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OSCILLATORY AND ASYMPTOTIC PROPERTIES OF FRACTIONAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. This article discusses the oscillatory and asymptotic properties of a test delay differential system involving a non-integer derivative order. We formulate corresponding criteria via explicit necessary and sufficient conditions that enable direct comparisons with the results known for classical integer-order delay differential equations. In particular, we shall observe that oscillatory behaviour of solutions of delay system with non-integer derivatives embodies quite different features compared to the classical results known from the integer-order case.

1. Introduction and preliminaries

Basic qualitative properties of the delay differential equation

$$y'(t) = Ay(t-\tau), \quad t \in (0,\infty), \tag{1.1}$$

where A is a constant real $d \times d$ matrix and $\tau > 0$ is a constant real lag, are well described in previous numerous investigations. While stability and asymptotic properties of (1.1) were reported in [8], answers to various oscillation problems regarding (1.1) were surveyed in [7].

A crucial role in these investigations was played by the associated characteristic equation

$$\det(sI - A\exp\{-s\tau\}) = 0, \tag{1.2}$$

where I is the identity matrix. More precisely, appropriate properties of (1.1) were first described via location of all roots of (1.2) in a specific area of the complex plane. Then, efficient criteria guaranteeing such root locations were formulated in terms of conditions imposed directly on the eigenvalues of A.

We recall some of relevant statements (reformulated in the above mentioned sense) along with their consequences to the scalar case when (1.1) becomes

$$y'(t) = ay(t-\tau), \quad t \in (0,\infty) \tag{1.3}$$

where a is a real number. Since we are primarily interested in discussions of oscillatory properties of appropriate fractional extensions of (1.1), we first state (see [7]) oscillation conditions for (1.1) (as it is customary, we say that a solution of (1.1) is oscillatory if every its component has arbitrarily large zeros; otherwise the solution is called non-oscillatory).

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Theorem 1.1. *Let $A \in \mathbb{R}^{d\times d}$ and $\tau \in \mathbb{R}^+$. Then the following statements are equivalent:*

(a) *All solutions of* (1.1) *oscillate;*
(b) *The characteristic equation* (1.2) *has no real roots;*
(c) *A has no real eigenvalues in $[-1/(\tau \mathrm{e}), \infty)$.*

Corollary 1.2. *Let $a \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$. All solutions of* (1.3) *oscillate if and only if*

$$a < -\frac{1}{\tau \mathrm{e}}\,.$$

As we shall see later, oscillatory properties of the corresponding fractional delay system are closely related to convergence of all its solutions to the zero solution. In the first-order case (1.1), this property was characterized in [8] via

Theorem 1.3. *Let $A \in \mathbb{R}^{d\times d}$ and $\tau \in \mathbb{R}^+$. Then the following statements are equivalent:*

(a) *Any solution y of* (1.1) *tends to zero as $t \to \infty$;*
(b) *The characteristic equation* (1.2) *has all roots with negative real parts;*
(c) *All eigenvalues λ_i ($i = 1, \dots, d$) of A satisfy*

$$\tau|\lambda_i| < |\arg(\lambda_i)| - \pi/2\,.$$

Moreover, the convergence of y to zero is of exponential type.

Remark 1.4. The condition (c) can be equivalently expressed via the requirement that all eigenvalues λ_i ($i = 1, \dots, d$) of A have to be located inside the region bounded by the curve

$$\Re(\lambda) = \omega\cos(\omega\tau), \quad \Im(\lambda) = -\omega\sin(\omega\tau), \quad -\frac{\pi}{2\tau} \le \omega \le \frac{\pi}{2\tau}$$

in the complex plane.

Corollary 1.5. *Let $a \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$. Any solution y of* (1.3) *tends to zero as $t \to \infty$ if and only if*

$$-\frac{\pi}{2\tau} < a < 0\,.$$

Extensions of previous results to the n-th order equation (n is a positive integer)

$$y^{(n)}(t) = Ay(t-\tau), \quad t \in (0,\infty) \tag{1.4}$$

yield different conclusions. In this case, the characteristic equation becomes

$$\det(s^n I - A\exp\{-s\tau\}) = 0\,. \tag{1.5}$$

If $n \ge 2$, then there is no analogue to Theorem 1.3. More precisely, the convergence of all solutions of (1.4) to zero is not possible whenever $n \ge 2$ (see, e.g. [6]). Regarding oscillatory properties of (1.4), equivalency of conditions (a) and (b) (with (1.2) replaced by (1.5)) of Theorem 1.1 remains preserved, but their conversion into an explicit form depends on parity of n (see [7]).

The main goal of this article is to discuss these oscillatory and related asymptotic properties of (1.1) with respect to their possible extension to the fractional delay differential equation

$$\mathrm{D}_0^\alpha y(t) = Ay(t-\tau), \quad t \in (0,\infty) \tag{1.6}$$

where $\alpha > 0$ is a real scalar and the symbol D_0^α is the Caputo derivative of order α introduced in the following way: First let y be a real scalar function defined on $(0, \infty)$. For a positive real γ, the fractional integral of y is given by

$$\mathrm{D}_0^{-\gamma} y(t) = \int_0^t \frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)} y(\xi) \mathrm{d}\xi, \quad t \in (0, \infty)$$

and, for a positive real α, the Caputo fractional derivative of y is given by

$$\mathrm{D}_0^\alpha y(t) = \mathrm{D}_0^{-(\lceil\alpha\rceil - \alpha)} \Big(\frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} y(t) \Big), \quad t \in (0, \infty)$$

where $\lceil\cdot\rceil$ means the upper integer part. As it is customary, we put $\mathrm{D}_0^0 y(t) = y(t)$ (for more on fractional calculus, see, e.g. [10, 15]). If y is a real vector function, the corresponding fractional operators are considered component-wise (similarly, if y is a complex-valued function, then these fractional operators are introduced for its real and imaginary part separately). We add that the initial conditions associated to (1.6) are

$$y(t) = \phi(t), \quad t \in [-\tau, 0], \tag{1.7}$$

$$\lim_{t \to 0^+} y^{(j)}(t) = \phi_j, \quad j = 0, \dots, \lceil\alpha\rceil - 1 \tag{1.8}$$

where all components of ϕ are absolutely Riemann integrable on $[-\tau, 0]$ and ϕ_j are real scalars. In the frame of our oscillatory and asymptotic discussions on (1.6), we are going not only to extend previous results to (1.6) but also discuss a dependence of relevant conditions on changing derivative order α (with a special attention to the case when α is crossing integer values).

The structure of this paper is following: Section 2 recalls some related special functions as well as the characteristic equation associated with (1.6). Some asymptotic expansions of the studied special functions are described as well. In Section 3, we discuss in detail distribution of roots of the characteristic equation in specific areas of the complex plane. Using these auxiliary statements, Sections 4 and 5 formulate a series of results describing oscillation and asymptotic properties of (1.6) in the vector and scalar case. More precisely, Section 4 presents analogues of Theorems 1.1 and 1.3, and Section 5 contains some additional oscillation results in the scalar case. Discussions on non-consistency of the obtained results with the above recalled classical properties of (1.1) and (1.3) are subject of Section 6 concluding the paper.

2. Special functions and their properties

In this section, we recall and extend some notions and formulae introduced in [3] in the frame of stability analysis of (1.6) with $0 < \alpha < 1$. As we shall see later, these tools turn out to be very useful also in oscillatory investigations of (1.6) with arbitrary real $\alpha > 0$. Since the proofs of auxiliary statements stated below are (essentially) analogous to the proofs of appropriate assertions from [3], we omit them.

In the sequel, the symbols $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and inverse Laplace transform of appropriate functions, respectively.

Definition 2.1. Let $A \in \mathbb{R}^{d \times d}$, let I be the identity $d \times d$ matrix and let $\alpha, \tau \in \mathbb{R}^+$. The matrix function $R : \mathbb{R} \to \mathbb{C}^{d \times d}$ given by

$$R(t) = \mathcal{L}^{-1} \left((s^\alpha I - A \exp\{-s\tau\})^{-1} \right)(t) \tag{2.1}$$

is called the fundamental matrix solution of (1.6).

Theorem 2.2. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha, \tau \in \mathbb{R}^+$ and let R be the fundamental matrix solution of* (1.6)*. Then the solution y of* (1.6)–(1.8) *is given by*

$$y(t) = \sum_{j=0}^{\lceil\alpha\rceil-1} \mathrm{D}_0^{\alpha-j-1} R(t)\phi_j + \int_{-\tau}^{0} R(t-\tau-u)A\phi(u)\mathrm{d}u.$$

Remark 2.3. Theorem 2.2 along with Definition 2.1 imply that the poles of the Laplace image of solution of (1.6) coincide with roots of

$$\det(s^\alpha I - A\exp\{-s\tau\}) = 0, \quad \text{equivalently} \quad \prod_{i=1}^{n}(s^\alpha - \lambda_i\exp\{-s\tau\})^{n_i} = 0\,, \tag{2.2}$$

where λ_i $(i = 1,\dots,n)$ are distinct eigenvalues of A and n_i are their algebraic multiplicities. This confirms the well-known fact that (2.2) is the characteristic equation associated to (1.6) (see, e.g. [5, 9, 11]).

The following notion of a generalized delay exponential function plays an important role in description of asymptotic expansions of the fundamental matrix solution of (1.6).

Definition 2.4. Let $\lambda \in \mathbb{C}$, $\eta, \beta, \tau \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+ \cup \{0\}$. The generalized delay exponential function (of Mittag-Leffler type) is introduced via

$$G_{\eta,\beta}^{\lambda,\tau,m}(t) = \sum_{j=0}^{\infty}\binom{m+j}{j}\frac{\lambda^j(t-(m+j)\tau)^{\eta(m+j)+\beta-1}}{\Gamma(\eta(m+j)+\beta)}h(t-(m+j)\tau)$$

where h is the Heaviside step function.

The relationship between the fundamental matrix solution R and the generalized delay exponential functions $G_{\eta,\beta}^{\lambda,\tau,m}$ can be specified via the following lemma.

Lemma 2.5. *The fundamental matrix solution* (2.1) *can be expressed as $R(t) = T^{-1}\mathcal{G}(t)T$, where T is a regular matrix and $\mathcal{G}$ is a block diagonal matrix with upper-triangular blocks B_j given by*

$$B_j(t) = \begin{pmatrix} G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,1}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,2}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-1}(t) \\ 0 & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,1}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-2}(t) \\ 0 & 0 & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-3}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) \end{pmatrix},$$

where $j = 1,\dots,J$ $(J \in \mathbb{Z}^+)$ and r_j is the size of the corresponding Jordan block of A.

As a next key auxiliary result, we describe asymptotic behaviour of $G_{\eta,\beta}^{\lambda,\tau,m}$ functions.

Lemma 2.6. *Let $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $\beta, \tau \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+ \cup \{0\}$. Further, let s_i $(i = 1, 2, \dots)$ be the roots of*

$$s^\alpha - \lambda\exp\{-s\tau\} = 0 \tag{2.3}$$

with ordering $\Re(s_i) \ge \Re(s_{i+1})$ $(i = 1, 2, \dots;$ in particular, s_1 is the rightmost root).

(i) *If* $\lambda = 0$, *then*

$$G^{0,\tau,m}_{\alpha,\beta}(t) = \frac{(t-m\tau)^{m\alpha+\beta-1}}{\Gamma(m\alpha+\beta)} h(t-m\tau).$$

(ii) *If* $\lambda \neq 0$, *then*

$$G^{\lambda,\tau,m}_{\alpha,\beta}(t) = \sum_{i=1}^{\infty}\sum_{j=0}^{m\cdot k_i} a_{ij}(t-m\tau)^j \exp\{s_i(t-m\tau)\} + S^{\lambda,\tau,m}_{\alpha,\beta}(t)\,,$$

where k_i *is a multiplicity of* s_i, a_{ij} *are suitable nonzero complex constants* ($j = 0,\dots,mk_i$, $i = 1,2,\dots$) *and the term* $S^{\lambda,\tau,m}_{\alpha,\beta}$ *has the algebraic asymptotic behaviour expressed via*

$$\begin{aligned} S^{\lambda,\tau,m}_{\alpha,\beta}(t) = {}& \frac{(-1)^{m+1}}{\lambda^{m+1}\Gamma(\beta-\alpha)}(t+\tau)^{\beta-\alpha-1} \\ &+ \frac{(-1)^{m+1}(m+1)}{\lambda^{m+2}\Gamma(\beta-2\alpha)}(t+2\tau)^{\beta-2\alpha-1} + \mathcal{O}(t^{\beta-3\alpha-1}) \quad \textit{as } t\to\infty. \end{aligned}$$

3. Distribution of characteristic roots

The aim of this section is to analyse (2.2) with respect to existence of its real roots as well as number of its roots with positive real parts. Doing this, it is enough to consider its partial form (2.3).

First, we characterize the set of all roots of (2.3) in terms of their magnitudes and arguments (we assume here $\lambda \neq 0$, i.e. $s \neq 0$). Using the goniometric forms of s and λ we obtain that (2.3) is equivalent to

$$\begin{aligned} &|s|^\alpha \cos[\alpha \arg(s)] - |\lambda| \exp\{-|s|\tau \cos[\arg(s)]\} \cos[\arg(\lambda) - |s|\tau \sin(\arg(s))] \\ &= 0, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &|s|^\alpha \sin[\alpha \arg(s)] - |\lambda| \exp\{-|s|\tau \cos[\arg(s)]\} \sin[\arg(\lambda) - |s|\tau \sin(\arg(s))] \\ &= 0. \end{aligned} \tag{3.2}$$

To solve (2.3), we consider (3.1)–(3.2) as a system with unknowns $|s|$ and $\arg(s)$. If $\alpha \arg(s) = \ell_1\pi$ for some integer ℓ_1, then $\arg(\lambda) - |s|\tau\sin[\arg(s)] = \ell_2\pi$ for some integer ℓ_2 and (3.1) yields

$$|s|^\alpha(-1)^{\ell_1} - |\lambda|\exp\{-|s|\tau\cos[\arg(s)]\}(-1)^{\ell_2} = 0,$$

i.e.

$$|s|^\alpha = (-1)^\ell|\lambda|\exp\{-|s|\tau\cos[\arg(s)]\} = 0 \quad \text{for some integer } \ell. \tag{3.3}$$

Thus (3.1)–(3.2) can be reduced to

$$\alpha\arg(s) - \arg(\lambda) - |s|\tau\sin[\arg(s)] = 2k\pi \quad \text{for some integer } k, \tag{3.4}$$

$$|s|^\alpha = |\lambda|\exp\{-|s|\tau\cos[\arg(s)]\}. \tag{3.5}$$

If $\alpha\arg(s) \neq \ell_1\pi$ for any integer ℓ_1, then $\arg(\lambda) - |s|\tau\sin[\arg(s)] \neq \ell_2\pi$ for any integer ℓ_2 and division (3.1) over (3.2) yields

$$\alpha\arg(s) = |\lambda|\exp\{-|s|\tau\cos[\arg(s)]\} + \ell\pi \quad \text{for some integer } \ell.$$

This, after substitution into (3.1), yields (3.3). Now, the same argumentation as above shows equivalency of (2.3) and (3.4)–(3.5).

Using the previous process, we can derive the following characterization of possible real roots of (2.3).

Proposition 3.1. *Let $\lambda \in \mathbb{C}$ and $\alpha, \tau \in \mathbb{R}^+$.*

(i) *The characteristic equation (2.3) has a positive real root if and only if λ is a positive real. This root is simple, unique and it is the rightmost root of (2.3).*

(ii) *The characteristic equation (2.3) has a negative real root if and only if*
$$0 < |\lambda| \le (\frac{\alpha}{\tau \mathrm{e}})^\alpha \quad \text{and} \quad \arg(\lambda) = (\alpha - 2k)\pi \quad \text{for some } k \in \mathbb{Z}\,.$$
More precisely, if
$$0 < |\lambda| = (\frac{\alpha}{\tau \mathrm{e}})^\alpha \quad \text{and} \quad \arg(\lambda) = (\alpha - 2k)\pi \quad \text{for some } k \in \mathbb{Z}\,,$$
then $s_{1,2} = -\alpha/\tau$ is double and the rightmost root of (2.3) (remaining roots of (2.3) are not real). If
$$0 < |\lambda| < (\frac{\alpha}{\tau \mathrm{e}})^\alpha \quad \text{and} \quad \arg(\lambda) = (\alpha - 2k)\pi \text{ for some } k \in \mathbb{Z}\,,$$
then (2.3) has a couple of simple real negative roots, the greater of them being rightmost (remaining roots of (2.3) are not real).

(iii) *The characteristic equation (2.3) has the zero root if and only if $\lambda = 0$.*

Furthermore, using (3.4)–(3.5) we can specify the distribution of characteristic roots of (2.3) with respect to the imaginary axis. Before doing this, we introduce the following areas in the complex plane.

For real parameters $0 < \alpha < 2$ and $\tau > 0$, we define the set $Q_0(\alpha,\tau)$ of all complex λ such that
$$|\arg(\lambda)| > \frac{\alpha\pi}{2} \quad \text{and} \quad |\lambda| < \Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2}}{\tau}\Big)^\alpha .$$
Further, for any positive integer m and real parameters $0 < \alpha < 4m+2$ and $\tau > 0$, we define the sets $Q_m(\alpha,\tau)$ of all complex λ such that either
$$\frac{\alpha\pi}{2} - 2m\pi < |\arg(\lambda)| \le \frac{\alpha\pi}{2} - (2m-2)\pi \quad \text{and} \quad |\lambda| < \Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2} + 2m\pi}{\tau}\Big)^\alpha ,$$
or $|\arg(\lambda)| > \frac{\alpha\pi}{2} - 2m\pi$ and
$$\Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2} + (2m-2)\pi}{\tau}\Big)^\alpha < |\lambda| < \Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2} + 2m\pi}{\tau}\Big)^\alpha .$$
We add that the sets $Q_m(\alpha,\tau)$ $(m = 0,1,\dots)$ are defined to be empty whenever $\alpha \ge 4m+2$.

Now, we can describe the location of the roots of (2.3) with respect to the imaginary axis in terms of the sets $Q_m(\alpha,\tau)$ (we utilize here the standard notation $\partial[Q_m(\alpha,\tau)]$ for their boundaries).

Proposition 3.2. *Let $\lambda \in \mathbb{C}$ and $\alpha, \tau \in \mathbb{R}^+$. Then there exist just m $(m = 0,1,\dots)$ characteristic roots of (2.3) with a positive real part (while remaining roots have negative real parts) if and only if $\lambda \in Q_m(\alpha,\tau)$. Moreover, (2.3) has a root with the zero real part if $\lambda \in \partial[Q_m(\alpha,\tau)]$ for some $m = 0,1,\dots$.*

The appropriate regions $Q_m(\alpha,\tau)$ are depicted in Figures 1 and 2.

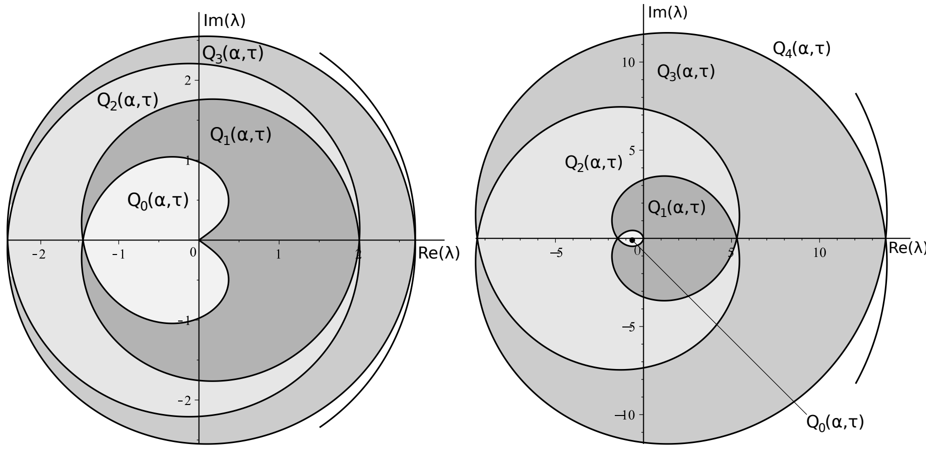

FIGURE 1. $\alpha = 0.4$ and $\tau = 1$ (left). $\alpha = 1.1$ and $\tau = 1$ (right)

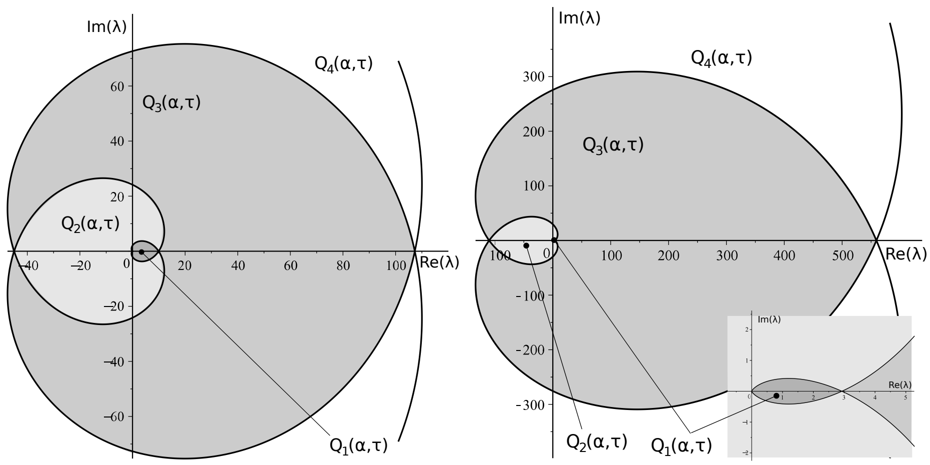

FIGURE 2. $\alpha = 2.1$ and $\tau = 1$ (left). $\alpha = 3.1$ and $\tau = 1$ (right)

Proof. We start with the proof of Proposition 3.1 and consider the characterization of roots s of (2.3) via (3.4)–(3.5). Obviously, (2.3) has a positive real root if $\arg(\lambda) = 0$ (i.e. λ is a positive real). In this case, the characteristic function

$$F(s) = s^\alpha - \lambda \exp\{-s\tau\}$$

is strictly increasing for all $s \geq 0$ with $F(0) = -\lambda < 0$ and $F(\infty) = \infty$, hence there is a unique positive real root s_1 of (2.3). To show its dominance, we consider remaining roots s_i of (2.3) with a positive real parameter λ. Then (3.5) yields

$$(s_1)^\alpha = \lambda \exp\{-s_1\tau\}, \quad |s_i|^\alpha = \lambda \exp\{-|s_i|\tau \cos[\arg(s_i)]\}.$$

From here, we obtain

$$\Big(\frac{s_1}{|s_i|}\Big)^\alpha = \exp\{(-s_1 + |s_i| \cos[\arg(s_i)])\tau\}\,. \tag{3.6}$$

Assume that s_1 is not the rightmost root of (2.3), i.e. $|s_i|\cos[\arg(s_i)] \geq s_1$ for some root s_i of (2.3). Then

$$\frac{s_1}{|s_i|} < 1 \quad \text{and} \quad \exp\{(-s_1 + |s_i|\cos[\arg(s_i)])\tau\} \geq 1$$

which contradicts (3.6). This proves Proposition 3.1(i).

Similarly, (3.4)–(3.5) imply that (2.3) has a negative real root s if and only if

$$\arg(\lambda) = (\alpha - 2k)\pi \quad \text{for some } k \in \mathbb{Z}$$

and

$$|s|^\alpha = |\lambda| \exp\{|s|\tau\}.$$

Put $r = |s|$ and $G(r) = r^\alpha - |\lambda|\exp\{r\tau\}$, $r \geq 0$. Then $G(0) = -|\lambda| < 0$, $G(\infty) = -\infty$ and G is increasing in $(0, r^*)$ and decreasing in (r^*, ∞) for a suitable $r^* > 0$. Thus G has (one or two) positive roots if and only if $G(r^*) \geq 0$. In particular, G has a unique positive root r^* if and only if $G(r^*) = G'(r^*) = 0$, i.e.

$$(r^*)^\alpha - |\lambda|\exp\{r^*\tau\} = \alpha(r^*)^{\alpha-1} - |\lambda|\tau\exp\{r^*\tau\} = 0\,.$$

From here, we obtain

$$r^* = \frac{\alpha}{\tau} \quad \text{and} \quad |\lambda| = \Big(\frac{\alpha}{\tau \mathrm{e}}\Big)^\alpha.$$

Obviously, if

$$|\lambda| < \Big(\frac{\alpha}{\tau \mathrm{e}}\Big)^\alpha,$$

then G has two real positive roots $r_1 < r_2$. We show that $s_1 = -r_1$ is the rightmost root of (2.3), i.e $s_1 > |s_i|\cos[\arg(s_i)]$ for all remaining roots s_i $(i = 2, 3, \dots)$ of (2.3). Indeed, by (3.5),

$$|s_1|^\alpha = |\lambda|\exp\{|s_1|\tau\} \quad \text{and} \quad |s_i|^\alpha = |\lambda|\exp\{-|s_i|\tau\cos[\arg(s_i)]\}\,.$$

Then $|s_1| < |s_i|$, i.e. $|s_1| + |s_i|\cos[\arg(s_i)] < 0$. Analogously we can show the dominance of a double real root $s_{1,2}$ (if exists). This proves Proposition 3.1 (ii). The assertion of Proposition 3.1(iii) is trivial.

Now, we show the validity of Proposition 3.2. Since the case of real characteristic roots of (2.3) has been discussed previously, we first search the roots s with $0 < \arg(s) \leq \pi/2$. Then (3.4)–(3.5) can be reduced to

$$|s| = \frac{\arg(\lambda) - \alpha\arg(s) + 2k\pi}{\tau\sin[\arg(s)]}, \tag{3.7}$$

$$\begin{aligned}\Big(\frac{\arg(\lambda) - \alpha\arg(s) + 2k\pi}{\tau\sin[\arg(s)]}\Big)^\alpha &- |\lambda|\exp\big\{(-\arg(\lambda) \\ &+ \alpha\arg(s) - 2k\pi)\,\mathrm{cotan}[\arg(s)]\big\} = 0\,.\end{aligned} \tag{3.8}$$

We denote the left-hand side of (3.8) by $H_k = H_k(\arg(s))$. Then

$$H_k(0^+) = \infty, \quad H_k(\pi/2) = \Big(\frac{\arg(\lambda) - \alpha\pi/2 + 2k\pi}{\tau}\Big)^\alpha - |\lambda|$$

and $H_k(\arg(s))$ decreases as $\arg(s)$ increases from 0 to $\pi/2$. This implies that (3.7)–(3.8) has just m couples of solutions with $|s| > 0$ and $0 < \arg(s) \leq \pi/2$ if and only if either

$$\frac{\alpha\pi}{2} - 2m\pi < \arg(\lambda) \leq \frac{\alpha\pi}{2} - (2m-2)\pi \quad \text{and} \quad H_m(\pi/2) > 0\,,$$

or

$$\arg(\lambda) > \frac{\alpha\pi}{2} - 2m\pi \quad \text{and} \quad H_m(\pi/2) > 0 > H_{m-1}(\pi/2)\,.$$

If $-\pi/2 \le \arg(s) < 0$, then we obtain the same conclusion with $\arg(\lambda)$ replaced by $-\arg(\lambda)$. This implies the main part of the assertion. The supplement on existence of purely imaginary roots of (2.3) follows from continuous dependence of roots s on parameter λ. Alternatively, it can be obtained via the standard D-decomposition method. □

4. Main results

In this section, we derive and formulate fractional-order analogues to Theorems 1.1 and 1.3.

Theorem 4.1. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ and $\tau \in \mathbb{R}^+$. Then the following statements are equivalent:*

- (a) *All non-trivial solutions of* (1.6) *are non-oscillatory;*
- (b) *The characteristic equation* (2.2) *admits only real roots or roots with a negative real part;*
- (c) *A has all eigenvalues lying in $Q_0(\alpha,\tau) \cup (Q_1(\alpha,\tau) \cap \mathbb{R}) \cup \{0\}$.*

Proof. Theorem 2.2 and Lemma 2.5 imply that every solution of (1.6)–(1.8) can be expressed as

$$y(t) = T^{-1}\sum_{j=0}^{\lceil\alpha\rceil-1} \mathrm{D}_0^{\alpha-j-1}\mathcal{G}(t)T\phi_j + T^{-1}\int_{-\tau}^{0}\mathcal{G}(t-\tau-u)JT\phi(u)\mathrm{d}u\,, \tag{4.1}$$

where $\mathcal{G}$ is a matrix function introduced in Lemma 2.5, J is a Jordan form of the system matrix A and T is the corresponding regular projection matrix, i.e. $A = TJT^{-1}$. Employing (4.1) and Lemma 2.5, we can see that every component of y is a linear combination of terms derived from elements of $\mathcal{G}$. We distinguish two cases with respect to (non)zeroness of eigenvalues λ_i of A.

First, let $\lambda_i \neq 0$ for all $i = 1,\dots,n$ (n being the number of distinct eigenvalues of A). Then the elements of matrices $\mathrm{D}_0^{\alpha-j-1}\mathcal{G}(t)$ ($j = 0,\dots,\lceil\alpha\rceil-1$) can be asymptotically expanded via the relation

$$\begin{aligned}
&\mathrm{D}_0^{\alpha-j-1}G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t) = G_{\alpha,j+1}^{\lambda_i,\tau,m}(t)\\
&= \sum_{w=1}^{N}\sum_{\ell=0}^{mk_w} t^{\ell}\exp\{s_w t\}b_{w,\ell}\big(1-\frac{m\tau}{t}\big)^{\ell}\exp\{-s_w m\tau\}\\
&\quad + t^{j-\alpha}\frac{(-1)^{m+1}(1+\tau/t)^{j-\alpha}}{\lambda_i^{m+1}\Gamma(j-\alpha+1)} + \mathcal{O}(t^{j-2\alpha}) \quad \text{as } t\to\infty\,,
\end{aligned} \tag{4.2}$$

where s_w ($w = 1,2,\dots,N$) are roots of (2.3) with the largest real parts ordered as $\Re(s_w) \ge \Re(s_{w+1})$, N is any positive integer satisfying $\Re(s_N) < 0$, k_w is multiplicity of s_w and $b_{w,\ell}$ are suitable real constants (see $a_{i,j}$ in Lemma 2.6(ii)). Similarly, the elements of the matrix $\int_{-\tau}^{0} \mathcal{G}(t-\tau-u)JT\phi(u)\mathrm{d}u$ have the expansions

$$\begin{aligned}
&\int_{-\tau}^{0} G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t-\tau-u)\hat{\phi}^p(u)\mathrm{d}u \\
&= \sum_{w=1}^{N}\sum_{\ell=0}^{mk_w} t^{\ell}\exp\{s_w t\}c_{w,\ell}\lambda_i \int_{-\tau}^{0}\Big(1-\frac{(m+1)\tau}{t}-\frac{u}{t}\Big)^{\ell}\mathrm{e}^{-s_w((m+1)\tau+u)}\hat{\phi}^p(u)\mathrm{d}u \\
&\quad + t^{-\alpha-1}\int_{-\tau}^{0}\frac{(-1)^{m+1}(m+1)(1+\tau/t-u/t)^{-\alpha-1}}{\lambda_i^{m+1}\Gamma(-\alpha)}\hat{\phi}^p(u)\mathrm{d}u + \mathcal{O}(t^{-2\alpha-1})
\end{aligned} \tag{4.3}$$

as $t \to \infty$, where $\hat{\phi}^p(u)$ is pth row of the vector $JT\phi(u)$ and $c_{w,\ell}$ are suitable real constants (see $a_{i,j}$ in Lemma 2.6(ii)).

If $\lambda_i = 0$ for some $i = 1, \dots, n$, then the appropriate analogues of (4.2)–(4.3) involve only algebraic terms (see Lemma 2.6(i)). Now, we can prove the presented equivalencies:

(a)⇔(b): The property (a) holds if and only if, for any choice of ϕ, the dominating terms involved in (4.2) and (4.3) are non-oscillatory. We can see that all the algebraic terms from (4.2) and (4.3) are non-oscillatory and eventually dominating with respect to all exponential terms with negative real parts of their arguments. Contrary, an exponential term is eventually dominating provided its argument has a non-negative real part. Clearly, if such a case does occur, the solution y of (1.6) is non-oscillatory only if the imaginary parts of the corresponding arguments are zero. By (4.2) and (4.3), the discussed arguments of the exponential terms are expressed via roots of (2.2), which yields equivalency of (a) and (b).

(b)⇔(c): This equivalency follows immediately from Propositions 3.1 and 3.2. □

In the scalar case, when (1.6) becomes

$$\mathrm{D}_0^{\alpha} y(t) = ay(t-\tau), \quad t \in (0,\infty)\,, \tag{4.4}$$

a being a real scalar, we obtain the following explicit characterization of non-existence of a non-trivial oscillatory solution.

Corollary 4.2. *Let $a \in \mathbb{R}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ and $\tau \in \mathbb{R}^+$. All non-trivial solutions y of* (4.4) *are non-oscillatory if and only if*

$$0 < \alpha < 2 \quad \text{and} \quad -\Big(\frac{(2-\alpha)\pi}{2\tau}\Big)^{\alpha} < a < \Big(\frac{(4-\alpha)\pi}{2\tau}\Big)^{\alpha},$$

or

$$2 < \alpha < 4 \quad \text{and} \quad 0 < a < \Big(\frac{(4-\alpha)\pi}{2\tau}\Big)^{\alpha}.$$

Remark 4.3. In the first-order case, the value $a = -1/(\tau\mathrm{e})$ is of a particular importance: crossing this value, the (negative) real roots of the associated characteristic equation disappear and all solutions of (1.3) become oscillatory for $a < -1/(\tau\mathrm{e})$. In the fractional-order case, the (negative) real roots disappear for $a < -(\alpha/(\tau\mathrm{e}))^{\alpha}$. However, such roots have no impact on oscillatory behaviour of the solutions of (4.4) because the exponential terms with negative arguments involved in the formulae (4.1)–(4.3) are eventually suppressed by algebraic terms.

By Theorem 4.1, if all roots of (2.2) have negative real parts, then all non-trivial solutions of (1.6) are non-oscillatory. Therefore, we give an explicit characterization of this assumption and thus provide a fractional-order analogue to Theorem 1.3.

Theorem 4.4. *Let $A \in \mathbb{R}^{d\times d}$ and $\alpha, \tau \in \mathbb{R}^+$. Then the following statements are equivalent*

(a) *Any solution y of (1.6) tends to zero as $t \to \infty$;*
(b) *The characteristic equation (2.2) has all roots with negative real parts;*
(c) *All eigenvalues λ_i $(i = 1, \dots, d)$ of A are nonzero and satisfy*

$$\tau|\lambda_i|^{1/\alpha} < |\arg(\lambda_i)| - \alpha\pi/2 .$$

Moreover, if $\alpha \notin \mathbb{Z}^+$, then the convergence to zero is of algebraic type; more precisely, for any solution y of (1.6) there exists a suitable integer $j \in \{0, \dots, \lceil\alpha\rceil\}$ such that $|y(t)| \sim t^{j-\alpha-1}$ as $t \to \infty$ (the symbol $\sim$ stands for asymptotic equivalency).

Proof. (a)$\Leftrightarrow$(b): If $\lambda_i = 0$ for some $i = 1, \dots, d$, then the appropriate analogues of (4.2) and (4.3) yield that there is always a constant term involved in these expansions (this constant is nonzero if ϕ_0 is nonzero), hence the property (a) is not true. Obviously, the property (b) cannot occur as well provided $\lambda_i = 0$ for some $i = 1, \dots, d$. Thus, without loss of generality, we may assume $\lambda_i \neq 0$ for all $i = 1, \dots, d$.

The statement (a) is valid if and only if (4.2) and (4.3) do not contain any terms with a non-negative real part of the argument, which directly yields the equivalency (see also [11]).

(b)$\Leftrightarrow$(c): It is a direct consequence of Proposition 3.2.

Consequently, since all the exponential terms in (4.2) and (4.3) have a negative argument, they are suppressed by the algebraic terms. The presence of the term behaving like $t^{j-\alpha-1}$ for $j = 1, \dots, \lceil\alpha\rceil$ as $t \to \infty$ is determined by values ϕ_{j-1}. If $\phi_{j-1} = 0$ for all $j = 1, \dots, \lceil\alpha\rceil$, the integral term (4.3) becomes dominant. The integrability of ϕ enables us to write

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t^{-\alpha-1}} \Big| \int_{-\tau}^{0} G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t-\tau-u)\hat{\phi}^p(u)\mathrm{d}u \Big| \\
&= \Big| \int_{-\tau}^{0} \lim_{t\to\infty} \frac{(-1)^{m+1}(m+1)(1+\tau/t-u/t)^{-\alpha-1}}{\lambda_i^{m+1}\Gamma(-\alpha)} \hat{\phi}^p(u)\mathrm{d}u \Big| \\
&= K \Big| \int_{-\tau}^{0} \hat{\phi}^p(u)\mathrm{d}u \Big|
\end{aligned}$$

for a suitable real K, therefore the integral term behaves like $t^{-\alpha-1}$ as $t \to \infty$. This completes the proof. □

For the case of scalar equation (4.4), we obtain the following result.

Corollary 4.5. *Let $a \in \mathbb{R}$ and $\alpha, \tau \in \mathbb{R}^+$. All solutions y of (4.4) tend to zero if and only if*

$$\alpha < 2 \quad \textit{and} \quad -\Big(\frac{(2-\alpha)\pi}{2\tau}\Big)^{\alpha} < a < 0 .$$

In particular, an interesting link between Theorems 4.1 and 4.4 is provided by the following assertion.

Corollary 4.6. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ and $\tau \in \mathbb{R}^+$. If (1.6) has a nontrivial oscillatory solution, then it has also a solution which does not tend to zero as $t \to \infty$.*

Remark 4.7. In fact, formulae (4.1)–(4.3) reveal that any non-trivial solution of (1.6) tending to zero is non-oscillatory. Moreover, the solutions tending to zero pose an algebraic decay (there is no solution with an exponential decay).

5. Other oscillatory properties of (4.4)

In the classical integer-order case, oscillation argumentation often uses the fact that $\exp(s_w t)$ is a solution of (1.3) for any root s_w of the corresponding characteristic equation

$$s - a\exp\{-s\tau\} = 0\,. \tag{5.1}$$

In particular, if (5.1) admits a real root, then (1.3) has (via appropriate choice of ϕ) a non-oscillatory solution. In the fractional-order case, no such a direct connection for the influence study of characteristic roots of

$$s^\alpha - a\exp\{-s\tau\} = 0 \tag{5.2}$$

on the oscillatory behaviour of (4.4) is available. Nevertheless, as we can see from (4.1)–(4.3), the exponential functions generated by characteristic roots of (5.2) again play an important role in qualitative analysis of solutions of (4.4). Using this fact, we are able to describe some oscillatory properties of (4.4) with respect to asymptotic relationship between the studied solutions and the corresponding exponential functions. To specify this relationship, we introduce the following asymptotic classifications of solutions of (4.4).

Definition 5.1. Let $a \in \mathbb{R}$ and $\alpha, \tau \in \mathbb{R}^+$. The solution y of (4.4) is called major solution, if it satisfies the asymptotic relationship

$$\limsup_{t\to\infty} \Big|\frac{y(t)}{t^{k_1}\exp\{s_1 t\}}\Big| > 0\,,$$

where s_1 is the rightmost root of (5.2) and k_1 its algebraic multiplicity.

Definition 5.2. Let $a \in \mathbb{R}$, $\alpha, \tau \in \mathbb{R}^+$, s_w $(w = 1, 2, \dots)$ be roots of (5.2) with ordering $\Re(s_w) \geq \Re(s_{w+1})$ and let k_w $(w = 1, 2, \dots)$ be the corresponding algebraic multiplicities. The solution y of (4.4) is called m-minor solution, if it satisfies the asymptotic relationships

$$\limsup_{t\to\infty} \Big|\frac{y(t)}{t^{k_m}\exp\{s_m t\}}\Big| = 0 \quad \text{and} \quad \limsup_{t\to\infty} \Big|\frac{y(t)}{t^{k_{m+1}}\exp\{s_{m+1} t\}}\Big| > 0\,.$$

Remark 5.3. The notions of the major and m-minor solutions are not just theoretical, but such solutions can be constructively obtained via appropriate choice of the initial function ϕ. For example, if s_1 is simple with a non-negative real part, then, by (4.1)–(4.3), the major solution occurs if ϕ meets the condition

$$\sum_{j=0}^{\lceil\alpha\rceil-1} \phi_j b_{1,j} + a c_{1,0} \int_{-\tau}^{0} \phi(u)\exp\{-s_1(\tau+u)\}\mathrm{d}u \neq 0$$

where $b_{1,j}$, $c_{1,0}$ have the same meaning as in (4.2)–(4.3). Clearly, such a condition is satisfied by infinitely many initial functions, e.g. by $\phi(u) = 1$, $\phi_j = 0$ $(j = 1, \dots, \lceil\alpha\rceil - 1)$ and $\phi_0 \neq -a c_{1,0}(1 - \exp\{-s_1\tau\})/(b_{1,0}s_1)$. Similarly, m-minor solution is characterized by the conditions

$$\sum_{j=0}^{\lceil\alpha\rceil-1} \phi_j b_{w,j} + a c_{w,0} \int_{-\tau}^{0} \phi(u)\exp\{-s_w(\tau+u)\}\mathrm{d}u = 0 \quad \text{for } w = 1, \dots, m\,,$$

$$\sum_{j=0}^{\lceil\alpha\rceil-1} \phi_j b_{m+1,j} + a c_{m+1,0} \int_{-\tau}^{0} \phi(u) \exp\{-s_{m+1}(\tau+u)\} \mathrm{d}u \neq 0$$

provided s_w $(w = 1, \dots, m+1)$ are simple roots and $b_{w,j}$, $c_{w,0}$, $b_{m+1,j}$, $c_{m+1,0}$ have the same meaning as in (4.2)–(4.3).

Using the notions of major and m-minor solutions, we can formulate in a more detail assertions revealing the relation between oscillatory properties of (4.4) and location of roots of (5.2) in the complex plane.

Lemma 5.4. *Let $a \in \mathbb{R} \setminus (Q_0(\alpha,\tau) \cup \{0\})$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $\tau \in \mathbb{R}^+$ and let s_w $(w = 1, 2, \dots)$ be roots of* (5.2) *with ordering $\Re(s_w) \geq \Re(s_{w+1})$. Then the major solutions of* (4.4) *do not tend to zero and there exists $M > 0$ such that all m-minor solutions of* (4.4) *are non-oscillatory and tend to zero as $t \to \infty$ for all $m \geq M$. Furthermore, it holds:*

- (i) *If $a \leq -((2-\alpha)\pi/(2\pi))^\alpha$ for $\alpha < 2$ or $a < 0$ for $\alpha > 2$, then all major solutions of* (4.4) *are oscillatory.*
- (ii) *If $\alpha < 4$ and $0 < a < ((4-\alpha)\pi/(2\pi))^\alpha$, then all non-trivial solutions of* (4.4) *are non-oscillatory.*
- (iii) *If $\alpha < 4$ and $a = ((4-\alpha)\pi/(2\pi))^\alpha$, then all major solutions of* (4.4) *are non-oscillatory. Moreover, all 1-minor solutions are oscillatory and bounded.*
- (iv) *If $a > ((4-\alpha)\pi/(2\pi))^\alpha$ for $\alpha < 4$ or $a > 0$ for $\alpha > 4$, then all major solutions of* (4.4) *are non-oscillatory. Moreover, all 1-minor solutions are oscillatory and unbounded.*

Proof. The first part of the assertion follows from the expansion of solution y of (4.4) based on (4.2)–(4.3). By Proposition 3.2, the rightmost root s_1 has a non-negative real part, therefore the major solutions involve, as a dominant term, an exponential function which does not tend to zero. Using a technique similar to that in Remark 5.3 we can always eliminate all terms in the asymptotic expansion of y corresponding to the characteristic roots with a non-negative real part, and, thus, construct non-oscillatory m-minor solutions algebraically tending to zero. Further utilization of this arguments enables us to obtain even more detailed results:

(i) The value $a \leq -((2-\alpha)\pi/(2\pi))^\alpha$ for $\alpha < 2$ or $a < 0$ for $\alpha > 2$ guarantees that the rightmost root s_1 has a non-negative real part and non-zero imaginary part (see Propositions 3.1 and 3.2), therefore the major solutions are oscillatory.

(ii)–(iv) If $a > 0$, Proposition 3.1(i) implies that the rightmost root s_1 is a positive real, therefore the major solutions are non-oscillatory. Eliminating the rightmost root s_1 as in Remark 5.3, the terms corresponding to s_2 become dominant and, again using Proposition 3.2, we obtain the parts (ii)–(iv). $\square$

Remark 5.5. For $a = 0$, (5.2) has the only root $s_1 = 0$ with multiplicity $\lceil\alpha\rceil$ and the qualitative behaviour is implied directly by Lemma 2.6(i). In particular, if $\alpha < 1$, then all non-trivial solutions of (4.4) are constant, i.e. they are bounded and non-oscillatory. If $\alpha > 1$, then all non-trivial solutions of (4.4) are non-oscillatory. Moreover, if $\phi_j = 0$ for all $j = 1, \dots, \lceil\alpha\rceil - 1$, then the solutions are bounded, otherwise being unbounded.

It is of a particular interest to emphasize that unlike the integer-order case, there is no combination of entry parameters such that all the solutions of (4.4) are oscillatory. In fact, (4.4) has always infinitely many non-oscillatory solutions.

6. Concluding remarks

We have discussed oscillatory and related asymptotic properties of solutions of the fractional delay differential system (1.6) as well as of the corresponding scalar equation (4.4). The obtained oscillation results qualitatively differ from those known from the classical oscillation theory of (integer-order) delay differential equations. We survey here the most important notes related to this phenomenon.

First, while the appropriate criteria from the classical theory (such as Theorem 1.1) formulate necessary and sufficient conditions for oscillation of all solutions, their fractional counterparts (Theorem 4.1) present conditions for non-oscillation of all non-trivial solutions. In particular, our analysis shows that (1.6) cannot admit only oscillatory solutions. Secondly, considering (1.6), one can observe a close resemblance between non-oscillation of all non-trivial solutions and convergence to zero of all solutions (this property defines asymptotic stability of the zero solution of (1.6)). The latter property is sufficient for non-oscillation of all non-trivial solutions of (1.6) and, moreover, it is not far from being also a necessary one. These features (along with some other precisions made in Section 5) demonstrate that (non)oscillatory properties of (1.6) qualitatively depend on the fact if the value α is integer or non-integer. In particular, Corollary 4.2 implies that the endpoints of corresponding non-oscillation intervals depend continuously on changing non-integer derivative order α; when α is crossing the integer-order value, a sudden change in oscillatory behaviour occurs (see Corollary 1.2). Note that despite of some introductory papers on oscillation of (1.6) and other related fractional delay differential equations (see, e.g. [1, 17]), these properties have not been reported yet.

On the other hand, one can observe that dependence of stability areas of (1.6) on changing derivative order is "continuous". As illustrated via Figures 1–4, this area is continuously becoming smaller, starting from the circle (corresponding to the non-differential case when $\alpha = 0$) to the empty set (when $\alpha = 2$). We add that the way to stability remains closed for all real $\alpha \geq 2$. From this viewpoint, considerations of (1.6) with non-integer derivative order enable a better understanding of classical stability results on (1.6) with integer α.

The method utilized in our oscillation analysis indicates that the main reason of a rather strange oscillatory behaviour of (1.6) with non-integer α is hidden in the algebraic rate of convergence of its solutions to zero (compared to the exponential rate known from the integer-order case). Since this type of convergence has been earlier described not only for other types of fractional delay equations (see [2, 9, 11, 12]), but also for fractional equations without delay (see [4, 13, 14, 16]), the above described oscillatory behaviour might be typical for a more general class of fractional differential equations.

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