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Partial exact controllability for the linear thermo-viscoelastic model

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Abstract

The problem of partial exact controllability for linear thermo-viscoelasticity is considered. Using classical multiplier techniques, a boundary observability inequality is established under smallness restrictions on coupling parameters and relaxation functions. Then, via the Hilbert Uniqueness method, the result of partial exact controllability is obtained with Dirichlet boundary controls acting on a part of the boundary of a domain.

§1. Introduction

The aim of this paper is to study the problem of partial exact controllability with Dirichlet boundary controls for the linear thermo-viscoelasticity model

$$
\begin{gathered}
u'' - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\nabla\theta \\
-\varepsilon\int_0^t g(t-s)[\mu\Delta u(x,s) + (\lambda+\mu)\nabla\operatorname{div}u(x,s)]\,ds = 0 \quad \text{in } Q, \\
\theta' - \Delta\theta + \alpha\operatorname{div} u' = 0 \quad \text{in } Q, \\
u = \phi, \quad \theta = 0 \quad \text{on } \Sigma, \\
u(x,0) = 0, \quad u'(x,0) = 0, \quad \theta(x,0) = 0 \quad \text{in } \Omega.
\end{gathered} \tag{1.1}
$$

System (1.1) is a model for a linear viscoelastic body Ω of Boltzmann type with thermal damping. The body Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$ and is assumed to be linear, homogeneous, and isotropic. $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ where $T > 0$. $u(x,t) = (u_1(x,t), \dots, u_n(x,t)), \theta(x,t)$ represent displacement and temperature deviation, respectively, from the natural state of the reference configuration at position x and time t. $\varepsilon > 0$ is a constant, λ, $\mu > 0$ the Lamé's constants and α the thermal strain parameter. $g(t)$ denotes the relaxation function and ϕ the control acting on a part of Σ. By $'$ we denote the derivative with respect to the time variable. Δ, ∇, div denote the Laplace, gradient, and divergence operator, respectively. We refer to [15] for the derivation of the model (1.1).

When $g \equiv 0$, (1.1) becomes the thermoelastic system. In this case, there have been extensive studies. The earliest results appear to be in the paper [14] of Narukawa, who proved the partial exact boundary controllability for the thermoelastic system. Later, Narukawa's result was improved by Lions [11, p.32-60] by introducing the Hilbert Uniqueness method. In [12], we proved the partial exact

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controllability for the higher-dimensional linear thermo-elasticity without the smallness restrictions on the coupling parameters. In these results, only the displacement is controlled, disregarding the values of temperature, which is the so-called partial exact controllability. This drawback was avoided by Hansen [5], who showed that, for at least the one-dimensional thermoelastic system, exact controllability of both the displacement and temperature is possible by only controlling the thermal or mechanical component on the boundary in the case where u and θ satisfy the Dirichlet-Neumann or Neumann-Dirichlet boundary conditions and the coupling parameters are small enough. Hansen's results were proved by making use of the method of moment problems and analysis of non-harmonic Fourier series. More recently, Zuazua [17] discussed the problem of distributed controllability and proved that if T is large enough then this system is exact-approximately controllable with a control supported in a neighborhood of the boundary of Ω. The method of Zuazua is based on multiplier techniques, compactness arguments and Holmgren's Uniqueness Theorem. In addition, Lebeau and Zuazua [9] considered another kind of system of thermo-elasticity and proved that the system is null controllable with a volume force located in a subset satisfying a geometric control condition. Their method of proof was based on a spectral decomposition of the system and its adjoint on the basis generated by the eigenfunctions of the Laplacian. The spectrum is split into a parabolic and a hyperbolic part, and then the system is decomposed into two weakly coupled systems, the first one behaving as a heat equation and the second one as a wave equation. Also, there has been some deep work by Lagnese [7] about the partial exact controllability for thermoelastic plates.

When $\alpha = 0$, system (1.1) is decoupled into the viscoelastic equation

$$\begin{gathered} u'' - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u \\ -\varepsilon\int_0^t g(t-s)[\mu\Delta u(x,s) + (\lambda+\mu)\nabla\operatorname{div} u(x,s)]\,ds = 0 \quad \text{in } Q, \\ u = \phi \quad \text{on } \Sigma, \\ u(x,0) = 0, \quad u'(x,0) = 0 \quad \text{in } \Omega, \end{gathered} \tag{1.2}$$

and the heat equation. The problem of exact controllability for such equations with memory has been considered by various people (see [6, 8, 16]). Under a smallness restriction on the relaxation function, the exact controllability has been proved. However, to our knowledge, little is known about the exact controllability for thermo-viscoelasticity.

The rest of this paper is organized as follows. We present the main result of this paper in Section 2. Using classical multiplier techniques, we establish the boundary observability inequality in Section 3. Then the main result is proved by means of the Hilbert Uniqueness method in Section 4.

§2. Main Result

In what follows, $H^s(\Omega)$ denotes the usual Sobolev space and $\|\cdot\|_s$ denotes its norm for any $s \in \mathbb{R}$ (see [1]). For $s \geq 0$, $H_0^s(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω with compact support in Ω. Let X be a Banach space. We denote by $C^k([0,T],X)$

the space of all k times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T],X)$ for $C^0([0,T],X)$.

Let $x^0 \in \mathbb{R}^n$ and $\nu = (\nu_1,\dots,\nu_n)$ denote the unit normal on Γ directed towards the exterior of Ω. Set

$$\begin{gathered}
m(x) = x - x^0 = (x_k - x_k^0),\\
\Gamma(x^0) = \{x \in \Gamma \ : \ m(x)\cdot\nu(x) = m_k(x)\nu_k(x) > 0\},\\
\Gamma_*(x^0) = \Gamma - \Gamma(x^0) = \{x \in \Gamma \ : \ m(x)\cdot\nu(x) \le 0\},\\
\Sigma(x^0) = \Gamma(x^0) \times (0,T),\\
\Sigma_*(x^0) = \Gamma_*(x^0) \times (0,T),\\
R(x^0) = \max_{x\in\bar\Omega} |m(x)| = \max_{x\in\bar\Omega} \left|\sum_{k=1}^n (x_k - x_k^0)^2\right|^{1/2}.
\end{gathered}$$

The partial exact controllability problem can be formulated as follows: *Given T large enough, for every state (u^0,u^1) in a suitable function space, is it possible to find corresponding controls ϕ such that the solution of (1.1) satisfies*

$$u(x,T) = u^0, \quad u'(x,T) = u^1 \quad \text{in } \Omega, \tag{2.1}$$

disregarding the values of temperature?

As stated in [11, p.34-35], this is equivalent to steering every initial state (u^0,u^1) of the displacement in the function space to rest, disregarding the values of temperature.

The main result of this paper is the following.

Theorem 2.1. *Let the boundary Γ of Ω be of class C^2. Suppose that $g(t) \in H^2(0,T)$ and $T > 2R(x^0)/\sqrt{\mu}$. Then there exist $\varepsilon_0,\alpha_0 > 0$ such that if $\varepsilon \le \varepsilon_0$ and $\alpha \le \alpha_0$, then for every state $(u^0,u^1) \in (L^2(\Omega))^n \times (H^{-1}(\Omega))^n$ there exists a control $\phi \in (L^2(\Sigma(x^0)))^n$ steering the displacement of the system (1.1) to the state (u^0,u^1).*

Remark 2.2. It can be seen from the proof of Theorem 2.1 in section 4 that the solution (u,θ) of (1.1) satisfies

$$\begin{gathered}
u \in C([0,T];(L^2(\Omega))^n) \cap C^1([0,T];(H^{-1}(\Omega))^n),\\
\theta \in C([0,T];L^2(\Omega)).
\end{gathered}$$

Remark 2.3. We may wish to prove the exact-approximate controllability for problem (1.1) as Zuazua [17] did for the thermo-elasticity. However, when we do so, we need a boundary uniqueness theorem, that is, if u,θ satisfy the equation

$$\begin{gathered}
u'' - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\nabla\theta\\
-\varepsilon\int_0^t g(t-s)[\mu\Delta u(x,s) + (\lambda+\mu)\nabla\operatorname{div} u(x,s)]\,ds = 0 \quad \text{in } Q,\\
\theta' - \Delta\theta + \alpha\operatorname{div} u' = 0 \quad \text{in } Q,\\
u = 0, \quad \theta = 0 \quad \text{on } \Sigma,\\
\frac{\partial u}{\partial\nu} + \varepsilon\int_0^t g(t-s)\frac{\partial u(s)}{\partial\nu}\,ds = 0 \quad \text{on } \Sigma(x^0),
\end{gathered} \tag{2.2}$$

then $u=0$ and $\theta=0$ in Q.

Such a uniqueness problem seems to be open, even in the case of thermoelasticity, as stated in [17]. In fact, for thermoelastic plates, Lagnese [7] has claimed that it is not possible to exactly control θ by means of the boundary displacement control alone, and it is physically unrealistic to use θ as an additional control variable.

Remark 2.4. Theorem 2.1 shows that the thermal effects do not affect the exact controllability of the displacement provided the thermal strains and relaxation functions are small enough. Whether a similar result is valid when the thermal strains and relaxation functions are large is an open problem. For thermoelastic plates, Lagnese [7] has claimed that it seems unlikely.

§3. Observability Inequalities

We consider the adjoint system of (1.1)

$$\begin{gathered}
\varphi''-\mu\Delta\varphi-(\lambda+\mu)\nabla\operatorname{div}\varphi+\alpha\nabla\psi' \\
-\varepsilon\int_t^T g(s-t)[\mu\Delta\varphi(x,s)+(\lambda+\mu)\nabla\operatorname{div}\varphi(x,s)]\,ds=0\quad\text{in } Q, \\
-\psi'-\Delta\psi-\alpha\operatorname{div}\varphi=0\quad\text{in } Q, \\
\varphi=0,\quad \psi=0\quad\text{on } \Sigma, \\
\varphi(x,T)=\varphi^0(x),\quad \varphi'(x,T)=\varphi^1(x),\quad \psi(x,T)=\psi^0(x)\quad\text{in } \Omega.
\end{gathered}\tag{3.1}$$

By changing t into $T-t$, (3.1) can be transformed into

$$\begin{gathered}
\varphi''-\mu\Delta\varphi-(\lambda+\mu)\nabla\operatorname{div}\varphi-\alpha\nabla\psi' \\
-\varepsilon\int_0^t g(t-s)[\mu\Delta\varphi(x,s)+(\lambda+\mu)\nabla\operatorname{div}\varphi(x,s)]\,ds=0\quad\text{in } Q, \\
\psi'-\Delta\psi-\alpha\operatorname{div}\varphi=0\quad\text{in } Q, \\
\varphi=0,\quad \psi=0\quad\text{on } \Sigma, \\
\varphi(x,0)=\varphi^0(x),\quad \varphi'(x,0)=\varphi^1(x),\quad \psi(x,0)=\psi^0(x)\quad\text{in } \Omega.
\end{gathered}\tag{3.2}$$

For the solution (φ,ψ) of (3.2), we set

$$E(\varphi,t)=\frac12\int_\Omega\Big[|\varphi'|^2+\mu|\nabla\varphi|^2+(\lambda+\mu)|\operatorname{div}\varphi|^2\Big]\,dx\,,\tag{3.3}$$

$$F(\psi,t)=\frac12\int_\Omega|\psi'|^2\,dx\,,\tag{3.4}$$

$$G(\psi,t)=\frac12\int_\Omega|\psi|^2\,dx\,.\tag{3.5}$$

Lemma 3.1. *Suppose that $g(t)\in H^1(0,T)$ and*

$$(\varphi^0,\varphi^1,\psi^0)\in(H_0^1(\Omega))^n\times(L^2(\Omega))^n\times(H^2(\Omega)\cap H_0^1(\Omega))\,.$$

Then for the solution (φ,ψ) of (3.2) we have

$$E(\varphi,t)+2F(\psi,t)+2\int_0^t\int_\Omega|\nabla\psi'|^2\,dx\,dt\le C[E(\varphi,0)+F(\psi,0)]\,,\tag{3.6}$$

$$G(\psi,t)+\int_0^t\int_\Omega|\nabla\psi|^2\,dx\,dt\le C[E(\varphi,0)+F(\psi,0)+G(\psi,0)]\,,\tag{3.7}$$

where $C = C(\varepsilon, g, T)$ is a constant dependent on ε, g, T only.

Proof. Multiplying the first equation of (3.2) by φ' and integrating over $\Omega \times (0,t)$, we obtain

$$\begin{aligned} &E(\varphi,t) - E(\varphi,0) - \alpha \int_0^t \int_\Omega \varphi' \nabla \psi' \, dx \, dt \\ &\qquad = -\varepsilon\mu \int_0^t \int_0^\tau \int_\Omega g(\tau - s) \nabla\varphi(x,s) \nabla\varphi'(x,\tau) \, dx \, ds \, d\tau \\ &\qquad\quad - (\lambda+\mu)\varepsilon \int_0^t \int_0^\tau \int_\Omega g(\tau - s) \mathrm{div}\varphi(x,s) \mathrm{div}\, \varphi'(x,\tau)] \, dx \, ds \, d\tau. \end{aligned} \tag{3.8}$$

Differentiating the second equation of (3.2) with respect to t, we obtain

$$\psi'' - \Delta\psi' - \alpha \,\mathrm{div}\, \varphi' = 0 \quad \text{in } Q. \tag{3.9}$$

Multiplying (3.9) by ψ' and integrating over $\Omega \times (0,t)$, we obtain

$$F(\psi,t) - F(\psi,0) + \int_0^t \int_\Omega |\nabla\psi'|^2 \, dx \, dt - \alpha \int_0^t \int_\Omega \psi' \mathrm{div}\, \varphi' = 0. \tag{3.10}$$

Since

$$\int_0^t \int_\Omega \varphi' \nabla\psi' \, dx \, dt = - \int_0^t \int_\Omega \psi' \mathrm{div}\, \varphi' \,, \tag{3.11}$$

it follows from (3.8) and (3.10) that

$$\begin{aligned} &E(\varphi,t) + F(\psi,t) + \int_0^t \int_\Omega |\nabla\psi'|^2 \, dx \, dt \\ &= E(\varphi,0) + F(\psi,0) \\ &\quad - \varepsilon\mu \int_0^t \int_0^\tau \int_\Omega g(\tau - s)\nabla\varphi(x,s)\nabla\varphi'(x,\tau) \, dx \, ds \, d\tau \\ &\quad - (\lambda+\mu)\varepsilon \int_0^t \int_0^\tau \int_\Omega g(\tau - s)\mathrm{div}\varphi(x,s)\mathrm{div}\varphi'(x,\tau)] \, dx \, ds \, d\tau \\ &= E(\varphi,0) + F(\psi,0) \\ &\quad - \varepsilon \int_0^t \int_\Omega g(t-s)[\mu\nabla\varphi(x,s)\nabla\varphi(x,t) + (\lambda+\mu)\mathrm{div}\varphi(x,s)\mathrm{div}\varphi(x,t)] \, dx \, ds \\ &\quad + \varepsilon g(0) \int_0^t \int_\Omega [\mu|\nabla\varphi(x,s)|^2 + (\lambda+\mu)|\mathrm{div}\varphi(x,s)|^2] \, dx \, ds \\ &\quad + \varepsilon\mu \int_0^t \int_0^\tau \int_\Omega g'(\tau - s)\nabla\varphi(x,s)\nabla\varphi(x,\tau) \, dx \, ds \, d\tau \\ &\quad + (\lambda+\mu)\varepsilon \int_0^t \int_0^\tau \int_\Omega g'(\tau - s)\mathrm{div}\, \varphi(x,s)\mathrm{div}\varphi(x,\tau)] \, dx \, ds \, d\tau \\ &\le E(\varphi,0) + F(\psi,0) + \frac{1}{4}[\mu\|\nabla\varphi(t)\|^2 + (\lambda+\mu)\|\mathrm{div}\varphi(t)\|_0^2] \\ &\quad + C(\varepsilon,g,T) \int_0^t [\mu\|\nabla\varphi(s)\|_0^2 + (\lambda+\mu)\|\mathrm{div}\, \varphi(s)\|_0^2] \, ds, \end{aligned} \tag{3.12}$$

since

$$\begin{aligned}
&\varepsilon \int_0^t \int_\Omega g(t-s)[\mu\nabla\varphi(x,s)\nabla\varphi(x,t) + (\lambda+\mu)\mathrm{div}\varphi(x,s)\mathrm{div}\,\varphi(x,t)]\,dx\,ds \\
&\le \frac{1}{4}[\mu\|\nabla\varphi(t)\|^2 + (\lambda+\mu)\|\mathrm{div}\,\varphi(t)\|_0^2] \\
&\quad + \varepsilon^2 \int_0^t |g(s)|^2 ds \int_0^t [\mu\|\nabla\varphi(s)\|_0^2 + (\lambda+\mu)\|\mathrm{div}\,\varphi(s)\|_0^2]\,ds\,,
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon \int_0^t \int_0^\tau \int_\Omega g'(\tau-s)[\mu\nabla\varphi(x,s)\nabla\varphi(x,\tau) + (\lambda+\mu)\mathrm{div}\varphi(x,s)\mathrm{div}\varphi(x,\tau)]\,dx\,ds\,d\tau \\
&\qquad \le \varepsilon\sqrt{t}\Big(\int_0^t |g'(s)|^2 ds\Big)^{1/2} \int_0^t [\mu\|\nabla\varphi(s)\|_0^2 + (\lambda+\mu)\|\mathrm{div}\,\varphi(s)\|_0^2]\,ds\,.
\end{aligned}$$

By Gronwall's inequality (see [4], p.35), (3.6) follows from (3.12)

Similarly, by multiplying the second equation of (3.2) by ψ and integrating over $\Omega \times (0,t)$, we can deduce (3.7). $\diamondsuit$

In order to establish observability inequalities for the system (3.2), we transform (3.2) into a thermoelastic system. For this, we set

$$\begin{aligned}
\Phi &= \varphi + \varepsilon \int_0^t g(t-s)\varphi(x,s)\,ds\,, \\
f &= \varepsilon\Big(\int_0^t g(t-s)\varphi(x,s)ds\Big)'', \\
h &= -\alpha\varepsilon\,\mathrm{div}\,\Big(\int_0^t g(t-s)\varphi(x,s)\,ds\Big)\,.
\end{aligned} \tag{3.13}$$

Then (3.2) becomes

$$\begin{gathered}
\Phi'' - \mu\Delta\Phi - (\lambda+\mu)\nabla\mathrm{div}\,\Phi - \alpha\nabla\psi' = f \quad \text{in } Q, \\
\psi' - \Delta\psi - \alpha\mathrm{div}\,\Phi = h \quad \text{in } Q, \\
\Phi = 0, \quad \psi = 0 \quad \text{on } \Sigma\,, \\
\Phi(x,0) = \Phi^0(x), \quad \Phi'(x,0) = \Phi^1(x), \quad \psi(x,0) = \psi^0(x) \quad \text{in } \Omega\,,
\end{gathered} \tag{3.14}$$

where

$$\Phi^0(x) = \varphi^0(x), \quad \Phi^1(x) = \varphi^1(x) + \varepsilon g(0)\varphi^0(x)\,.$$

For the thermoelastic system (3.14), the following a priori boundary estimates are well known (see [11, Chap.1]). These estimates were established in [11, Chap.1] in the case where there is no term $(\lambda+\mu)\nabla\mathrm{div}\,\Phi$, but there is no added difficulty in the present case.

Lemma 3.2. *Let the boundary Γ of Ω be of class C^2. Suppose that $(\Phi^0, \Phi^1, \psi^0) \in (H_0^1(\Omega))^n \times (L^2(\Omega))^n \times (H^2(\Omega) \cap H_0^1(\Omega))$, $f \in L^1(0,T; L^2(\Omega))$, and*

$h \in W^{1,1}(0,T;L^2(\Omega))$. Then there exists a constant $C > 0$ such that for all solutions of (3.14)

$$\begin{aligned}&\int_\Sigma \Big(\mu\Big|\frac{\partial\Phi}{\partial\nu}\Big|^2 + (\lambda+\mu)|\mathrm{div}\,\Phi|^2\Big)\,d\Sigma \\ \le& C\Big[\|\Phi^0\|_1^2 + \|\Phi^1\|_0^2 + \|\psi^0\|_2^2 \\ &+ \|h(0)\|_0^2 + \|f\|^2_{L^1(0,T;L^2(\Omega))} + \|h'\|^2_{L^1(0,T;L^2(\Omega))}\Big].\end{aligned} \tag{3.15}$$

Lemma 3.3 (The observability inequality). *Suppose the boundary Γ of Ω is of class C^2. Suppose that $T > 2R(x^0)/\sqrt{\mu}$. Then there exists $\alpha_0 > 0$ such that if $\alpha \le \alpha_0$, then there exists a constant $C = C(\alpha,T) > 0$ such that for every weak solution of (3.14) with $(\Phi^0,\Phi^1) \in (H_0^1(\Omega))^n \times (L^2(\Omega))^n$ and $f = h \equiv 0$, $\psi^0 = 0$ we have*

$$\int_{\Sigma(x^0)} \Big(\mu\Big|\frac{\partial\Phi}{\partial\nu}\Big|^2 + (\lambda+\mu)|\mathrm{div}\,\Phi|^2\Big)\,d\Sigma \ge CE(\Phi,0)\,. \tag{3.16}$$

By Lemma 3.2, we obtain the following boundary regularity for the solution of (3.2).

Lemma 3.4. *Let the boundary Γ of Ω be of class C^2. Suppose that $(\varphi^0,\varphi^1,\psi^0) \in (H_0^1(\Omega))^n \times (L^2(\Omega))^n \times (H^2(\Omega)\cap H_0^1(\Omega))$.*
Then there exists a constant $C = C(\alpha,\varepsilon,g,T) > 0$ such that for all solutions of (3.2)

$$\int_\Sigma \Big(\mu\Big|\frac{\partial\Phi}{\partial\nu}\Big|^2 + (\lambda+\mu)|\mathrm{div}\Phi|^2\Big)\,d\Sigma \le C\Big[\|\varphi^0\|_1^2 + \|\varphi^1\|_0^2 + \|\psi^0\|_2^2\Big], \tag{3.17}$$

where Φ is given by (3.13)

Proof. We first deduce that

$$\begin{aligned}&\|\Big(\int_0^t g(t-s)\varphi(x,s)ds\Big)''\|_0 \\ &\le |g(0)|\|\varphi'(t)\|_0 + |g'(0)|\|\varphi(t)\|_0 + \Big(\int_\Omega |\int_0^t g''(t-s)\varphi(x,s)ds|^2dx\Big)^{1/2} \\ &\le |g(0)|\|\varphi'(t)\|_0 + |g'(0)|\|\varphi(t)\|_0 \\ &\quad + \Big(\int_0^t |g''(s)|^2ds\Big)^{1/2}\Big(\int_0^t\int_\Omega |\varphi(x,s)|^2\,dx\,ds\Big)^{1/2} \\ &\le C(g,T)[E(\varphi,0)+F(\psi,0)]^{1/2},\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}&\|\mathrm{div}\Big(\int_0^t g(t-s)\varphi(x,s)ds\Big)'\|_0 \\ &\le |g(0)|\|\mathrm{div}\varphi(t)\|_0 + \Big(\int_0^t |g'(s)|^2ds\Big)^{1/2}\Big(\int_0^t\int_\Omega |\mathrm{div}\varphi(x,s)|^2\,dx\,ds\Big)^{1/2} \\ &\le C(g,T)[E(\varphi,0)+F(\psi,0)]^{1/2}.\end{aligned} \tag{3.19}$$

Consequently, (3.17) follows from (3.15), (3.18) and (3.19). ♢

By Lemma 3.3, we have the following observability inequality for (3.2).

Lemma 3.5 (The observability inequality). *Let the boundary Γ of Ω be of class C^2. Suppose that $g(t) \in H^2(0,T)$ and $T > 2R(x^0)/\sqrt{\mu}$. Then there exist $\varepsilon_0, \alpha_0 > 0$ such that if $\varepsilon \le \varepsilon_0$ and $\alpha \le \alpha_0$, then there exists a constant $C = C(\alpha, \varepsilon, g, T) > 0$ such that for every weak solution of (3.2) with $(\varphi^0, \varphi^1) \in (H_0^1(\Omega))^n \times (L^2(\Omega))^n$ and $\psi^0 = 0$ we have*

$$\int_{\Sigma(x^0)} \Big(\mu\Big|\frac{\partial \Phi}{\partial \nu}\Big|^2 + (\lambda+\mu)|\mathrm{div}\Phi|^2\Big)\, d\Sigma \ge C\Big[\|\varphi^0\|_1^2 + \|\varphi^1\|_0^2\Big]. \tag{3.20}$$

Proof. The solution (Φ, ψ) of (3.14) can be written as

$$(\Phi, \psi) = (v, w) + (y, z)\,,$$

where (v, w) is the solution of

$$\begin{gathered} v'' - \mu\Delta v - (\lambda+\mu)\nabla \mathrm{div}\, v - \alpha \nabla w' = 0 \quad \text{in } Q, \\ w' - \Delta w - \alpha \mathrm{div}\, v = 0 \quad \text{in } Q, \\ v = 0, \quad w = 0 \quad \text{on } \Sigma\,, \\ v(x,0) = \Phi^0(x), \quad v'(x,0) = \Phi^1(x), \quad w(x,0) = 0 \quad \text{in } \Omega, \end{gathered} \tag{3.21}$$

and (y, z) is the solutions of

$$\begin{gathered} y'' - \mu\Delta y - (\lambda+\mu)\nabla \mathrm{div}\, y - \alpha \nabla z' = f \quad \text{in } Q, \\ z' - \Delta z - \alpha \mathrm{div}\, y = h \quad \text{in } Q, \\ y = 0, \quad z = 0 \quad \text{on } \Sigma\,, \\ y(x,0) = 0, \quad y'(x,0) = 0, \quad z(x,0) = 0 \quad \text{in } \Omega. \end{gathered} \tag{3.22}$$

Thus, by Lemma 3.2, we have

$$\begin{aligned} &\int_{\Sigma(x^0)} \Big(\mu\Big|\frac{\partial v}{\partial \nu}\Big|^2 + (\lambda+\mu)|\mathrm{div}\, v|^2\Big)\, d\Sigma \\ &\le 2\int_{\Sigma(x^0)} \Big(\mu\Big|\frac{\partial \Phi}{\partial \nu}\Big|^2 + (\lambda+\mu)|\mathrm{div}\, \Phi|^2\Big)\, d\Sigma \\ &\quad + 2\int_{\Sigma(x^0)} \Big(\mu\Big|\frac{\partial y}{\partial \nu}\Big|^2 + (\lambda+\mu)|\mathrm{div}\, y|^2\Big)\, d\Sigma \\ &\le 2\int_{\Sigma(x^0)} \Big(\mu\Big|\frac{\partial \Phi}{\partial \nu}\Big|^2 + (\lambda+\mu)|\mathrm{div}\, \Phi|^2\Big)\, d\Sigma \\ &\quad + 2\varepsilon^2 C(\alpha, T)\|\Big(\int_0^t g(t-s)\varphi(x,s)\, ds\Big)''\|^2_{L^1(0,T,L^2(\Omega))} \\ &\quad + 2\varepsilon^2\alpha^2 C(\alpha, T)\|\mathrm{div}\Big(\int_0^t g(t-s)\varphi(x,s)\, ds\Big)'\|^2_{L^1(0,T,L^2(\Omega))}. \end{aligned}$$

Hence, (3.20) follows from (3.16), (3.18) and (3.19). $\diamondsuit$

§4. Proof of Main Result

Proof. We apply the Hilbert Uniqueness method. Given $(\varphi^0, \varphi^1) \in (H_0^1(\Omega))^n \times (L^2(\Omega))^n$, we consider

$$
\begin{gathered}
\varphi'' - \mu\Delta\varphi - (\lambda+\mu)\nabla\mathrm{div}\,\varphi + \alpha\nabla\psi' \\
-\varepsilon\int_t^T g(s-t)[\mu\Delta\varphi(x,s) + (\lambda+\mu)\nabla\mathrm{div}\varphi(x,s)]\,ds = 0 \quad \text{in } Q, \\
-\psi' - \Delta\psi - \alpha\mathrm{div}\,\varphi = 0 \quad \text{in } Q, \\
\varphi = 0, \quad \psi = 0 \quad \text{on } \Sigma, \\
\varphi(x,T) = \varphi^0(x), \quad \varphi'(x,T) = \varphi^1(x), \quad \psi(x,T) = 0 \quad \text{in } \Omega,
\end{gathered}
\tag{4.1}
$$

Using the solution φ of (4.1), we then consider

$$
\begin{gathered}
u'' - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}\,u + \alpha\nabla\theta \\
-\varepsilon\int_0^t g(t-s)[\mu\Delta u(x,s) + (\lambda+\mu)\nabla\mathrm{div}u(x,s)]\,ds = 0 \quad \text{in } Q, \\
\theta' - \Delta\theta + \alpha\mathrm{div}\,u' = 0 \quad \text{in } Q, \\
\theta = 0 \quad \text{on } \Sigma, \\
u = \begin{cases} \frac{\partial\varphi}{\partial\nu} + \varepsilon\int_t^T g(s-t)\frac{\partial}{\partial\nu}\varphi(x,s)\,ds & \text{on } \Sigma(x^0) \\ 0 & \text{on } \Sigma_*(x^0) \end{cases} \\
u(x,0) = 0, \quad u'(x,0) = 0, \quad \theta(x,0) = 0 \quad \text{in } \Omega.
\end{gathered}
\tag{4.2}
$$

By transposition methods (see [11]), one can show that (4.2) has a unique solution (u, θ) with

$$
\begin{gathered}
u \in C([0,T]; (L^2(\Omega))^n) \cap C^1([0,T]; (H^{-1}(\Omega))^n), \\
\theta \in C([0,T]; L^2(\Omega)).
\end{gathered}
$$

Define the operator Λ by

$$
\Lambda(\varphi^0, \varphi^1) = (-u'(T), u(T))\,.
$$

Set

$$
\Phi = \varphi + \varepsilon\int_t^T g(s-t)\varphi(x,s)\,ds\,.
$$

Multiplying the second equation of (4.1) by θ and integrating over Q, we obtain

$$
\alpha\int_Q \psi'\mathrm{div}\,u\,dx\,dt - \alpha\int_Q \theta\mathrm{div}\,\varphi\,dx\,dt = 0\,.
$$

Multiplying the first equation of (4.1) by u and integrating over Q, we obtain

$$
\begin{aligned}
\langle\Lambda(\varphi^0,\varphi^1), (\varphi^0,\varphi^1)\rangle =& \mu\int_\Sigma u\frac{\partial\Phi}{\partial\nu}\,d\Sigma + (\lambda+\mu)\int_\Sigma u\cdot\nu\mathrm{div}\Phi\,d\Sigma \\
&+ \alpha\int_Q \psi'\mathrm{div}u\,dx\,dt - \alpha\int_Q \theta\mathrm{div}\varphi\,dx\,dt \\
=& \mu\int_\Sigma u\frac{\partial\Phi}{\partial\nu}\,d\Sigma + (\lambda+\mu)\int_\Sigma u\cdot\nu\mathrm{div}\Phi\,d\Sigma.
\end{aligned}
$$

Since $\Phi = 0$ on Σ, we have

$$\frac{\partial \Phi_i}{\partial \nu} \nu_i = \frac{\partial \Phi_i}{\partial x_i}, \quad \text{on } \Sigma \,.$$

Noting that $u = \dfrac{\partial \Phi}{\partial \nu}$ on Σ, we deduce that

$$\langle \Lambda(\varphi^0, \varphi^1), (\varphi^0, \varphi^1) \rangle = \mu \int_\Sigma |\frac{\partial \Phi}{\partial \nu}|^2 \, d\Sigma + (\lambda + \mu) \int_\Sigma |\mathrm{div}\Phi|^2 \, d\Sigma \,.$$

Therefore, it follows from Lemmas 3.4 and 3.5 that Λ is an isomorphism from $(H_0^1(\Omega))^n \times (L^2(\Omega))^n$ onto $(H^{-1}(\Omega))^n \times (L^2(\Omega))^n$. This completes the proof of Theorem 2.1.

References

[1] R. Adams, "Sobolev Spaces," Academic Press, New York, 1975.

[2] C.M. Dafermos, *An abstract Volterra equation with applications to linear viscoelasticity,* J. Differential Equations 7 (1970), 554–569.

[3] C.M. Dafermos, *Asymptotic stability in viscoelasticity,* Arch. Rational Mech. Anal. 37 (1970), 297–308.

[4] J.K. Hale, "Ordinary Differential Equations," Wiley-interscience, New York, 1969.

[5] S.W. Hansen, *Boundary control of a one-dimensional linear thermoelastic rod,* SIAM J. Control Optim. (4) 32 (1994), 1054–1074.

[6] J.U. Kim, *Control of a plate equation with large memory,* Differential Integral Equations (2) 5 (1992), 261–279.

[7] J. Lagnese, *The reachability problem for thermoelastic plates,* Arch. Rational Mech. Anal. 112 (1990), 223–267.

[8] J. Lagnese and J.L. Lions, "Modelling Analysis and Control of Thin Plates," Masson, Paris, 1989.

[9] G. Lebeau and E. Zuazua, *Null controllability of a system of linear thermoelasticity,* Preprint.

[10] G. Leugering, *Exact controllability in viscoelasticity of fading memory type,* Applicable Anal. (3) 18 (1984), 221–243.

[11] J.L. Lions, "Contrôlabilité Exacte Perturbations et Stabilisation de Sys- tèmes Distribués," Tome 2, Perturbations, Masson, Paris, Milan, Barcelona, Mexico, 1988.

[12] W.J. Liu, *Partial exact controllability and exponential stability of the higher-dimensional linear thermoelasticity,* ESAIM Contrôle Optim. C- alc. Var., 3 (1998), 23–48.

[13] Z. Liu and S. Zheng, *On the exponential stability of linear viscoelasticity and thermoviscoelasticity,* Quart. Appl. Math. LIV (1996), 21–31.

[14] K. Narukawa, *Boundary value control of thermoelastic systems,* Hiroshima Math. J. 13 (1983), 227–272.

[15] C.B. Navarro, *Asymptotic stability in linear thermoviscoelasticity,* J. Math. Anal. Appl. 65 (1978), 399–431.

[16] J.H. Yan, *Contrôlabilité exacte pour des systèmes à mêmoire,* Rev. Mat. Univ. Complut. Madrid 5 (1992), 319–327.

[17] E. Zuazua, *Controllability of the linear system of thermoelasticity,* J. Math. Pres. Appl. (9) 74 (1995), 291–315.

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