. When asked how the journals could be changed to be more effective, Caleb suggested having one per class and a weekly summary journal. He felt the feedback motivated him and provided encouragement (therapeutic benefit) because he "wasn't writing for nothing". Finally, Caleb said he would recommend keeping a journal for every class, and indicated that he planned to keep a journal in the upcoming semester.

Caleb's analogy of a proof to a puzzle highlights his key ideas about proof writing. In order to complete a puzzle, you have to consider all the individual pieces and have a plan for putting them together (procedural ideas), and also think about the big picture (heuristic ideas). Also, Caleb's least favorite aspect could be interpreted as the instances when he was unable to connect his heuristic ideas with the procedure of how to structure the proof.

Part II: Task

Caleb started off declaring that the statement was true, incorrectly stating, "You add the same thing on both sides, it really doesn't change the two variables. So, I mean, if you wanted to, you could go ahead and like write an example so you could have like, you know, *a* equals 1, *b* equals 2. So, you know, you still have your *a* is greater than *b*. I mean, you have *c* equals 3, so if you have your 1 times 3, um, and then you have your 2 times 3, and you just work through it, 3 is still going to be less than 6. And, it, it doesn't matter. So you, so you know it's true." Caleb confused the multiplication by c on both sides with adding a positive value to both sides. Once he decided the statement was true, he said "Um, and then, first thing you do is figure out what kind of proof you want to do with this." Caleb's process was to first try to understand the situation, and then think about the proof. However, as he was thinking about which proof method to use, Caleb realized "Because there's a whole lot thrown at you. Um, let's see. Um ... A and B. B, real numbers. Because I can think of a counterexample where it's not going to be true. ... Yeah, because once you go into the negative numbers, it's not going to be true." Although he said he was thinking about how to prove the statement, Caleb was still trying to grasp the situation, and in doing so corrected his earlier mistake. Caleb modified the statement to have a, b, and *c* positive real numbers. Then, although I gave him the number theory hint, he proceeded to write a complete, correct proof. He noted that since $a < b$, then $0 < b - a$. Since $c > 0$, $0 < c(b-a)$ and he distributed, getting $0 < cb - ca$, which implies that $ca < cb$. After he finished writing the proof, he quickly said, "I did direct proof". For Caleb, the process of writing the proof relied on his constantly reaching back to his heuristic understanding of the situation. Even after he began writing the proof, he continued to refine his understanding of the problem. Also, he explicitly stated that he would think about what proof method was best for the situation. For these reasons, I believe he demonstrated key ideas about proof-writing.

Diana

Unfortunately, Diana did not fill out the Pre-Proof Survey, so I am unable to describe her experiences prior to the course. Diana completed 9 journals and received an 88 in the course.

Part I: Impressions of the Course and Journals

When asked her overall impressions of the course, Diana noted that it was "definitely not a regular math course" and that it was new and stressful because it was hard to get help. The course influenced Diana's beliefs about mathematics because it was "an eye opener," and she was "glad [she] didn't have to do that [write proofs] before. However, Diana felt the course influenced her beliefs about proof by improving her confidence. To Diana, a proof was a way to show something is true, a process-oriented definition. Diana favorite aspect of proof was "getting it right…seeing it all tied together," and her least favorite aspect of proof was "not knowing where to start". Diana liked the journals, saying they "show where you were from beginning to end," (improvements in learning benefit) and she felt they influenced her learning to prove by allowing her to "cheer myself on" (therapeutic benefit) and pinpoint problems (improvements in learning benefit). Diana felt the journals influenced her opinion on proofs by helping her pinpoint issues, and also listed pinpointing issues as the benefit she received from the journals. Diana suggested the journals could be more effective by using repeated prompts to show how your opinions change over time (views benefit). Diana did not realize there was feedback on the journals, although she figured her instructor read them because Diana observed instances when she would make a comment in her journal and the instructor would address that comment the following class, which Diana appreciated. When asked if Diana would recommend keeping a journal to a friend about to take the course, Diana responded "Absolutely."

Part II: Task

Diana began by stating, "Looking at proof the first thing I think is if I can contradict it...i mean, like find a counterexample where it's not true." The counterexample she used was if a = -2, b =1, and c = -1, and she correctly realized that the statement failed. However, when modifying the statement, Diana only required c to be positive, suggesting that she understood that her choice of a = -2 in the counterexample was irrelevant, and suggesting that Diana had a deep understanding of the situation. As she began her proof, she said, "I would start out with writing 'proof". [The professor] likes when we do that. And so you start with your hypothesis, so that is, you assume that a is less than b and then you have c>0." Diana focused heavily on the formatting of her proof, mentioning her professor's preference for how proofs are written. After writing the assumption, Diana said "So, I know there's, like, a theorem that says multiplying number on each size will keep it the same but I don't know which one it is, it's in the book... But I wasn't given any theorems, but pretty much that's how I would do it." Diana’s proof-writing procedure was to set up the assumptions and then use previous theorems to arrive at the conclusion, which is not incorrect. However, Diana was unable to think back to her understanding of the situation and implement a heuristic or key idea to help her write the proof. She was unable to complete her procedure because she didn't have access to a list of theorems, and therefore quickly gave up on the proof. It is also interesting to note that Diana showed the deepest understanding of the original statement since she was the only participant to just restrict c to positive values and not a, b, and c. However, ultimately, her procedural focus on writing the proof hindered her ability to relate her understanding of the situation to writing the proof.

Alicia

Alicia was a senior applied mathematics and computer science double major. Her favorite part of mathematics is the confidence she gets from being good at a subject so many people struggle with, and her least favorite part of mathematics was feeling stupid when she does not understand a problem. Before enrolling in the introduction to advanced mathematics course, Alicia had proving experience in two discrete mathematics courses. At the beginning of the semester, she believed a proof is a process and that people prove to validate and explain. Alicia had some previous negative journaling experience. Alicia completed 10 journals and received a 92 in the course.

Part I: Impressions of the Course and Journals

Alicia found the course fun and enjoyed the problem solving and logic in the course, although she noted she felt the clas moved slowly. However, she said that there were students who were struggling more than she was, so the pace was appropriate for them. She did not feel the course changed her beliefs about mathematics or proving because she had already taken discrete math, and had learned proof methods in that course. To Alicia, a proof is “a method used to show that something is true or false” (process). In this definition, a proof is a process instead of a product. Alicia enjoyed logic and rules in proofs, and did not have a least favorite aspect of proof.

Alicia's overall impression of the journal was that it was helpful in organizing her thoughts, and she said she was planning to keep one in analysis in the following semester. Alicia felt the journals supported her learning to prove because she saw the instructor make adjustments in class based on Alicia’s journal comments. Alicia did not think the journal influenced her opinion about proofs. The benefits Alicia described from the journal were content (poses good questions about the material), therapeutic (helpful to write informally, get thoughts on paper), and problem solving (keep you organized, think about how to improve in your proof writing). Alicia did not realize there was feedback, and her suggestion for improving the journals was to include more questions about how the students stay organized to encourage the students to stay organized in the class. She said that, yes she would recommend keeping a journal to a friend about to take the course, especially if they were struggling.

Part II: Task

Alicia was the most successful at completing the task. She immediately recognized that the statement was incorrect, noting "well if c is 0 it's not true" before I finished reading the statement to her. Once she decided the statement was incorrect, Alicia spent the most time of all participants thinking of modifications, noting "well you could put an equal because you see you have real numbers, so ... Um, if *c* is a negative fraction then it's not true, either. Or just a negative, I think...Yeah, just a negative in general, it's not true. You'd have to change, um ... I guess you could change the real numbers to naturals and then that could happen." Her recognition of multiple ways to modify the statement shows that Alicia had a sophisticated understanding of the problem, and thought heuristically about the problem statement. However, Alicia also thought of how her modification would affect her proof, and decided "Yeah, because then you have to think about negatives and whatnot. So if I just do naturals, then that's an easier thing". Thus, Alicia displayed a key idea about proof by connecting her knowledge of the problem situation with her knowledge of how to prove the statement. As she wrote her proof, Alicia got stuck after writing the assumptions. However, rather than getting frustrated like Bill or giving up like Diana, Alicia said "Well, let me think about it". She continued to struggle, so I gave her the number theory hint that $m < n$ is equivalent to saying $m + k = n$ for some $k>0$. Alicia continued to struggle, saying, "You could just multiply both sides by a natural number for *c* and it's still true. That's some algebra thing that you can do, I don't see why, okay...so I can think of a couple ways to do this. So, I mean, this is still true, but that's what you want to prove that this algebra statement is true." Then, however, she used the hint, deciding, "Okay, so let's see if we can use this [the hint]. If m is less than n and it's true that $m + k$ is equal to *n* right, so that means that is *a* is less than b, then *a* + some *k* is equal to *b*, right?" From this point, Alicia quickly finished writing a complete, correct proof. By employing key ideas about writing proofs, Alicia was able to understand the situation, modify the statement appropriately, persevere past getting temporarily stuck, and correctly structure and write a proof.

Emma

Emma was a senior mathematics major at the time of the introduction to advanced mathematics course. She had previously written proofs in her linear algebra and history of mathematics courses. Emma is an English Language Learner, and her first language is Spanish. She finds mathematics to be “both easy and difficult, but always fun”. Her goal for the introduction to advanced mathematics course was to learn to write proofs better. Her favorite part of mathematics was solving problems, and her least favorite part was memorizing formulas. In the Pre-Proof Views Survey, Emma expressed that she did not know enough about proofs yet to give a definition of a proof or describe why people prove. Student E completed 11 journals and received a 66 in the course.

Part I: Impressions of the Course and Journals

Emma enjoyed the course, noting she “started out struggling, but it’s not that bad”. She didn’t think the course changed her views about mathematics or proof because she had “been in similar classes before”. When asked her definition of a mathematical proof, she said “Hmm, let's see, that's a hard one, because I don't know… Like you're basically describing, you know, like the steps on how to get to that answer and how that works. Saying how, why it works.” Although she was not confident, her definition was focused on proof as a process of describing. Her favorite aspect of proof was proving by induction and her least favorite aspect was proof by contrapositive. Both of these responses represent specific proof methods, not general aspects of proof. After responding with contrapositive as her least favorite aspect of proof, Emma said, “Yeah, those in addition to the one that I forgot the name of it but I can't ... I think I better look in my journal. I can't stand that one.” In this response, Emma shows that she considers her journal entries as a resource she can refer back to. Emma enjoyed the journal, using it as a record of her struggles and progress, a way to set goals, and a way to pinpoint how she works in class. When asked if the journal supported her learning to prove, Emma replied, “It made me like okay I was kind of embarrassing. I was not good at this. And then by next time I write it I want to be able to say I improved that.” Emma appreciated the feedback because “It’s helpful to have somebody behind you.” She did not have any recommendations to make the journals more effective, and said she would recommend keeping a journal to a friend about to take the course.

Part II: Task

Emma spent the most time of all participants deciding if the statement was true. She began by looking at examples, saying "So let's say we have four is less than six, right? Then CA is less than CB. And let's say that C is equal to, I don't know, three. So three times four, that's twelve less than ... What is this, eighteen. So that one's true but then do we know if it works on all the numbers." She then tried using the example $-3 < 2$ and multiplying both sides by -2, and realized that the statement was incorrect. At this point, Emma said "So, then this is false" and stopped. I prompted her to modify the conditions to salvage the statement. She said, "Let's say that they are pos., for any positive integers. Then that’s the only way that it would be true". When she began writing the proof, Emma got stuck immediately, and even with the hint that saying $m < n$ is equivalent to saying $m + k = n$ for some $k>0$, Emma was unable to continue. With heavy coaching and help, Emma was eventually able to write the proof. After she wrote each line, she asked me if she was correct; it seemed she did not have the confidence in her procedural ability to write the proof. She was much more confident with heuristically understanding the situation of the problem, and with talking about what she was thinking than with actually writing the proof. In Emma's case, her heuristic ideas about proof writing hindered her from being able to complete the procedure of writing the proof.

Research Question 1: Journal Use and Benefits

Benefits of the Unstructured Journals

Altogether, the journals consisted of 6 unstructured prompts and 7 structured prompts. Overall, students completed an average of 8 journal assignments. In their Post-Proof surveys, many students section mentioned forgetting about the journals and felt they would have been more helpful if they would have remembered to complete more of them.

In the structured prompt journals, the students were given specific prompts to write about in order to focus their reflection on certain aspects of mathematical proof (ex: the use of definitions in proof writing, proof writing strategies, judging the completeness of a proof, etc.). Since I was interested in how each type of journal influenced the students' learning, I coded the unstructured and structured journal entries separately. I also did this because I wanted to look at what the students chose to write about in the unstructured journal assignments. This gave me an idea of how the students used the journals.

There were a total of 150 unstructured journal entries over the course of the semester. I coded the unstructured prompt responses (weeks 1, 3, 5, 10, and 12) according to the Borasi & Rose (1989) framework of 4 benefits students experience from journaling in a mathematics course: improvements in learning and problem solving (coded as Learning), improvements in content knowledge (coded as Content), improvements in views towards mathematics (coded as Views), and therapeutic benefits (coded as Therapeutic). In 71 instances, evidence of more than one benefit was present in the writing. When this occurred, I included the code for each benefit present; in fact, I was happy when it occurred, as it showed the student getting multiple benefits from the journal. This brought the total occurrences of the Borasi and Rose (1989) benefits in the students' journals to 221. The unstructured journal responses helped me examine research question 1 (What beneficial themes arise in undergraduate students' journals in an introduction to proofs course? How do students use the journals?).

The summary of the 221 occurrences of benefits supported in the students' writing is given in the table and figure below:

Table 9

Occurrences of benefits present in students' unstructured journals

Therapeutic	Problem Solving	Content	Views
111 (49%)	81 (36%)	23 (11%)	8 (4%)

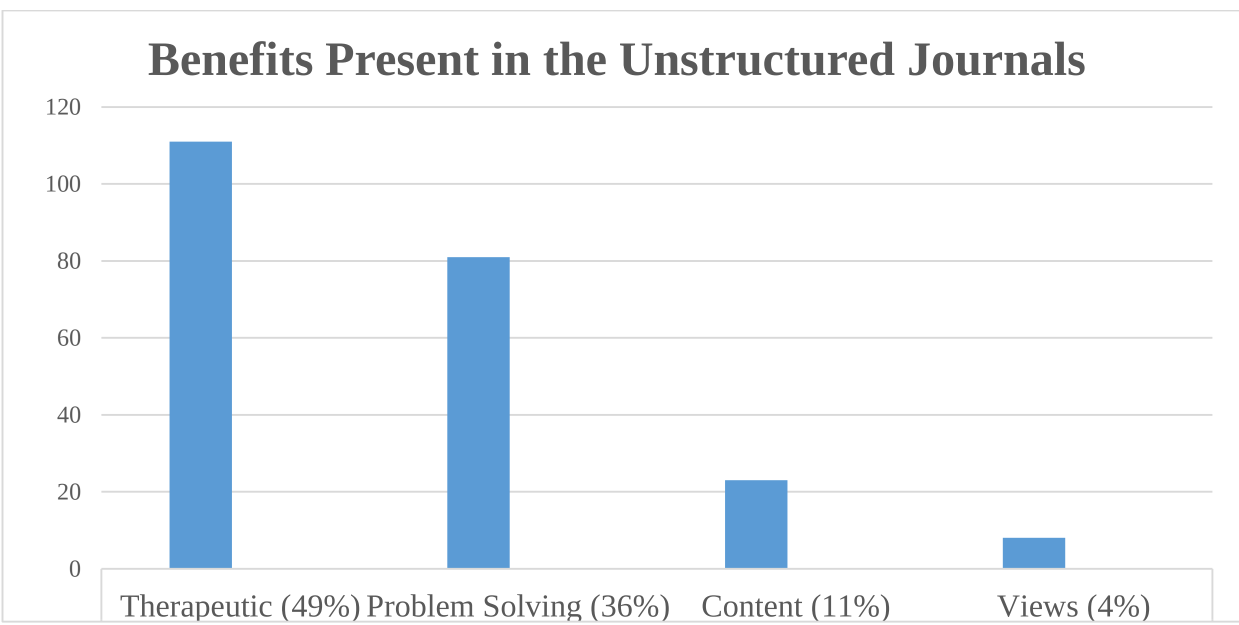

Figure 6. Benefits Present in the Unstructured Journals.

Therapeutic responses were most prevalent in the unstructured journals, followed by problem solving, then content, and views. For the therapeutic responses, I also considered whether the responses were positive, negative, or neutral. Of the therapeutic responses, 47% were positive, 20% were negative, and 33% were neutral ("The pace and difficulty seem like it should be for this time of the semester". – Pam).

Below, I give two examples of unstructured journal text from the 150 unstructured journal entries that support each of the four benefits to demonstrate my coding;

<u>Benefit 1: Therapeutic</u>

Example 1, from Caleb:

> I am working on it along with all the work from my other classes. I get an overwhelming feeling and sometime become paralyzed. Then I start taking baby steps to keep moving. Dr. [professor] is great and really wants to see his students succeed and is very helpful. I appreciate that.

Example 2, from Alicia:

> Going over the proofs in class as a discussion, and showing everyone where we get stuck is nice. it's like working as a team! It's fun. I think it would be nice to see an example of a complete and well written proof just as a good reference. But I guess we can still kind of figure it out without that.

<u>Benefit 2: Learning and Problem Solving</u>

Example 1, from Bill:

> The class pace was good for the content that we are going over. some of the content i was having problems with understanding but under careful review of the

> text book i understand it. the difficulty was only causing me issues because i wasn't understanding how it worked.

Example 2, from Mike:

> I enjoy the roulette of doing problems in class in addition to the assigned homework. It helps you confirm or correct any issues with the homework by using similar examples.

<u>Benefit 3: Content</u>

Example 1, from Diana:

> My main questions concern proof of absolute values, by induction, and proof by contrapositive. I feel like we didn't spend enough time on these during class so I never really got a good understanding of them. The way these proofs are worded in the book are weird and need to be explained in a simpler way in class.

Example 2, from Ryan:

> I really like the idea, even though it is slightly uncomfortable, to recite a problem using mathematical terminology. One thing that I think is going to be somewhat concerning, is learning how to identify the form of a preposition and knowing when there's another form that will be easier to use as a proof. Changing the implication form into an or statement, or in the conclusion, and such, as well as using the Empirical Truths.

The discussions of content were generally only a few sentences in the unstructured responses, as opposed to the longer (approximately 1/3 of a page) content discussions in the structured prompts. Further, in the unstructured responses, the discussions of content were generally related to therapeutic professions of like/dislike or perceived difficulty level.

Benefit 4: Views about Mathematics and Proof:

Example 1, from Diana:

> It's [the course is] really getting me to think differently about the meaning of definitions. I'm starting to realize that I can't always assume that the obvious answer is the correct answer. When it comes to thinking like a mathematician, I have to learn how to really think outside the box. I think the pace is just right and wouldn't ask for it to change.

Example 2, from Kyle:

> The pace of the class is fine, so far this is the most difficult math course I have taken thus far. Before this class all my math classes revolved around solving an equation, and this one has much more to do with proving why certain equations and theorems in mathematics hold true. I guess it will just take some getting used to this new study of mathematics for me to really understand the point behind it.

The unstructured responses generally tended to follow a similar pattern:

1. Overall course impression (generally a therapeutic statement. ‘The class is going really well...’ or ‘I am not enjoying the course...’)
2. Statement(s) to expand upon the course impression (could be any kind of the four benefits. ‘X helps me learn best’, ‘it’s getting me to think differently about X’, and ‘I do not like X types of proofs’

There is a third, less common component of the unstructured journal responses that appeared in approximately one fourth of the responses:

3. Discuss future plan or study goals. This is generally a problem solving, content, or therapeutic statement, such as "I took off work Tuesdays and Thursdays so I can now come to office hours" –Kyle, "I know though that everything will work out in the end" –Rachel, or "I want to be guided in a way that I can think on my own and accept critiques" – Travis

A common theme that emerged in the journals was the students' perceived ambiguity of explaining mathematics, demonstrated in the following journal excerpts:

> "I am a math major and proving is by far the most difficult. Not difficult because I cannot do it, but I have difficulty explaining. My goal of the course is to better my explanations in proving" –Kyle.

"There will often be questions that stump me, but hearing others explain what they got makes me understand it." – Amy

> "As we hit the quarter mark of the semester, the class is coming into its own. It's not learning new information as other classes are but the expanding of our minds and the conceptual thinking to cement previous knowledge. Hearing how other people tackle problems and having to be able to explain and defend my own thinking really forces us to think about the content." –Sam

Students discussed their difficulties putting their ideas into words or explaining their reasoning. They particularly struggled with proofs of theorems that seemed to be simple ideas but were not obvious proofs to them, such as "the product of two even integers will be an even integer."

Another common theme was the idea of getting stuck and not knowing where to go. Examples of these responses are:

“I don’t find anything particularly difficult except just getting started on my logical thinking. Once I figure out which direction I’m heading, I get more confident as I go.” –Hannah

“I also have no idea how to even start writing a proof even though apparently our whole class has the basic tools needed to do this.” -Diana

“I knew how to start the proofs but I always had a problem with choosing what my next step would be and how to pick the best next step” -Connor

“I struggle with not seeing the obvious and thinking "what else do i know that i can use?" It will take me some practice and im a slow learner but im hoping to get to that point.” – Violet

Many students described their frustration at getting stuck in a proof and not knowing how to proceed. They often described a procedure of writing the skeleton of a proof: starting with the word ‘Proof:’, writing the assumptions, writing what you are trying to show, and using definitions and previous theorems to get from the assumption to the conclusion. However, once they had the skeleton many students reported getting “stuck” and not knowing how to connect the assumption and conclusion. This feeling of being stuck and not knowing how to proceed is something my literature review identifies as a common struggle for students learning proof, and I anticipated this theme in the unstructured journals. I will discuss this theme further with the structured journals below.

A third theme that emerged in the journals was organization. Students discussed learning goals, study habits, and how they prepared for the course in their journals. For example:

"The only problem I am having with this class is finding the time to practice the way I would like to and completing homework assignments on time." -Travis

> "the difficulty for my isn't bad i just have to ensure i read the book and review the assignments before class. the only issue have been having is getting induction proof down but i think i understand now." – Amy

This theme is broader, and overlaps with the Problem Solving benefit from Borasi and Rose's (1989) framework. However, I felt it was worthy of noting because the organizational component provides a record to the students and the instructor of progress and goals, as the students see them, throughout the semester. These themes are:

Table 10
Themes generated in unstructured journals.

Theme	Description	Examples
Ambiguity of explaining mathematics	Students discussed their unease with explaining their thinking or writing using correct mathematical notation and conventions.	"I am a math major and proving is by far the most difficult. Not difficult because I cannot do it, but I have difficulty explaining." "There will often be questions that stump me, but hearing others explain what they got makes me understand it."
Feeling stuck in a proof	Students described their frustration with feeling stuck in a proof and not being able to connect their understanding of the ideas with how to write the proof.	"I knew how to start the proofs but I always had a problem with choosing what my next step would be and how to pick the best next step." "I struggle with not seeing the obvious and thinking "what else do i know that i can use?" It will take me some practice and im a slow learner but im hoping to get to that point."
Organization	Students discussed learning goals and steps they were taking to achieve their goals.	"The difficulty for my [me] isn't bad i just have to ensure I read the book and review the assignments before class." "The only problem I am having with this class is finding the time to practice the way I would like to and completing homework assignments on time."

Benefits of the Structured Journals

My aim in this section is to provide examples of student responses and a discussion of the types of responses students gave to the structured prompts. Further, I do so in a way that demonstrates how the journal responses provided different information to the instructor (via my weekly summaries) than submitted homework proofs and supplemented the course content.

The structured journals made up 7 of the 13 total journal entries that students completed, and thus compose a significant part of the students' experience with the journals. In addition to seeing the three themes described above in the unstructured journals and in the research literature on students' experiences learning proof and reflective journaling in mathematics, I began noticing these themes starting in the first structured journal entry. One of the goals of the structured prompts was to help students think about and become aware of times when they managed to overcome their feelings of being stuck and figure the proof out, how they understood various proof-related ideas, and the types of activities that helped them in the course. Another goal was to help students think about the process of proving.

The structured journal prompts are presented in the table below:

Table 11
Structured prompt used and associated benefit.

Week	Prompt	Benefit
2	Discuss the role that definitions play in mathematics and writing proofs. How are definitions important? How might you use definitions when writing proofs? Please also describe any other questions or comments you have for the course that you think are important.	Problem Solving, Views
4	Choose a definition that you have recently been using in class - it can be one that you understand well or one that you are struggling with. Write the definition using formal terminology, and then write in words how you interpret that definition. How would you describe it to a friend? Please also describe any other questions or comments you have for the course that you think are important.	Content, Problem Solving
6	When given a theorem to prove, what is your proving strategy? How do you judge the completeness of a proof?	Problem Solving
7	Pick a proof or problem that you recently completed and copy this into your journal. What did you think about and what was your process for solving that problem? Please also describe any other questions or comments you have for the course that you think are important.	Content, Problem Solving
9	Please pick a proof that you recently completed and copy this into your journal What did you think about and what was your process for solving that problem.	Content, Problem Solving
11	So far in the course, what is your favorite proof technique? Why? What is your least favorite proof technique? Why?	Views, Content
13	For this final journal, please reflect on the course and your progress throughout the course this semester. What advice would you give to a student about to take an introduction to proofs course? How, if at all, did your ideas about mathematics and proofs change during the semester? Please also mention any other comments or questions you have.	Views, Problem Solving

Consider, for example, the week 2 prompt about definition use in proofs. I used open coding to look for emergent themes in the students' responses, which are summarized in the table below:

Table 12
Themes emerging from the week 2 journal responses.

Theme	Example	Frequency
Tool to construct proofs	"Definitions are a tool used in proofs. It is what we use to base assumptions. Definitions is how we group numbers and theories to place them in our little containers of types."	3
Communicate with the reader	"Definitions are important for the reader to be able to understand and follow what you are describing in your proof."	2
Tools to build knowledge	"Definition are essential to proofs because they are stated facts and are needed to derive a true or false statements. If you didn't have definitions and axioms. There would be no sense of truth."	7
Organizational tool	"Definitions are very important when writing proofs, they are important because if you did not use them then your work would be messy, unorganized and hard to follow."	2

In addition to submitting the weekly homework assignment, which consisted of theorems the students were to prove and turn in (as finished products), this journal helped the students think about the importance of definitions in the process of writing proofs.

This focus on the process of writing proofs continued to be apparent in the students' responses to other journal prompts, as well. For example, consider the following excerpts of responses to the week 9 prompt:

> [This is what the student is trying to prove] $6 \mid n^3 - n$
> Base Case:
> n= 1
> $6 \mid 1^3 - 1 \;=\; 6 \mid 0$ TRUE
> Assume there exists a k such that:

$6 \mid k^3 - k$
Which by defn means
$k^3 - k = 6m$
Then prove the statement holds for k+1.
$(k+1)^3 - (k+1) = 6m$
$(k^2 + 2k + 1)(k+1) - (k+1) = 6m$
$k^3 + 2k^2 + k + k^2 + + 2k + 1 - k - 1 = 6m$
$k^3 + 3k^2 + 3k + 1 - k - 1 = 6m$
$(k^3 - k) + (3k^2 + 3k) = 6m$
$(k^3 - k) + 3(k^2 + k) = 6m$
$(6m) + 3(k^2 + k) = 6m$
If k is even $k^2 + k$
so 2m
if k is an odd
Then $k^2 + k = 2q$
Using induction for this problem has opened up my mind that you can use it for problems other than just Summation problems. During the k+1 problem, I got very confused about what to do next but for this problem you really just need to look at it as a jigsaw puzzle, and work around with what you got to make the pieces fit together. I enjoy working on problems like this that challenge my brain to think outside the box.
-Connor

Here, the student discusses how this problem changed his views and understandings of mathematical induction. While it is not clear whether the realization of this shift in views occurred in the proof-writing process or in the after-the-fact reflection, but the journal entry allowed the student to be aware of and express this shift in his understanding. Connor expressed a neutral impression of the journals, saying that they were "just another assignment". Although he did not believe he influenced his learning to prove, he felt they could be very useful to a struggling student.

Another example of a response to journal 9 is:

> Prove or disprove: If n is a natural number then n^2+n+41 is prime.
>
> **PROOF:** To prove this, we would have to test every natural number and see if the result is prime. Instead let's see if we can disprove by finding a value n where this is false, that is is not prime.

If we start by using the value 41 for *n*, then we get 41^2+41+41. This can be rewritten as 41(41+1+1), or 41(43). 41(43) is an integer by closure, and by the definition of a prime number, is not prime.
Therefore there exists a natural number *n* where is not prime.
Instead of proving for all the numbers that it did work, it was easier to show that it didn't work and could be shown as false. Hence a lot of plug and play to see if we could get a number that would allow us to pull out a common factor and find a multiple of the number.
-Sam

This student is discussing their strategy in deciding which proof method to use. They realized that a direct proof would be inefficient ("we would have to test every natural number and see if the result is prime"). Instead, Sam decided to look for a contradiction, and did "lots of plug and chug" until he found that *n* = 41 would work for the proof. Sam gave a positive impression of the journals, and felt that the journals helped him record his progress through the course and identify what he understood and didn't understand.

Joe used this structured journal to show an example of a proof that helped him solidify his understanding, as shown below:

Show that if x is rational and y is irrational that x+y is also irrational by contradiction
Proof
Assume that x is rational and y is irrational and that x+y is rational

$$x + y = \frac{m}{n} \; for\ some\ intergers\ m\ and\ n\ where\ does\ not = 0$$

So x is also rational so

$$\frac{p}{q} + y = \frac{m}{n} \; for\ some\ intergers\ p\ and\ q\ where\ q\ does\ not\ equal\ 0$$

Get y by itself and combine the terms to get

$$y = \frac{r}{s} \; where\ r\ and\ s\ are\ intergers\ and\ s\ does\ not\ equal\ 0$$

We can do this because integers are closed under multiplication and addition so after it would still be two integers
Now it says that y is also rational but that is not possible because y is irrational so it is a contradiction.
For me this helped because I did not understand the full way of doing proofs by contradiction until this point.

-Joe

In addition to giving reasoning for each step of the proof, this student identified the proof he chose as the proof that helped him gain a better understanding of proof by contradiction. When he said, “For me this helped because I did not understand the full way of doing proofs by contradiction until this point”, it is unclear whether he was referring to the proof or the journal, but like Connor the journal helped him be aware of and express his understanding. Joe gave a positive journal impression, and said the journal helped him record his progress and identify issues in his understandings.

Some students provided less reflection and explanation of their thought processes, which can be seen by looking at Travis’ response to prompt 9:

> [Prove that] $|x| = c$ iff $x = c$ or $x = -c$ and c greater than or equal to 0.
>
> Let $|x| = c$.
>
> Then $|x| = \{x$, x greater than or equal to 0
>
> $\{-x, x < 0$.
>
> So, $c = \{x$, x greater than or equal to 0
>
> $\{-x, x < 0$.
>
> Since $c = x$, x greater than or equal to 0 $\Rightarrow$ c greater than or equal to 0.
>
> Since $c = -x$, $x < 0 \Rightarrow c > 0 \Rightarrow$ c greater than or equal to 0, then $-c = x$.
>
> Let $x = c$ and c greater than or equal to 0.
>
> Since $x = c$ and c greater than or equal to 0, then x is greater than or equal to zero.
>
> So, $|x| = x = c$,
>
> Therefore $|x| = c$.
>
> Let $x = -c$ and c greater than or equal to 0.

So, x is less than or equal to 0 [if x = 0 then c = 0 and |0| = 0].

So, |x| = -x = -(-c) = c.

Therefore, |x| = c.

When I thought about this problem, I thought about what I wanted to show. The above proof is my thinking process.

-Travis

This student did not separate his thinking process from the final, written proof. Generally, mathematicians do not write proofs without any scratch work or first attempts, but students first learning proof often think that they do. Unfortunately, Travis did not complete the Post-Proof Views survey, so I am unable to say how he felt about the journals.

Here I have given examples of the types of responses students gave to the week 9 prompt, and how these responses connected to their learning and impressions of the journals. This was just one of the seven structured prompts, each of which provided the students with an opportunity to focus on their awareness of their own proof-related understandings and pinpoint their progress. While it would be unfeasible to describe all responses to each of the prompts, these examples give an idea of the flavor of the responses and highlight the different benefits to the students.

Overall, while the unstructured prompts provided students opportunities to reflect on their overall understandings and progress in the course, the structured prompts helped to focus the students' reflections on specific proof-related ideas, particularly related to understanding and explaining proof logic, the process of proving, and the students' views about proof.

Survey Data: Impressions of the Course and Journals

As part of the Pre-Proof Views survey, I asked the students the following questions that I used to answer Research Question 1:

- How do you feel about keeping a weekly math journal this semester?
- What do you feel are the benefits of journal writing in a proof-based mathematics course?
- How could journal writing be changed to be more effective?
- Would you recommend keeping a math journal to a friend about to start a proofs course in the future? Why or why not?

As I described in Chapter 3, the Pre- and Post- Proof Views Surveys were coded using open coding, and also deductively using the research framework. I first coded the post survey using open coding. However, for the post survey, I was interested in seeing how students felt about the course and journals, if and how their views changed, and what benefits they experienced from the journals. I therefore introduced multiple codes to responses to the related questions.

For example, for the question, "How did the journaling support your learning to prove in this class, if at all?" I created a three-dimensional code, including (yes/no, codes from the open coding, benefit described). For "benefit described", I am referring to the four benefits to students of journaling in math classes that Borasi and Rose (1989) identified. So, to illustrate the coding, consider the response "By writing things down, I was able to jog memories of items that I need to work on". This response indicates that the journaling did support the student's learning to prove. In addition, the code from the open coding phase was Identify Issues, and the benefit described was Improvements in

Learning and Problem Solving. Coding this way allowed me to still capture the precise benefits described by the participants, but also categorize them based on the Borasi and Rose framework.

For the question “How has the course affected your beliefs about proof?” I created the two-dimensional code (positive/negative/neutral, codes from open coding) to allow me to quickly identify not only the code from open coding, but also get a larger picture sense of whether the responses were positive, negative, or neutral. For example, consider the response: “I enjoy them now. I see they are useful”. This response was given the code (positive, realized their importance) because they indicated a positive change in their beliefs, and specifically indicated the change occurred to their perception of the importance of proofs. However, in the response, “It did not really change any of my thoughts”, the code was (neutral, Not) because the participant indicated that no change had occurred. It was useful to have that additional distinction when counting occurrences of different responses.

Overall, 12 students wrote positive impressions of the journals in the Post-Proof Views survey, 8 wrote negative impressions, and 8 wrote neutral impressions. Thirteen participants said they would recommend keeping a journal to a friend about to take the course, 7 said they would not, and 8 said they were unsure or only under certain circumstances, which included “if the student is interested”, “if the student is struggling”, or “if the prompts are more focused on content”. Further, 12 felt the journals supported their learning of proof, 12 did not, and 2 were not sure.

The vast majority of students preferred the structured prompts to the unstructured, with 16 suggesting to use more prompts related to proofs, 4 suggesting to write more proofs directly into the journals and evaluate their thinking behind writing the proof, and only 1 student suggesting more open-ended (unstructured) prompts. Further, 2 suggested a need for more reminders about the journals; 2 thought feedback directly from the instructor would be useful; 1 thought the journals should be daily; and 3 had no suggestions for improvement.

When asked about the benefits of journaling in an introduction to proofs course, 5 students mentioned benefits related to mathematical content understanding, 13 said benefits to learning and problem solving, 2 said benefits to their views about mathematics, 5 mentioned communication with the instructor and feedback, 2 said there were no benefits, and 2 said there were benefits to struggling students. The students that discussed benefits to learning and problem solving mentioned things like “a record of your progress”, “a way to identify issues in understanding”, “a way to practice writing mathematically”, and “a way to look back and see how your proofs developed”.

Interviews: Impressions of the Course and Journals

In the first portion of the interview, the participants were asked about their impressions of the course, proofs, and the journals. The interview responses were coded twice, first using open coding and then again according to Borasi and Rose’s (1989) framework of benefits students experience journaling in mathematics courses.

For the open coding, each question was coded separately, looking for similarities, differences, and themes in the responses. Then, the questions were compared and used for triangulation. For instance, when asked, “How has the course affected your beliefs about proof, if at all?” 3 participants said no. Later, when asked, “How did the journaling tasks influence your opinions about proof in mathematics, if at all?” four participants said no. These are consistent, and therefore the validity of the results is supported.

In the second round of coding, the group of questions regarding the journals were coded using Borasi and Rose’s (1989) framework of the following four benefits of journal writing: 1) Therapeutic effect; 2) Increase in content knowledge; 3 Improvements in learning and problem solving skills; and 4) Refinement of views towards mathematics. I included each benefit mentioned by the participants, including those not mentioned in the response to “What are the benefits of journal writing in a proof-based mathematics course?” For instance, when asked “How do you feel about keeping a weekly math journal this semester?” Emma replied:

> it was a good idea because it helped you, you know, like stay on track and like record like everything that you're struggling with and then go back and see like oh I was struggling with that and now I'm not.

This response was coded as benefit 3 and was included because, although it was not mentioned during the question explicitly asking about benefits of journaling, it represents a benefit Emma observed.

After the open coding was completed, I tallied the occurrences of each benefit mentioned by the participants, yielding the following counts, which are given in the table below:

Table 13
Frequencies of each journal benefit mentioned by interview participants.

Benefit	Therapeutic	Content	Problem Solving	Views
Frequency (out of 20 total)	4	3	12	1

It was clear that Benefit 3 (Improvements in learning and problem solving skills) was mentioned overwhelmingly more often than any of the other benefits. To further investigate exactly what the participants were referring to, I created the following sub-codes within benefit 3: Organizing work and goal setting, Record of progress over time, and Pinpointing understanding (predominantly struggle).

Table 14

Interview subcodes of the problem solving benefit

Benefit 3 Subcode	Organizing and Goal Setting	Record of Progress Over Time	Pinpointing Understanding
Frequency (out of 9)	2	2	5

Next, I compared the results of the open coding and second round of coding to identify any discrepancies or connections between the results. Finally, I looked at each participants' responses across all questions and analyzed their responses as cases.

In addition to the benefits from the Borasi and Rose framework, three interview participants (2 from Section 1 and 1 from Section 2) described instances of noticing changes in instruction in classes following their journals, and that the changes seemed to address what they had written about. The participants attributed the changes to the professor reading their journals and acting on the content of their journals. Upon member-checking with both professors, I learned that they did read the journal summaries I sent each week and made conscious efforts to address comments that students had, both in the form of making changes to instruction, adding additional explanations, or discussing her reasoning with the class as to why they made certain instructional decisions. So, the students' observations that their instructor made changes according to their journals was correct, and the changes did not happen by chance.

Rather, the journals provided formative assessment for the instructor and a means for the students to dialogue with the instructor.

In the interviews, 2 students did not realize there was feedback to their journal entries. Unfortunately, I did not ask students directly about feedback in the Post-Proof Views survey, so I am unable to say how many total students in the course did not realize there was feedback. However, with 5 students mentioning communication with the instructor and feedback in the Post-Proof Survey and the interview comments about appreciating the instructor making changes as a result of the journals, it seems that the students also used the journals as a means to communicate with their instructor. I am curious how many more students would have discussed feedback if they had realized that there was journal feedback.

Overall, the responses were positive towards the journals (4/5 participants) and identified Benefit 3 as most present in the journals (9/17 benefits mentioned), but also saw therapeutic (4/17), content (3/17), and views about mathematics (1/17) benefits as present. Each participant described problem solving benefits of the journals. Participants from each section described noticed the instructor address a comment from their journal in the next class. Two students didn't realize there was feedback to the journals, one never checked it, and the two who did use the feedback said they appreciated it for therapeutic reasons.

Alicia also mentioned using the journal to practice writing mathematically in an informal setting. By writing about the proofs and proof topics, Alicia was creating a narrative of her understanding, which she was able to use to create her formal arguments and written proofs.

Summary of Research Question 1

In the unstructured journals, students wrote about therapeutic topics most often (49% of all entries), followed by problem solving (36%), content (11%), and then views (4%). Three themes emerged in the unstructured journals: students' perceived ambiguity of explaining mathematics, getting stuck when writing a proof and not knowing how to proceed, and organization. In the structured journals, students were prompted to address these themes more explicitly and also reflect on specific aspects of proof writing, while reflecting on their awareness of their own proof-related understandings and pinpoint their progress. In the surveys, students expressed mostly positive impressions of the course and the journals (12), then negative (8), and neutral (8). Half of the students who left neutral responses felt the journals would have been more beneficial if the prompts were all structured.

The students reported experiencing mostly problem solving benefits (organizing and goal setting, record of progress over time, and pinpointing understanding), then content, and views. The students overwhelmingly suggested that using more structured, proof-related prompts would make journaling more effective. Some students also felt the journals provided an opportunity to dialogue with their instructor, and appreciated feedback, while some may not have realized there was feedback to their journal responses.

Research Question 2: Ideas about Proof Writing

Ideas about Proof Writing in the Structured Journals

The structured interviews were coded using directed content analysis (Zhang & Wildemuth, 2009), in which the journal entries were first coded deductively according to the framework related to that research topic, and then were inductively examined to identify any remaining themes that emerge.

There were three prompts in particular (weeks 6, 7, and 9) in which I asked the students to describe their proof writing strategy (week 6) and write a proof in the journal and explain what they thought about as they worked on the proof (weeks 7 and 9). Thirty students completed at least one of these journals. For these prompts, I used Raman's (2003) framework for ideas about proof writing to code the journal responses. The types of ideas about proof writing the students expressed in each journal are given in the table below:

Table 15
Ideas about proof writing.

	Week 6	Week 7	Week 9	Totals
Heuristic	5	5	2	12
Procedural	7	7	17	31
Key	6	5	2	13

It is important to note that these weekly totals are not necessarily the same students, since not every student submitted a response each week. It is also important to note the variation between weeks, particularly with week 9 compared to the others. In week 9, one section of the course had just completed a unit on induction, a fairly procedural proof method, which may have influenced why so many more students described procedural ideas in week 9 as compared to weeks 6 and 7. It appears that the type of proof the students chose may have played a role in their ideas about how to approach that proof.

When coding, I also noticed that most students' proof ideas were not consistent across all three weeks. Here I do not include the 10 students who completed only one of the three structured assignments used in this section; rather, I only looked at students who completed at least two of the three journals. Only three of the thirty students consistently exhibited the same proof writing ideas, and all of them were procedural. Of those three students, 1 reported positive impressions of the journals, 1 neutral, and 1 did not complete the survey. One reason for the inconsistency could be the students' professed difficulty with explaining their thinking. This could have caused them to focus more on procedural aspects of proof writing in their write up ("I started by writing 'proof' and then wrote my assumptions"). Another reason could be the possible interaction of the type of proof chosen with the students' ideas about how to approach that proof. These are both areas to investigate further in future studies. In the next section, I use the students' survey responses to explore their experiences with the journals in relation to the types of proof writing ideas the students demonstrated in their weeks 6, 7, and 9 journals.

Survey Data: Pre- and Post- Ideas about Proof Writing

Once I identified the types of proof ideas present in students' structured journals, I examined their journal completion and survey responses related to the journal tasks. Of the 8 students who demonstrated heuristic ideas in their structured journals, the journal completion was varied: 2 participants completed 7; 1 completed 8; 1 completed 9; 3 completed 10; and 1 completed 12. Of these 8 students, 4 gave positive journal impressions; 2 gave negative; 1 neutral; and 1 did not complete the survey.

Of the 10 students who demonstrated key ideas in their structured journals, the minimum total number of journals any one of them completed is 6 (1 student); 3 completed 10; 1 completed 11; and 6 completed 12. Further, 7 of these students described positive journal impressions; 3 described negative; 2 neutral; and 1 did not complete the survey. Of the students who described a negative impression, 1 said he disliked the journals because he is “not a fan of writing”, though he noted that the journals did provide him with a record of his progress. He also found the course frustrating because of the proof-writing component. The other said he disliked the journals and left no additional comments. The students who had a neutral journal impression both said the journals were useful to help identify issues in understanding, but they felt that the journals were just another assignment. Overall, the 10 students described benefits of the journaling as helping them with content (2), communication with the instructor (2), identifying issues (2), keeping a record and reflecting on their work and progress (4), organizing their thoughts about a problem (2), practicing writing mathematically in an informal setting (1), and none (1). Here, 4 students described more than one benefit of the journals, which is why the counts in the previous sentence add up to more than 10. Their suggestions for improving the journals were to include more prompts about proof writing (4), write more proofs directly into the journals (1), do a better job of reminding students about the journals (1), and none (3)

It appears that throughout the semester the students who demonstrated key ideas in the journals tended to have higher journal completion and an appreciation for the journals as a tool to support their learning to prove in a variety of ways, mostly associated with the learning and problem solving benefit from Borasi and Rose’s (1989) framework. In particular, identifying issues in understanding represents one function of journals to influence ideas about proof writing.

As I was reading the participants’ definitions of mathematical proof in the Pre-Proof Survey, two themes emerged: proof as a product or proof as a process. These are related to Raman’s (2003) ideas about writing proofs because the definition of proof as a product corresponds to the procedural focus on proof writing and the definition of proof as a process corresponds to the heuristic proof writing idea. Raman's key ideas represent the successful merging of the process and product definitions of proof.

In the Pre-Proof Views survey, 14 (45%) students gave process-focused definitions of proof, while 8 (25%) gave product-focused definitions of proof. Eight (25%) participants left the questions blank, and two (5%) said they did not know.

In the Post Proof Views survey, 4 students defined proof as an explanation, which could either be taken as a product or a process, and I was unable to determine their intent based on the context. At the end of the semester, 4 students defined proof as an explanation (13%), 14 as a process (48%), and 11 as a product (38%). Of the students who completed both the Pre and Post Views Surveys, 6 started out believing proofs are a process, 7 defined proofs as a product, and 6 left the problem blank or wrote "I don't know". Of those students, 3 defined proof as an explanation, 8 as a process, and 9 as a product at the end of the semester. The students' definitions of proof were not necessarily consistent throughout the semester, since 2 students switched from process to explanation, 1 switched from process to product, 3 remained with process, 2 switched from product to explanation, 3 switched from product to process, and 1 switched from product to explanation. This suggests that students' ideas about proof and what it means to prove changed throughout the course, but not necessarily in the same ways.

I also asked the students outright whether and how the journals supported their learning to prove. The participants were split about whether or not the journals directly supported their learning to prove: 12 said yes, 12 said no, and 2 were unsure. Of the participants who said no, 6 suggested more proof-related prompts should be used, 1 said more proofs should be written directly into the journal, 2 had no suggestions for improvement, 2 wanted instructor feedback, and 1 wanted more reminders to do the journals. The students who said yes overwhelmingly described the problem solving, content, and communication benefits of the journals as helping them learn to prove.

Another set of survey questions asked the students how the course influenced their beliefs about proof and how the journals influenced their beliefs about proof. From the 27 students who completed the Post-Proof Views Survey, sixteen students felt the course positively influenced their beliefs about proof (gave them more confidence in their proof-writing, helped them see the importance of proof, gave them new way of viewing mathematics), 7 said negative (learned they don't like proofs, proofs are a hassle), and 4 were neutral (didn't change). Of the 16 who noted positive changes from the course, 11 also felt the journals positively influenced their beliefs, 2 did not, and 3 weren't sure. Overall, 13 students felt the journals positively influenced their beliefs about proof, 8 said the journals did not (including the 4 who did not think the class influenced their beliefs at all), and 6 were not sure (either not sure if their beliefs changed at all or not sure if the journals influenced the change). The students who did feel they experienced a change in their beliefs resulting from the journals claimed the change was because the journals helped them communicate with the instructor and get feedback (3), identify issues in their understandings (3), record and reflect on their progress (5), motivate themselves to succeed in the course (1), and solidify their thoughts (1).

So, of the students who felt the course influenced their ideas and beliefs about proof, 68 percent of them also felt the journals influenced their beliefs about proof. Of the benefits these students described, one centered on communication with the instructor, 3 were related to reflection and self-awareness (identify issues, record and reflect, and solidify), and 1 was therapeutic (motivation).

Part II of the Interviews: Ideas about Proof Writing in the Task

The task given to the students was:

Consider the statement: For real numbers a, b, and c, if $a < b$, then $ca < cb$.

- If the statement is true, construct a proof.
- If the statement is not true, please modify the statement to describe the correct relationship between ca and cb and then construct a proof of the modified statement.

The statement is false, because if c is 0 or negative, then the inequality fails. One of the reasons this problem was chosen is that the statement can be modified in numerous ways to become true. One modification is to restrict c to positive real numbers, which one participant did. The other participants restricted a, b, and c to positive reals (1 participant) or natural numbers (3 participants).

For the task portion of the interviews, I first looked at the participants' written work from the interview, identifying errors and insights in their proofs and scratch work. I then read the transcripts as I listened to the audio recording of the interviews, taking observation notes about how the participants progressed through the task, identifying whether they displayed heuristic, procedural, or key proof writing ideas according to Raman's (2003) framework.

All participants correctly identified that the statement was false as written; 2 used examples to make sense of the statement, and 3 immediately recognized that a negative c value would make the statement false. However, only 1 participant modified the statement to require *c* to be positive. The other 4 changed the conditions on *a* and *b* as well, limiting the scope of the lemma. None of them recognized that by changing limiting *a* and *b*, they were limiting the applicability of the theorem they proved. All participants except the first participant had to be given the lemma that for real numbers m and n, $m < n$ is equivalent to saying $n = m + k$, where k is a positive real number. The first participant set up the proof and said that the rest would follow from previous theorems, but since she didn't have a list of theorems she couldn't continue. Once the remaining 4 participants had the hint, 2 were able to write a correct proof, 1 used the lemma but assumed the conclusion in their proof, and 1 was able to write a correct proof, but with lots of hints and guidance from the interviewer about how to structure the proof.

The two participants who successfully completed the proof displayed key ideas about proof writing, and also described the most varied benefits of the journals, each listing therapeutic, problem solving, and content benefits. They are also the two participants who, unprompted, said they were planning to keep a math journal in future classes, and said they would have gotten more from the journals if they had written more often. One of them completed 10 journal entries, and the other completed 5 (although he went back and completed the other 7 late). The one participant who did not find the journals beneficial displayed procedural ideas about proof-writing, completed 7 journals, and incorrectly assumed the conclusion in his proof. He also did not like the course and felt he did not learn as much since he had taken a discrete mathematics course before enrolling in introduction to advanced mathematics. The participant who gave up displayed procedural ideas towards proof writing, noted a positive journal impression, and completed 9 journal entries. The student who was able to complete the proof but only with step-by-step guidance from the interviewer displayed heuristic ideas about proof writing, wrote 11 journal entries, and liked the journals and the course.

Summary of Research Question 2

At the beginning of the semester, most students defined proof as a process, and at the end of the semester most students defined proof as a product. Although students displayed all three types of proof-writing ideas in the structured journals that explicitly asked students to describe their proof-writing process, procedural was the most common (31 instances), then key (13) and heuristic (12). It appears the students' choice of which proof to write in the journal may have interacted with the type of proof idea displayed. Of the 10 students who demonstrated key ideas in their structured journals, all but one of them completed at least 10 journals; 7 of these students described positive journal impressions, 3 described negative, 2 neutral, and 1 did not complete the survey. Also, 9 of these 10 students found the journals beneficial, and they described content, problem solving, and instructor dialogue benefits as a result of the journals. In the interviews, the two students who successfully completed the task demonstrated key ideas and described positive journal impressions. The two students who were unsuccessful displayed procedural proof ideas, and one of them had a positive journal impression and the other negative. The fifth interview participant displayed heuristic proof ideas and mentioned positive journal impressions, and was able to correctly complete the task but only with lots of researcher guidance about the structure of the proof.

Sixteen of the 27 respondents to the Post-Proof Survey felt the course positively influenced their beliefs about proof, 7 said the course influenced their beliefs about proof in a negative way, and 4 reported that their beliefs had not changed. In the interviews, 2 of the 5 participants did not think the course influenced their beliefs about proof, and gave their previous experiences with proofs in prior courses as the reason their beliefs did not change. Of the 16 who noted positive changes from the course, 11 also felt the journals positively influenced their beliefs, 2 did not, and 3 weren't sure. The students who did feel they experienced a change in their beliefs resulting from the journals claimed the change was because the journals helped them communicate with the instructor and get feedback (3), identify issues in their understandings (3), record and reflect on their progress (5), motivate themselves to succeed in the course (1), and solidify their thoughts (1).

Research Question 4: Proof Writing Performance

Next, I will discuss research question 4 because it is related to the first two research questions. I will then discuss research question 3.

Survey Data: Student Impressions of Journals, Performance

As I described above under Research Question 2, 12 students answered yes, they felt the journals directly supported their learning to prove, 12 said no, and 1 was unsure. Of the no responses, 6 said more proof-related prompts should be used; 1 said more proofs should be written directly into the journal; 2 had no suggestions for improvement; 2 wanted instructor feedback; and 1 wanted more reminders to do the journals.

The participants overwhelmingly discussed the problem solving benefit from the journals being helpful, saying the journals acted as "a record of your progress", "a way to identify issues in understanding", "a way to look back and see how your proofs developed". Participants also cited the content benefits, saying the journals were a way to practice writing mathematically" or "a way to practice vocab/definitions". For the students who felt that the journals did influence their learning to prove, the journals did so by helping the students develop an awareness of their problem solving process and learning. In particular, the structured prompts helped to direct the students' reflections away from therapeutically discussing their feelings about the course, and instead focus on proof-related topics. By helping the students be aware of their understandings and misunderstandings, progress, and successful and unsuccessful learning strategies related to proof topics, the journals helped the students solidify their understandings, which in turn, they felt, benefitted their proof writing.

Interviews: Student Impressions of Journals and Performance

As I mentioned above, 2 students (Alicia and Caleb) completed the interview task successfully and independently of my help, 1 (Bill) assumed the conclusion, 1 (Diana) gave up because she didn't have a list of previous theorems, and 1 (Emma) was able to complete the proof but only with step-by-step guidance from the interviewer. The successful students noticed the greatest number of benefits from the journals, describing therapeutic, problem solving, and content benefits. The student who assumed the conclusion recognized that there could be problem solving benefits as a result of the journals, but he did not experience those benefits himself. The student who completed the proof with my help and the student who gave up both felt they experienced therapeutic and problem solving benefits from the journal. It is interesting that the two successful students at the interview task were the only two interview participants who felt they observed content-related benefits, which would improve their proof writing performance. This supports the observation that improvements in proof writing as a result of the journals occur when the students' writing is focused on particular course content.

In my analysis of the first two research questions, I learned that the students were split about their impressions of the journals, the course, and whether the journals directly influenced their learning to prove. More students had positive impressions of the journals than negative, and students were split about whether the journals directly helped them learn to prove. However, the biggest benefits the students mentioned as a result of the journals were related to their own awareness of their understandings and progress in the course. Further, the students who did not believe the journals influenced their learning to prove seemed to agree that the journals could be helpful for struggling students, and that the journals might have helped them more if more of the prompts were proof-related (structured). These data suggest that there is a relationship between the journals, particularly the structured journals, and students' learning of proof. For this research question, I wanted to look quantitatively at what relationships may or may not be present between course performance and journal completion, taking into account my previous analysis of the journals, interviews, and the surveys.

Quantitative Data: Journals and Course Performance

Correlation

I would like to first mention that this study is exploratory in nature, and is not attempting to use journal completion as a predictor for course performance. With the design of my study, I am unable to imply causation between journal completion and course grade. Instead, I can investigate if a correlation appears to be present between journal completion and course grade that warrants further investigation in future studies. Further, I attempted to look at correlations between different groups of students' course performance and journal completion. I present the different correlations that I calculated below, along with a reasoning behind why I calculated each correlation and an interpretation of the correlation coefficient. Although the number of journals completed alone does not determine the quality of the writing, it is a good place to start exploring what, if any, relationship exists between the journals and proof writing performance. In future research, I would like to build a Journal Quality Index (JQI) to better quantify the journal writing. Also, because the course centers on learning proof and proof writing techniques, the course grade is an appropriate choice of variable.

To explore whether the journal completion and course grades were related, I began by calculating a correlation coefficient between the number of journals completed and the course grade for all 42 students still enrolled in the course at the end of the semester. The scatter plot for the data is given below:

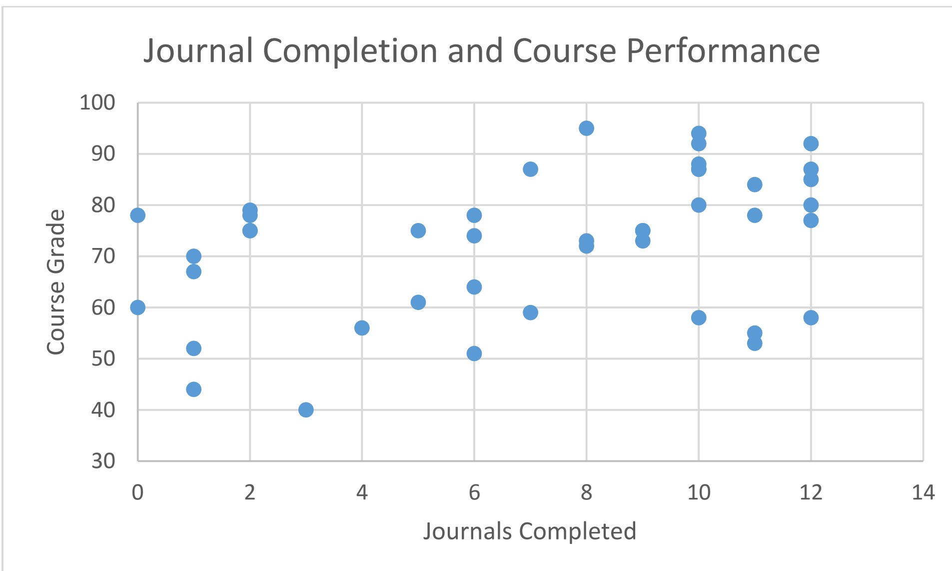

Figure 7. Journal Completion and Course Performance Scatter Plot.

The Pearson Correlation Coefficient for this set of data was r = 0.4003, which was significant with a p-value of 0. 0078. This tell us that there is a positive and significant relationship between journal completion and proof-writing performance, suggesting that the journals are having an impact on student learning or that high achievement students tend to complete journals.

When I created the scatter plot for this data, two groups of outliers became apparent to me: the students who completed a high number of journals, but ended up with a course grade in the 50s, and the students who completed a low number of journals, but ended up with a course grade of 70 or above. I became curious about what characteristics these students shared and consulted their survey and interview data to help get a better picture of these students' experiences. I summarize their course experiences in the table below:

Table 16

Experiences of low performing, high journal students.

Student	Number of journals	Course grade	Previous proof experience	Course impressions	Journal impressions
Emma	11	53	none	Positive. “I enjoyed the challenge, but it is very frustrating…that it takes me a long time to work even the simplest proof”. Shealso frequently mentioned her math anxiety during the interview. She completed 5 of 6 homework assignments with a homework average of 68%.	Positive. “It was a good idea! Will definitely consider doing it myself for future math classes”. The journals helped her stay organized and motivate herself.
Amy	12	58	none	Neutral. “It was so so”. She completed 5 of 6 homework assignments with a homework average of 70%.	Neutral. “It was ok, helped me take a look back at what I did weekly”. “It often helped explain my strengths and weaknesses”.
Travis	10	58	none	Did not complete Post-Proof Views survey. They completed 10 of the 14 homework assignments and earned a 53% homework average.	Did not complete Post-Proof Views survey
Ryan	11	55	none	Negative. “I did not care for it. Proofs are very hard for me…because the work is tedious”. They completed 6 of the 14 homework assignments and earned a 33% homework average.	Neutral. “It was the easiest part of the class” but the benefits are “none at all if only for feedback from the professor”.

A unifying piece of information between these students is that they represent 4 of the 10 students who reported no previous proof experience on the pre-survey. Also, they earned a maximum of 70% for their homework average and turned in 70.225% of the homework assignments. So, although they completed at least 78% of the journal assignments, their homework completion and homework averages were lower. Perhaps, although they completed a high number of journals, the fact that they had not taken a proofs course before made it more difficult for them to succeed in the course.

I also wanted to examine the experiences of the students in the second group of outliers. These 5 students completed 2 or fewer journals and earned a course grade of a 70 or above. Their course experiences are summarized in the table below.

Table 17
Experiences of passing, low journal students

Student	Number of journals	Course grade	Previous proof experience	Course impressions	Journal impressions
Julie	0	78	yes	Neutral. "It was slow in the beginning and fast-paced toward the end. I went from taking baby steps to making a few leaps". She completed 13 of the 14 homework assignments and earned an 81% homework average.	Negative. She found them "a bit tedious. Had a heavy course load this semester". However, she said she would recommend keeping a journal "if they have the time".
Tyler	1	70		Did not complete the post survey. They completed 6 of the 14 homework assignments and earned a 35% homework average.	
Adam	2	75	Previous attempt at the introduction to advanced mathematics course	Did not complete the post survey. Completed 3 of 6 homework assignments during the course and earned a homework average of a 40%.	
Dylan	2	78	yes	Did not complete the post survey. Completed 0 of 6 homework assignments during the course.	
Violet	2	79	yes	Did not complete the post survey. Completed all homework assignments with a final homework average of 70%.	

This data set is more incomplete than the first group of outliers, since 4 of the 5 members of this group did not complete the post-survey. However, unlike the first group of outliers, all 4 of the students in this group that I have the data for described previous proof experience. I did not have information about course or journal impressions for 4 of the students, but was able to look at their homework performance to compare it with their journal performance. These students were split based on their homework completion, with three completing fewer than half of the assignments and 2 completing at least 70% of the assignments. Perhaps their previous experiences with proof were enough to help them pass the course despite not completing much of their homework or journal assignments.

When I remove the 10 outlying students from the analysis, the correlation coefficient for journals completed and course grade becomes $r = 0.7188$, which is significant at $p < 0.00001$. The scatter plot is shown below:

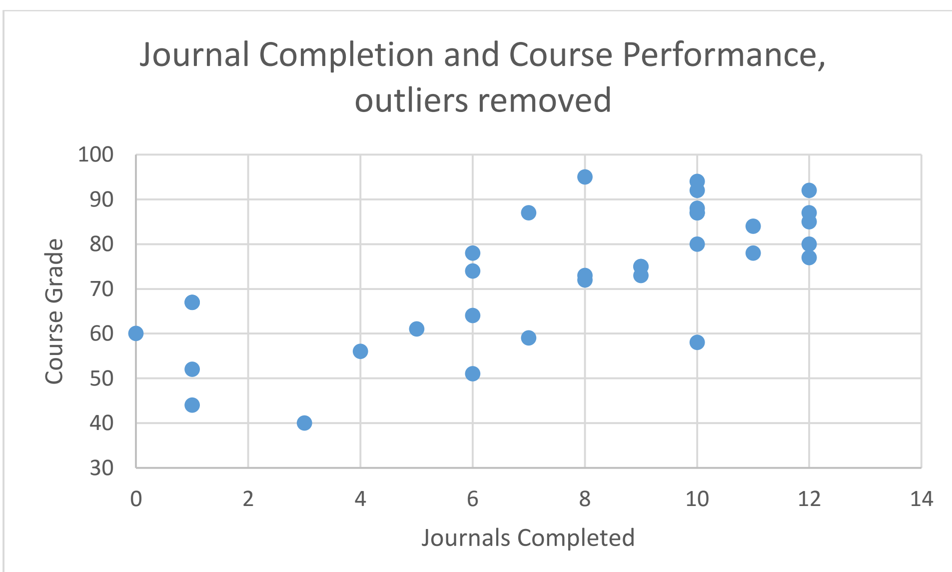

Figure 8. Journal Completion and Course Performance, Outliers Removed.

Since the students mentioned the structured journals as being overwhelmingly more beneficial than the unstructured, I also wanted to look at the relationship between journal completion and course grade when I only considered the number of structured journals completed. I began with the entire set of 42 students still enrolled in the course at the end of the semester, counted the number of structured journals each student completed, and calculated the correlation coefficient between the number of structured journals completed and course grade. The scatter plot for this set of data is given below:

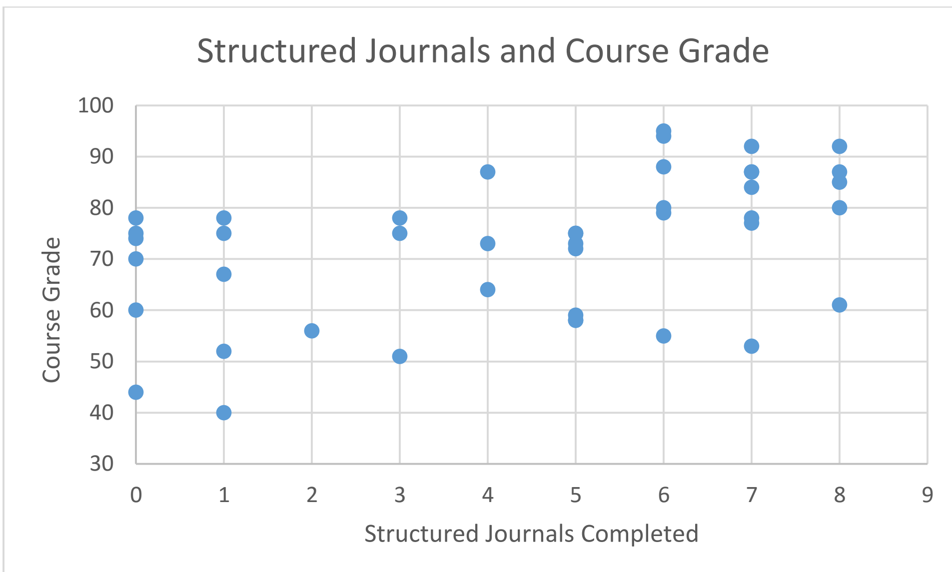

Figure 9. Structured Journals and Course Grade.

Considering only structured journal completion yielded a correlation coefficient of $r = 0.4544$, which was significant with $p = 0.0022$. By only including the structured journals, the correlation increased slightly. This supports the students' claims that the structured journals were more beneficial to them than the unstructured journals.

Next, I wanted to get an idea of how strong the relationship may be between journal completion and course grade for the students who had positive journal impressions. On the Post-Proof Views Part I survey, 13 students out of the 28 who completed the post survey who said that yes, they would recommend keeping a journal to someone about to take the course, 7 said no, and 8 said maybe if certain conditions were met (mostly if the prompts became all proof-related, if the student was struggling, or if the student enjoys writing). Additionally, there was one interview participant who did not complete the Post-Proof Survey. This participant answered yes to the question about recommending the journals to a friend in the interview, so I included them in the "yes" group, bringing the total to 14. I considered only the 14 students who said yes, and calculated the correlation coefficient for their journal completion and course grade. This yielded an r value of r = 0.5524, which had a significant p-value of p = 0.0405. The scatter plot for this set of data is given below:

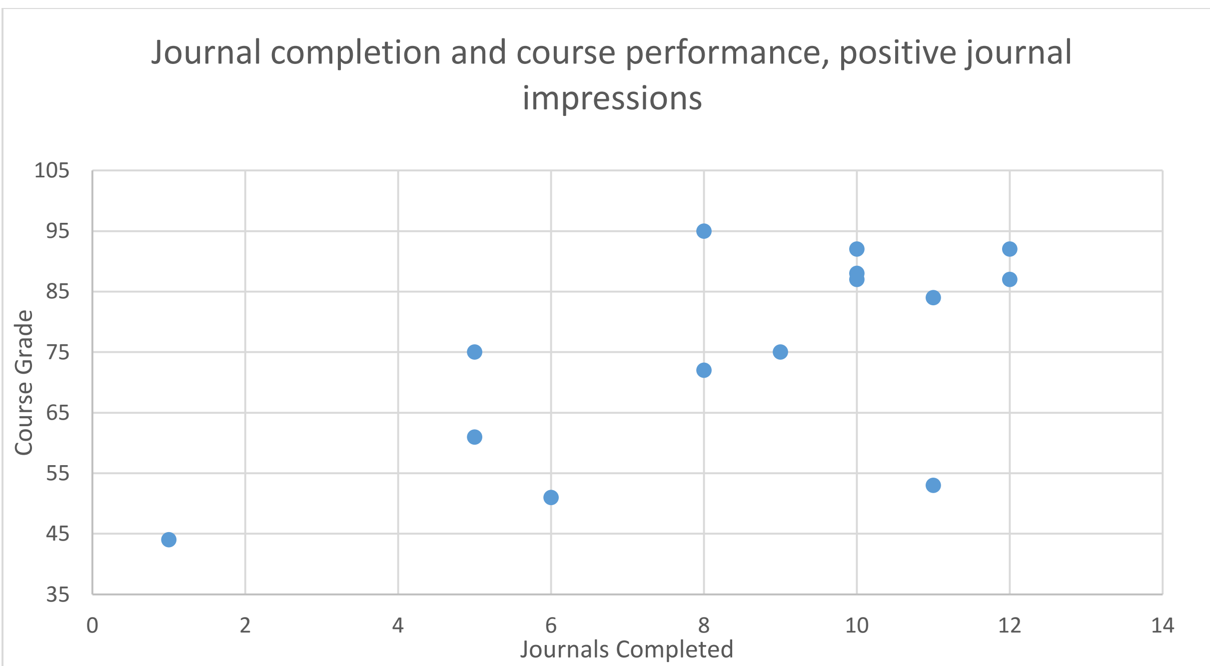

Figure 10. Correlation for Positive Journals Impressions Group.

In total, I calculated four different correlation coefficients using different sorting methods for the data. All four correlation coefficients were significant and positive, and varied in strength. A summary of the quantitative performance analysis is given below:

Table 18
Quantitative course performance analysis.

Relationship measured	Data sorting method	Number of data points	P-value	Correlation coefficient
Journal completion and course performance	Whole class, all journals completed	43	$p = 0.0078$	$r = 0.4003$
Journal completion and course performance, outliers removed	Eliminated the low performance, high journal and high performance, low journal outliers	33	$p < 0.00001$	$r = 0.7188$
Structured journals and course grade	Whole class, just looked at the number of structured journals completed	43	$p = 0.0022$	$r = 0.4544$
Journal completion and course performance, positive journal impressions	Only considered students who said yes they would recommend keeping a journal to a friend about to take the course, along with number of journals completed	14	$p = 0.0405$	$r = 0.5524$

Summary of Research Question 4

In the Post-Proof Views survey, 12 students said their journals directly influenced their learning to prove, 12 said no, and 1 was unsure. Of the no responses, 6 said more proof-related prompts should be used, 1 said more proofs should be written directly into the journal, 2 had no suggestions for improvement, 2 wanted instructor feedback, and 1 wanted more reminders to do the journals. The students who did believe the journal directly influenced their learning to prove said they felt that way because of the problem solving and content benefits of the journals. In the task portion of the interviews, the two successful students noticed the greatest number of benefits from the journals, describing therapeutic, problem solving, and content benefits. They were the only interview participants who described content benefits from the journals.

I also quantitatively examined the data. There exists a significant positive correlation ($r = 0.4003$) between the number of journals completed and course grade. Two groups of outliers became apparent: students who completed a high number of journals (10 or more), but ended up with a course grade in the 50s, and the students who completed a low number of journals (2 or fewer), but ended up with a course grade of 70 or above. All of the students in the low journal passing grade group for whom I had Pre-Proof Views survey data reported previous experience with proofs in other courses, and their homework grades varied between passing (2) and not passing (3). The students in the high journal, low grade group had no proving experience before the course and completed an average of 70.225% of their homework assignments. Removing these outliers yielded a significant correlation of $r = 0.7188$.

When considering only the number of structured journals completed (students overwhelmingly found these more beneficial than the unstructured) and course grade, the significant correlation becomes stronger, with $r = 0.4544$. I also found that, for the students who felt the journals were worthwhile (answered “yes” they would recommend keeping a journal to a friend about to take an introduction to proofs course), the correlation between the number of journals completed and course grade was significant with $r = 0.5524$.

Research Question 3: Views about the Functions of Proof

Views about the Functions of Proof in the Journal Entries

Students rarely discussed their views in the unstructured journals, with explicit discussions of views composing only 4% of unstructured journal entries. When students did discuss their views, the views were generally related to the course structure rather than their views about mathematics and proof. In the unstructured journals, mentions of proof were often in the context of therapeutic discussions and not reflections on the functions of proof.

There were three structured journals that prompted students to discuss their views about mathematics. The first occurred in week 2 and asked students to:

> Discuss the role that definitions play in mathematics and writing proofs. How are definitions important? How might you use definitions when writing proofs? Please also describe any other questions or comments you have for the course that you think are important.

Some examples of responses to this journal are:

> Definitions are important because they are something like common knowledge in math, well that's how I see them. And when you use them in proofs, it helps the person who writes the proof in several ways, first of all it shortens the proof because all you have to write is, "by definition", instead of having to make a separate little side proof, proofing the definition which can be tedious or hard to do. All you have to do is write by definition because the definition is knowledge that is already understood. - Hannah

> Definitions are a tool used in proofs. It is what we use to base assumptions. Definitions is how we group numbers and theories to place them in our little containers of types. –Alicia

> Definitions provide one who is trying to prove certainty that what they are saying is correct. Definitions are the main key in proving because a definition is always true and is used to back up a statement stated in the proof. In a proof, a definition is a sort of easy way to get away with saying something because the definition has already been proven. In some ways, a definition may be more important than a proof because the definition had to have been proven at once too without a previous definition. –Bill

The responses to this prompt provide an interesting glimpse into how the students view definitions in mathematics. This was the prompt in which students most directly discussed their views about the nature of mathematics and proof. In the other two structured prompts focused on views about mathematics, the students tended to discuss their views related to their own understandings of the content and course progress, as I show below.

The next structured journal asking students to reflect on their views about proof occurred in week 11 and asked students, “So far in the course, what is your favorite proof technique? Why? What is your least favorite proof technique? Why?” Some examples of student responses to this prompt are:

> My favorite proof technique has been induction, because it is a more direct approach to proof. In just a matter of steps it can be proved. My least proof

technique would have to be set theory. At first I liked it until all the families and unions and complements and others became mixed in together. –Amy

My favorite proof technique so far is proof by contradiction. I seem to grasp it a little easier than the rest. My least favorite is proof by induction. I always get tripped up in the steps. I know what steps I need to do but when I try to add the "n+1", it starts getting murky. –Caleb

I'm enjoying the Union and Intersection problems and I'm really understanding it. The only problem I have with these problems is that I don't know what the correct way to write a proof for this kind of problem looks like. Like an example proof just to see how to put it into words. My least favorite are the subsets, proper subsets, and power sets. They're not too too bad they are just thinkers a little. I understand the simple problems but then its hard to see it when you get more in depth. I've found that drawing out the picture, though, has really helped me see a problem better and what its actually saying. Work in progress going great!
– Violet

In this journal, most students did not discuss their favorite or least favorite proof in relation to their views about mathematics, and instead discussed the proof techniques in relation to how well they understood them.

The other structured journal that prompted students to reflect on their views was the last journal entry, which asked

Please reflect on the course and your progress throughout the course this semester. What advice would you give to a student about to take an introduction to proofs course? How, if at all, did your ideas about mathematics and proofs change during the semester? Please also mention any other comments or questions you have.

In this week, similarly to week 11, students also did not tend to discuss their views about the nature of mathematics or proving, and instead focused on the progression of their understanding of the content throughout the semester.

For example, Veronica responded with:

> I am absolutely satisfied with this course. It was a struggle in the beginning, which is to be expected since I've had no proofs-writing experience, but with practice and our group work it didn't turn out to be so bad. I would definitely advise a future student to try their best to not miss any classes. Missing one class can throw everything completely off. Another piece of advise I would give would be to make sure to write their definitions in a notecard booklet when you go over it in class so when you do your homework or practice problems, their right there. And then when testing time comes around you dont have to worry about that part.
> – Veronica

Hannah said:

> This course has been an interesting and new way to look at mathematics, I feel like all the proofs are like puzzles and I actually grew to enjoy it even though I wasn't sure about it in the beginning. I feel like I have grown mathematically because I've seen a new side of things.
>
> I think the advance I would give them is to not stress out because things don't come out right in the very beginning, you just need to step back and take your

> time. And another thing is to even try to solve it algebraically at first and then go from there and add words to it.
>
> I feel like I have grown stronger at proofs, because I was pretty horrible in the beginning. But I have gotten better. – Hannah

Daniel said:

> When I first got in this class the only proof experience I had was from the Modern Geometry class I had taken last semester. As I sat through that class I was mostly confused on what to do and how to start and what direction to go in these proofs. Then I came into this class and it cleared up a lot. I now have an idea of how to start a proof, and how to go about completing the proof. If I had to give advice to a new student about to take an introduction to proofs class, I would tell the simply be patient, practice, its not easy to do the work, but you get a good feeling when you finally get something. The feeling of actual accomplishment, the best reward in the class. - Daniel

It appears that, even when given prompts related to students' views about mathematics, students may discuss their views of their own progress and understandings in the course, not their views about the nature of mathematics or proving. In future journal implementation, prompts that explicitly ask students to consider their view about the nature of mathematics or proof should be used.

Survey Data: Student Impressions

One of the items on the Pre- and Post- Proof surveys reads "In mathematics, people write proofs to…" I coded the participants' responses to this item according with De Villiers' 5 functions of mathematical proof. At the beginning of the semester, participants mentioned all 5 functions of proof, though not equally: verify (13), explain (13), discover (1), systematize (1), and communicate (7). Also, at the beginning of the semester, there were 2 students who listed two functions of mathematical proof. At the end of the semester, students believed people write proofs to communicate (1), discover (1), verify (11), explain (7), and systematize (2). Also, at the end of the semester, 4 students wrote multiple functions of proof. These responses are summarized in the table and figure below:

Table 19
Students' views in the pre- and post- proof surveys, part I.

Function	Convince	Explain	Systematize	Discover	Communicate	Row Total
Pre-Proof	13 (37%)	13 (37%)	1 (3%)	1 (3%)	7 (20%)	35
Post-Proof	11 (50%)	7 (31%)	2 (9%)	1 (5%)	1 (5%)	22

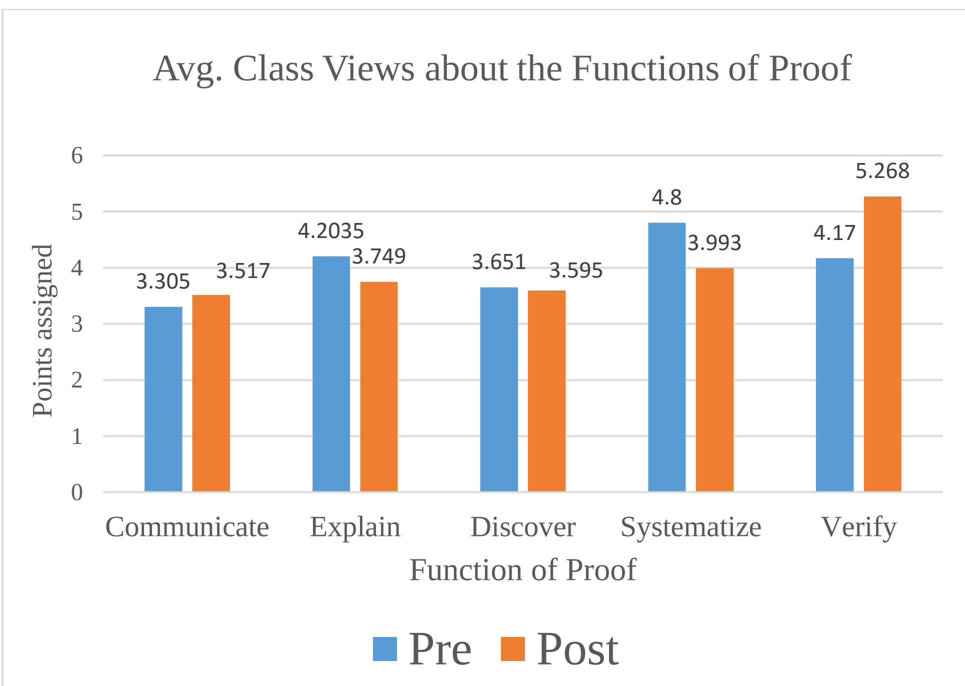

Figure 11. Average Class Views about the Functions of Proof.

Though the numbers of respondents to the pre- and post-surveys differ, when looking at the percentages, it appears that students believed more strongly in the convincing function of proof and students also believed less strongly in the communication function at the end of the semester than at the beginning. Although it appears students' views about the functions of proof did appear to change throughout the semester, the students generally did not describe changes to their views about mathematics of proof as a benefit of the journals (consistent with Borasi & Rose's 1989 findings). Only two students reported that the journals influenced their views about mathematics and proof.

At the beginning of the semester, 21 out of 32 students who completed the Pre-Proof Survey had already taken a proof-based course before enrolling in Introduction to Advanced Mathematics, predominantly discrete mathematics. These students mentioned goals for the course such as "improving my proof writing", "learning to write more concise proofs", and "learning more diverse proof techniques". For the most part, these students did not expect their views to change during the course. Alternatively, the students who had not previously taken proofs classes mentioned goals like "learn a new way to look at mathematics" and "learn what a proof is and how to write one". These students appeared to expect their beliefs about proof to change.

This distinction between the students who had written proofs in previous classes and those who had not was also noticed by students in the course. In the first journal, Stacey said, "A concern I have is that a lot of the people in the class seem to have some sort of experience with proving or some and I feel like I am already a little bit behind, so I am a little concerned that I am not where I need to be right now in order to succeed in the class." Although there is no prerequisite requirement of proof experience for the course, the fact that many students did have this previous experience became a cause for concern for numerous students. Future research is needed to explore how the mixture of experienced and inexperienced students in the course affects students' experiences. I had insufficient data to determine how the differences between previous proof experiences affected the students' impressions of the journals and the benefits the students received from the journals, but it is something I plan to explore in future studies.

Quantitative Data: Students' Views about Functions of Proof

In this instrument, students had 20 points to distribute among 20 statements about the reasons why people write proofs. Each of the five functions of proof was represented by four statements in the survey. A perfectly balanced view of the functions of proof would have each function, verification, explanation, communication, systematization, and discovery, each earning 4 points. I created three levels of balanced-ness, balanced, somewhat balanced, and unbalanced, based on the largest difference between any two functions of proof. I considered views to be balanced if the largest difference between any two functions was less than two points, somewhat balanced if the largest difference was between 2-4 points, and unbalanced if the largest difference was more than 4 points. To consider how the students' views about the functions of proof changed, I looked at both the course averages and individual students' scores.

I began by examining the course averages. Tables summarizing the section averages are given below.

Table 20

Views on the functions of proof, Section 1 averages for pre and post.

Function	Communicate	Explain	Discover	Systematize	Verify	Degree of Balance
Pre	3.340	3.740	3.603	5.667	3.740	Somewhat balanced
Post	3.751	3.378	3.646	4.565	4.842	Balanced

Table 21

Views on the functions of proof, Section 2 averages for pre and post.

Function	Communicate	Explain	Discover	Systematize	Verify	Degree of Balance
Pre	3.267	4.667	3.70	3.933	4.600	Balanced
Post	3.284	4.121	3.544	3.421	5.695	Somewhat Balanced

At the beginning of the semester, the students' views about the functions of proof seemed fairly balanced. In Section 1, systematization scored the highest, with 5.667 of the 20 points, followed by explaining (3.740 points), verifying (3.740 points), discovery (3.603), and then communicating (3.340). The largest difference between two groups was 2.064 points (10.32 %) between systematization and communication, giving the section a somewhat balanced distinction. In the other section (Section 2), explaining had the most points (4.667), then verifying (4.600), systematization (3.933), discovering (3.70), and then communicating (3.267). In that section the largest difference was between explaining and communicating, with explaining earning an average of 1.4 more points (7%), giving the section a balanced distinction.

At the end of the semester, the averages seemed to still be balanced. In Section 1, verifying had the most points (4.842), then systematizing (4.565), communicating (3.751), discovery (3.646), then explaining (3.378). The largest difference between two functions was 1.464 points (7.32%) between verifying and explaining, giving Section 1 a balanced view of the functions of proof. In section 2, verifying had 5.695 points, then explaining (4.121), discovery (3.544), systematization (3.421), and communicating (3.284). The largest difference here (2.411 points; 12.055 %) was between verifying and communicating, giving Section 2 a somewhat balanced view of the functions of proof.

Taken as a whole, the students' views about the functions of proof did not appear to change much throughout the course. The largest average change between both sections was the change in the verification function, which increased by 1.202 points out of 20 (6.01% increase). Further, explanation decreased by an average of .454 points (-2.27%), systematization decreased by an average of 0.807 points (4.035%), communication increased by 0.214 points (1.07%), and discovery decreased by 0.072 points (0.36%). At the end of the semester, most students in both sections gave verifying the most points and verifying had the highest number of points in each section.

However, when we look at individual students, their views appear to be less balanced, and changes in their views become more apparent. A total of 15 students completed both the Pre- and Post- Functions of Proof Surveys. Below, I give two tables and three figures that summarize these 15 students' views about the functions of proof and journal and course impressions. The first table give the point breakdown of the students' responses on the Pre- and Post- Proof Views Part II surveys, as well as their degree of balance for each. Underneath the column for each function of proof, the table cells contain the number of points given by the student to that function in the Pre-Proof Views survey and then the number of points given by the student to that function in the Post-Proof Views survey in parenthesis. For example, if a cell in the Systematize column contained the values 4 (7), then that student gave “Systematize” 4 points at the beginning of the semester and then 7 points at the end of the semester. The second table gives the degree of balance level for each student, as well as relevant information about their journal completion and impressions of the course and journals. There are two students, B14 and B5 who did not assign a total of 20 points when they completed Part II of the Post-Proof Views Survey. B5 assigned a total of 18 points and B14 assigned a total of 21 points. For each of those students, I calculated the percentage of points they gave each function and converted the percentage to the number of points out of 20. So for example, B4 gave verifying 3 points out of 18, which I converted to 3.333 by the following calculation: (3/18)*20. The figures show the points assigned to each function by the students grouped into three groups: those whose degree of balanced shifted towards being more balanced, those whose degree of balance shifted towards being less balanced, and those whose degree of balance stayed the same.

Table 22

Views on the functions of proof, individual students' pre-scores and (post scores).

Student	Verify	Explain	Systematize	Communicate	Discover	Pre-Degree of Balance	Post-Degree of Balance
Mike	5 (8)	3 (5)	1 (3)	3 (3)	8 (1)	Unbal.	Unbal.
Lizzie	3 (8)	7 (4)	7 (3)	0 (0)	3 (5)	Unbal.	Unbal.
Saul	6 (4)	5 (0)	3 (4)	3 (8)	3 (4)	Unbal.	Unbal.
Alicia	4 (4)	4 (4)	4 (4)	4 (4)	4 (4)	Bal.	Bal.
Amy	4 (4)	4 (4)	4 (4)	4 (4)	4 (4)	Bal.	Bal.
Sheila	5 (5)	4 (5)	5 (1)	3 (5)	4 (4)	Some.	Some.
Lauren	4 (4)	4 (5)	2 (4)	4 (3)	6 (4)	Some.	Some.
Steven	1 (5)	2 (3)	14 (5)	1 (3)	1 (4)	Unbal.	Some.
Rachel	6 (11)	8 (7)	3 (2)	3 (0)	0 (0)	Some.	Unbal.
Maliah	6 (3.333)	5 (4.444)	5 (4.444)	1 (4.444)	3 (3.333)	Unbal.	Bal.
Roberto	6 (4)	5 (4)	3 (4)	3 (4)	3 (4)	Some.	Bal.
Emma	2 (4)	4 (4)	6 (4)	4 (4)	4 (4)	Some.	Bal.
Jaime	3 (5)	6 (7)	4 (4)	5 (3)	2 (1)	Some.	Unbal.
Sam	4 (2)	4 (5)	4 (5)	4 (4)	4 (4)	Bal.	Some.
Navarro	4 (7.619)	2 (2.857)	3 (3.810)	5 (0.952)	6 (4.762)	Some.	Unbal.
Average points given to each function by all 15 students	4.2 (5.263)	4.467 (4.287)	4.533 (3.684)	3.133 (3.360)	3.667 (3.400)		

Table 23
Summary of individual students' views and course experiences.

Student	Number of unstruc. journals	Number of struc. journals	Course impressions (positive, negative, or neutral)	Journal impressions (positive, negative, or neutral)	Reported journal benefits	Pre-degree of balance	Post - degree of balance
Mike	3	3	Pos	Pos	Feedback	Unbal.	Unbal.
Lizzie	5	7	Pos	Pos	Content	Unbal.	Unbal.
Saul	5	7	Pos	Pos	Feedback	Unbal.	Unbal.
Alicia	4	6	Pos	Pos	Content, problem solving, & therap.	Bal.	Bal.
Amy	5	7	Neg	Neu	Only to struggling students	Bal.	Bal.
Sheila	5	5	Neu	Pos	Problem solving	Somew.	Somew.
Lauren	5	5	Pos	Pos	Problem solving	Somew.	Somew.
Steven	5	7	Neu	Neu	Problem solving	Unbal.	Somew.
Rachel	5	3	Pos	Pos	Problem solving	Unbal.	Bal.
Maliah	5	7	Pos	Pos	Content	Unbal.	Bal.
Roberto	5	6	Pos	Pos	Feedback	Somew	Bal.
Emma	5	7	Neg	Pos	Problem solving	Somew	Bal.
Jaime	0	1	Neg	Neg	Content	Somew	Unbal.
Sam	5	5	Pos	Neg	Content	Bal.	Somew.
Navarro	5	7	Pos	Neg	None	Some.	Unbal.

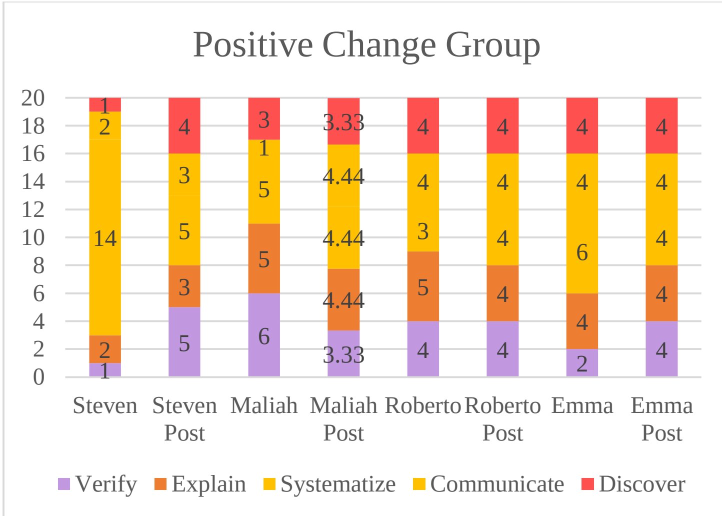

Figure 12. Positive Views Change Group.

The above students' pre- and post- responses are given side by side, with the post-views of functions of proof being more balanced than the pre- views.

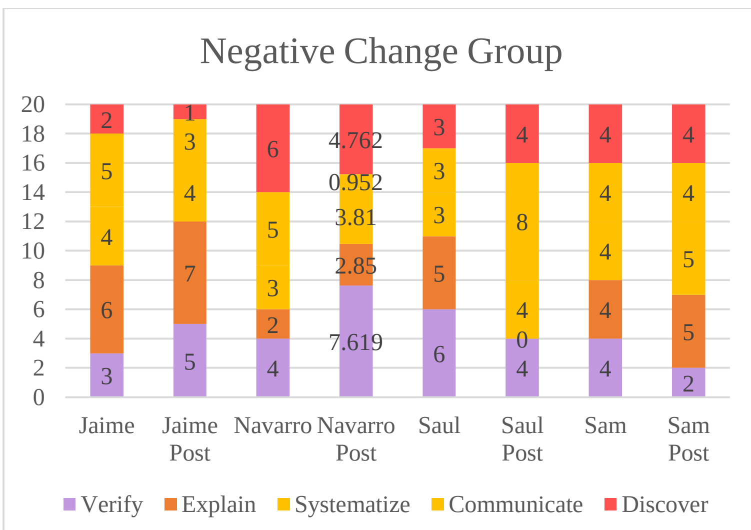

Figure 13. Negative Views Change Group.

The above students' pre- and post- responses are given side by side, with the post-views of functions of proof being less balanced than the pre- views.

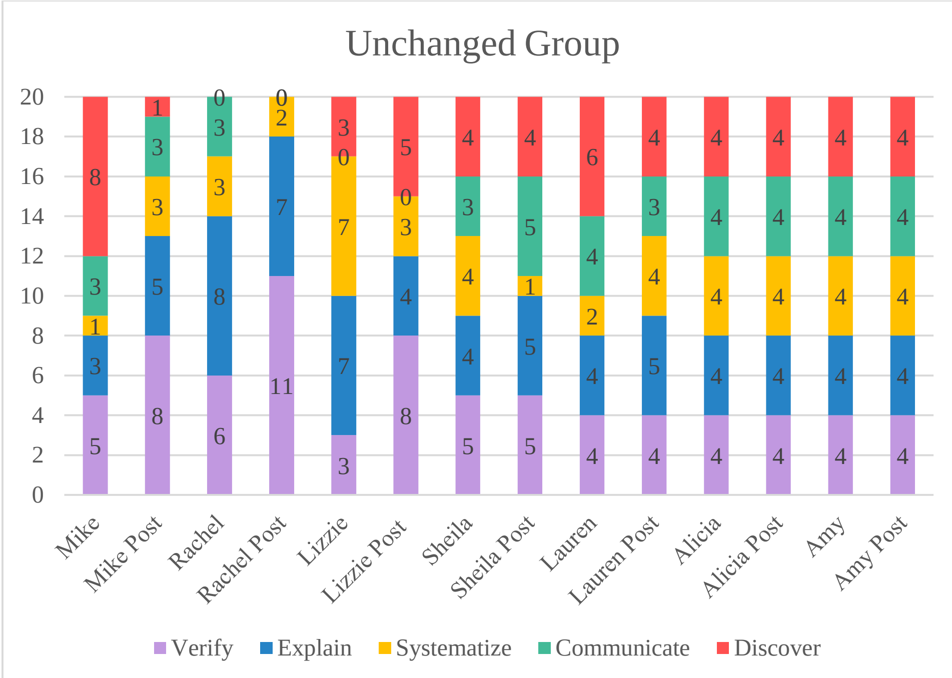

Figure 14. Unchanged Views Group.

The above students' pre- and post- responses are given side by side, and for each student, their pre- and post- views of the functions of proof had the same degree of balance.

Of those 15 students, 3 were considered balanced, 6 were somewhat balanced, and 6 were unbalanced at the beginning of the semester. At the end of the semester, 5 students were considered balanced, 4 were somewhat balanced, and 6 were unbalanced. Five students transitioned in a positive direction (moving towards being more balanced), 3 transitioned in a negative direction (moving towards being less balanced), and 7 remained unchanged in their degree of balance. It is important to note, however, that some students, like Mike, gave particular functions very different total points between the pre- and post- survey without the degree of balance changing. In Mike's case, he gave Discover 8 points at the beginning of the semester and 1 at the end. For the pre- survey, he was considered unbalanced because the maximum difference between functions was 7 points between Discover and Systematize. At the end of the semester, he gave Verify 8 points, which gave a maximum difference between functions of 7 points (between Discover and Verify), earning him another unbalanced label. So although he was unbalanced at both the beginning and end of the semester, the functions he considered most important changed. More research is needed to establish a framework for looking at students' views of the functions of proof in a way that examines changes in views and relative weights of the functions.

Of the students who transitioned in a positive direction, 4 had positive journal impressions, and 1 had a negative journal impression. Also, two of the students had previous proof experience and two did not. Students whose views shifted towards being more balanced mentioned problem solving, content, and feedback benefits.

Students whose views remained the same degree of balance mentioned problem solving, content, therapeutic, and feedback benefits of the journals. These students had mostly positive journal impressions (5 positive and 1 neutral). Of these students, three had previous proving experience, 3 did not, and 1 did not complete the Pre-Proof Views survey.

Of the 3 students who transitioned in a negative direction of balance, all three had negative journal impressions. Students in this group mentioned content benefits or said they experienced no benefits as a result of the journals, and all three of these students had previous proving experience.

Overall, it appears that the students who had negative impression of the journals tended to be in the group of students whose views shifted towards being less balanced. Further, it does not appear that any students without previous proving experience shifted towards less-balanced views. There does not seem to be a major pattern in how students in any group distributed points among the five functions of proof. Also, the majority of students in each group of students completed 10 or more journals. As a result of conflicting data, it is difficult to detect a strong journal influence on how students' beliefs about the functions of proof changed or did not change.

Interviews: Student Impressions of Journals and Views

Similar to the survey results, the interview participants were not in agreement about whether they thought the journals influenced their views about proof. Two of the five participants had taken a previous proof-based course and said that the introduction to advanced mathematics course therefore did not affect their views about proof. For example, when asked if the course influenced her beliefs about proof, A said "Not really, no. Um okay, so I took discrete math, so ...I had already done some proofing and whatnot. Um, not in this course, I did more proving (laughing) I didn't really think that it would change my perspective." Similarly, Bill said "It's just now everything that I've seen in the past, uh, from notation and everything, now I, I, I learned all the notation before. I mean, now we just ... it's very okay." Alicia's response highlights the expectation of many students entering the course that their views would not change, and instead they would just learn more diverse proving techniques. Of the three students who did think the course influenced their views about proof, Diana said that although the course made her realize how tedious proofs are, it strengthened her confidence in her proof writing, Emma said the course helped her see how important definitions are to the structure of proofs, even though they "seem basic", and Caleb aid the course helped him see the reasoning behind formulas and procedures. In the survey, I used the word "beliefs" instead of views, which seems to have impacted the students' responses, as many of them discussed their views of themselves as provers and not their views about what it means to prove. In future research, I will clarify my survey items to focus specifically on views about the nature of proving.

When asked how, if at all, the journals influenced their beliefs about proof, 4 of the 5 participants said they did not think the journals influenced their beliefs. Although 3 of these 4 students mentioned positive journal impressions and benefits from the journals to their proof writing, they did not believe the journals directly influenced their views about mathematics and proof. Caleb, the student who did believe the journals influenced his beliefs about proof gave the following explanation as to how: "It, um, it made me humble myself a little bit, because there's going- there's times where you would just, you know, obsess over a problem and it's, not that it's so difficult, it's just trying to find the right steps, and then, um, sometimes you just need to ask for help." Here, Caleb is discussing his views about himself as a mathematician, not necessarily his views about mathematics. By encouraging him to reflect on his understanding and struggles, the journal assignments helped C develop the belief that asking for help is ok, and is helpful.

Overall, the interview participants restated many of the sentiments of the students in the Post- Proof Survey, namely:

1. The students who had previously taken a proof-based course tended to not believe their views about proof changed as a result of the course or the journals
2. The students who did believe their views about proof changed as a result of the journals tended to describe changes to their views of themselves as

learners or mathematicians, and not changes to their views on the functions of proof or nature of mathematics.

Summary of Research Question 3

At the beginning of the semester, many students (21 out of 32 respondents) had taken a previous proof- heavy course (mostly discrete mathematics). These students did not express changes to their views as part of their course goals. The 11 students who had not taken a previous proof-heavy course mentioned views-related goals. Students rarely discussed their views about mathematics and proof in the unstructured journals, and when prompted to do so in the structured journals, the students tended to discuss their views of their own progress and understandings in the course, not their views about the nature of mathematics or proving. At the end of the semester, students believed the most important function of proof was to verify the truthfulness of a statement, followed by explain, systematize, and discover and communicate. However, students generally did not describe changes to their views about mathematics of proof as a benefit of the journals.

At both the beginning and end of the semester, the course averages of the students' scores on the quantitative instrument were mostly balanced in their beliefs about the relative importance of the 5 functions of proof. 15 students completed both the Pre- and Post- Proof Views Part II surveys, and looking at their individual scores revealed more variation, with five students transitioning towards being more balanced, 3 transitioning towards being less balanced, and 7 remaining unchanged in their degree of balance (3 unbalanced, 2 balanced, and 2 somewhat balanced). Of the students who transitioned towards being more balanced or remained the same degree of balance, 9 had positive journal impressions and 2 had negative. Of the students who transitioned towards being less balanced, all three had negative journal impressions. However, there did not appear to be a relationship between the number of journals completed and a change in the degree of balance, because the majority of students in each group of students completed 10 or more journals. In chapter 5, I will connect these results to research on how view students view the functions of proof as they learn mathematical proof.

V. DISCUSSION

Current mathematics education researchers highlight many challenges students face when learning proof, including deficits in student proof writing, proof comprehension, and appreciation for proof (De Villiers, 1990; Weber, 2002; Stylanides & Stylanides, 2009). Further, research has highlighted ways in which traditional, non-interactive lecture-based instruction in introduction to proofs courses reinforces procedural notions of proof writing, and conceptualization of a proof as a formal, finished product. Such instruction often includes lectures that do not require students to be active in proof construction, and the instructors require students to turn in only "polished" proofs in their homework (Weber, 2014; Lai & Weber, 2013; Yoo, 2008; Jones, 2000; Blanton, Stylianou, & David, 2009). There is less research on ways to support students' learning proof and how students develop more sophisticated notions of proof, and even less research about the merits of written reflections in proof writing.

This study begins to address this gap in the research literature by examining how the use of reflective journaling enhances students' learning of proof. In this research, I explored the benefits students received from the journals, and the relationships between the journals and students' proof writing ideas, course performance, and views about the functions of proof. Below, I summarize the findings presented in Chapter 4 and situate them in the field of research literature on the learning of proof and writing to learn in mathematics.

Summary of Findings

Benefits

This study found that the participating students experienced numerous benefits as a result of the reflective journaling tasks. In terms of the Borasi and Rose (1989) framework on benefits of journaling in mathematics courses, there was a discrepancy between the benefits apparent in the students' journal entries and the benefits the students reported experiencing. In 150 the unstructured journals, the therapeutic benefit occurred in almost half of the journals (49%), followed by problem solving (36%), content (11%), and then views (4%). However, in the surveys and interviews, students reported experiencing mostly problem solving benefits, then content, and only three students mentioned the views benefit. Further examination of the students' survey and interview responses reveals a possible reason why. Although students wrote therapeutically in the journals to complete the assignments, most students found the unstructured journal prompts to be "a hassle". They did not recognize a significant value in reflecting about the course unless the reflections were directed towards proof topics. This was not consistent with the pilot study results, in which students overwhelmingly appreciated the therapeutic benefit of the journals. However, the students in the pilot study were Honor's College students, and were more diverse in their majors. Many of the pilot study students were not science majors. Also, the instructor read the journals directly and responded personally to each one. Though it is worth noting that in the pilot study, the most frequent recommendation to improve the effectiveness of the journals was to include more proof related prompts, which is why this study included 7 structured prompts instead of 4.

When asked for suggestions for how to make the journals more effective, two-thirds of the respondents in the current study suggested including more prompts that are directly related to proofs or to write proofs directly into their journals and reflect on the process. To understand why, I looked at the benefits the students felt were most present in the journals: problem solving and content. Within the problem solving benefit, students described using the journals for organization and goal setting, a record of their progress over time, and to pinpoint their understandings and misunderstandings. Students often wrote in the journals about their unease at explaining their thinking and described their frustration at getting stuck on a proof and not knowing how to proceed. They appreciated the structured prompts because, by writing about specific proof topics, students were able to "practice writing informally" and "identify issues" as well as put their understandings into words.

This notion of getting stuck on a proof and not knowing where to continue may be related to students' focus on proof as a formal object (Weber, 2014). In lower level mathematics courses, many students develop computationally-based, procedural notions of mathematics, and they bring these ideas with them into proof-based courses (Dreyfus, 1999). When describing their proving strategy, students listed procedurally-focused behaviors like "I write 'Proof:'", or "I write my assumptions and the conclusion". Fewer students wrote about building an understanding of what they were trying to prove or understanding the ideas behind the proof method. By reflecting on how they understand the proof or proof-related ideas, students were able to begin to identify exactly where they were "stuck" in their understandings.

This recognition of where the students were stuck and the exercise of putting their understandings into words in an informal setting was particularly helpful because, in the journals, students did not have to worry about presenting a complete, formal write up of an argument as they would on a homework assignment. When students submit completed proofs, they do not submit their scratch work, just as when presenting proofs to their students, university instructors often do not write down their scratch work or write informal explanations. Generally, the explanations are given verbally as the instructor writes the formal proof (Lai & Weber, 2013; Alcock, 2012). However, mathematicians hardly ever write a complete, finished proof the first time around. In the journals, the students were able to create a sort of first draft narrative of their understandings or explanations, which they felt helped them in the course.

Because of the problem solving and content benefits, 14 responding students said they would recommend keeping the journals to a friend about to take the course, 7 said no, and 8 said maybe under certain conditions, which included: if the prompts are all proof-related (4), if the student is struggling in the course (3), or if the student is interested in journaling (1). There did appear to be a difference between the sections, however, on how the students responded to this question. In Section 1, the section that was more lecture-based, the students did express a greater appreciation for the journals, with 7 answering yes, 2 no, and 2 unsure to the question about recommending the journals to a friend. Although the instructor did call on students often in class to give next steps in the proofs the professor presented, the students did not present and explain proofs in class. In Section 2, which was inquiry-based, and in which the students had a presentation requirement in their course grade, 6 answered yes, 5 no, and 6 were unsure about recommending the journals. The sample sizes are too small to determine if the findings are significant, but the apparent difference could be because students in Section 2 were repeatedly required to “defend” their proofs to the class and explain their reasoning, as well as to critique their classmates’ proofs. This may have allowed them to reflect on their understandings and address their “getting stuck” in class. By reflecting on their understandings and proof writing process, students created a narrative of their understandings, which supported their proof writing, itself a form of a narrative.

In addition to using the journals to supplement their homework by using the journals to become aware of their own conceptual understandings, which in turn influenced their procedural proof writing, students used the journals as a vehicle for communicating questions or concerns with the instructor. Although the professors did not read all of the journals, the researcher read and responded to the entries and submitted weekly summaries to the professors. Both participating instructors read the summaries, reflected on the comments, and made conscious attempts to address the concerns in class, either directly or indirectly. Students in both sections noticed the professors’ incorporation of their journal comments into the class instruction and expressed appreciation for that in the surveys and interviews. In this way, a form of discourse between the professors and students evolved, and the students felt their voices mattered. However, this feedback benefit was mentioned less frequently than the problem solving benefit, so it appears that feedback is a supporting benefit but not the students’ primary use of the journals. Borasi and Rose (1989) also provide a framework of benefits to the instructor and benefits to the student-instructor discourse from students journaling in mathematics, which should be explored more in-depth with this population of students in future studies.

Supporting Ideas about Writing Proofs and Performance

In the journals and task-based interviews, students displayed all three types of proof ideas (procedural, heuristic, and key) identified in Raman's (2003) framework on proof writing ideas. However, the frequencies were far from equal. Procedural ideas occurred more than twice as many times as key or heuristic, which appeared to occur with approximately equal frequency. This is consistent with Raman's finding that undergraduate proof learners tend to display procedural ideas in their proof writing. I considered the relationship between the journals and proof writing ideas in numerous ways. Of the 10 students who demonstrated key ideas in their journals, 9 completed at least 10 journal assignments and 7 described positive journal impressions. Also, 9 of the students found the journals to be beneficial to their learning to prove and they described content, problem solving, and instructor dialogue benefits from the journals. As I discussed above, a benefit students experienced as a result of the journals was becoming aware of their own understandings and misunderstandings of proof topics. It appears that this benefit played a role in the students' development of key proof writing ideas. By reflecting on their proof writing process and explaining the proof in informal terminology, these 10 students were reminded to consider their conceptual understandings in addition to the procedural aspects of writing a proof.

The participants were split on whether they felt the journals directly supported their learning to prove: 12 said yes, 12 said no, and 2 were unsure. However, of the 12 who said no, 6 said the journals would have been helpful if they included more prompts directly related to proof writing. The journal's relationship to proof writing ideas is also apparent in the ways students described how the journals supported their learning to prove: communicate with the instructor and get feedback (3), identify issues in their understandings (3), record and reflect on their progress (5), motivate themselves to succeed in the course (1), and solidify their thoughts (1). Although these benefits seem like they may be apparent in homework assignments as well, at least half of the participants found them to have a component unique to the journals, and it appears that this quality is related to the informal, rough draft-like aspect of the journal writing. For students who are new to proof, a proof does not necessarily connect to their ideas about writing the proof. Instead, the proof and proof-writing ideas are thought to be separate (Raman, 2003). Each of the described benefits were related to the students' *ideas* about proof, expressed in their own words, not their formal proof attempts, which the students did not seem to recognize as directly related to their ideas about proof or their homework proofs.

The interviews also supported the claim that the journals supported students' proof ideas and proof writing; the two participants who successfully completed the task displayed key ideas, positive journal impressions, and described the most varied journal benefits (both mentioned therapeutic, problem solving, and content benefits). These two participants also, unprompted, said they were planning to keep a journal in future mathematics courses. Of the three students who were unable to complete the proof in the task independent of my help, two displayed procedural ideas about proof writing and one displayed heuristic ideas. One of the procedurally focused students did not find the journals helpful at all. However, this student also found the course to be subpar in terms of difficulty (although he earned a B, not an A), and felt his ideas about proof did not change at all during the course. He, like many students in both sections, had previously taken a course that required him to write proofs, and therefore felt like he already knew how to prove.

In the surveys, I noticed this closed-mindedness about the introduction to mathematics course from many students who had previous proving experiences. They failed to realize that within mathematics, different disciplines employ different proof techniques. For example, a proof in a discrete mathematics course may be quite different from a proof in an analysis course. When I quantitatively explored the relationship between journaling and proof writing performance by calculating the correlation between the number of journals completed and the course grade, one group of outlying data was composed entirely of students who had previous proving experience, who did fewer than 2 journal assignments and still passed the course. These students also completed an average of fewer than half of the homework assignments, but scored high enough on exams to carry their course grade into passing. The other group of outlying data consisted of four students who completed at least 10 journal assignments but earned a course grade of 60 or below. These four students had no previous proving experience, and one of them mentioned in her first journal already feeling behind compared to her classmates since she had no previous experiences proving. I believe future research is needed into how the students' previous experiences with proving affect their expectations and experiences with introduction to proofs courses, and how the journals may affect the learning for both populations of students (previous proving experience vs no previous proving experience).

The quantitative correlation analysis also supported the claim that journal completion is related to proof writing success. All of the correlations presented are significant with $p < 0.05$. A positive correlation of 0.4003 between the number of journal assignments completed and course grade was calculated when considering the entire sample of students. This correlation increased to 0.4544 when considering only the number of structured journals completed. Also, for the students who felt the journals directly supported their learning to prove, the correlation between journals completed was $r = 0.5524$. It appears that the benefits to their proof ideas that students reported experiencing are supported with the data. Although the correlations do not imply causation, they suggest there is a positive relationship present worth exploring further.

Views about the Functions of Proof

Of the four benefits in the Borasi and Rose (1989), students displayed the views benefit least frequently in their journals, with discussions of views comprising only 4% of unstructured journal entries. This is consistent with Borasi and Rose's findings about the content of unstructured journal writing in mathematics. In the three views-related structured journals, students generally discussed their views related to their understandings or the perceived difficulty of proof topics, and not their views about the nature of mathematics or proving. In this way, the journals did not appear to influence students' views about the nature or functions of proof since the students did not write about their views related to these topics. Perhaps more journals explicitly asking about views would have triggered a larger change in views or recognition of change in views.

The students' views did appear to change throughout the course, however, as demonstrated in the quantitative Pre- and Post- Proof Views Part II instrument, that asked the students to distribute 20 points among 20 statements about the functions of proof. This instrument allowed me to measure the relative importance each student ascribed to DeVillier's (1990) five functions of proof (verify, explain, communicate, systematize, and discover), as well as determine the degree of balance of their views. I chose to look at the degree of balance of the views because there is not agreement within the mathematical community about the relative importance of each function. Of the 15 respondents who completed both the pre- and post- instrument, 8 of them appeared to have the balance of their views change, 5 towards being more balanced and 3 towards being less balanced. The students whose views about the functions of proof remained the same degree of balance or transitioned towards being more balanced had mostly positive journal and course impressions, and recognized problem solving, therapeutic, content, and feedback benefits of the journals. The three students whose views transitioned towards being negatively balanced all reported negative journal impressions. However, there was no apparent relationship between the number of journals and a shift in the degree of balance of students' views.

In terms of the relative weights of the functions, most students in both sections gave the verify function more points at the end of the semester than they did at the beginning of the semester. Students repeatedly expressed frustration at having to prove theorems which they already accepted to be true, such as the product of two even integers will be even. For the students, proofs like this did not seem to communicate, explain, or discover anything new. Weber (2014) points out that the types of proofs presented to students influence the students' views about the purpose of proof. It seems that in the introduction to advanced mathematics course, the focus is more on being introduced to proof ideas and learning basic proof techniques, which appears to shift students' views about the functions of proof towards verifying or convincing. Perhaps it is later, in advanced mathematics courses such as analysis, that students use proof as a tool to explore content that the appreciation for the various other functions of proof develops.

In both sections, students described other affective benefits from the journals, such as increases in confidence, motivation, goal setting, and the realization that it is ok to ask questions. Loud's experimental study (1999) found that being reflective helped university mathematics students develop more positive beliefs and attitudes about mathematics than the students in the control group; however, Loud's sample consisted of all female students enrolled in a contemporary mathematics course. In her book, *Overcoming Math Anxiety,* Sheila Tobias (1995) describes keeping a math diary as a way to confront and overcome mathematics anxiety. The interview participant who had the heuristic focus on writing the proof task described having severe math anxiety and found the journals to be a way to motivate herself and become aware of her progress. In this way, she used the journal assignments as an outlet for her anxiety. These affective benefits were not part of the framework for my study, but are worth mentioning here. Perhaps by reflecting on their understandings, and confronting and overcoming their feelings of getting stuck, the journals may have helped students build their confidence and motivation to continue. The journaling effect on students' affecting qualities should be investigated further in future studies. Although there was no clear indication of the journals affecting students' views about the nature of proving and the functions of proof, the journals did seem to provide students with a record of their personal process of becoming a prover. Future research can investigate how the journals affect the development of students' identities as provers.

Implications for Teaching an Introduction to Proofs Course

A traditional obstacle in the teaching of this course is balancing the students' focus on procedural ideas, such as notation, logical structure of arguments, clarity in writing, and formatting the structure of the proofs, with heuristic ideas, such as understanding the ideas behind the theorems, recognizing connections, and understanding the concepts behind the different proof techniques. When students just submit their finished proofs in homework, the instructor's feedback is centered on the write-up of the proof (the product) instead of the process of how the student thought about the ideas. Because of this, students traditionally become more procedurally focused and get bogged down with the formatting and notation for proofs.

One of the main implications from this study is that providing students opportunities to reflect on their understandings and to practice writing informally about proof ideas help students focus on the process of proof writing and related concepts. The analysis of the data in this study suggests that the reflective writing assignments helped students make sense of what they were learning and become aware of and keep track of their understandings and progress throughout the course. The journals also were positively correlated with students' performance in the course. More students found the journals to be worthwhile than not, and the students overwhelmingly suggested that including more proof-related prompts and reflections about their process for writing specific proofs would make the journals more effective. The journals connect to proofs via the idea of forming a narrative: a proof is itself a form of a narrative, and the journals allow students to create their personal narrative for the course. The students' recommendations to include more prompts requiring them to write a proof and then reflect on their process for constructing that proof suggest they recognize the connection between creating their personal narratives for a proof and constructing the formal narrative (proof).

Another implication of this study is how the journals provide students an opportunity to demonstrate to the instructor their understanding in a way that just turning in proofs does not allow for. Students repeatedly described their frustrations at getting stuck on a proof, particularly when they felt like they understood the ideas behind the proof but just did not know how to structure and write the proof. The journals allowed students to communicate this heuristic understanding to the professor, and get credit (in the form of a journal grade) for that heuristic idea. This also allowed the instructor to have an idea of how the students conceptualize proof-related ideas and topics, which the instructor might not see from reading turned-in proof attempts alone.

Another finding that affects the teaching of this course is whether or not students have previously taken a course that required proofs on the class dynamic and students' experiences in the course, primarily discrete mathematics. Some of the students who did not have previous proving experience noticed that many of their classmates did, and described in their journals that they felt behind at the start of the semester because they had no proving experience, although it was not required to enroll in the course. Some of the students who did have previous proving experience felt the course moved too slowly for them, and did not appear to have an open mind about how their views and understandings could progress and develop in the introduction to proofs course. Rather, they just wanted to learn more proof techniques and felt they already understood the ideas behind proof and the importance of proof. University mathematics departments should be aware of the impact of the sequencing of the courses on the students' experiences and take that into consideration when designing degree plans.

Implementation Suggestions

As a result of my observations, discussions with the participating instructors in this study, and analysis of the students' comments in the journals, interviews, and surveys, I give some suggestions for how to effectively implement reflective journaling in an introduction to proofs course.

Assigning, Collecting, and Giving Feedback

The students appreciated being able to submit their reflections online, but many complained that they often forgot to do the journals. Also, some students did not realize there was feedback to their journals or know how to check the feedback. Further, the students appreciated the prompts that directly related to proof writing and proof ideas more than the unstructured, open-ended prompts. For some students, the idea of keeping a "journal" was a hassle, and they did not recognize how "describing your day" could be helpful. Therefore, I recommend that instead of having the journal be a separate grade and assignment, make it part of the weekly or bi-weekly homework assignment. Perhaps the last 2-3 questions each week could be "reflection questions" that students write up by hand or type, print, and attach to their homework assignment to turn it in. This will also encourage reflection, as students would be able to think about the homework proofs they recently completed or struggled with as they write their reflections. Further, this will show students that writing the proofs and reflecting on their understandings and proving process are related and complement each other.

While it is preferable for the instructor to read and respond to the journals, the strategy of acting as a teaching assistant or grader for the course, reading the journals, grading and responding to them, and then submitting the summary to the instructor seemed successful in both sections. Both instructors read the summaries and consciously attempted to consider and address the students' comments, and students in both sections recognized and appreciated these responses to their particular struggles.

One instructor suggested using a free, file-sharing website and having each student have one file that he or she adds to each week. This way, the student could type in a journal entry, and the instructor or grader could type in a response into the same document, and the process could continue this way with all of the student's journals and feedback in one document. This would help the students see the narrative they are constructing of their course experiences, because they would scroll through previous journals to write the new journal entry.

Prompts and Journal Content

The students overwhelmingly found the structured journals to be the most beneficial. However, they did write often about therapeutic topics in their unstructured journals. Therefore, I recommend the "reflection questions" being structured, but with the additional comment at the end of the last prompt to "Please also describe any other questions or comments you have for the course that you think are important." This way, students realize they have the option to write therapeutically if they would like and can dialogue with the instructor about their impressions of the course and their learning in the course.

Incorporating the journals into the homework assignments also helps students connect the prompts to class topics. In addition, while it is good for students to reflect on instances when they were successful at figuring out a proof, I also suggest including prompts asking students to reflect on an instance where they were unsuccessful and how they might go about resolving the issue they are having. The students could also possibly be given this prompt as a follow up once the graded homework is returned. Finally, I recommend providing prompts that address the problem solving, content, and views benefits identified by Borasi and Rose (1998). For the views benefits, it is important to be explicit with students and ask questions that directly address their views on the nature of mathematics and proof, and reasons why people prove. Below, I provide a table with possible prompts to use, along with the benefits they may address.

Table 24

Suggestions of structured prompts and associated benefit.

Prompt	Benefit
Discuss the role that definitions play in mathematics and writing proofs. How are definitions important? How might you use definitions when writing proofs? Please also describe any other questions or comments you have for the course that you think are important.	Problem Solving, Views
Choose a definition that you have recently been using in class - it can be one that you understand well or one that you are struggling with. Write the definition using formal terminology, and then write in words how you interpret that definition. How would you describe it to a friend?	Content, Problem Solving
When given a theorem to prove, what is your proving strategy? How do you judge the completeness of a proof?	Problem Solving
Please pick a proof that you recently completed and copy this into your journal What did you think about and what was your process for writing the proof, starting with when you first read the theorem to prove?	Content, Problem Solving
So far in the course, what is your favorite proof technique? Why? What is your least favorite proof technique? Why?	Views, Content
For this final journal, please reflect on the course and your progress throughout the course this semester. What advice would you give to a student about to take an introduction to proofs course? How, if at all, did your ideas about mathematics and proofs change during the semester? Please also mention any other comments or questions you have.	Views, Problem Solving
Please pick a proof that you were unable to complete and copy your attempt into your journal. Identify where you got stuck and what you might do to resolve your issue.	Content, Problem Solving
Why do people write proofs? What is the value of proving theorems?	Views
Why do you think mathematicians place so much emphasis on the importance of being precise with language and definitions in proofs? For instance, how are the statements "For every x" and "For some x" different?	Views Content
Although in most cases an example is not a proof, many mathematicians use them to help with proof writing. What are your thoughts or experiences on how examples can be used to aid proof writing?	Views, Problem Solving
How do you understand the ______ proof technique? How would you describe it to a friend?	Content, Problem Solving

Limitations

This study focused on two sections of an introduction to advanced mathematics course in central Texas that was taught by two different instructors with two different teaching styles. I took the instructional styles into consideration when any differences between sections occurred. However, future research is needed to examine exactly how the course setup and instruction influence students' experiences journaling in an instruction to proofs course. Thus, generalizing should be done cautiously.

In the interviews, a student from each section reported not knowing there was feedback to the journals. Although the students were not explicitly asked about feedback on the Post-Proof Views survey, multiple students in both sections said that having feedback available would make the journals more effective. I am unable to say how many students knew and did not know there was feedback, so I am unable to make claims about the effects of written feedback on the students' learning in the course.

The five students selected for the interviews were chosen based on maximum variation sampling using the number of journals completed and course grade. While the five students demonstrated notable differences, a student with low journaling and a low grade did not participate. Therefore, the interviews do not necessarily reflect all of the variations
among the students.

To quantitatively explore the relationship between the journals and proof writing performance, I used the number of journals completed and course grade to calculate the correlation coefficient. I then used the survey results to sort the data and identify other correlations that may be present. My results suggest that there is a relationship present;

however, I am not implying causation, and there are other variables present that may be affecting the relationship. For example, course attendance, depth of the journal responses, and previous experiences with proof likely also interact with the journals. Future research should develop a more accurate measure of depth of journal writing than the number of journals completed, and should explore the quantitative relationship between the journals and course performance using more sophisticated statistical analysis. Also, I did not have a control group in this study. It would be interesting to do a comparison study to further explore the relationship between reflective journaling and proof-writing performance.

Future Research Directions

As I finished analyzing my data, I was left with many questions I would like to explore further. Because of the exploratory nature of my study, there was no existing framework on reflective journaling in an introduction to proofs course. More research is needed to develop solid theories on the phenomena associated with reflection in learning to prove. The students' descriptions of self-awareness as a benefit of the journals highlight a possibility for research on the cognitive value of journaling about learning to prove and influences of the journals on students' affective qualities such as self-efficacy and confidence. Further, similar studies can develop a more accurate instrument to measure the depth of students' journal writing, which will be useful in exploring the effects of the journals on students' learning of proof.

A limitation of my study was the miscommunication with some students about the presence of feedback to their journals. Although I discussed the feedback at the beginning of the semester and in reminder emails, some students did not realize there was feedback to their journals. Future studies can examine the impact that feedback has on the journal entries, students' experiences and impressions with the journals, and the effects of the journals on the students' views about proof and proof writing performance. My study suggests that the journals may not be appreciated as much by students who are taught using an inquiry-based approach than students who are taught using interactive lectures. Future research is needed to investigate the role of the class instruction on students' impressions of the journals. Further, both instructors in my study had an interactive component to their class sessions, in which students were responsible for contributing to the class dialogue, although in very different ways. It would be interesting to replicate this study in a purely lecture-based section of the course and see how the results compare or differ.

Although I did meet with the instructors about their use of the weekly summaries and future directions for the journals, I did not analyze their experiences in detail because of feasibility concerns with this study. Therefore, my study was only aimed at investigating the students' experiences with the journals, but future research can investigate the instructors' experiences with the journals as well.

Finally, my study brings up questions not only related to journaling during the learning of proof, but other areas related to learning proof as well, particularly in the area of examining the functions of proof and how students' views about the functions of proof develop. My study supports other research that has shown that students often become more procedurally focused and place stronger emphasis on the verification function of proof during introduction to advanced mathematics courses. However, somewhere along the line, the students who continue studying mathematics develop more balanced and flexible views about the functions of proof and key ideas about writing proofs. Perhaps it happens in courses such as analysis, topology, or abstract algebra where the proofs are not the main content of the course, but rather a tool. The quantitative Part II of my Pre- or Post- Proof Views instrument can be used in future studies to examine students' views about proof longitudinally, or compare how students' views about proof compare to those of their instructor. My quantitative instrument could also be used in conjunction with Weber's (2002) framework on the types of proof presented to students, and the functions of proof they represent, to investigate how the proofs presented to students affect their views about proof.

APPENDIX SECTION

APPENDIX A: Pilot Study Honors Elementary Number Theory Syllabus

Syllabus

Course Information

Semester – Fall 2014
Course – Honors 3392v
Section – 001
Class Time – M, W 3:30-4:50 pm
Class Room – Lampasas 502-B

Instructor Information

Name – Max Warshauer
Office – ASB 110
Telephone – (512) 245-2935
Email – max@txstate.edu
Office Hours – M,W 2:00-3:00, T, Th 5-6 and by appointment

Course Title – Elementary Number Theory

Course Description – Elementary Number Theory allows students at different levels of mathematical maturity to participate and work together. Students will study simple ideas about the integers, where they already have a well-developed intuition. To paraphrase David Gries (Science of Programming), one should never take basic principles for granted, for it is only through careful application of simple fundamental ideas that progress is made. The division algorithm is studied in detail, and is seen to have far-reaching consequences throughout the course; it yields Euclid's algorithm and the solution to linear Diophantine equations. Properties of divisibility also lead naturally to modular arithmetic. Students will explore ideas in number theory that have modern applications such as coding theory.

Objectives –

1. This class will provide each student with the opportunity to learn how to explore problems deeply and give careful, rigorous mathematical proofs
2. This class will provide each student with a foundation in elementary number theory, and the background needed for more advanced mathematics courses.
3. The student will learn how to explain their ideas both orally and in writing, and how to apply the mathematics learned to different types of problems. In short, the students will connect the ideas from number theory to thinking critically in a wide variety of areas.

Course Outline – See detailed topics below. The heart of the course is elementary number theory. We begin by agreeing on axioms for the integers, where the students already have a well-developed intuition. We then briefly cover logic so that students will be able to give rigorous mathematical proofs. The course then moves on to cover fundamental ideas in number theory, including divisibility, modular arithmetic, and applications such as public key encryption.

Students will present their solutions to problems in class. Problems include a rich mixture of questions that develop both computational fluency and theoretical understanding. The problems are to prove or disprove and salvage if possible. The main difference between this and many other math courses is that students will often be asked to explore ideas before they are discussed in class.

A detailed list of topics is given at the end of the syllabus. These topics will be connected to more general critical thinking that can be applied in a wide variety of areas.

Textbook

1. "The 5 Elements of Effective Thinking," by Burger and Starbird.

Daily Notes and Homework

Students should write up their daily notes and keep these in a "Number Theory Notebook" (NB). This notebook should contain both class notes and homework problems. This should be carefully organized, and not put together as an afterthought. A table of contents, copy of each problem, and work done on each problem is required. **The Number Theory Notebook (NB) will count 50 points towards the final grade.**

Homework

Each homework assignment should be kept in the Number Theory Notebook described above. Approximately 5 designated homework problems will be assigned each class, and returned the class after they are turned in. These homework problems should be organized as part of the Number Theory Notebook. **The homework (HW) will count 100 points towards the final grade.** Students may turn in Redos of the problems that are returned in the following class, along with the new assignment. A copy of the original problem submitted should be turned in with the Redos. Bonus points will be given based on class participation and class presentations.

Journal

Students will submit a journal each week describing their progress in the course, along with any problems or suggestions. Each journal submission should be no longer than 1 or 2 pages. The idea is to reflect on what you are learning, relate the mathematics to the 5 elements of effective thinking, and discuss any problems or concerns encountered. **The Journal (J) will count 50 points towards the final grade.**

Weekly quizzes

We will have short weekly quizzes (QZ) each Wednesday that will for 100 points towards the final grade. These quizzes will cover definitions, computations, and proofs.

Grading

As indicated above, the grade will be based on a variety of factors:

Number Theory Notebook (NB):	50 pts
Homework (HW)	100 pts
Journal (J)	50 pts
Weekly Quizzes (QZ)	100 pts

Midterm (MT)	100 pts
Final Exam (Final)	100 pts
Total points	500 pts

Average = [NB + HW +J + QZ + MT + Final]/5

Letter grades are given by the usual plan. We may curve the grades depending on the class, but in any case the usual cut-offs are guaranteed.

Test Dates:
Mideterm (MT) Oct. 15
Final: Monday, Dec 8 2-4:30 PM

Attendance Policy – Attendance is strongly encouraged. I hope that you will all enjoy class and want to come every day.

Withdrawal – The final date to withdraw with 100% refund is Sept. 10. The final date to drop with automatic W is Oct. 23.

Academic Integrity – No academic dishonesty is allowed in this course. Learning and teaching take place best in an atmosphere of intellectual fair-minded openness. All members of the academic community are responsible for supporting freedom and openness through rigorous personal standards of honesty and fairness. Plagiarism and other forms of academic dishonesty undermine the very purpose of Texas State and diminish the value of an education. Specific sanctions for academic dishonesty are outlined in the student handbook, UPPS No. 07.10.01. Students are encouraged to work together on the homeworks, but then should write up their own solutions independently.

Cellular Telephones – No telephones may be used in class. No telephones may be on the table top during examinations. Telephones should not ring in class.

Special Needs – Students with special needs, as documented by the Office of Disability Services, should identify themselves at the beginning of the semester.

Class Day	Descriptive Comments
1 Aug 25	The integers—bases
2 Aug 27	Axioms for the integers, ordering, divisibility
3 Sept 1	Labor day

3 Sept 3	Logic
4 Sept 8	Trichotomy, factorial notation, perfect squares
5 Sept 10	Properties of multiplication, positives, and negatives
6 Sept 15	Modular arithmetic, properties of congruences
7 Sept 17	Weak induction, Well-ordering
8 Sept 22	The division algorithm
9 Sept 24	Euclid's algorithm and the magic table
10 Sept 29	Greatest common divisor, least common multiple
11 Oct 1	Strong induction, primes, prime factorization
12 Oct 6	Uniqueness
13 Oct 8	Relations, equivalence relations
14 Oct 13	Review
15 Oct 15	Test 1
16 Oct 20	Complete and reduced residue systems
17 Oct 22	Fermat's little theorem, Wilson's theorem
18 Oct 27	Chinese Remainder Theorem
19 Oct 29	Polynomials, division algorithm
20 Nov 3	Factor theorem, roots of polynomials
21 Nov 5	Order of elements
22 Nov 10	Z mod p is cyclic
23 Nov 12	Units and Euler's phi function
24 Nov 17	Coding Theory
25 Nov 19	Raising numbers to large powers
26 Nov 24	Weak, strong pseudo-prime tests
27 Nov 26	Thanksgiving
28 Dec 1	Electronic coin toss
29 Dec 3	Last class day, review
Dec 8	Final exam, 2:00-4:30 PM

APPENDIX B – Pilot Study Initial Survey

Dear Number Theory Students,

Thank you for agreeing to participate in this research study. We ask you to please take about 10 – 15 minutes to complete the survey below. Your responses will greatly help our research.

Initial Survey

Answer the following questions as honestly as you can. Give some thought to the questions and please give detailed answers.

Name:

Major:

Classification: Freshman Sophomore Junior Senior

Gender: _____________

What math courses have you taken? (Approximate date and institution):

Why are you taking HON 3392V?

What are your expectations for this course?

High school math to me was…

Math to me is…

My favorite part of math is…

My *least* favorite part of math is…

A mathematical proof to me is…

APPENDIX C – Pilot Study Post Survey

Dear Number Theory Students,
Thank you for your participation in our research study. We ask you to please take about 10 – 15 minutes to complete the survey below and bring it with you to submit in class by Monday, November 24. Your responses will greatly help our research. Part 2 is to be completed online. The link will be provided.

Post-Survey Part 1

Answer the following questions as honestly as you can. Give some thought to the questions and please give detailed answers.

Name:

Math to me is…

My favorite part of math is…

My *least* favorite part of math is…

A mathematical proof to me is…

Please describe your overall impressions of the course this semester.

How has the course affected you beliefs about proof?

How do you feel about keeping a weekly math journal this semester?

How did the journaling support your learning to prove in this class, if at all?

What do you feel are the benefits of journal writing in a proof-based mathematics course?

How could journal writing be changed to be more effective?

How valuable, if at all, do you find instructor feedback on your journals submissions?

Would your recommend keeping a math journal to a friend about to start a proofs course in the future?

Thank you for your participation in our study this semester! We really appreciate it.

APPENDIX D – Pilot Study Interview Protocol

Honors Number Theory Study
Interview Protocol

INTRODUCTION
Hello, my name is Christina. Thank you so much for taking the time to be here. This is a personal interview, in which I will ask you about your experiences as a student in your Honors Number Theory course this semester. The purpose is to get your perceptions of your experiences learning to prove, especially your thoughts on the journaling component of the course. There are no right or wrong or answers; I would like you to feel comfortable with saying what you really think and how you really feel. The first part of our interview will consist of me asking you some open ended questions about your experiences this semester. Then, I'll ask you to attempt to prove a statement while explaining your thoughts as your write the proof.

LIVESCRIBE RECORDER INSTRUCTIONS
If it is okay with you, I will be recording our conversation (with a LiveScribe). The purpose of this is so that I can get all the details but at the same time be able to carry on an attentive conversation with you. I assure you that all your comments will remain confidential. I will be compiling a report which will contain all students' comments without any reference to individuals.

CONSENT FORM INSTRUCTIONS
Before we get started, please take a few minutes to read and sign the consent form.

PART I - Questions:

1. Questions about the course in general.
    a. Please describe your overall impressions of the course so far.
    **If they do not mention any negative aspects of the course, ask*: What are some aspects of the course you dislike?

2. Questions about proof.
    a. How has the course affected your beliefs about mathematics?
    b. What is a mathematical proof?
    c. How has the course affected you beliefs about proof?
    d. What is your favorite aspect of proving?
    e. What is your least favorite aspect of proving?

3. Questions about the journals.
    a. How do you feel about keeping a weekly math journal this semester?
    b. How did the journaling tasks affect your learning in the course?

c. How did the journaling support your learning to prove in this class, if at all?
d. How did the journaling tasks influence your opinions about proof in mathematics?
e. What are the benefits of journal writing in a proof-based mathematics course?
f. How could journal writing be changed to be more effective?
g. How valuable, if at all, do you find instructor feedback on your journals submissions?
h. Would your recommend keeping a math journal to a friend about to start a proofs course in the future?

PART II – Proof Task

For this part of the interview, I'd like you to attempt the following problem: (*Hand them the prompt, Livescribe pen, and Livescribe paper.)* Please do all of your scratch work and proof writing on the Livescribe paper so that the pen can record your written work. As you work on this task on the paper, please "think aloud" and explain your thinking for each part of the problem.

<u>Prompt:</u>

"Consider the statement: For real numbers $a, b,$ and $c,$ if $a < b$, then $ca < cb$.

- If the statement is true, construct a proof.
- If the statement is not true, please modify the statement to describe the correct relationship between ca and cb and then construct a proof of the modified statement."

Part III – Conclusion.

Ok, ____, that's all the questions I have for you. Is there anything else you would like to add or say? Thank you again for taking the time to help me with this research project. I appreciate it very much. Have a great day!

APPENDIX E – Pre-Proof Views Part I

Dear Introduction to Advanced Mathematics Students,

Thank you for agreeing to participate in this research study. I ask you to please take about 10 – 15 minutes to complete the survey below. Your responses will greatly help my research.

Pre-Proof Views: Part I

Answer the following questions as honestly as you can. Give some thought to the questions and please give detailed answers.

Name:

Major:

Classification: Freshman Sophomore Junior Senior

Gender: ______________

What math courses have you taken? (Approximate date and institution):

Why are you taking Math 3330?

What are your expectations for this course?

High school math to me was…

Math to me is…

My favorite part of math is…

My *__least__* favorite part of math is…

To me, the definition of a mathematical proof is…

In mathematics, people write proofs to…

APPENDIX F– Post-Proof Views Part I

Dear Introduction to Advanced Mathematics Student,

Thank you for your participation in my research study. Please take about 10 – 15 minutes to complete the survey below and bring it with you to submit in class. Your responses will greatly help our research.

Post-Proof Views Part 1

Answer the following questions as honestly as you can. Give some thought to the questions and please give detailed answers.

Name:

Math to me is…

My favorite part of math is…

My *__least__* favorite part of math is…

To me, the definition of a mathematical proof is…

In mathematics, people write proofs to…

Please describe your overall impressions of the course this semester.

How has the course affected you beliefs about proof?

How do you feel about keeping a weekly math journal this semester?

How did the journaling support your learning to prove in this class, if at all?

What do you feel are the benefits of journal writing in a proof-based mathematics course?

How could journal writing be changed to be more effective?

Would your recommend keeping a math journal to a friend about to start a proofs course in the future? Why or why not?

Thank you for your participation in my study this semester! I really appreciate it.
Christina Starkey

APPENDIX G – Journal Suggestions

Weekly Journal: Topic Suggestions

As part of your course requirements, you will be asked to write and submit a weekly reflective journal entry about the course. Sometimes you will be given specific prompts to respond to, and sometimes you will be asked to free write about your perceptions of the course and the course material. During those unstructured journal prompts, students have expressed difficulties with deciding what to write about. To help this, I have compiled a list of possible topics. These are not required, and are just suggestions if you are having trouble getting started writing.

Possible journal topics:

- What is your proof strategy?
- What does (definition) mean to you? How would you explain it to a friend?
- How do you understand the ______ proof technique?
- Reflect on your ideas or feelings about math
- What does a mathematician do?
- Is mathematics invented or discovered?
- What is the value of proving theorems?
- How do you judge the completeness of a proof?
- How are you feeling about the class?
- What do you understand well in the class?
- What are you struggling with in the class?
- Pick a proof or problem that you recently completed. How did you do that proof?
- What did you learn today in class?
- Respond to a particular class or topic
- How do you read the textbook?
- Describe your favorite math class

APPENDIX H – Department Introduction to Advanced Mathematics Syllabus

Syllabus

Course Information

Semester –
Course – MATH 3330
Section –
Class Time –
Class Room –

Instructor Information

Name –
Office –
Telephone –
Email –
Office Hours –

Course Title – Introduction to Advanced Mathematics

Course Description – An introduction to the theory of sets, relations, functions, finite and infinite sets, and other selected topics. Algebraic structure and topological properties of Euclidean Space, and an introduction to metric spaces. Prerequisite: MATH 2471 with a grade of "C" or higher.

Objectives – The goal of Introduction to Advanced Mathematics is to provide students an opportunity to learn to prove mathematical theorems. This course provides an introduction to higher level abstraction in mathematics. This is achieved within the following framework:

- Logic
- Set theory
- Number Theory
- Properties of real numbers
- Functions

Textbook – *A Transition to Advanced Mathematics,* 6th edition, by Smith, Eggen, St.Andre, pub. Thomson Brooks/Cole.

Brief Course Outline – *Course material will be selected from the first four chapters of this text. Supplemental material from other sources may also be covered as deemed appropriate by the instructor.*

Attendance Policy - *Students are expected to attend each scheduled class meeting. Regular attendance is essential to your success in this course. If you are not in class you are responsible for the material covered. NO MAKE UP EXAMS WILL BE GIVEN FOR ANY REASON. Should you miss an exam contact the instructor ASAP!*

Important Dates:

Exams - *see course calendar*
Final Exam –

Drop Dates -

Drop with no record -
Drop with an automatic W – by 5:00 pm on
Last day to withdraw from the University – at the office of the Registrar by 5 pm on

Grading – *Your final grade will be computed using the following percentages.*

Exam 1-4	*20% each*
Final Exam	*20%*
Total	*100%*

Letter grades are assigned using the following percentages:

A (90-100)
B (80-89)
C (70-79)
D (60-69)
F (0-59)

Academic Honor Code

As members of a community dedicated to learning, inquiry and creation, the students, faculty and administration of our university live by the principles in this Honor Code. These principles require all members of this community to be conscientious, respectful and honest.

We are conscientious.

We complete our work on time and make every effort to do it right. We come to class and meetings prepared and are willing to demonstrate it. We hold ourselves to doing what is required, embrace rigor, and shun mediocrity, special requests, and excuses.

We are respectful.

We act civilly toward one another and we cooperate with each other. We will strive to create an environment in which people respect and listen to one another, speaking when appropriate, and permitting other people to participate and express their views.

We are honest.

We do our own work and are honest with one another in all matters. We understand how various acts of dishonesty, like plagiarizing, falsifying data, and giving or receiving assistance to which one is not entitled, conflict as much with academic achievement as with the values of honesty and integrity.

The Pledge for Students

Students at our university recognize that, to ensure honest conduct, more is needed than an expectation of academic honesty, and we therefore adopt the practice of affixing the following pledge of honesty to the work we submit for evaluation:

Honor Code web site http://txstate.edu/effective/upps/upps-07-10-01.html

Electronic Devices - *Cellular Telephones**,** Pagers, Palm Pilots or any device that may distract from the class should be turned off before class begins and may not be on the desk during class or tests.*

Special Needs – *Students with special needs, as documented by the Office of Disability Services, should identify themselves at the beginning of the semester.*

Resources -

SLAC (see website for times)
Math Lab (see website for times)
Check with SLAC and Math Lab to ascertain if they can provide help with this advanced course

Texas State Endorses Wingspread Journal's Seven Principles for Good Practice in Undergraduate Education:

1. *Student-faculty intellectual interaction*
2. *Intellectual interaction with fellow students, except when it interferes with assignments to be completed on an independent basis*
3. *Active Learning*
4. *Prompt feedback*
5. *Timely completion of tasks*
6. *High expectations, and*
7. *Respect for diverse talents and ways of learning*

Notes:

1. The instructor reserves the right to deviate from the syllabus in a short term basis to better serve the students enrolled in the course.
2. Due to diverse background of students, instructor may be required to devote more time on reviews and consequently deviate from the following calendar.
3. The instructor may select a different textbook or other ancillary material, however, the same concepts will be covered.
4. The instructor may deviate from the sequential order presented below, however the outlying concepts will be covered.
5. Some concepts, like logic, may be integrated within other contexts and therefore covered accordingly.
6. Instructor may deviate in scheduling tests and reviews depending on the pace. Moreover, some instructors review material in an ongoing basis and thereby the following schedule will be adjusted accordingly.
7. Some instructors give a daily/weekly test in an ongoing basis, and therefore the following test schedule will not necessarily be applicable. The sequential order of the material will be adjusted accordingly.

The following daily schedule was designed to coordinate with one possible textbook. Should a professor select a different text, the order in which the topics are covered might need to be adjusted to better fit the text. Adjustments will also need to be made to fit a MWF class versus a T/Th or MW class. Various topics can be used when teaching basic proof methods, for example, number theory, properties of integers, and properties of inequalities of real numbers can be used, as determined by the instructor. Items in parentheses are suggestions of topics within the broader heading.

Class Day	Descriptive Comments
1	Mathematical language and notation
2	Basic Proof Methods (direct, biconditional)
3	Basic Proof Methods (counterexamples, contrapositive, contradicition)
4	Basic Proof Methods (Quantifiers)
5	Proofs :Strategies for selecting an appropriate method
6	Additional Proofs and examples
7	Basics of Set Theory and Set Operations
8	Proofs involving sets (set equality, DeMorgan's laws, etc.)
9	Indexed Families of Sets
10	Principle of Mathematical Induction
11	Induction Continued
12	Equivalent Forms of Induction (strong induction, well-ordering principal)
13	Exam 1
14	Principles of Counting
15	Cartesian Products and Relations
16	Equivalence Relations
17	Equivalence Relations
18	Partitions
19	Ordering Relations
20	Functions
21	Properties of functions (proofs involving 1-1 and Onto Functions)
22	Functions and sets (proofs involving images of sets and inverse images of sets)
23	Exam 2 (Thursday)
24	Equivalent Sets and Counting (introduction to cardinality, Cantor's argument)
25	Real numbers (proofs involving upper and lower bounds of subsets)
26	Open and Closed Sets in the Reals
27	Boundary and Accumulation Points and Sequences of real numbers
28	Review

APPENDIX I

Dear Introduction to Advanced Mathematics Student,

Thank you for your participation in my research study. Please take about 10 – 15 minutes to complete the survey below and bring it with you to submit in class. Your responses will greatly help our research.

Post-Proof Views Part 1

Answer the following questions as honestly as you can. Give some thought to the questions and please give detailed answers.

Name:

Math to me is…

My favorite part of math is…

My ***least*** favorite part of math is…

To me, the definition of a mathematical proof is…

In mathematics, people write proofs to…

Please describe your overall impressions of the course this semester.

How has the course affected you beliefs about proof?

How do you feel about keeping a weekly math journal this semester?

How did the journaling support your learning to prove in this class, if at all?

What do you feel are the benefits of journal writing in a proof-based mathematics course?

How could journal writing be changed to be more effective?

Would you recommend keeping a math journal to a friend about to start a proofs course in the future? Why or why not?

Post-Proof Views Part 2: **Functions of Proof Survey**

Read the following 20 statements and consider which ones you think represent the purpose of proof in mathematics. You are given a total of 20 points to distribute among 20 statements however you would like (assign a value between 0 and 20 to each item). You must use all 20 points. For example, assigning one point to each statement would mean you think each statement is equally important. Or, assigning 20 points to one statement and 0 to all others means you think only one statement is important. Please make sure the points add up to 20 by including a total count at the end.

The purpose of proofs in mathematics is…

to test whether certain statements are true or not _______

to convince someone the truthfulness of a mathematical statement. _______
to remove doubts about whether a mathematical statement is valid. _______
to show that a bad conjecture cannot be true. _______
to provide insight into why a certain proposition is true. _______
to gain a deeper understanding of the conjecture you are proving. _______
to show the reader why the statement is true or false. _______
to explain your thinking about a conjecture. _______
to identify inconsistencies and circular arguments. _______

to unify mathematical theories by integrating unrelated statements, theorems, and concepts with one another. _______

to provide a useful global perspective of a topic by exposing the underlying axiomatic structure of that topic. _______

to organize true mathematical statements into a coherent unified whole. _______
to create a forum for critical debate about mathematics. _______
to create a formal means for interacting with other mathematicians. _______
to share ideas and compare thinking within the mathematical community. _______
to communicate new mathematical results to others in the mathematical community. ______
to discover new results in the process of writing a proof. _______
to uncover previously unknown connections between ideas. _______
to observe and create new conjectures. _______
to get new ideas and insights about a topic. _______

Total Points (should be 20): _______

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