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CAUCHY PROBLEMS FOR CHEMOTAXIS SYSTEMS WITH CHEMO ATTRACTANT AND REPELLENT

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Abstract. We study the existence of global solutions and their asymptotic behavior for a chemotaxis system with chemo-attractant and repellent in three dimensions. To accomplish this, we use the Fourier transform and energy method. First, we establish L^q time-decay for the linear homogeneous system using Fourier transform and finding Green's matrix. Then, we find L^q time-decay for the nonlinear system using the Duhamel's principle and the time-weighted estimates.

1. Introduction

Chemotaxis refers to the biological process of cells or microscopic organisms moving toward a favorable chemical concentration gradient. There are many examples that describe this biological phenomena, such as the oriented movement of certain cells from higher or lower concentrations of chemicals in the case of immunology, or the movement of endothelial cells toward the higher concentration of chemoattractant that cancer cells produce [1, 4]. A preeminent model for describing the chemotaxis of cells was proposed by Keller and Segel [11, 12], and is now one of the most studied and utilized models in mathematical biology. Many mathematical approaches have since emerged to describe chemotaxis using systems of partial differential equations. Detailed introductions to mathematical aspects of Keller-Segel models are available in [8, 9].

In this article, we use the equations for continuum mechanics to describe the movement of cells and use the diffusion equations for the chemo-attractant and repellent. Thus, we consider the initial value problem of the system in $\mathbb{R}^3$

$$\begin{gathered}
\partial_t n + \nabla\cdot(nu) = 0\\
\partial_t(nu) + \nabla\cdot(nu\otimes u) + \nabla p(n) = n(\chi_1(c_1)\nabla c_1 - \chi_2(c_2)\nabla c_2) - \nu nu\\
\partial_t c_1 = \Delta c_1 - a_{11}c_1 + a_{12}n\kappa_1(c_1)\\
\partial_t c_2 = \Delta c_2 - a_{21}c_2 + a_{22}n\kappa_2(c_2)
\end{gathered}\tag{1.1}$$

with the initial data

$$(n,u,c_1,c_2)\big|_{t=0} = (n_0,u_0,c_{1,0},c_{2,0})(x),\quad x\in\mathbb{R}^3. \tag{1.2}$$

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Here, $n(x,t), u(x,t), c_1(x,t), c_2(x,t)$ and $p(n)$ for $t > 0, x \in \mathbb{R}^3$, are the cell concentration, velocity of cells, chemo-attractant, chemo-repellent and pressure of the cells, respectively. A positive constant ν is the coefficient of the damping. Subtracting the first equation from the second equation, we will consider the simplified chemotaxis fluid equations taking the form

$$\begin{gathered}
\partial_t n + \nabla\cdot(nu) = 0\\
\partial_t u + u\cdot\nabla u + \frac{\nabla p(n)}{n} = \chi_1(c_1)\nabla c_1 - \chi_2(c_2)\nabla c_2 - \nu u\\
\partial_t c_1 = \Delta c_1 - a_{11}c_1 + a_{12}n\kappa_1(c_1)\\
\partial_t c_2 = \Delta c_2 - a_{21}c_2 + a_{22}n\kappa_2(c_2)
\end{gathered} \tag{1.3}$$

with the initial data

$$(n,u,c_1,c_2)\big|_{t=0} = (n_0,u_0,c_{1,0},c_{2,0})(x),\ x\in\mathbb{R}^3 \tag{1.4}$$

satisfying

$$(n_0,u_0,c_{1,0},c_{2,0})(x) \to (n_\infty,0,0,0) \quad \text{as} \quad |x|\to\infty$$

for a given constant $n_\infty > 0$. Throughout this paper, we assume the following: $p(\cdot)$ is the smooth functions of n and $p'(n) > 0$. Also, $\chi_1(\cdot)$, $\chi_2(\cdot)$, $\kappa_1(\cdot)$, $\kappa_2(\cdot)$ are the smooth functions of c_1, c_2, and $\kappa_i(0) = 0, \chi_i(0) = 0, i = 1,2$. $n\kappa_1(c_1)$ and $n\kappa_2(c_2)$ accelerate the rate of change of c_1 and c_2, respectively.

The main goals of this article are to show the existence of local and global solutions in $H^N(\mathbb{R}^3)$ and L^q time-decay rates of solutions for the Cauchy problem of the above system (1.1)-(1.2). The main result of this paper is stated as follows. Notations are explained at the end of the section.

Theorem 1.1. *Let $N \ge 4$. There exists a positive number ϵ_0 such that if*

$$\|(n_0-n_\infty,u_0,c_{1,0},c_{2,0})\|_N \le \epsilon_0,$$

the Cauchy problem (1.3)-(1.4) has a unique solution $(n,u,c_1,c_2)(t)$ globally in time which satisfies

$$\begin{gathered}
(n-n_\infty,u)(t) \in C([0,\infty);H^N(\mathbb{R}^3)) \cap C^1([0,\infty);H^{N-1}(\mathbb{R}^3)),\\
(c_1,c_2)(t) \in C([0,\infty);H^N(\mathbb{R}^3)) \cap C^1([0,\infty);H^{N-2}(\mathbb{R}^3))
\end{gathered}$$

and there are constants $\lambda_1 > 0, \lambda_2 > 0$, $\lambda_3 > 0$ and $C_0 > 0$ such that

$$\begin{aligned}
&\|(n-n_\infty,u,c_1,c_2)\|_N^2 + \lambda_1\int_0^t \|\nabla(n-n_\infty)\|_{N-1}^2\\
&+ \lambda_2\int_0^t \|\nabla(c_1,c_2)\|_N^2 + \lambda_3\int_0^t \|(u,c_1,c_2)\|_N^2\\
&\le C_0\|(n_0-n_\infty,u_0,c_{1,0},c_{2,0})\|_{H^N}^2.
\end{aligned} \tag{1.5}$$

Moreover, the global solution $[n,u,c_1,c_2]$ obtained above satisfies for a sufficiently large time $t > 0$

$$\|n-n_\infty\|_{L^q} \le C(1+t)^{\frac{-3}{2}+\frac{3}{2q}}, \tag{1.6}$$

$$\|u\|_{L^q} \le C(1+t)^{-2+\frac{3}{2q}}, \tag{1.7}$$

$$\|(c_1,c_2)\|_{L^q} \le C(1+t)^{-3/2}, \tag{1.8}$$

with $2 \le q < \infty$, where $C > 0$ is a positive constant independent of time.

The proof of the existence of global solutions in Theorem 1.1 is based on the local existence and the a priori estimates. The local existence can be proved by constructing a sequence of approximation functions based on iteration method discussed, for example, in Kato [10]. The a priori estimates can be obtained by the energy method. Moreover, our approach to obtain the time-decay rate in the L^q norm of solutions in Theorem 1.1 is a combined analysis of Green's function of the linear system and the refined energy estimates with the help of Duhamel's principle. We obtain Green's matrix of the linear system by Fourier transform.

Concerning the chemotaxis based on the Keller-Segel model, Wang [17] explored the interactions between the nonlinear diffusion and logistic source on the solutions of the attraction-repulsion chemotaxis system in three dimensions. E. Lankeit and J. Lankeit [13] proved global existence of classical solutions to a chemotaxis system with singular sensitivity. Liu and Wang [14] established the existence of global classical solutions and steady states to an attraction-repulsion chemotaxis model in one dimension based on the energy method. For the long time behavior of solutions, Tan and Zhou [16] proved the global existence and time-decay estimate of solutions to the Keller-Segel system in R^3 with small initial data.

Modeling the cell movements based on the incompressible or compressible fluid dynamics has been studied as well. For the incompressible case, Chae, Kang and Lee [3], and Duan, Lorz, and Markowich [7] showed the global-in-time existence for the incompressible chemotaxis equations near constant states, if the initial data is sufficiently small. Rodriguez, Ferreira and Villamizar-Roa [15] showed the global existence for an attraction-repulsion chemotaxis fluid model with a logistic source. For the compressible case, D. Ambrosi, F. Bussolino and L. Preziosi [2] discussed the vasculogenesis using the compressible fluid dynamics for the cells and the diffusion equation for the attractant. There are many related approaches that use Fourier transform, and we only mention that Duan [5], and Duan, Liu and Zhu [6] proved the time-decay rates combining the energy estimates and spectral analysis.

We summarize the notation used in the paper here. C denotes a positive constant, λ_i, where $i = 1, 2$, denote positive (generally small) constants, where both C and λ_i may take different values in different places. For any integer $m \geq 0$, we use H^m to denote the Sobolev space $H^m(\mathbb{R}^3)$ and $\dot{H}^m$ the m^{th}-order homogeneous Sobolev space. Set $L^2 = H^0$. For simplicity, the norm of H^m is denoted by $\|\cdot\|_m$ with $\|\cdot\| = \|\cdot\|_0$. The L^q norms are given by $\|\cdot\|_{L^q}$. We set $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ for a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. The length of α is $|.| = \alpha_1 + \alpha_2 + \alpha_3$ and we also set $\partial_j = \partial_{x_j}$for $j = 1, 2, 3$. For an integrable function $f : \mathbb{R}^3 \to \mathbb{R}$, its Fourier transform is defined by $\hat{f} = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx,\ x \cdot \xi = \sum_{i=0}^{3} x_j\xi_j,\ x \in \mathbb{R}^3$, where $i = \sqrt{-1}$ is the imaginary unit. Let us denote the space

$$\begin{aligned} X(0,T) = \big\{&(\rho,u) \in C([0,T]; H^N(\mathbb{R}^3)) \cap C^1([0,T]; H^{N-1}(\mathbb{R}^3)), \\ &(c_1,c_2) \in C([0,T]; H^N(\mathbb{R}^3)) \cap C^1([0,T]; H^{N-2}(\mathbb{R}^3))\big\}. \end{aligned}$$

This article is organized as follows. In section 2, we reformulate the Cauchy problem under consideration. In section 3, we prove the global existence and uniqueness of solutions for the reformulated problem using the energy method. In section 4, we linearize the reformulated problem and obtain the L^q time-decay property for the linearized equations by the Fourier transform. In section 5, we use the results of sections 3 and 4 to study the L^q time-decay rates of solutions to the reformulated nonlinear system and complete the proof of Theorem 1.1.

2. Reformulation of the system (1.3)

Let $U(t) = (n, u, c_1, c_2)$ be a smooth solution to the Cauchy problem of the chemotaxis fluid equations (1.3) with initial data $U_0 = (n_0, u_0, c_{1,0}, c_{2,0})$. Set

$$n(x,t) = \rho(x,t) + n_\infty. \tag{2.1}$$

Then the Cauchy problem (1.3)-(1.4) is reformulated as

$$\begin{gathered}\partial_t \rho + n_\infty \nabla \cdot u = -\nabla \cdot (\rho u)\\ \partial_t u + u \cdot \nabla u + \nu u + \frac{p'(\rho + n_\infty)}{\rho + n_\infty} \nabla \rho = \chi_1(c_1)\nabla c_1 - \chi_2(c_2)\nabla c_2\\ \partial_t c_1 - \Delta c_1 + (a_{11} - a_{12} n_\infty \kappa_1'(0))c_1 = a_{12}(\rho + n_\infty)\kappa_1(c_1) - a_{12} n_\infty \kappa_1'(0)c_1\\ \partial_t c_2 - \Delta c_2 + (a_{21} - a_{22} n_\infty \kappa_2'(0))c_2 = a_{22}(\rho + n_\infty)\kappa_2(c_2) - a_{22} n_\infty \kappa_2'(0)c_2\end{gathered} \tag{2.2}$$

with the initial data

$$(\rho, u, c_1, c_2)\big|_{t=0} = (\rho_0, u_0, c_{1,0}, c_{2,0}), \tag{2.3}$$

where $\rho_0 = n_0 - n_\infty$ and

$$(\rho_0, u_0, c_{1,0}, c_{2,0}) \to (0,0,0,0) \quad \text{as} \quad |x| \to \infty.$$

We assume that $(a_{11} - n_\infty a_{12}\kappa_1'(0)) > 0$ and $(a_{22} - n_\infty a_{21}\kappa_2'(0)) > 0$. In the following, we set $N \geq 4$.

Concerning the reformulated Cauchy problem (2.2)-(2.3), one has the following global existence result.

Proposition 2.1. *Suppose that $\|(\rho_0, u_0, c_{1,0}, c_{2,0})\|_N$ is sufficiently small. Then the Cauchy problem (2.2)-(2.3) has a unique solution $U(t) = (\rho, u, c_1, c_2)(t)$ globally in time which satisfies $U(t) \in X(0,\infty)$ and for any $t \geq 0$,*

$$\mathcal{E}_N(U(t)) + \lambda_1 \int_0^t \mathcal{D}_N(U(t))ds + \lambda_2 \int_0^t \mathcal{D}_N^h(U(t))ds \leq C_0 \mathcal{E}_N(U_0), \tag{2.4}$$

where

$$\mathcal{E}_N(U(t)) = \|(\rho, u, c_1, c_2)\|_N^2, \tag{2.5}$$

is the energy functional, and

$$\mathcal{D}_N(U(t)) = \|\nabla(c_1, c_2)\|_N^2, \tag{2.6}$$

$$\mathcal{D}_N^h(U(t)) = \|\nabla \rho\|_{N-1}^2 + \|(u, c_1, c_2)\|_N^2 \tag{2.7}$$

are the dissipation rates.

The solutions obtained in Proposition 2.1 indeed have the decay rates in time under additional conditions on the initial data. Given $U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0})$, set $\epsilon_N(U_0)$ as

$$\epsilon_N(U_0) = \|U_0\|_N + \|(\rho_0, u_0)\|_{L^1}, \tag{2.8}$$

for $N \geq 4$. Then, we have the following two Propositions:

Proposition 2.2. *Let $U(t) = (\rho, u, c_1, c_2)$ be the solution to the Cauchy problem (2.2) with the initial data $U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0})$. If $\epsilon_{N+1}(U_0) > 0$ is sufficiently small, then the solution $U(t) = (\rho, u, c_1, c_2)$ satisfies*

$$\|U(t)\|_N \leq \epsilon_{N+1}(U_0)(1+t)^{-3/4}, \tag{2.9}$$

$$\|\nabla U(t)\|_N \leq \epsilon_{N+1}(U_0)(1+t)^{-5/4}, \tag{2.10}$$

for any $t \geq 0$.

Proposition 2.3. *Let $2 \le q \le \infty$. Suppose that $U(t) = (\rho, u, c_1, c_2)$ be the solution to the Cauchy problem (2.2)-(2.3) obtained in Proposition 2.1. Then the solution $U(t) = (\rho, u, c_1, c_2)$ satisfies the following L^q-time decay estimates:*

$$\|\rho\|_{L^q} \le C(1+t)^{-2+\frac{3}{2q}}, \tag{2.11}$$

$$\|u\|_{L^q} \le C(1+t)^{\frac{-3}{2}+\frac{3}{2q}}, \tag{2.12}$$

$$\|(c_1, c_2)\|_{L^q} \le C(1+t)^{-3/2} \tag{2.13}$$

for any $t \ge 0$, $2 \le q \le \infty$.

The existence of global solutions in Theorem 1.1 is obtained directly from Proposition 2.1 and the derivation of rates in Theorem 1.1 is based on Proposition 2.3.

3. Global solution of the nonlinear system (2.2)

The goal of this section is to prove the existence of global solutions to the Cauchy problem (2.2), when initial data is a small, smooth perturbation near the steady state $(n_\infty, 0, 0, 0)$. The proof is based on uniform a priori estimates combined with the local existence that will be shown in subsections 3.1 and 3.2.

3.1. **Existence of local solutions.** The existence of smooth local solutions for symmetrizable hyperbolic equations $(2.2)_1$ and $(2.2)_2$ can be proved as in [10]. Since $(2.2)_3$ and $(2.2)_4$ are the heat equations, the local solutions obviously exist. We construct a solution sequence $(\rho^j, u^j, c_1^j, c_2^j)_{j\ge 0}$ by iteratively solving the Cauchy problem on the following system

$$\begin{gathered}
\partial_t \rho^{j+1} + n_\infty \nabla\cdot u^j = -\nabla\cdot(\rho^{j+1}u^j),\\
\begin{aligned}
&\partial_t u^{j+1} + \nu u^{j+1} + \frac{p'(n_\infty)}{n_\infty}\nabla\rho^{j+1}\\
&= -u^j\cdot\nabla u^{j+1} + \chi_1(c_1^j)\nabla c_1^j - \chi_1(c_2^j)\nabla c_2^j - \Big(\frac{p'(\rho^j+n_\infty)}{\rho^j+n_\infty} - \frac{p'(n_\infty)}{n_\infty}\Big)\nabla\rho^{j+1},
\end{aligned}\\
\partial_t c_1^{j+1} - \Delta c_1^{j+1} + (a_{11} - a_{12}n_\infty\kappa_1'(0))c_1^{j+1}\\
= a_{12}(\rho^j+n_\infty)c_1^{j+1} - a_{12}n_\infty\kappa_1'(0)c_1^{j+1},\\
\partial_t c_2^{j+1} - \Delta c_2^{j+1} + (a_{21} - a_{22}n_\infty\kappa_2'(0))c_2^{j+1}\\
= a_{22}(\rho^j+n_\infty)c_2^{j+1} - a_{22}n_\infty\kappa_2'(0)c_2^{j+1},
\end{gathered} \tag{3.1}$$

with initial data

$$(\rho^{j+1}, u^{j+1}, c_1^{j+1}, c_2^{j+1})\big|_{t=0} = (\rho_0, u_0, c_{1,0}, c_{2,0}), \tag{3.2}$$

for $j \ge 0$, where $(\rho^0, u^0, c_1^0, c_2^0) \equiv (0,0,0,0)$ holds. For simplicity, in what follows, we write $U^j = (\rho^j, u^j, c_1^j, c_2^j)$ and $U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0})$.

Lemma 3.1. *There are constants $T_1 > 0$, $\epsilon_0 > 0$, $B > 0$ such that if the initial data $U_0 \in H^N(\mathbb{R}^3)$ and $\|U_0\|_N \le \epsilon_0$, then for each $j \ge 0$, $U^j \in C([0,T_1] : H^N(\mathbb{R}^3))$ is well-defined and*

$$\sup_{0\le t\le T_1} \|U^j(t)\|_N \le B, \quad j \ge 0. \tag{3.3}$$

Moreover, $(U^j)_{j\geq 0}$ *is a Cauchy sequence in the Banach space* $C([0,T_1];H^N(\mathbb{R}^3))$, *and the limit function* $U(x,t)$ *of* $(U^j)_{j\geq 0}$ *satisfies*

$$\sup_{0\leq t\leq T_1}\|U(t)\|_N\leq B, \tag{3.4}$$

and $U=(\rho,u,c_1,c_2)$ *is a solution over* $[0,T_1]$ *to the Cauchy problem* (2.2)-(2.3). *Finally, the Cauchy problem* (2.2)-(2.3) *admits at most one solution* $U\in C([0,T_1]:H^N(\mathbb{R}^3))$ *satisfying* (3.2).

3.2. **A Priori Estimates.** In this subsection we provide some estimates for the solutions for any $t>0$. We establish the uniform-in-time a priori estimates for smooth solutions to Cauchy problem (2.2)-(2.3) applying basic energy estimates.

Lemma 3.2 (a priori estimates)**.** *Suppose that there exists a solution*

$$U(t)=(\rho,u,c_1,c_2)\in C([0,T];H^N(\mathbb{R}^3)$$

to the Cauchy problem (2.2)-(2.3), *with*

$$\sup_{0\leq t\leq T}\|(\rho,u,c_1,c_2)(t)\|_N\leq\epsilon \tag{3.5}$$

for $0<\epsilon\leq 1$. *Then, there are* $\epsilon_0>0$, $C_0>0$ *and* $\lambda>0$ *such that for any* $\epsilon\leq\epsilon_0$,

$$\mathcal{E}_N(U(t))+\lambda_1\int_0^t\mathcal{D}_N(U(t))ds+\lambda_2\int_0^t\mathcal{D}_N^h(U(t))ds\leq C_0\mathcal{E}_N(U_0) \tag{3.6}$$

holds for any $t\in[0,T]$.

Proof. First, we find the zero-order estimates. For the estimate of ρ, multiplying ρ to both sides of the first equation of (2.2) and integrating in $x\in\mathbb{R}^3$, we obtain

$$\int_{\mathbb{R}^3}\rho\rho_t\mathrm{d}x+n_\infty\int_{\mathbb{R}^3}\rho\nabla\cdot u\mathrm{d}x=-\int_{\mathbb{R}^3}\rho\nabla\cdot(\rho u)\mathrm{d}x.$$

Using integration by parts and the Cauchy-Schwarz inequality, we have

$$\frac{1}{2}\int_{\mathbb{R}^3}\left(\rho^2\right)_t\mathrm{d}x+n_\infty\int_{\mathbb{R}^3}\rho\nabla\cdot u\mathrm{d}x\leq C\|\rho\|_2\int_{\mathbb{R}^3}|u|^2+|\nabla\rho|^2\mathrm{d}x. \tag{3.7}$$

For the estimate of u, multiplying by u on both sides of the second equation of (2.2) and integrating in $x\in\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3}uu_t\mathrm{d}x+\int_{\mathbb{R}^3}u(u\cdot\nabla u)\mathrm{d}x+\nu\int_{\mathbb{R}^3}u^2\mathrm{d}x+\frac{p'(n_\infty)}{n_\infty}\int_{\mathbb{R}^3}u\cdot\nabla\rho\mathrm{d}x\\
&=\int_{\mathbb{R}^3}u\chi_1(c_1)\nabla c_1\mathrm{d}x-\int_{\mathbb{R}^3}u\chi_2(c_2)\nabla c_2\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}u\Big(\frac{p'(\rho+n_\infty)}{\rho+n_\infty}-\frac{p'(n_\infty)}{n_\infty}\Big)\nabla\rho\mathrm{d}x.
\end{aligned}$$

Using integration by parts and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}\left(u^2\right)_t\mathrm{d}x+\nu\int_{\mathbb{R}^3}|u|^2\mathrm{d}x-\frac{p'(n_\infty)}{n_\infty}\int_{\mathbb{R}^3}\rho\nabla\cdot u\mathrm{d}x\\
&\leq\|u\|_3\int_{\mathbb{R}^3}|u|^2\mathrm{d}x+C\|c_1\|_2\int_{\mathbb{R}^3}|\nabla c_1|^2+|u|^2\mathrm{d}x+C\|c_2\|_2\int_{\mathbb{R}^3}|\nabla c_2|^2+|u|^2\mathrm{d}x\\
&\quad+C\|\rho\|_2\int_{\mathbb{R}^3}|\nabla\rho|^2+|u|^2\mathrm{d}x.
\end{aligned}$$

For the estimates of c_1, multiplying c_1 to both sides of the equation of c_1, using $\kappa_1(c_1) - \kappa_1'(0)c_1 = O(c_1^2)$, and integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} (c_1^2)_t \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla c_1|^2 \mathrm{d}x + (a_{11} - n_\infty a_{12}\kappa_1'(0)) \int_{\mathbb{R}^3} |c_1|^2 \mathrm{d}x \\
&\le C(\|\rho\|_2 + \|c_1\|_2) \int_{\mathbb{R}^3} |c_1|^2 \mathrm{d}x.
\end{aligned} \tag{3.8}$$

Similarly, for c_2, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} (c_2^2)_t \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla c_2|^2 \mathrm{d}x + (a_{21} - n_\infty a_{22}\kappa_2'(0)) \int_{\mathbb{R}^3} |c_2|^2 \mathrm{d}x \\
&\le C(\|\rho\|_2 + \|c_2\|_2) \int_{\mathbb{R}^3} |c_2|^2 \mathrm{d}x.
\end{aligned} \tag{3.9}$$

Then, as long as $\mathcal{E}_N^{1/2}(U)$ is small so that

$$\begin{gathered}
(a_{11} - n_\infty a_{12}\kappa_1'(0)) > C\mathcal{E}_N^{1/2}(U), \\
(a_{21} - n_\infty a_{22}\kappa_2'(0)) > C\mathcal{E}_N^{1/2}(U)
\end{gathered}$$

are satisfied, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \Big(|u|^2 + \frac{p'(n_\infty)}{n_\infty^2}|\rho|^2 + |c_1|^2 + |c_2|^2\Big) \mathrm{d}x \\
&+ \nu \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla c_1|^2 \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla c_2|^2 \mathrm{d}x \\
&+ (a_{11} - n_\infty a_{12}\kappa_1'(0)) \int_{\mathbb{R}^3} |c_1|^2 \mathrm{d}x + (a_{21} - n_\infty a_{22}\kappa_2'(0)) \int_{\mathbb{R}^3} |c_2|^2 \mathrm{d}x \\
&\le C\|\rho\|_2 \int_{\mathbb{R}^3} |\nabla\rho|^2 \mathrm{d}x.
\end{aligned} \tag{3.10}$$

Now, we estimate the higher-order derivatives of (ρ, u, c_1, c_2). Take α with $1 \le |\alpha| \le N$. Applying ∂^α to the second equation of (2.2), multiplying by $\partial^\alpha u$, and integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} (\partial^\alpha u)_t^2 \, \mathrm{d}x + \nu \int_{\mathbb{R}^3} |\partial^\alpha u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \partial^\alpha u (\frac{p'(\rho + n_\infty)}{\rho + n_\infty}) \partial^\alpha \nabla\rho \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \partial^\alpha u \sum_{\beta=1}^{\alpha} C_\alpha^\beta \partial^\beta (\frac{p'(\rho + n_\infty)}{\rho + n_\infty}) \partial^{\alpha-\beta} \nabla\rho \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} \partial^\alpha u \sum_{\beta=0}^{\alpha} C_\alpha^\beta (\partial^{\alpha-\beta} u \cdot \nabla \partial^\beta u) \mathrm{d}x + \int_{\mathbb{R}^3} \partial^\alpha u \partial^\alpha (\chi_1(c_1)\nabla c_1) \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} \partial^\alpha u \partial^\alpha (\chi_2(c_2)\nabla c_2) \mathrm{d}x.
\end{aligned} \tag{3.11}$$

Using the first equation in (2.2) and the integration by parts, we rewrite the third term on the left-hand side as follows.

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \partial^\alpha u\ (\frac{p'(\rho+n_\infty)}{\rho+n_\infty})\partial^\alpha \nabla\rho \mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \frac{p'(\rho+n_\infty)}{(\rho+n_\infty)^2}(\partial^\alpha \rho^2)_t\ \mathrm{d}x + \int_{\mathbb{R}^3} \frac{p'(\rho+n_\infty)}{(\rho+n_\infty)}\partial^\alpha(\frac{1}{\rho+n_\infty})\rho_t \partial^\alpha \rho \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \frac{1}{\rho+n_\infty}\partial^\alpha(\nabla\rho\cdot u)\frac{p'(\rho+n_\infty)}{(\rho+n_\infty)}\partial^\alpha\rho \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \partial^\alpha(\frac{1}{\rho+n_\infty})(\nabla\rho\cdot u)\frac{p'(\rho+n_\infty)}{(\rho+n_\infty)}\partial^\alpha \rho\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} \partial^\alpha u \nabla(\frac{p'(\rho+n_\infty)}{\rho+n_\infty})\partial^\alpha\rho \mathrm{d}x.
\end{aligned}
\tag{3.12}
$$

Therefore, combining (3.11) and (3.12), we obtain

$$
\begin{aligned}
&\frac{1}{2}\|\partial^\alpha u\|^2 + C_1\|\partial^\alpha \rho\|^2 + \nu\int_0^t \|\partial^\alpha u\|^2 \mathrm{d}s \\
&\le C\|\partial^\alpha u_0\| + C\|\partial^\alpha \rho_0\| + C\|\rho\|_N \int_0^t (\|\partial^\alpha u\|^2 + \|\partial^\alpha\rho\|^2)\mathrm{d}s \\
&\quad + C\|u\|_N \int_0^t \|\partial^\alpha \rho\|^2 \mathrm{d}s + C\|u\|_N \int_0^t \|\partial^\alpha u\|^2\mathrm{d}s \\
&\quad + C\|c_1\|_N \int_0^t (\|\partial^\alpha u\|^2 + \|\partial^\alpha \nabla c_1\|^2)\mathrm{d}s \\
&\quad + C\|c_2\|_N \int_0^t (\|\partial^\alpha u\|^2 + \|\partial^\alpha \nabla c_2\|^2)\mathrm{d}s.
\end{aligned}
\tag{3.13}
$$

Similarly, we estimate the higher-order derivatives of c_1 and c_2 as follows:

$$
\begin{aligned}
&\frac{1}{2}\|\partial^\alpha c_1\|^2 + \int_0^t \|\nabla\partial^\alpha c_1\|^2\mathrm{d}s + (a_{11} - n_\infty a_{12}\kappa_1'(0))\int_0^t \|\partial^\alpha c_1\|^2\mathrm{d}s \\
&\le C\|\partial^\alpha c_{1,0}\| + C\|\rho\|_N \int_0^t \|\partial^\alpha c_1\|^2\mathrm{d}s + C\|c_1\|_N\int_0^t (\|\partial^\alpha c_1\|^2 + \|\partial^\alpha\rho\|^2)\mathrm{d}s,
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\|\partial^\alpha c_2\|^2 + \int_0^t \|\nabla\partial^\alpha c_2\|^2\mathrm{d}s + (a_{21} - n_\infty a_{22}\kappa_2'(0))\int_0^t \|\partial^\alpha c_2\|^2\mathrm{d}s \\
&\le C\|\partial^\alpha c_{2,0}\| + C\|\rho\|_N \int_0^t (\|\partial^\alpha c_2\|^2\mathrm{d}s \\
&\quad + C\|c_2\|_N\int_0^t (\|\partial^\alpha c_2\|^2 + \|\partial^\alpha\rho\|^2)\mathrm{d}s.
\end{aligned}
\tag{3.15}
$$

Then, summing (3.13)-(3.15) over $|\alpha| \le N$, we have

$$
\begin{aligned}
&\frac{1}{2}(\|u\|_N^2 + C_1\|\rho\|_N^2 + \|c_1\|_N^2 + \|c_2\|_N^2) + \nu\int_0^t \|u\|_N^2 \mathrm{d}s + \int_0^t \|\nabla c_1\|_N^2 \mathrm{d}s \\
&+ \int_0^t \|\nabla c_2\|_N^2 \mathrm{d}s + (a_{11} - n_\infty a_{12}\kappa_1'(0))\int_0^t \|c_1\|_N^2 \mathrm{d}s \\
&+ (a_{21} - n_\infty a_{22}\kappa_1'(0))\int_0^t \|c_2\|_N^2 \mathrm{d}s \\
&\le C_0\|U_0\|_N + C\|\rho\|_N\int_0^t (\|u\|_N^2 + \|\rho\|_N^2 + \|c_1\|_N^2 + \|c_2\|_N^2)\mathrm{d}s \\
&\quad + C\|u, c_1, c_2\|_N\int_0^t \|\rho\|_N^2 \mathrm{d}s + C\|u\|_N\int_0^t \|u\|_N^2 \mathrm{d}s \\
&\quad + C\|c_1\|_N\int_0^t (\|u\|_N^2 + \|c_1\|_N^2 + \|\nabla c_1\|_N^2)\mathrm{d}s \\
&+ C\|c_2\|_N\int_0^t (\|u\|_N^2 + \|c_2\|_N^2 + \|\nabla c_2\|_N^2)\mathrm{d}s.
\end{aligned}
\tag{3.16}
$$

For the estimates of ρ, let $|\alpha| \le N-1$. Applying ∂^α to $(2.2)_2$, multiplying it by $\partial^\alpha\nabla\rho$ and integrating in x yield

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha u_t \mathrm{d}x + \nu\int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha u \mathrm{d}x + \frac{p'(n_\infty)}{n_\infty}\int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha\nabla\rho \mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha(u\cdot\nabla u)\mathrm{d}x + \int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha(\chi_1(c_1)\nabla c_1)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha(\chi_2(c_2)\nabla c_2)\mathrm{d}x - \int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha(\frac{p'(\rho+n_\infty)}{\rho+n_\infty} - \frac{p'(n_\infty)}{n_\infty})\nabla\rho)\mathrm{d}x,
\end{aligned}
$$

which can be simplified further by (2.2)

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (\partial^\alpha\nabla\rho\partial^\alpha u)_t \mathrm{d}x + \frac{p'(n_\infty)}{n_\infty}\int_{\mathbb{R}^3} |\partial^\alpha\nabla\rho|^2 \mathrm{d}x \\
&= -\nu\int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha u \mathrm{d}x - \int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha(u\cdot\nabla u)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha(\chi_1(c_1)\nabla c_1)\mathrm{d}x - \int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha(\chi_2(c_2)\nabla c_2)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} \partial^\alpha\nabla\rho\partial^\alpha(\frac{p'(\rho+n_\infty)}{\rho+n_\infty} - \frac{p'(n_\infty)}{n_\infty})\nabla\rho)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} \partial^\alpha\nabla\cdot u\partial^\alpha\nabla\cdot((\rho+n_\infty)u)\mathrm{d}x.
\end{aligned}
$$

Finally, applying the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} (\partial^\alpha\nabla\rho\partial^\alpha u)\mathrm{d}x + \lambda_2\|\partial^\alpha\nabla\rho\|^2 \\
&\le C(\|\nabla\cdot\partial^\alpha u\|^2 + \|\partial^\alpha u\|^2) + C(\|(c_1,c_2)\|\|\partial^\alpha\nabla(\rho,c_1,c_2)\|^2) \\
&\quad + C(\|(\rho,u)\|_N\|\nabla\cdot\partial^\alpha(\rho,u)\|^2).
\end{aligned}
$$

Summing over $|\alpha| \le N-1$ and integrating with respect to t, we have

$$\begin{aligned} &\sum_{|\alpha|\le N-1}\int_{\mathbb{R}^3}\partial^\alpha\nabla\rho\partial^\alpha u\mathrm{d}x+\lambda_2\int_0^t\|\nabla\rho\|_{N-1}^2\mathrm{d}s\\ &\le \sum_{|\alpha|\le N-1}\int_{\mathbb{R}^3}\partial^\alpha\nabla\rho\partial^\alpha u\ \mathrm{d}x\big|_{t=0}+C\int_0^t\|u\|_N^2\mathrm{d}s\\ &\quad + C\|(c_1,c_2)\|_N\int_0^t\|\nabla(\rho,c_1,c_2)\|_{N-1}^2\mathrm{d}s\\ &\quad + C\|(\rho,u)]\|_N\int_0^t\|\nabla\cdot(\rho,u)]\|_{N-1}^2\mathrm{d}s. \end{aligned}\tag{3.17}$$

Taking linear combination (3.10) + (3.16) + k(3.17), we see that

$$\begin{aligned} &\|U\|_N^2+k\sum_{|\alpha|\le N-1}\int_{\mathbb{R}^3}\partial^\alpha\nabla\rho\partial^\alpha u\ \mathrm{d}x+\lambda_1\int_0^t\|\nabla(c_1,c_2)\|_N^2\mathrm{d}s\\ &+\lambda_2\int_0^t(\|\nabla\rho\|_{N-1}^2+\|(u,c_1,c_2)\|_N^2)\mathrm{d}s\le C_0\|U_0\|_N^2 \end{aligned}$$

for constant $0<k\ll 1$. Since the second term on the left hand side is absorbed in $\|U\|_N^2$, we obtain the energy estimate (3.6) in Lemma 3.2. □

Proof of Proposition 2.1. Choose a positive constant $\bar\epsilon=\min\{\epsilon_0,\epsilon_1\}$, where $\epsilon_0>0$ and $\epsilon_1>0$ are given in Lemma 3.1 and Lemma 3.2. Let $U_0\in H^N(\mathbb{R}^3)$ satisfy

$$\|U_0\|_N\le\frac{\bar\epsilon}{2\sqrt{C_0+1}}.$$

Now, let us define

$$T=\{t\ge 0:\sup_{0\le s\le t}\|U(s)\|_{H^N}\le\bar\epsilon\}.$$

Note that

$$\|U_0\|_N\le\frac{\bar\epsilon}{2\sqrt{C_0+1}}\le\frac{\bar\epsilon}{2}<\bar\epsilon\le\epsilon_0.$$

Then $T>0$ holds true from the local existence result. If T is finite, from definition of T, we have

$$\sup_{0\le s\le t}\|U\|=\bar\epsilon.\tag{3.18}$$

On the other hand, by Lemma 3.2 we have

$$\sup_{0\le s\le t}\|U(s)\|_N\le\sqrt{C_0}\|U_0\|_N\le\frac{\bar\epsilon\sqrt{C_0}}{2\sqrt{C_0+1}}\le\frac{\bar\epsilon}{2},$$

which is a contradiction to 3.18. Therefore, $T=\infty$ holds true. This implies that local solution $U(t)$ obtained in Lemma 3.1 can be extended to infinity in time. Thus, we have a global solution $(\rho,u,c_1,c_2)(t)\in C([0,\infty);H^N)$. This completes the proof. □

4. Linearized homogeneous system

In this section, we study the time-decay property of solutions to the nonlinear system (2.2). For this purpose, we separate the system into the linear and the nonlinear parts around the constant state $(n_\infty, 0, 0, 0)$ and we write the system as follows: Then $U = (\rho, u, c_1, c_2)$ satisfies

$$
\begin{aligned}
&\partial_t \rho + n_\infty \nabla \cdot u = g_1 \\
&\partial_t u + \nu u + \frac{p'(n_\infty)}{n_\infty} \nabla \rho = g_2 \\
&\partial_t c_1 - \Delta c_1 + (a_{11} - a_{12} n_\infty \kappa_1'(0)) c_1 = g_3 \\
&\partial_t c_2 - \Delta c_2 + (a_{21} - a_{22} n_\infty \kappa_2'(0)) c_2 = g_4,
\end{aligned}
\tag{4.1}
$$

where the nonlinear source terms are

$$
\begin{aligned}
&g_1 = -\nabla \cdot (\rho u) \\
&g_2 = -u \cdot \nabla u + \chi_1(c_1) \nabla c_1 - \chi_2(c_2) \nabla c_2 - \Big(\frac{p'(\rho + n_\infty)}{\rho + n_\infty} - \frac{p'(n_\infty)}{n_\infty}\Big) \nabla \rho \\
&g_3 = a_{12} \rho \kappa_1(c_1) + a_{12} n_\infty (\kappa_1(c_1) - \kappa_1'(0) c_1) \\
&g_4 = a_{21} \rho \kappa_2(c_2) + a_{22} n_\infty (\kappa_2(c_2) - \kappa_2'(0) c_2)
\end{aligned}
\tag{4.2}
$$

and the initial data are

$$
(\rho, u, c_1, c_2)\big|_{t=0} = U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0}). \tag{4.3}
$$

To obtain the time-decay rates of the solution to the system (4.1), we first study the Cauchy problem for the corresponding linearized homogeneous system

$$
\begin{aligned}
&\partial_t \rho + n_\infty \nabla \cdot u = 0 \\
&\partial_t u + \nu u + \frac{p'(n_\infty)}{n_\infty} \nabla \rho = 0 \\
&\partial_t c_1 - \Delta c_1 + (a_{12} - a_{11} n_\infty \kappa_1'(0)) c_1 = 0 \\
&\partial_t c_2 - \Delta c_2 + (a_{22} - a_{21} n_\infty \kappa_2'(0)) c_2 = 0.
\end{aligned}
\tag{4.4}
$$

For this purpose, we define $U_1 = (\rho, u)$ to be the solution of the linearized homogeneous equations

$$
\begin{aligned}
&\partial_t \rho + n_\infty \nabla \cdot u = 0 \\
&\partial_t u + \nu u + \frac{p'(n_\infty)}{n_\infty} \nabla \rho = 0,
\end{aligned}
\tag{4.5}
$$

with the initial data

$$
U_1\big|_{t=0} = U_{1,0} = (\rho_0, u_0), \quad x \in \mathbb{R}^3. \tag{4.6}
$$

4.1. **Representation of solutions for** (4.5) **and** (4.6).

We first find the explicit representation of the Fourier transform of the solution $U_1 = (\rho, u) = e^{tB} U_{1,0}$ to the system (4.5) with the initial data (4.6), where e^{tB} is the linear solution operator.

First, we take the time derivative for the first equation of (4.5) and using the second equation to replace $\partial_t u$, we have

$$
\partial_{tt} \rho + \nu \partial_t \rho - p'(n_\infty) \Delta \rho = 0. \tag{4.7}
$$

Initial data are given by

$$
\rho\big|_{t=0} = \rho_0, \quad \partial_t \rho\big|_{t=0} = -n_\infty \nabla \cdot u_0. \tag{4.8}
$$

Then, taking the Fourier transform of the above equation, we get the second-order ODE as

$$\begin{gathered}\partial_{tt}\hat{\rho}+\nu\partial_t\hat{\rho}+p'(n_\infty)|\xi|^2\hat{\rho}=0\\ \hat{\rho}\big|_{t=0}=\hat{\rho}_0,\ \partial_t\rho\big|_{t=0}=-n_\infty i\xi\cdot\hat{u}_0.\end{gathered}\tag{4.9}$$

It is straightforward to obtain

$$\hat{\rho}=\frac{\lambda_1e^{\lambda_2t}-\lambda_2e^{\lambda_1t}}{\lambda_1-\lambda_2}\hat{\rho}_0-in_\infty\xi\frac{e^{\lambda_1t}-e^{\lambda_2t}}{\lambda_1-\lambda_2}\hat{u}_0,\tag{4.10}$$

where $\lambda_{1,2}=-\frac{\nu}{2}\pm\frac{1}{2}\sqrt{\nu^2-4p'(n_\infty)|\xi|^2}$ are the roots of the characteristic equation $\lambda^2+\nu\lambda+p'(n_\infty)|\xi|^2=0$. Similarly, taking the time derivative for the second equation of (4.5) and using the first equation to replace $\partial_t\rho$, we have

$$\partial_{tt}u+\nu\partial_tu-p'(n_\infty)\nabla(\nabla\cdot u)=0.\tag{4.11}$$

Further taking the divergence, one has

$$\partial_{tt}(\nabla\cdot u)+\nu\partial_t(\nabla\cdot u)-p'(n_\infty)\Delta(\nabla\cdot u)=0,\tag{4.12}$$

$$\nabla\cdot u\big|_{t=0}=\nabla\cdot u_0,\tag{4.13}$$

$$\partial_t\nabla\cdot u\big|_{t=0}=-\nu\nabla\cdot u_0-\frac{p'(n_\infty)}{n_\infty}\Delta\rho_0.\tag{4.14}$$

Here and in the sequel we define $\tilde{\xi}=\xi/|\xi|$ for $|\xi|\neq0$. By taking the Fourier transform of (4.12), (4.13) and (4.14), we get the second-order ODE

$$\begin{gathered}\partial_{tt}(\tilde{\xi}\cdot\hat{u})+\nu\partial_t(\tilde{\xi}\cdot\hat{u})+p'(n_\infty)|\xi|^2(\tilde{\xi}\cdot\hat{u})=0\\ (\tilde{\xi}\cdot\hat{u})\big|_{t=0}=\tilde{\xi}\cdot\hat{u}_0,\\ \partial_t(\tilde{\xi}\cdot\hat{u})\big|_{t=0}=-\nu\tilde{\xi}\cdot\hat{u}_0-i\frac{p'(n_\infty)}{n_\infty}|\xi|\hat{\rho}_0.\end{gathered}\tag{4.15}$$

Therefore,

$$\tilde{\xi}\cdot\hat{u}=\frac{e^{\lambda_1t}-e^{\lambda_2t}}{\lambda_1-\lambda_2}\Big(-i\frac{p'(n_\infty)}{n_\infty}|\xi|\hat{\rho}_0\Big)+\frac{(\lambda_1+\nu)e^{\lambda_2t}-(\lambda_2+\nu)e^{\lambda_1t}}{\lambda_1-\lambda_2}\tilde{\xi}\cdot\hat{u}_0.\tag{4.16}$$

Now, by taking the curl for the second equation of (4.5), we have

$$\partial_t(\nabla\times u)+\nu(\nabla\times u)=0.$$

Taking the Fourier transform of the above equation, we have

$$\partial_t(\tilde{\xi}\times\hat{u})+\nu\tilde{\xi}\times\hat{u}=0.\tag{4.17}$$

Initial data is given as

$$(\tilde{\xi}\times\hat{u})\big|_{t=0}=\tilde{\xi}\times\hat{u}_0.\tag{4.18}$$

Solving the initial value problem (4.17) and (4.18), we have

$$(\tilde{\xi}\times\hat{u})=e^{-\nu t}\tilde{\xi}\times\hat{u}_0.\tag{4.19}$$

For $t\geq0$ and $\xi\in\mathbb{R}^3$ with $|\xi|\neq0$, one has the decomposition $\hat{u}=\tilde{\xi}\tilde{\xi}\cdot\hat{u}-\tilde{\xi}\times(\tilde{\xi}\times\hat{u})$. It is straightforward to obtain

$$\begin{aligned}\hat{u}=&\frac{e^{\lambda_1t}-e^{\lambda_2t}}{\lambda_1-\lambda_2}(-i\frac{p'(n_\infty)}{n_\infty}\xi\hat{\rho}_0)\\ &+\frac{(\lambda_1+\nu)e^{\lambda_2t}-(\lambda_2+\nu)e^{\lambda_1t}}{\lambda_1-\lambda_2}\tilde{\xi}\tilde{\xi}\cdot\hat{u}_0-e^{-\nu}\tilde{\xi}\times(\tilde{\xi}\times\hat{u}_0).\end{aligned}\tag{4.20}$$

Then

$$
\begin{aligned}
\hat{u} = & \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(-i\frac{p'(n_\infty)}{n_\infty}\xi\hat{\rho}_0) \\
& + \Big(\frac{(\lambda_1+\nu)e^{\lambda_2 t} - (\lambda_2+\nu)e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\Big)\frac{\xi\otimes\xi}{|\xi|^2}\hat{u}_0 + e^{-\nu t}(I_3 - \frac{\xi\otimes\xi}{|\xi|^2})\hat{u}_0.
\end{aligned}
\tag{4.21}
$$

Therefore, if we define the Fourier transform $\hat{G}(t,\xi)$ of Green's function $G(t,\xi) = e^{tB}$ to be

$$
\begin{aligned}
& \hat{G}(t,\xi) \\
& = \begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} \end{bmatrix} \\
& = \begin{bmatrix} \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} & (-in_\infty\xi)\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{(\lambda_1 - \lambda_2)} \\ \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(-i\frac{p'(n_\infty)}{n_\infty}\xi) & \frac{(\lambda_1+\nu)e^{\lambda_2 t} - (\lambda_2+\nu)e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\frac{\xi\otimes\xi}{|\xi|^2} + e^{-\nu t}(I_3 - \frac{\xi\otimes\xi}{|\xi|^2}) \end{bmatrix},
\end{aligned}
\tag{4.22}
$$

then (4.10) and (4.21) can be written as

$$
\begin{bmatrix} \hat{\rho}(t,\xi) \\ \hat{u}(t,\xi) \end{bmatrix} = \hat{G}(t,\xi)\begin{bmatrix} \hat{\rho}(0,\xi) \\ \hat{u}(0,\xi) \end{bmatrix}.
$$

4.2. **Refined $L^2 - L^q$ time-decay property.** In this subsection, we use (4.22) to obtain some refined $L^2 - L^q$ time-decay property for $U_1 = (\rho, u)$. To do so, we need to find the time-frequency pointwise estimate on $\hat{\rho}$ and $\hat{u}$ in the following Lemma:

Lemma 4.1. *Suppose $U_1 = (\rho, u)$ is the solution to the linear homogeneous system (4.5) with the initial data $U_1\big|_{t=0} = (\rho_0, u_0)$. Then, there are constants $\epsilon > 0$, $C > 0$, $\lambda > 0$ such that for all $t > 0, |\xi| \le \epsilon$,*

$$
|\hat{\rho}(t,\xi)| \le C(e^{-\lambda|\xi|^2 t} + |\xi|^2 e^{-\nu\lambda t})|\hat{\rho}_0(\xi)| + C(|\xi|e^{-\lambda|\xi|^2 t} + |\xi|e^{-\nu\lambda t})|\hat{u}_0(\xi)|, \tag{4.23}
$$

$$
|\hat{u}(t,\xi)| \le C|\xi|(e^{-\lambda|\xi|^2 t} + e^{-\nu\lambda t})|\hat{\rho}_0(\xi)| + C(|\xi|^2 e^{-\lambda|\xi|^2 t} + e^{-\nu\lambda t})|\hat{u}_0(\xi)|, \tag{4.24}
$$

and for all $t > 0, |\xi| \ge \epsilon$,

$$
|\hat{\rho}(t,\xi)| \le Ce^{-\lambda t}|\hat{\rho}_0(\xi), \hat{u}_0(\xi)|, \tag{4.25}
$$

$$
|\hat{u}(t,\xi)| \le Ce^{-\lambda t}|\hat{\rho}_0(\xi), \hat{u}_0(\xi)|. \tag{4.26}
$$

Proof. To obtain the upper bound of $\hat{\rho}(t,\xi)$ and $\hat{u}(t,\xi)$, we estimate $\hat{G}_{11}, \hat{G}_{12}, \hat{G}_{21}$, and $\hat{G}_{22}$ in (4.22). If $\nu^2 - 4p'(n_\infty)|\xi|^2 \ge 0$, then $\lambda_{1,2} = -\frac{\nu}{2} \pm \frac{1}{2}\sqrt{\nu^2 - 4p'(n_\infty)|\xi|^2}$ are real. It is straightforward to obtain

$$
\begin{gathered}
\lambda_1 \sim -O(1)|\xi|^2, \\
\lambda_2 \sim -\nu + O(1)|\xi|^2,
\end{gathered}
$$

as $|\xi| \to 0$.

On other hand, if $\nu^2 - 4p'(n_\infty)|\xi|^2 \le 0$, then $\lambda_{1,2} = -\frac{\nu}{2} \pm \frac{\nu}{2}i\sqrt{\frac{4p'(n_\infty)}{\nu^2}|\xi|^2 - 1}$ are complex conjugates. Moreover, we have

$$
\begin{gathered}
|\lambda_{1,2}| \sim O(1)|\xi|, \\
\lambda_1 - \lambda_2 \sim iO(1)|\xi|.
\end{gathered}
$$

as $|\xi| \to \infty$. Then, there exists $\epsilon \le \sqrt{\frac{\nu^2}{4p'(n_\infty)}} \le R$, with $0 < \epsilon \ll 1 \ll R < \infty$ such that one can estimate $\hat{G}(t,\xi)$ as follows:

$$\begin{gathered}
|\hat{G}_{11}| \le C(e^{-\lambda|\xi|^2 t} + |\xi|^2 e^{-\nu\lambda t}) \\
|\hat{G}_{12}| \le C|\xi|(e^{-\lambda|\xi|^2 t} + e^{-\nu\lambda t}) \\
|\hat{G}_{21}| \le C|\xi|(e^{-\lambda|\xi|^2 t} + e^{-\nu\lambda t}) \\
|\hat{G}_{22}| \le Ce^{-\nu t} + C(|\xi|^2 e^{-\lambda|\xi|^2 t} + e^{-\nu\lambda t}) \le C|\xi|^2 e^{-\lambda|\xi|^2 t} + Ce^{-\nu\lambda t},
\end{gathered}$$

as $|\xi| \le \epsilon$, and

$$\begin{gathered}
|\hat{G}_{11}| \le Ce^{-\frac{\nu}{2}t} \le Ce^{-\lambda t} \\
|\hat{G}_{12}| + |\hat{G}_{21}| \le Ce^{-\frac{\nu}{2}t} \le Ce^{-\lambda t} \\
|\hat{G}_{22}| \le Ce^{-\nu} + Ce^{-\frac{\nu}{2}t} \le Ce^{-\lambda t},
\end{gathered}$$

as $|\xi| > R$. When the eigenvalues coalesce, since the real part is negative, we have $te^{-\frac{\nu}{2}t}$ in the solution, but this decays exponentially. Then we have $te^{-\frac{\nu}{2}t} \le e^{-\lambda t}$ and over $\epsilon \le |\xi| \le R$ we have

$$\begin{gathered}
|\hat{G}_{11}| \le C(e^{-\lambda|\xi|^2 t} + |\xi|^2 e^{-\nu\lambda t}) \\
|\hat{G}_{12}| \le C|\xi|(e^{-\lambda|\xi|^2 t} + e^{-\nu\lambda t}) \\
|\hat{G}_{21}| \le C|\xi|(e^{-\lambda|\xi|^2 t} + e^{-\nu\lambda t}) \\
|\hat{G}_{22}| \le C|\xi|^2 e^{-\lambda|\xi|^2 t} + Ce^{-\nu\lambda t}.
\end{gathered}$$

Now, we can estimate $\hat{\rho}$, $\hat{u}$ as follows

$$\begin{aligned}
|\hat{\rho}(t,\xi)| &= |\hat{G}_{11}\hat{\rho}_0 + \hat{G}_{12}\hat{u}_0| \\
&\le |\hat{G}_{11}||\hat{\rho}_0| + |\hat{G}_{12}||\hat{u}_0| \\
&\le C(e^{-\lambda|\xi|^2 t} + |\xi|^2 e^{-\nu\lambda t})|\hat{\rho}_0(\xi)| + C(|\xi|e^{-\lambda|\xi|^2 t} + |\xi|e^{-\nu\lambda t})|\hat{u}_0(\xi)|,
\end{aligned}$$

and

$$\begin{aligned}
|\hat{u}(t,\xi)| &= |\hat{G}_{21}\hat{\rho}_0 + \hat{G}_{22}\hat{u}_0| \\
&\le |\hat{G}_{21}||\hat{\rho}_0| + |\hat{G}_{22}||\hat{u}_0| \\
&\le C|\xi|(e^{-\lambda|\xi|^2 t} + e^{-\nu\lambda t})|\hat{\rho}_0(\xi)| + C(|\xi|^2 e^{-\lambda|\xi|^2 t} + e^{-\nu\lambda t})|\hat{u}_0(\xi)|,
\end{aligned}$$

for $|\xi| \le \epsilon$. Finally, (4.25) and (4.26) can be proved in the same way as for (4.23) and (4.24). This completes the proof of Lemma (4.1). □

Theorem 4.2. *Let $2 \le q \le \infty$, and let $m \ge 0$ be an integer. Assume $U_1 = e^{Bt}U_{1,0}$ is the solution to the Cauchy problem (4.5)-(4.6). Then for any $t \ge 0$, $U_1 = (\rho, u)$ satisfies:*

$$\begin{aligned}
\|\nabla^m \rho(t)\|_{L^q} &\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{q})-\frac{m}{2}}\|\rho_0, u_0\|_{L^1} \\
&\quad + e^{-\lambda t}\|\nabla^{m+[3(\frac{1}{2}-\frac{1}{q})]_+}(\rho_0, u_0)\|_{L^2},
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
\|\nabla^m u(t)\|_{L^q} &\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{q})-\frac{m+1}{2}}\|\rho_0, u_0\|_{L^1} \\
&\quad + e^{-\lambda t}\|\nabla^{m+[3(\frac{1}{2}-\frac{1}{q})]_+}(\rho_0, u_0)\|_{L^2},
\end{aligned} \tag{4.28}$$

where $C = C(m,q)$ and $[3(\frac{1}{2}-\frac{1}{q})]_+$ are defined by

$$[3(\frac{1}{2}-\frac{1}{q})]_+ = \begin{cases} 0 & \text{if } q = 2 \\ [3(\frac{1}{2}-\frac{1}{q})]_- + 1 & \text{otherwise.} \end{cases}$$

Here, $[\cdot]_-$ denotes the integer part of the argument.

Proof. Take $2 \le q \le \infty$ and let $m \ge 0$ be an integer. Set $U_1 = e^{Bt}U_{1,0}$. Using the Hausdorff-Young inequality and (4.23) we prove (4.27) as follows,

$$\begin{aligned} \|\nabla^m \rho(t)\|_{L^q(\mathbb{R}^3_x)} &\le C\||\xi|^m \hat{\rho}(\xi,t)\|_{L^{q'}(\mathbb{R}^3_\xi)} \\ &\le C\||\xi|^m \hat{\rho}(\xi,t)\|_{L^{q'}(|\xi|\le\epsilon)} + C\||\xi|^m \hat{\rho}(\xi,t)\|_{L^{q'}(|\xi|\ge\epsilon)}, \end{aligned} \tag{4.29}$$

where $\frac{1}{q}+\frac{1}{q'} = 1$. We estimate the first term on the right-hand side of (4.29) as

$$\begin{aligned} &\||\xi|^m \hat{\rho}(t,\xi)\|^{q'}_{L^{q'}(|\xi|\le\epsilon)} \\ &\le C\int_{|\xi|\le\epsilon} |\xi|^{mq'} e^{-q'\lambda|\xi|^2 t} + |\xi|^{(m+2)q'} e^{-q'\nu\lambda t})|\hat{\rho}_0(\xi)|^{q'} \\ &\quad + C(|\xi|^{mq'+q'} e^{-q'\lambda|\xi|^2 t} + |\xi|^{mq'+q'} e^{-q'\nu\lambda t})|\hat{u}_0(\xi)|^{q'} d\xi \\ &\le C\sup_\xi |\hat{\rho}_0|^{q'} \int_{|\xi|\le\epsilon} |\xi|^{mq'} e^{-q'\lambda|\xi|^2(1+t)+q'\lambda|\xi|^2} + |\xi|^{mq'+2q'} e^{-q'\nu\lambda t})d\xi \\ &\quad + C\sup_\xi |\hat{u}_0|^{q'} \int_{|\xi|\le\epsilon} (|\xi|^{mq'+q'} e^{-q'\lambda|\xi|^2(1+t)+q'\lambda|\xi|^2} + |\xi|^{mq'+q'} e^{-q'\nu\lambda t})d\xi \\ &\le C(1+t)^{-\frac{mq'+3}{2}}\|\rho_0\|^{q'}_{L^1} + C(1+t)^{-\frac{mq'+q'+3}{2}}\|u_0\|^{q'}_{L^1} + Ce^{-q'\nu\lambda t}\|(\rho_0,u_0)\|^{q'}_{L^1}. \end{aligned}$$

Thus,

$$\begin{aligned} \||\xi|^m \hat{\rho}(t,\xi)\|_{L^{q'}(|\xi|\le\epsilon)} &\le C(1+t)^{-\frac{3}{2q'}-\frac{m}{2}}\|\rho_0\|_{L^1} \\ &\quad + C(1+t)^{-\frac{3}{2q'}-(\frac{m+1}{2})}\|u_0\|_{L^1} + Ce^{-\nu\lambda t}\|(\rho_0,u_0)\|_{L^1} \\ &\le C(1+t)^{-\frac{3}{2}[1-\frac{1}{q}]-\frac{m}{2}}\|(\rho_0,u_0)\|_{L^1}. \end{aligned} \tag{4.30}$$

Now, we estimate the second term of (4.29) using the Hölder inequality with sufficiently small $\epsilon > 0$. Then we obtain

$$\begin{aligned} \||\xi|^m \hat{\rho}(t,\xi)\|_{L^{q'}(|\xi|\ge\epsilon)} &\le C\int_{|\xi|\ge\epsilon} |\xi|^{mq'} e^{-q'\lambda t}|(\hat{\rho}_0(\xi),\hat{u}_0(\xi))|^{q'} d\xi \\ &\le Ce^{-\lambda t}\||\xi|^{-(3+\epsilon)}\|^{\frac{2-q'}{2q'}}\||\xi|^{(3+\epsilon)\frac{2-q'}{2q'}+m}(\hat{\rho}_0(\xi),\hat{u}_0(\xi))\|_{L^2} \\ &\le Ce^{-\lambda t}\|\nabla^{m+3[\frac{1}{2}-\frac{1}{q}]_-}(\rho_0,u_0)\|_{L^2}. \end{aligned} \tag{4.31}$$

Substituting (4.30) and (4.31) to (4.29), we have (4.27).

To prove (4.28), it similarly holds that

$$\begin{aligned} \|\nabla^m u(t)\|_{L^q(\mathbb{R}^3_x)} &\le C\||\xi|^m \hat{u}(t,\xi)\|_{L^{q'}(\mathbb{R}^3_\xi)} \\ &\le C\||\xi|^m \hat{u}(t,\xi)\|_{L^{q'}(|\xi|\le\epsilon)} + C\||\xi|^m \hat{u}(t,\xi)\|_{L^{q'}(|\xi|\ge\epsilon)}. \end{aligned} \tag{4.32}$$

Using (4.24), we estimate the first term of (4.32) as

$$\||\xi|^m \hat{u}(t,\xi)\|^{q'}_{L^{q'}(|\xi|\le\epsilon)}$$

$$
\begin{aligned}
&\le C\int_{|\xi|\le\epsilon}(|\xi|^{mq'+q'}(e^{-q'\lambda|\xi|^2(t+1)}+e^{-q'\nu\lambda t})|\hat\rho_0(\xi)|^{q'})d\xi\\
&\quad + C\int_{\xi\le\epsilon}(|\xi|^{(m+2)q'}e^{-q'\lambda|\xi|^2(t+1)}+e^{-q'\nu\lambda t})|\hat u_0(\xi)|^{q'}d\xi\\
&\le C(1+t)^{-\frac{mq'+q'+3}{2}}\|\rho_0\|_{L^1}^{q'}+(1+t)^{-\frac{mq'+2q'+3}{2}}\|u_0\|_{L^1}^{q'}\\
&\quad + Ce^{-q'\nu\lambda t}\|(\rho_0,u_0)\|_{L^1}^{q'}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\||\xi|^m\hat u(t,\xi)\|_{L^{q'}(|\xi|\le\epsilon)}\\
&\le C(1+t)^{-\frac{3}{2q'}-\frac{m+1}{2}}\|[\rho_0\|_{L^1}+(1+t)^{-\frac{3}{2q'}-\frac{m+2}{2}}\|u_0\|_{L^1}\\
&\quad + Ce^{-\nu\lambda t}\|(\rho_0,u_0)\|_{L^1} \qquad (4.33)\\
&\le C(1+t)^{-\frac32[1-\frac1q]-\frac{m+1}{2}}\|\rho_0\|_{L^1}+(1+t)^{-\frac32[1-\frac1q]-\frac{m+2}{2}}\|u_0\|_{L^1}\\
&\le C(1+t)^{-\frac32[1-\frac1q]-\frac{m+1}{2}}\|(\rho_0,u_0)\|_{L^1}.
\end{aligned}
$$

Similarly to (4.31), one has

$$
\||\xi|^m\hat u(t,\xi)\|_{L^{q'}(|\xi|\ge\epsilon)}\le Ce^{-\lambda t}\|\nabla^{m+3[\frac12-\frac1q]_-}(\rho_0,u_0)\|_{L^2}. \tag{4.34}
$$

Thus, substituting (4.34) and (4.33) in (4.32), we obtain (4.28). This completes the proof. □

Corollary 4.3. *Assume that $U_1=e^{Bt}U_{1,0}$ is the solution to the Cauchy problem (4.6) with initial data $U_{1,0}=(\rho_0,u_0)$. Then, $U_1=(\rho,u)$ satisfies the following:*

$$
\|\rho(t)\|_{L^2}\le C(1+t)^{-3/4}\|(\rho_0,u_0)\|_{L^1}+e^{-\lambda t}\|(\rho_0,u_0)\|_{L^2}, \tag{4.35}
$$
$$
\|u(t)\|_{L^2}\le C(1+t)^{-\frac54}\|(\rho_0,u_0)\|_{L^1}+e^{-\lambda t}\|(\rho_0,u_0)\|_{L^2}, \tag{4.36}
$$
$$
\|\rho(t)\|_{L^\infty}\le C(1+t)^{-\frac32}\|(\rho_0,u_0)\|_{L^1}+e^{-\lambda t}\|\nabla^2(\rho_0,u_0)\|_{L^2}, \tag{4.37}
$$
$$
\|u(t)\|_{L^\infty}\le C(1+t)^{-2}\|(\rho_0,u_0)\|_{L^1}+e^{-\lambda t}\|\nabla^2(\rho_0,u_0)\|_{L^2}. \tag{4.38}
$$

5. Time-decay rates for the nonlinear system

In this section, we prove Propositions 2.2 and 2.3. The main idea is to combine the energy estimates and the spectral analysis. We will apply the linear L^2-L^q time-decay property of the homogeneous system (4.5) studied in the previous section to the nonlinear case. We need the mild form of the original nonlinear Cauchy problem (2.2). Throughout this section, we suppose that $U=(\rho,u,c_1,c_2)$ is the solution to the Cauchy problem (2.2) with initial data $U_0=(\rho_0,u_0,c_{1,0},c_{2,0})$ satisfying (2.3). Then, by Duhamel's principle, the solution $U=(\rho,u,c_1,c_2)$ can be formally written as

$$
U(t)=e^{Bt}U_0+\int_0^t e^{(t-s)B}(g_1,g_2,g_3,g_4)ds, \tag{5.1}
$$

where e^{Bt}, $t\ge 0$, is called the linear solution operator and the nonlinear source term takes the form (4.2).

5.1. **Decay rates for the energy functional and high-order energy functional.** In this subsection, we prove Proposition 2.2, *i.e.*, the decay rates for the energy functional $\|U(t)\|_N^2$ and for the high-order energy functional $\|\nabla U(t)\|_N^2$. For that, we investigate the time-decay rates of solutions in Proposition 2.1 under the extra condition (2.8).

Proof of Proposition 2.2. Suppose $\epsilon_{N+1}(U_0)$ is sufficiently small. Then, from Proposition 2.1 the solution $U = (\rho, u, c_1, c_2)$ satisfies

$$\frac{d}{dt}\mathcal{E}_N(U(t)) + \lambda_1 \mathcal{D}_N(U(t)) + \lambda_2 \mathcal{D}_N^h(U(t)) \le 0 \tag{5.2}$$

for any $t \ge 0$.

Now, we begin with the time-weighted estimates and iteration for inequality (5.2). Let $l \ge 0$. Multiplying (5.2) by $(1+t)^l$ and integrating over $[0,t]$ give

$$\begin{aligned}
&(1+t)^l \mathcal{E}_N(U(t)) + \lambda_1 \int_0^t (1+s)^l \mathcal{D}_N(U(s))ds + \lambda_2 \int_0^t (1+s)^l \mathcal{D}_N^h(U(s))ds \\
&\le \mathcal{E}_N(U_0) + l \int_0^t (1+s)^{l-1} \mathcal{E}_N U(s) ds \\
&\le \mathcal{E}_N(U_0) + Cl \int_0^t (1+s)^{l-1} (\mathcal{D}_{N-1}(U(s)) + \mathcal{D}_N^h(U(s)) + \|\rho(s)\|^2) ds,
\end{aligned}$$

where we have used

$$\mathcal{E}_N(U(t)) \le C\mathcal{D}_{N-1}(U(t)) + C\mathcal{D}_N^h(U(t)) + \|\rho(t)\|^2.$$

Using (5.2) again, we have

$$\mathcal{E}_{N+1}(U(t)) + \lambda_1 \int_0^t \mathcal{D}_{N+1}(U(t)) + \lambda_2 \int_0^t \mathcal{D}_{N+1}^h(U(t)) \le \mathcal{E}_{N+1}(U_0),$$

and

$$\begin{aligned}
&(1+t)^{l-1} \mathcal{E}_{N+1}(U(t)) + \lambda_1 \int_0^t (1+s)^{l-1} \mathcal{D}_{N+1}(U(s))ds \\
&+ \lambda_2 \int_0^t (1+s)^{l-1} \mathcal{D}_{N+1}^h(U(s))ds \\
&\le \mathcal{E}_{N+1}(U_0) + C(l-1) \int_0^t (1+s)^{l-2} \mathcal{E}_{N+1}(U(s))ds \\
&\le \mathcal{E}_{N+1}(U_0) + C(l-1) \int_0^t (1+s)^{l-2} (\mathcal{D}_N(U(s)) \\
&\quad + C\mathcal{D}_{N+1}^h(U(s)) + \|\rho(s)\|^2) ds.
\end{aligned}$$

Then, for $1 < l < 2$, iterating the previous estimates, we obtain

$$\begin{aligned}
&(1+t)^l \mathcal{E}_N(U(t)) + \lambda_1 \int_0^t (1+s)^l \mathcal{D}_N(U(s))ds + \lambda_2 \int_0^t (1+s)^l \mathcal{D}_N^h(U(s))ds \\
&\le \mathcal{E}_{N+1}(U_0) + C \int_0^t (1+s)^{l-1} \|\rho(s)\|^2 ds.
\end{aligned} \tag{5.3}$$

On the other hand, to estimate the integral term on the right-hand side of the previous inequality, let us define

$$\mathcal{E}_{N,\infty}(U(t)) = \sup_{0 \le s \le T} (1+t)^{\frac{3}{2}} \mathcal{E}_N(U(t)).$$

Then, applying the linear estimate on ρ in (4.35) to the mild form (5.1), one has

$$\begin{aligned}\|\rho(t)\| \le C(1+t)^{-3/4}\|\rho_0,u_0\|_{L^1} + Ce^{-\lambda t}\|(\rho_0,u_0)\| \\ + C\int_0^t (1+t-s)^{-3/4}\|(g_1(s),g_2(s))\|_{L^1}ds \\ + C\int_0^t e^{-\lambda(t-s)}\|(g_1(s),g_2(s))\|ds.\end{aligned} \tag{5.4}$$

Recall the definitions (4.2) of g_1 and g_2. It is easy to check that for any $0 \le s \le t$,

$$\|(g_1(s),g_2(s))\|_{L^1\cap L^2} \le C\mathcal{E}_N(U(t)) \le C(1+s)^{-3/2}\mathcal{E}_{N,\infty}(U(t)).$$

Substituting this into (5.4) gives

$$\|\rho(t)\| \le C(1+t)^{-3/4}(\mathcal{E}^2_{N,\infty}(U(t)) + \|(\rho_0,u_0)\|^2_{L^1\cap L^2}). \tag{5.5}$$

Next, we prove the uniform-in-time bound of $\mathcal{E}_{N,\infty}(U(t))$ which implies the decay rates of the energy functional $\mathcal{E}_N(U(t))$. In fact, taking $l = \frac{3}{2} + \epsilon$ in (5.3), where $\epsilon > 0$ is small enough, we see that

$$\begin{aligned}&(1+t)^{\frac{3}{2}+\epsilon}\mathcal{E}_N(U(t)) + \lambda_1\int_0^t (1+s)^{\frac{3}{2}+\epsilon}\mathcal{D}_N(U(s))ds + \lambda_2\int_0^t (1+s)^{\frac{3}{2}+\epsilon}\mathcal{D}^h_N(U(s))ds \\ &\le \mathcal{E}_{N+1}(U_0) + C\int_0^t (1+s)^{\frac{1}{2}+\epsilon}\|\rho(s)\|^2 ds.\end{aligned}$$

For ρ, using (5.5), we obtain

$$\int_0^t (1+s)^{\frac{1}{2}+\epsilon}\|\rho(t)\|^2 ds \le C(1+t)^{\epsilon}(\mathcal{E}^2_{N,\infty}(U(t)) + \|(\rho_0,u_0)\|^2_{L^1\cap L^2}).$$

Therefore,

$$\begin{aligned}&(1+t)^{\frac{3}{2}+\epsilon}\mathcal{E}_N(U(t)) + \lambda_1\int_0^t (1+s)^{\frac{3}{2}+\epsilon}\mathcal{D}_N(U(s))ds + \lambda_2\int_0^t (1+s)^{\frac{3}{2}+\epsilon}\mathcal{D}^h_N(U(s))ds \\ &\le \mathcal{E}_{N+1}(U_0) + C(1+t)^{\epsilon}(\mathcal{E}^2_{N,\infty}(U(t)) + \|(\rho_0,u_0)\|^2_{L^1\cap L^2}),\end{aligned}$$

which implies

$$(1+t)^{\frac{3}{2}}\mathcal{E}_N(U(t)) \le C(\mathcal{E}_{N+1}(U_0) + \|(\rho_0,u_0)\|^2_{L^1} + \mathcal{E}^2_{N,\infty}(U(t))),$$

and thus

$$\mathcal{E}_{N,\infty}(U(t)) \le C(\epsilon^2_{N+1}(U_0) + \mathcal{E}^2_{N,\infty}(U(t))).$$

Since $\epsilon^2_{N+1}(U_0) > 0$ is sufficiently small, it holds that $\mathcal{E}_{N,\infty}(U(t)) \le C\epsilon^2_{N+1}(U_0)$ for any $t \ge 0$, which gives $\|U(s)\|_N \le C(\mathcal{E}_N(U(t)))^{1/2} \le C\epsilon_{N+1}(U_0)(1+t)^{-3/4}$. This proves (2.9).

Now, we estimate the high-order energy functional. By comparing the definitions of $\mathcal{E}_N(U(t))$, $\mathcal{D}_N(U(t))$ and $\mathcal{D}^h_N(U(t))$, it follows from (5.2) that

$$\frac{d}{dt}\|\nabla U(t)\|^2_N + \lambda\|\nabla U(t)\|^2_N \le C\|\nabla\rho(t)\|^2,$$

which implies

$$\|\nabla U(t)\|^2_N \le e^{-\lambda t}\|\nabla U_0\|^2_N + C\int_0^t e^{-\lambda(t-s)}\|\nabla\rho(s)\|^2 ds, \tag{5.6}$$

for any $t \geq 0$. To estimate the time integral term on the right hand side of the above inequality, one can apply the linear estimate (4.27) to the mild form (5.1) of the solution $U(t)$ so that

$$\begin{aligned}\|\nabla\rho(t)\| \leq\, & C(1+t)^{-5/4}\|(\rho_0,u_0)\|_{L^1} + Ce^{-\lambda t}\|\nabla(\rho_0,u_0)\| \\ & + C\int_0^t (1+t-s)^{-5/4}\|(g_1(s),g_2(s))\|_{L^1}ds \\ & + C\int_0^t e^{-\lambda(t-s)}\|\nabla(g_1(s),g_2(s))\|ds.\end{aligned} \tag{5.7}$$

Recall the definition (4.2) of g_1 and g_2. It is straightforward to check that for any $0 \leq s \leq t$

$$\|(g_1(s),g_2(s))\|_{L^1\cap\dot{H}^1} \leq C\mathcal{E}_N(U(s)) \leq C\epsilon_{N+1}^2(U_0)(1+s)^{-3/2}.$$

Putting this into (5.7) gives

$$\|\nabla\rho(t)\| \leq C\epsilon_{N+1}(U_0)(1+t)^{-5/4}. \tag{5.8}$$

Then, by using (5.8) in (5.6), we have

$$\|\nabla U(t)\|_N^2 \leq e^{-\lambda t}\|\nabla U_0\|_N^2 + C\epsilon_{N+1}^2(U_0)(1+t)^{-5/2},$$

which implies (2.10). The proof of Proposition 2.2 is complete. □

5.2. **Decay rates in L^q.** In this subsection, we prove Proposition 2.3 for time-decay rates in L^q corresponding to (1.6)-(1.8) in Theorem 1.1. For $N \geq 4$, Proposition 2.2 shows that if $\epsilon_{N+1}(U_0)$ is small enough,

$$\|U(s)\|_N \leq C\epsilon_{N+1}(U_0)(1+t)^{-3/4}, \tag{5.9}$$

$$\|\nabla U(t)\|_N \leq C\epsilon_{N+1}(U_0)(1+t)^{-5/4}. \tag{5.10}$$

First, we consider the L^q estimates on ρ. We use the $L^2 - L^\infty$ interpolation inequality. For the L^2 rate, it is easy to see from (5.5) and (5.9) that

$$\|\rho(t)\| \leq C\epsilon_{N+1}(U_0)(1+t)^{-3/4} \leq C(1+t)^{-3/4}.$$

For the L^∞ rate, applying the L^∞ linear estimate on ρ in (4.37) to the mild form (5.1), we have

$$\begin{aligned}\|\rho(t)\|_\infty \leq\, & C(1+t)^{-3/2}\|(\rho_0,u_0)\|_{L^1} + Ce^{-\lambda t}\|\nabla^2(\rho_0,u_0)\| \\ & + C\int_0^t (1+t-s)^{-3/2}\|(g_1(s),g_2(s))\|_{L^1}ds \\ & + C\int_0^t e^{-\lambda(t-s)}\|\nabla^2(g_1(s),g_2(s))\|ds \\ \leq\, & C(1+t)^{-3/2}\|(\rho_0,u_0)\|_{L^1\cap\dot{H}^2} \\ & + C\int_0^t (1+t-s)^{-3/2}\|(g_1(s),g_2(s))\|_{L^1\cap\dot{H}^2}ds.\end{aligned} \tag{5.11}$$

Since by (5.9),

$$\|(g_1(s),g_2(s))\|_{L^1\cap\dot{H}^2} \leq C\|\nabla U(t)\|_N\|U(s)\|_N \leq C\epsilon_{N+1}^2(U_0)(1+s)^{-2},$$

substituting the above inequality into (5.11), we obtain

$$\|\rho(t)\|_{L^\infty} \leq C\epsilon_{N+1}(U_0)(1+t)^{-3/2}.$$

Then, by the $L^2 - L^\infty$ interpolation, we have

$$\|\rho\|_{L^q} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-3}{2}+\frac{3}{2q}} \tag{5.12}$$

for $2 \le q \le \infty$.

Next, we show the estimates on $\|u(t)\|_{L^q}$. For the L^2 rate, utilizing the L^2 estimate on u in (4.36) to (5.1), we have

$$\begin{aligned} \|u(t)\| \le\, & C(1+t)^{-5/4}\|(\rho_0,u_0)\|_{L^1} + Ce^{-\lambda t}\|(\rho_0,u_0)\| \\ & + C\int_0^t (1+t-s)^{-5/4}\|(g_1,g_2)\|_{L^1} ds \\ & + \int_0^t e^{-\lambda(t-s)}\|(g_1(s),g_2(s))\| ds. \end{aligned} \tag{5.13}$$

Since (5.9) implies

$$\|(g_1(s),g_2(s))\|_{L^1\cap L^2} \le C\|U(s)\|_N^2 \le C\epsilon_{N+1}(U_0)(1+t)^{-3/2},$$

(5.13) yields a slower decay estimate

$$\|u(t)\| \le C\epsilon_{N+1}(U_0)(1+t)^{-5/4} \le C(1+t)^{-5/4}. \tag{5.14}$$

For the L^∞ rate, applying the L^∞ estimate on u in (4.38) to (5.1), we have

$$\begin{aligned} \|u(t)\|_{L^\infty} \le\, & (1+t)^{-2}\|\rho_0,u_0\|_{L^1\cap\dot H^2} \\ & + C\int_0^t (1+t-s)^{-2}\|(g_1(s),g_2(s))\|_{L^1\cap\dot H^2} ds. \end{aligned} \tag{5.15}$$

By the estimates (5.9) and (5.10), we obtain

$$\|(g_1(s),g_2(s))\|_{L^1\cap\dot H^2} \le C\|\nabla U(t)\|_N\|U(s)\|_N \le C\epsilon_{N+1}^2(U_0)(1+s)^{-2},$$

Therefore, from (5.15) it follows that

$$\|u(t)\|_{L^\infty} \le C\epsilon_{N+1}(U_0)(1+s)^{-2},$$

and consequently, the L^2-L^∞ interpolation theorem implies

$$\|u(t)\|_{L^q} \le C\epsilon_{N+1}(U_0)(1+t)^{-2+\frac{3}{2q}} \tag{5.16}$$

for $2 \le q \le \infty$.

Lastly, we estimate the time-decay rates of (c_1, c_2). We start with the estimate on $\|c_1(t)\|_{L^q}$. For the L^2 rate,

$$
\begin{aligned}
\|c_1\|_{L^2} &\le C\|\hat{c}_1\|_{L^2(\xi)}\\
&\le C\Big[\int_\xi e^{-2(|\xi|^2+(a_{11}-a_{12}n_\infty\kappa_1'(0)))t}|\hat{c}_0|^2d\xi\Big]^{1/2}\\
&\quad + a_{11}\int_0^t\Big[\int_\xi e^{-2(|\xi|^2+(a_{11}-a_{12}n_\infty\kappa_1'(0))(t-s)}|\hat{g}_3|^2d\xi\Big]^{1/2}ds\\
&\le e^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))t}\Big[\int_\xi e^{-2|\xi|^2(t)}|\hat{c}_0|^2d\xi\Big]^{1/2}\\
&\quad + C\int_0^t e^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))(t-s)}\Big[\int_\xi e^{-2|\xi|^2(t-s+1)}|\hat{g}_3|^2d\xi\Big]^{1/2}ds\\
&\le Ce^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))t}\|\hat{c}_0\|_{L^2}\\
&\quad + C\int_0^t e^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))(t-s)}(\sup_\xi e^{-|\xi|^2(t-s+1)})\|g_3\|_{L^2}ds
\end{aligned}
\tag{5.17}
$$

By (5.12), we see that

$$
\|g_3\|_{L^2} \le (\|\rho\kappa_1(c_1)\|_{L^2} + \|c_1\|_{L^\infty}\|c_1\|_{L^2}^2) \le C\|U(s)\|_N^2 \le C\epsilon_{N+1}^2(U_0)(1+t)^{-3/2}.
$$

This and (5.17) imply the decay estimate for c_1

$$
\|c_1\|_{L^2} \le C\epsilon_{N+1}(U_0)(1+t)^{-3/2}. \tag{5.18}
$$

For L^∞ rates, we estimate the low frequency and high frequency separately. From the Hausdorff-Young inequality and Hölder inequality, we have

$$
\begin{aligned}
&\|c_1\|_{L^\infty}\\
&\le C\|\hat{c}_1\|_{L^1}\\
&\le C\int_{\xi\le\epsilon} e^{-(|\xi|^2+(a_{11}-a_{12}n_\infty\kappa_1'(0)))t}|\hat{c}_{1,0}|d\xi\\
&\quad + C\int_0^t\int_{\xi\le\epsilon} e^{-(|\xi|^2+(a_{11}-a_{12}n_\infty\kappa_1'(0)))(t-s)}|\hat{g}_3|d\xi ds\\
&\quad + C\int_{|\xi|\ge\epsilon} e^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))t}|\hat{c}_{1,0}|d\xi\\
&\quad + C\int_0^t\int_{|\xi|\ge\epsilon} e^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))(t-s)}|\hat{g_3}|d\xi ds\\
&\le Ce^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))t}(1+t)^{-3/2}\|c_0\|_{L^1} + C\int_0^t e^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))(t-s)}\|\hat{g_3}(s)\|_{L^1}\\
&\quad + Ce^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))t}\Big[\int_{|\xi|\ge\epsilon}|\xi|^{-4}d\xi\Big]^{1/2}\Big[\int_{|\xi|\ge\epsilon}|\xi|^4|\hat{c}_{1,0}|^2d\xi\Big]^{1/2}\\
&\quad + C\int_0^t e^{-(a_{11}-a_{12}n_\infty\kappa_1'(0)))(t-s)}\Big[\int_{|\xi|\ge\epsilon}|\xi|^{-4}d\xi\Big]^{1/2}\Big[\int_{|\xi|\ge\epsilon}|\xi|^4|\hat{g}_3|^2d\xi\Big]^{1/2}ds\\
&\le Ce^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))t}(1+t)^{-3/2}\|c_0\|_{L^1}\\
&\quad + C\int_0^t e^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))(t-s)}\|g_3(s)\|_{L^1}ds
\end{aligned}
$$

$$
\begin{aligned}
&+ Ce^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))t}\|\nabla^2 c_0\|_{L^2} \\
&+ C\int_0^t e^{-(a_{11}-a_{12}n_\infty\kappa_1'(0))(t-s)}\|\nabla^2 g_3(s)\|_{L^2}ds. \qquad (5.19)
\end{aligned}
$$

By (5.12), we see that

$$\|g_3(s)\|_{L^1\cap\dot{H}^2} \le C\|U(s)\|_N^2 \le C\epsilon_{N+1}^2(U_0)(1+t)^{-3/2}.$$

Then (5.19) implies the decay estimate

$$\|c_1\|_{L^\infty} \le C\epsilon_{N+1}(U_0)(1+t)^{-3/2}. \qquad (5.20)$$

The similar estimates hold for c_2. Therefore, by the L^2-L^∞ interpolation, we obtain

$$\|(c_1,c_2)\|_{L^q} \le C\epsilon_{N+1}(U_0)(1+t)^{-3/2} \qquad (5.21)$$

for $2 \le q \le \infty$. This completes the proof of Proposition 2.2 and hence of Theorem 1.1.

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