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UNIQUENESS OF ENTROPY SOLUTIONS TO NONLINEAR ELLIPTIC-PARABOLIC PROBLEMS

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Abstract. We study the Cauchy problem associated with the nonlinear elliptic - parabolic equation

$$b(u)_t - a(u, \varphi(u)_x)_x = f.$$

We prove an L^1-contraction principle and hence the uniqueness of entropy solutions, under rather general assumptions on the data.

1. Introduction

We consider the Cauchy problem

$$\begin{gathered} b(u)_t - a(u, \varphi(u)_x)_x = f \quad \text{in } \mathbb{Q} =]0, T[\times\mathbb{R} \\ b(u(0, .)) = v_0 \quad \text{on } \mathbb{R}, \end{gathered} \tag{1.1}$$

where $f \in L^1(\mathbb{Q})$, $T > 0$, $v_0 \in L^1(\mathbb{R})$, $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $a(k, .)$ is nondecreasing, φ and $b : \mathbb{R} \to \mathbb{R}$ are continuous, nondecreasing and b is surjective.

Whenever u is such that $b(u)$ is constant, (1.1) degenerates into an elliptic problem of the following form, with t as a parameter,

$$\begin{gathered} -a(u, \varphi(u)_x)_x = f \quad \text{in } \mathbb{Q} =]0, T[\times\mathbb{R} \\ b(u(0, .)) = v_0 \quad \text{on } \mathbb{R}. \end{gathered} \tag{1.2}$$

If the function b is one to one, on each part where u is such that $\varphi(u)$ is constant, (1.1) degenerates to a scalar conservation law of the form

$$\begin{gathered} v_t - \widetilde{a}(v, 0)_x = f \quad \text{in } \mathbb{Q} =]0, T[\times\mathbb{R} \\ v(0, .) = v_0 \quad \text{on } \mathbb{R}, \end{gathered} \tag{1.3}$$

with $v = b(u)$, $\widetilde{a}(k, \xi) = a(b^{-1}(k), \xi)$.

It is then clear that we include in (1.1), some first order hyperbolic problems, for which (even under assumptions of regularity on data) there is no hope of getting classical global solutions. It is well known that, for such equations, the above problems are ill-posed in the sense that there is no uniqueness. It is necessary to

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introduce Kruzhkov solutions in order to obtain existence and uniqueness results (see [15]).

Since b and φ are not increasing, the above formulations include Stefan problems, filtration problems, etc. in the one dimensional case. In the case where $b = id$, (1.1) was studied by Bénilan and Touré [7] in a bounded domain of $\mathbb{R}$, and by Bénilan and Wittbold [8], Carrillo and Wittbold [13] in a bounded domain of $\mathbb{R}^N$. Related problems were studied in [2, 3, 9, 10, 14, 16]. See also [4, 6, 11, 12] and the corresponding references for semigroup approach.

In this paper, using nonlinear semigroups theory in L^1, we established existence and uniqueness of mild solutions for the "evolution problem" (1.1) from the properties of the associated "stationary problem" in sense of Benilan:

$$b(u) - a(u, \varphi(u)_x)_x = f \quad \text{on } \mathbb{R}. \tag{1.4}$$

Under additional assumption on the data, we proved that mild solutions are entropy solutions. Furthermore, we established L^1-contraction principle for the entropy solution from which uniqueness arises.

2. Preliminaries

In what follows, a, b and φ are given functions such that

$$\begin{gathered} a : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ \varphi : \mathbb{R} \to \mathbb{R},\ b : \mathbb{R} \to \mathbb{R} \text{ are continuous,} \\ a(k, \xi) \text{ is nondecreasing in } \xi, \\ b(k),\ \varphi(k) \text{ are nondecreasing and } b \text{ is surjective.} \end{gathered} \tag{2.1}$$

Define

$$H(k) = a(k, 0) \quad \text{for } k \in \mathbb{R}, \quad h = a(u, \varphi(u)_x);$$

$$\operatorname{sign}_0(r) = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ -1 & \text{if } r < 0 \end{cases} \quad \operatorname{sign}_0^+(r) = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{otherwise;} \end{cases}$$

$$\operatorname{sign}^+(r) = \begin{cases} 1 & \text{if } r > 0 \\ [0,1] & \text{if } r = 0 \\ 0 & \text{if } r < 0 \end{cases} \quad \operatorname{sign}(r) = \begin{cases} 1 & \text{if } r > 0 \\ [-1,1] & \text{if } r = 0 \\ -1 & \text{if } r < 0; \end{cases}$$

$$H_\epsilon(r) = \min\Big(\frac{r^+}{\epsilon}, 1\Big); \quad H_0(r) = \operatorname{sign}_0^+(r).$$

Our main assumption is the coerciveness of a with respect to ξ, for k bounded; more precisely:

(H1) $\lim_{|\xi| \to \infty} \inf_{|k| < R} |a(k, \xi)| = +\infty$ for all $R > 0$.

We now define an $L^1(\mathbb{R})$ operator A_b associated with the evolution problem (1.1) by $A_b b(u) = -a(u, \varphi(u)_x)_x$ and it satisfies:

$$\begin{gathered} v \in A_b \Leftrightarrow b(u) \in L^1(\mathbb{R}), v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \\ u \text{ is an entropy solution of (1.4) with } f = v + b(u). \end{gathered}$$

Lemma 2.1. *Suppose that conditions* (2.1) *and (H1) are satisfied. Then the operator A_b defined above satisfies the following:*

(1) *A_b is T-accretive in $L^1(\mathbb{R})$; i.e.,*
$$\|(x-\widetilde{x})^+\|_{L^1} \le \|(x-\widetilde{x}+\lambda(A_b x - A_b\widetilde{x}))^+\|_{L^1}$$
for all $\lambda \ge 0$ and $x, \widetilde{x} \in D(A_b)$.

(2) *For each $\lambda > 0$, the range $R(I+\lambda A_b)$ of $I+\lambda A_b$ is dense in $L^1(\mathbb{R})$.*

(3) *The domain $D(A_b)$ of A_b is dense in $L^1(\mathbb{R})$.*

We now recall the definition of weak and entropy solution of (1.1).

Definition 2.2. Let $f \in L^2(0,T;H^{-1}(\mathbb{R}))$ and $v_0 \in L^1(\mathbb{R})$. A weak solution of problem (1.1) is a function u such that $\varphi(u) \in L^2(0,T;H^1(\mathbb{R}))$, $b(u) \in L^1(\mathbb{Q})$ and which also satisfies the following:

(i) $b(u)_t \in L^2(0,T;H^{-1}(\mathbb{R}))$, $h = a(u,\varphi(u)_x) \in L^1(\mathbb{Q}) \cap L^2(\mathbb{Q})$,

(ii) $b(u)_t - h_x = f$ in $D'(\mathbb{Q})$ and $b(u(0,.)) = v_0$.

The condition above should be understood in the sense
$$\int_0^T \langle b(u)_t, \xi\rangle dt = -\int_{\mathbb{Q}} b(u)\xi_t dxdt - \int_{\mathbb{R}} v_0\xi(0)dx$$
for any $\xi \in L^2(0,T;D(\mathbb{R})) \cap W^{1,1}(0,T;L^\infty(\mathbb{R}))$, such that $\xi(T) = 0$, where $\langle , \rangle$ represent the duality pairing between $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$.

Definition 2.3. Let $f \in L^2(0,T;H^{-1}(\mathbb{R})) \cap L^1(\mathbb{Q})$ and $v_0 \in L^1(\mathbb{R})$. An entropy solution of problem (1.1) is a weak solution u which satisfies the following:
$$\begin{aligned}&\int_{\mathbb{Q}} H_0(u-k)\{\xi_x(h-H(k)) - (b(u)-b(k))\xi_t - f\xi\}\,dx\,dt\\ &- \int_{\mathbb{R}} (b(u_0)-b(k))^+\xi(0)dx \le 0\end{aligned} \tag{2.2}$$
$$\begin{aligned}&\int_{\mathbb{Q}} H_0(k-u)\{\xi_x(h-H(k)) - (b(u)-b(k))\xi_t - f\xi\}\,dx\,dt\\ &+ \int_{\mathbb{R}} (b(u_0)-b(k))^-\xi(0)dx \ge 0\end{aligned} \tag{2.3}$$
for all $\xi \in D^+([0,T]\times\mathbb{R})$, $k \in \mathbb{R}$, and $\xi(T) = 0$.

To obtain regularity results of the mild solution which will enable us link up mild and entropy solutions, we also assume that

(H2) $v_0 \in \widehat{D}(A_b) \cap L^\infty(\mathbb{R})$, $f \in BV(0,T;L^1(\mathbb{R}))$ and
$$\int_0^T \|f(t,.)\|_{L^\infty(\mathbb{R})}dt < +\infty,$$
where $\widehat{D}(A_b)$ is the generalized domain of operator A_b defined by
$$\begin{aligned}\widehat{D}(A_b) = \big\{&v_0 \in L^1(\mathbb{R}) : \text{ there exist } v_n = b(u_n) \in D(A_b) \text{ such that}\\ &v_n \to v_0 \text{ in } L^1(\mathbb{R}) \text{ and } A_b v_n \text{ is bounded in } L^1(\mathbb{R})\big\}.\end{aligned}$$

Remark 2.4. (i) By [7, Proposition 1.4] (see also , [20, Proposition 6]), if $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $f \in L^1(\mathbb{Q})$ and u is the mild solution of (1.1), then $u \in L^\infty(\mathbb{Q})$ and
$$\|b(u)\|_{L^\infty(\mathbb{Q})} \le \|v_0\|_{L^\infty(\mathbb{R})} + \int_0^T \|f(t,.)\|_{L^\infty(\mathbb{R})}dt.$$

(ii) If $f \in BV(0,T;L^1(\mathbb{R}))$ and $v_0 \in \widehat{D}(A_b)$, then by nonlinear semigroup theory (see [4, 7]), the mild solution u of (1.1) satisfies $b(u) \in Lip([0,T];L^1(\mathbb{R}))$.

For the next lemma, we use following assumption (see [20])

(H3) $b+\varphi$ is a one-to-one function.

Lemma 2.5. *Under assumptions (H1)–(H3), Problem* (1.1) *has at least an entropy solution.*

3. Uniqueness

Our interest in this section is to study the uniqueness of the entropy solution to the evolution problem (1.1). We make the following additional assumptions

(H4) For all r, s, ξ, $\eta \in \mathbb{R}$,

$$\begin{aligned}&(a(r,\xi)-a(s,\eta)).(\xi-\eta)+M(r,s)(1+|\xi|^2+|\eta|^2)|\varphi(r)-\varphi(s)|\\&\geq \Gamma(\varphi(r),\varphi(s)).\xi+\widehat{\Gamma}(\varphi(r),\varphi(s)).\eta,\end{aligned}$$

where $M:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$, Γ, $\widehat{\Gamma}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

Remark 3.1. (i) Assumption (H4) implies $\Gamma(\varphi(r),\varphi(r))=\widehat{\Gamma}(\varphi(r),\varphi(r))=0$ for all $r\in\mathbb{R}$. Indeed, choosing $r=s$, $\eta=0$, $\xi=t\nu$, $t>0$, $\nu\in\mathbb{R}$ in (H4), we get $t\nu[a(r,t\nu)-a(r,0)]\geq\Gamma(\varphi(r),\varphi(r))t\nu$. Dividing by t and taking limit as $t\to 0$, we get $\Gamma(\varphi(r),\varphi(r))\nu\leq 0$ for all $\nu\in\mathbb{R}$; hence $\Gamma(\varphi(r),\varphi(r))=0$. Using the same argument we obtain the corresponding result for $\widehat{\Gamma}$.

(ii) Assumption (H4) implies that a is monotone with respect to the second variable (for the proof see [13, Remark 2.2]).

For the proof of uniqueness, we use a method developed by Carrillo (see [10]) and Carrillo-Wittbold (see [13]) for parabolic degenerated problems. We start by showing that entropy solutions satisfy Kato's inequality (cf [1]); more precisely we show that entropy solutions satisfy the following inequality.

Theorem 3.2 (Kato's Inequality)**.** *For all $i=1,2$, $f_i\in L^2(0,T;H^{-1}(\mathbb{R}))\cap L^1(\mathbb{Q})$, $v_{0_i}=b(u_{0_i})\in L^1(\mathbb{R})$, $u_{0_i}\in L^\infty(\mathbb{R})$ and u_i an entropy solution of* (1.1) *with respect to data (f_i,v_{0_i}), we have*

$$\begin{aligned}&\int_{\mathbb{Q}}H_0(u_1-u_2)(h_1-h_2)\xi_x\,dx\,dt-\int_{\mathbb{Q}}(b(u_1)-b(u_2))^+\xi_t\,dx\,dt\\&-\int_{\mathbb{R}}(v_{0_1}-v_{0_2})^+\xi(0)dx\\&\leq\int_{\mathbb{Q}}H_0(u_1-u_2)(f_1-f_2)\xi\,dx\,dt,\end{aligned}\tag{3.1}$$

for all $\xi\in D^+([0,T)\times\mathbb{R})$.

For the proof of Theorem 3.2, we need to prove the following lemma.

Lemma 3.3. *If u is a weak solution of* (1.1)*, then we have:*

$$\begin{aligned}
&\int_{\mathbb{Q}} H_0(u-k)\left\{(h-H(k))\xi_x - f\xi\right\}\,dx\,dt - \int_{\mathbb{Q}} (b(u)-b(k))^+\xi_t\,dx\,dt \\
&- \int_{\mathbb{R}} (v_0-b(k))^+\xi(0)dx \\
&= -\lim_{\epsilon\to 0}\int_{\mathbb{Q}} (h-H(k))H_\epsilon(\varphi(u)-\varphi(k))_x\xi\,dx\,dt
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\int_{\mathbb{Q}} H_0(k-u)\left\{(h-H(k))\xi_x - f\xi\right\}\,dx\,dt + \int_{\mathbb{Q}} (b(k)-b(u))^+\xi_t\,dx\,dt \\
&+ \int_{\mathbb{R}} (b(k)-v_0)^+\xi(0)dx \\
&= -\lim_{\epsilon\to 0}\int_{\mathbb{Q}} (h-H(k))H_\epsilon(\varphi(k)-\varphi(u))_x\xi\,dx\,dt
\end{aligned} \tag{3.3}$$

for all $\xi \in D^+([0,T)\times\mathbb{R})$ and all $k \in \mathbb{R}$ such that $\varphi(k)\notin E$, where

$$E = \{r \in \operatorname{Im}(\varphi)/(\varphi^{-1})_0 \text{ is discontinuous into } r\}.$$

Proof. We observe that for all k such that $\varphi(k)\notin E$, we have

$$H_0(u-k) = H_0(\varphi(u)-\varphi(k)) \quad \text{on } \mathbb{Q}.$$

Since $\varphi(u)\in L^2(\mathbb{Q})$ and $\varphi(u(t))\in H^1(\mathbb{R})$ then

$$H_\epsilon(\varphi(u)-\varphi(k))\xi \in L^2(0,T;H^1(\mathbb{R})).$$

Now, put

$$\psi_\epsilon(z) = H_\epsilon(z-\varphi(s)), \quad B_{\psi_\epsilon}(z) = \int_0^z H_\epsilon(\varphi\circ((b^{-1})_0(r)) - \varphi(k))dr.$$

Since ψ_ϵ is bounded, we have

$$B_{\psi_\epsilon}(v_0)\in L^1(\mathbb{R}), \quad B_{\psi_\epsilon}(b(u))\in L^\infty(0,T;L^1(\mathbb{R})),$$

$$\int_{\mathbb{Q}} B_{\psi_\epsilon}(b(u))\xi_t\,dx\,dt + \int_{\mathbb{R}} B_{\psi_\epsilon}(v_0)\xi(0)dx = -\int_0^T \langle b(u)_t, H_\epsilon(\varphi(u)-\varphi(k))\xi\rangle dt.$$

Moreover, since u is weak solution and $H_\epsilon(\varphi(u)-\varphi(k))\xi \in L^2(0,T;H^1(\mathbb{R}))$, it follows that

$$\begin{aligned}
&-\int_0^T \langle b(u)_t, H_\epsilon(\varphi(u)-\varphi(k))\xi\rangle dt \\
&= \int_{\mathbb{Q}} \{(h-H(k))[H_\epsilon(\varphi(u)-\varphi(k))\xi]_x - fH_\epsilon(\varphi(u)-\varphi(k))\xi\}\,dx\,dt.
\end{aligned}$$

This equality gives

$$\begin{aligned}
&\int_{\mathbb{Q}} B_{\psi_\epsilon}(b(u))\xi_t\,dx\,dt + \int_{\mathbb{R}} B_{\psi_\epsilon}(v_0)\xi(0)dx \\
&= \int_{\mathbb{Q}} \left\{(h-H(k))\left[H_\epsilon(\varphi(u)-\varphi(k))\xi\right]_x - fH_\epsilon(\varphi(u)-\varphi(k))\xi\right\}\,dx\,dt.
\end{aligned} \tag{3.4}$$

To obtain (3.2), it is sufficient to show that

$$\begin{aligned} &\lim_{\epsilon\to 0}\Big(\int_{\mathbb{Q}} B_{\psi_\epsilon}(b(u))\xi_t\,dx\,dt + \int_{\mathbb{R}} B_{\psi_\epsilon}(v_0)\xi(0)dx\Big) \\ &= \int_{\mathbb{Q}} (b(u)-b(k))^+\xi_t\,dx\,dt + \int_{\mathbb{R}} (v_0-b(k))^+\xi(0)dx, \end{aligned} \tag{3.5}$$

for all $k\in\mathbb{R}$ such that $\varphi(k)\notin E$, where $B_{\psi_\epsilon}(b(u))$ is defined by

$$B_{\psi_\epsilon}(b(u)) = \int_0^{b(u)} H_\epsilon(\varphi\circ((b^{-1})_0(r)) - \varphi(k))dr.$$

Step 1. For $k\geq 0$.

$$B_{\psi_\epsilon}(b(u)) = \int_{b(k)}^{b(u)} H_\epsilon(\varphi\circ((b^{-1})_0(r)) - \varphi(k))dr \to (b(u)-b(k))^+$$

as $\epsilon\to 0$. Since b is continuous, and $\varphi(k)\notin E$, we have that $\varphi\circ((b^{-1})_0(r))-\varphi(k)>0$, for all $r>b(k)$ and then

$$H_\epsilon(\varphi\circ((b^{-1})_0(r)) - \varphi(k))dr \to 1$$

as $\epsilon\to 0$ for all $r>b(k)$. Thus, in a similar way, we obtain

$$\lim_{\epsilon\to 0} B_{\psi_\epsilon}(v_0) = (v_0-b(k))^+.$$

It is clear that $|B_{\psi_\epsilon}(b(u))|\leq |b(u)|$ and $|B_{\psi_\epsilon}(v_0)|\leq |v_0|$, which implies

$$\begin{aligned} &\lim_{\epsilon\to 0}\Big(\int_{\mathbb{Q}} B_{\psi_\epsilon}(b(u))\xi_t\,dx\,dt + \int_{\mathbb{R}} B_{\psi_\epsilon}(v_0)\xi(0)dx\Big) \\ &= \int_{\mathbb{Q}} (b(u)-b(k))^+\xi_t\,dx\,dt + \int_{\mathbb{R}} (v_0-b(k))^+\xi(0)dx. \end{aligned}$$

Step 2. For $k\leq 0$.

$$B_{\psi_\epsilon}(b(u)) = \int_0^{b(k)} H_\epsilon(\varphi\circ((b^{-1})_0(r)) - \varphi(k))dr + \int_{b(k)}^{b(u)} H_\epsilon(\varphi\circ((b^{-1})_0(r)) - \varphi(k))dr.$$

Therefore,

$$\lim_{\epsilon\to 0} B_{\psi_\epsilon}(b(u)) = (b(u)-b(k))^+ + b(k).$$

In a similar way,

$$\lim_{\epsilon\to 0} B_{\psi_\epsilon}(v_0) = (v_0-b(k))^+ + b(k).$$

Consequently,

$$\begin{aligned} &\lim_{\epsilon\to 0}\Big(\int_{\mathbb{Q}} B_{\psi_\epsilon}(b(u))\xi_t\,dx\,dt + \int_{\mathbb{R}} B_{\psi_\epsilon}(v_0)\xi(0)dx\Big) \\ &= \int_{\mathbb{Q}} (b(u)-b(k))^+\xi_t\,dx\,dt + \int_{\mathbb{Q}} b(k)\xi_t\,dx\,dt \\ &\quad + \int_{\mathbb{R}} (v_0-b(k))^+\xi(0)dx + \int_{\mathbb{R}} b(k)\xi(0)dx. \end{aligned}$$

Since

$$\int_{\mathbb{Q}} b(k)\xi_t\,dx\,dt + \int_{\mathbb{R}} b(k)\xi(0)dx = \int_{\mathbb{R}} b(k)\Big(\int_0^T \xi_t dt\Big)dx + \int_{\mathbb{R}} b(k)\xi(0)dx = 0,$$

it follows that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\Big(\int_{\mathbb{Q}} B_{\psi_\epsilon}(b(u))\xi_t\,dx\,dt+\int_{\mathbb{R}} B_{\psi_\epsilon}(v_0)\xi(0)dx\Big)\\
&=\int_{\mathbb{Q}}(b(u)-b(k))^+\xi_t\,dx\,dt+\int_{\mathbb{R}}(v_0-b(k))^+\xi(0)dx.
\end{aligned}$$

Hence, (3.5) is established. Again, taking limit as $\epsilon\to 0$ in (3.4) and using (3.5), we obtain

$$\begin{aligned}
&\int_{\mathbb{Q}}(b(u)-b(k))^+\xi_t\,dx\,dt+\int_{\mathbb{R}}(v_0-b(k))^+\xi(0)dx\\
&=\int_{\mathbb{Q}} H_0(u-k)\big[(h-H(k))\xi_x-f\xi\big]\,dx\,dt\\
&\quad+\lim_{\epsilon\to 0}\int_{\mathbb{Q}}(h-H(k))H_\epsilon(\varphi(u)-\varphi(k))_x\xi\,dx\,dt;
\end{aligned}$$

from which we deduce (3.2). The inequality (3.3) is obtained in a similar way. □

Proof of theorem 3.2. We use the method of doubling variables, which was introduced by Kruzkhov [15] for scalar conservation laws. Let (s,y) and (t,x) be two pairs of variables in $\mathbb{Q}$. We set $u_1=u_1(s,y)$, $f_1=f_1(s,y)$, $v_{0_1}=v_{0_1}(y)$ and $u_2=u_2(t,x)$, $f_2=f_2(t,x)$, $v_{0_2}=v_{0_2}(x)$. Let ξ be a positive test function of $D(\mathbb{Q}\times\mathbb{Q})$, then for all $(t,x)\in\mathbb{Q},(s,y)\in\mathbb{Q}$:

$$\begin{aligned}
&(s,y)\mapsto\xi(t,x,s,y)\in D^+([0,T)\times\mathbb{R})\quad\forall(t,x)\in\mathbb{Q},\\
&(t,x)\mapsto\xi(t,x,s,y)\in D^+([0,T)\times\mathbb{R})\quad\forall(s,y)\in\mathbb{Q}.
\end{aligned}\tag{3.6}$$

Let

$$\begin{aligned}
\mathbb{Q}_1&=\{(s,y)\in\mathbb{Q}/\varphi(u_1(s,y))\in E\},\\
\mathbb{Q}_2&=\{(t,x)\in\mathbb{Q}/\varphi(u_2(t,x))\in E\}.
\end{aligned}$$

We deduce that

$$\begin{aligned}
\varphi(u_1)_y&=0\quad\text{on }\mathbb{Q}_1,\\
\varphi(u_2)_x&=0\quad\text{on }\mathbb{Q}_2.
\end{aligned}\tag{3.7}$$

Then

$$H_0(u_1-u_2)=H_0(\varphi(u_1)-\varphi(u_2))\quad\text{in }\big[(\mathbb{Q}\setminus\mathbb{Q}_1)\times\mathbb{Q}\big]\cup\big[\mathbb{Q}\times(\mathbb{Q}\setminus\mathbb{Q}_2)\big].\tag{3.8}$$

Replace u by u_1 and k by u_2 in (3.2) and integrate over $\mathbb{Q}\setminus\mathbb{Q}_2$. Also replace u by u_1 and k by u_2 in (2.2) and integrate over $\mathbb{Q}_2$. Then adding the two inequalities, we obtain

$$\begin{aligned}
&\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)\left\{(h_1-a(u_2,0))\xi_y-(b(u_1)-b(u_2))\xi_s-f_1\xi\right\}\,dy\,ds\,dx\,dt\\
&-\int_{\mathbb{Q}\times\mathbb{R}}(v_{0_1}-b(u_2))^+\xi(0)dy\,dx\,dt\\
&\le-\lim_{\epsilon\to 0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times\mathbb{Q}}(h_1-a(u_2,0))H_\epsilon(\varphi(u_1)-\varphi(u_2))_y\xi\,dy\,ds\,dx\,dt.
\end{aligned}\tag{3.9}$$

In the same way, we replace k by u_1 and u by u_2 in (3.3) and integrate over $\mathbb{Q}\setminus\mathbb{Q}_1$. Furthermore, replace k by u_1 and u by u_2 in (2.3) and integrate over $\mathbb{Q}_1$. Again, adding the two inequalities gives

$$\begin{aligned}
&\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)\left\{(h_2-a(u_1,0))\xi_x-(b(u_2)-b(u_1))\xi_t-f_2\xi\right\}\,dy\,ds\,dx\,dt\\
&+\int_{\mathbb{R}\times\mathbb{Q}}(b(u_1)-v_{0_2})^+\xi(0)\,dy\,ds\,dx\\
&\geq -\lim_{\epsilon\to 0}\int_{\mathbb{Q}\times(\mathbb{Q}\setminus\mathbb{Q}_1)}(h_2-a(u_1,0))H_\epsilon(\varphi(u_1)-\varphi(u_2))_x\xi\,dy\,ds\,dx\,dt.
\end{aligned}\tag{3.10}$$

From (3.9), we deduce that

$$\begin{aligned}
&\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)\left\{h_1(\xi_y+\xi_x)-(b(u_1)-b(u_2))\xi_s-f_1\xi\right\}\,dy\,ds\,dx\,dt\\
&-\int_{\mathbb{Q}\times\mathbb{R}}(v_{0_1}-b(u_2))^+\xi(0)dy\,dx\,dt\\
&\leq \int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)a(u_2,0)\xi_y\,dy\,ds\,dx\,dt+\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)h_1\xi_x\,dy\,ds\,dx\,dt\\
&\quad -\lim_{\epsilon\to 0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times\mathbb{Q}}(h_1-a(u_2,0))H_\epsilon(\varphi(u_1)-\varphi(u_2))_y\xi\,dy\,ds\,dx\,dt.
\end{aligned}\tag{3.11}$$

¿From (3.10), we deduce that

$$\begin{aligned}
&\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)\left\{h_2(\xi_x+\xi_y)-(b(u_2)-b(u_1))\xi_t-f_2\xi\right\}\,dy\,ds\,dx\,dt\\
&+\int_{\mathbb{R}\times\mathbb{Q}}(b(u_1)-v_{0_2})^+\xi(0)\,dy\,ds\,dx\\
&\geq \int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)a(u_1,0)\xi_x\,dy\,ds\,dx\,dt+\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)h_2\xi_y\,dy\,ds\,dx\,dt\\
&\quad -\lim_{\epsilon\to 0}\int_{\mathbb{Q}\times(\mathbb{Q}\setminus\mathbb{Q}_1)}(h_2-a(u_1,0))H_\epsilon(\varphi(u_1)-\varphi(u_2))_x\xi\,dy\,ds\,dx\,dt.
\end{aligned}\tag{3.12}$$

Subtracting (3.12) from (3.11) gives

$$
\begin{aligned}
&\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)\big\{(h_1-h_2)(\xi_x+\xi_y)+(b(u_2)-b(u_1))(\xi_s+\xi_t)\\
&+(f_2-f_1)\xi\big\}\,dy\,ds\,dx\,dt\\
&-\int_{\mathbb{Q}\times\mathbb{R}}(v_{0_1}-b(u_2))^+\xi(0)dy\,dx\,dt-\int_{\mathbb{R}\times\mathbb{Q}}(b(u_1)-v_{0_2})^+\xi(0)\,dy\,ds\,dx\\
&\le\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)[h_1-a(u_1,0)]\xi_x\,dy\,ds\,dx\,dt\\
&\quad-\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)[h_2-a(u_2,0)]\xi_y\,dy\,ds\,dx\,dt\\
&\quad-\lim_{\epsilon\to0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times\mathbb{Q}}[h_1-a(u_2,0)]H_\epsilon(\varphi(u_1)-\varphi(u_2))_y\xi\,dy\,ds\,dx\,dt\\
&\quad+\lim_{\epsilon\to0}\int_{\mathbb{Q}\times(\mathbb{Q}\setminus\mathbb{Q}_1)}[h_2-a(u_1,0)]H_\epsilon(\varphi(u_1)-\varphi(u_2))_x\xi\,dy\,ds\,dx\,dt.
\end{aligned}
\tag{3.13}
$$

Using (3.7), we obtain

$$
\begin{aligned}
&\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)[h_1-a(u_1,0)]\xi_x\,dy\,ds\,dx\,dt\\
&=\int_{\mathbb{Q}\times(\mathbb{Q}\setminus\mathbb{Q}_1)} H_0(u_1-u_2)[h_1-a(u_1,0)]\xi_x\,dy\,ds\,dx\,dt\\
&=\lim_{\epsilon\to0}\int_{\mathbb{Q}\times(\mathbb{Q}\setminus\mathbb{Q}_1)}[h_1-a(u_1,0)]H_\epsilon(\varphi(u_1)-\varphi(u_2))\xi_x\,dy\,ds\,dx\,dt\\
&=-\lim_{\epsilon\to0}\int_{\mathbb{Q}\times(\mathbb{Q}\setminus\mathbb{Q}_1)}[h_1-a(u_1,0)]H_\epsilon(\varphi(u_1)-\varphi(u_2))_x\xi\,dy\,ds\,dx\,dt
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)[h_2-a(u_2,0)]\xi_y\,dy\,ds\,dx\,dt\\
&=\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times\mathbb{Q}} H_0(u_1-u_2)[h_2-a(u_2,0)]\xi_y\,dy\,ds\,dx\,dt\\
&=\lim_{\epsilon\to0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times\mathbb{Q}}[h_2-a(u_2,0)]H_\epsilon(\varphi(u_1)-\varphi(u_2))\xi_y\,dy\,ds\,dx\,dt\\
&=-\lim_{\epsilon\to0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times\mathbb{Q}}[h_2-a(u_2,0)]H_\epsilon(\varphi(u_1)-\varphi(u_2))_y\xi\,dy\,ds\,dx\,dt.
\end{aligned}
\tag{3.15}
$$

Substituting (3.14) and (3.15) in (3.13), we obtain

$$
\begin{aligned}
&\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)\big\{(h_1-h_2)(\xi_x+\xi_y)+(b(u_2)-b(u_1))(\xi_s+\xi_t)\\
&+(f_2-f_1)\xi\big\}\,dy\,ds\,dx\,dt\\
&-\int_{\mathbb{Q}\times\mathbb{R}}(v_{0_1}-b(u_2))^+\xi(0)dy\,dx\,dt-\int_{\mathbb{R}\times\mathbb{Q}}(b(u_1)-v_{0_2})^+\xi(0)\,dy\,ds\,dx\\
&\le-\lim_{\epsilon\to0}\int_{\mathbb{Q}\times(\mathbb{Q}\setminus\mathbb{Q}_1)} h_1H_\epsilon(\varphi(u_1)-\varphi(u_2))_x\xi\,dy\,ds\,dx\,dt\\
&\quad+\lim_{\epsilon\to0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times\mathbb{Q}} h_2H_\epsilon(\varphi(u_1)-\varphi(u_2))_y\xi\,dy\,ds\,dx\,dt\\
&\quad-\lim_{\epsilon\to0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times\mathbb{Q}} h_1H_\epsilon(\varphi(u_1)-\varphi(u_2))_y\xi\,dy\,ds\,dx\,dt\\
&\quad+\lim_{\epsilon\to0}\int_{\mathbb{Q}\times(\mathbb{Q}\setminus\mathbb{Q}_1)} h_2H_\epsilon(\varphi(u_1)-\varphi(u_2))_x\xi\,dy\,ds\,dx\,dt.
\end{aligned}
$$

Moreover, using (3.7) in the inequality above, we obtain

$$
\begin{aligned}
&\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)\big\{(h_1-h_2)(\xi_x+\xi_y)+(b(u_2)-b(u_1))(\xi_s+\xi_t)\\
&+(f_2-f_1)\xi\big\}\,dy\,ds\,dx\,dt\\
&-\int_{\mathbb{Q}\times\mathbb{R}}(v_{0_1}-b(u_2))^+\xi(0)dy\,dx\,dt-\int_{\mathbb{R}\times\mathbb{Q}}(b(u_1)-v_{0_2})^+\xi(0)\,dy\,ds\,dx\\
&\le\lim_{\epsilon\to0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times(\mathbb{Q}\setminus\mathbb{Q}_1)}[h_2-h_1]\operatorname{div}H_\epsilon(\varphi(u_1)-\varphi(u_2))\xi\,dy\,ds\,dx\,dt.
\end{aligned}
\tag{3.16}
$$

Now, put

$$
I=\lim_{\epsilon\to0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times(\mathbb{Q}\setminus\mathbb{Q}_1)}[h_2-h_1]\operatorname{div}H_\epsilon(\varphi(u_1)-\varphi(u_2))\xi\,dy\,ds\,dx\,dt.
$$

Then by (H4),

$$
\begin{aligned}
I&=-\lim_{\epsilon\to0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times(\mathbb{Q}\setminus\mathbb{Q}_1)}[a(u_1,\varphi(u_1)_y)-a(u_2,\varphi(u_2)_x)](\varphi(u_1)_y\\
&\quad-\varphi(u_2)_x)H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi\,dy\,ds\,dx\,dt\\
&\le\lim_{\epsilon\to0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times(\mathbb{Q}\setminus\mathbb{Q}_1)}M(u_1,u_2)(1+|\varphi(u_1)_y|^2\\
&\quad+|\varphi(u_2)_x|^2)|\varphi(u_1)-\varphi(u_2)|H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi\\
&\quad-\lim_{\epsilon\to0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times(\mathbb{Q}\setminus\mathbb{Q}_1)}\Gamma(\varphi(u_1),\varphi(u_2))\varphi(u_1)_yH'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi\\
&\quad-\lim_{\epsilon\to0}\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times(\mathbb{Q}\setminus\mathbb{Q}_1)}\widehat{\Gamma}(\varphi(u_1),\varphi(u_2))\varphi(u_2)_xH'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi\\
&=\lim_{\epsilon\to0}I_1-\lim_{\epsilon\to0}I_2-\lim_{\epsilon\to0}I_3.
\end{aligned}
$$

It is easy to see that $\lim_{\epsilon\to 0} I_1 = 0$. Set

$$F_\epsilon(z) = \int_0^z \Gamma(r,\varphi(u_2))H'_\epsilon(r-\varphi(u_2))dr.$$

Then we have

$$\begin{aligned} I_2 &= \int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times(\mathbb{Q}\setminus\mathbb{Q}_1)} \operatorname{div_y} F_\epsilon(\varphi(u_1))\xi\, dy\, ds\, dx\, dt \\ &= -\int_{(\mathbb{Q}\setminus\mathbb{Q}_2)\times(\mathbb{Q}\setminus\mathbb{Q}_1)} F_\epsilon(\varphi(u_1))\xi_y\, dy\, ds\, dx\, dt. \end{aligned}$$

Note that

$$F_\epsilon(z) = \frac{1}{\epsilon}\int_{\min(z,\varphi(u_2))}^{\min(z,\varphi(u_2)+\epsilon)} \Gamma(r,\varphi(u_2))dr.$$

The function $\Gamma \in C(\mathbb{R}^2)$ and attains its maximum and minimum on any compact subset of $\mathbb{R}$; in particular on $[\varphi(u_2),\varphi(u_2)+\epsilon]$ since $\|u_2\|_\infty$ is finite. Again, there exists m_ϵ and M_ϵ such that

$$m_\epsilon \le \frac{1}{\epsilon}\int_{\min(z,\varphi(u_2))}^{\min(z,\varphi(u_2)+\epsilon)} \Gamma(r,\varphi(u_2))dr \le M_\epsilon.$$

By the intermediate value theorem, there exists $r_1(\epsilon)$ and $r_2(\epsilon)$ in $[\varphi(u_2),\varphi(u_2)+\epsilon]$ such that:

$$m_\epsilon = \Gamma(r_1(\epsilon),\varphi(u_2)), \quad M_\epsilon = \Gamma(r_2(\epsilon),\varphi(u_2)).$$

Since $r_1(\epsilon)$ and $r_2(\epsilon) \in [\varphi(u_2),\varphi(u_2)+\epsilon]$, there exists θ_1 and $\theta_2 \in]0,1[$ such that

$$\begin{aligned} r_1(\epsilon) &= \theta_1(\varphi(u_2)) + (1-\theta_1)(\varphi(u_2)+\epsilon), \\ r_2(\epsilon) &= \theta_2(\varphi(u_2)) + (1-\theta_2)(\varphi(u_2)+\epsilon). \end{aligned}$$

Consequently,

$$\lim_{\epsilon\to 0} r_1(\epsilon) = \varphi(u_2), \quad \lim_{\epsilon\to 0} r_2(\epsilon) = \varphi(u_2).$$

Thus, we obtain:

$$\lim_{\epsilon\to 0} m_\epsilon = \Gamma(\varphi(u_2),\varphi(u_2)) = 0, \quad \lim_{\epsilon\to 0} M_\epsilon = \Gamma(\varphi(u_2),\varphi(u_2)) = 0.$$

This implies that $F_\epsilon \to 0$ as $\epsilon \to 0$, and so $\lim_{\epsilon\to 0} I_2 = 0$. Similarly, we get that $\lim_{\epsilon\to 0} I_3 = 0$. Consequently, $I \le 0$ and, from (3.16), we deduce the following inequality:

$$\begin{aligned} &\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)\big\{(h_1-h_2)(\xi_x+\xi_y) + (b(u_2)-b(u_1))(\xi_s+\xi_t) \\ &+ (f_2-f_1)\xi\big\}\, dy\, ds\, dx\, dt \\ &- \int_{\mathbb{Q}\times\mathbb{R}} (v_{0_1}-b(u_2))^+\xi(0)dy\, dx\, dt - \int_{\mathbb{R}\times\mathbb{Q}} (b(u_1)-v_{0_2})^+\xi(0)\, dy\, ds\, dx \le 0. \end{aligned} \tag{3.17}$$

Now let $\xi \in D(]0,T[\times\mathbb{R})$ such that $\xi \ge 0$; let (ρ_n) and (ρ_l) be classical sequences of mollifiers in $\mathbb{R}$ such that $\rho_l(s) = \rho_l(-s)$ and $\rho_n(s) = \rho_n(-s)$. Define

$$\xi^{l,n}(t,x,s,y) = \xi(\frac{t+s}{2},\frac{x+y}{2})\rho_n(\frac{x-y}{2})\rho_l(\frac{t-s}{2});$$

then $\xi^{l,n}$ is a positive function satisfying (3.6) for n and l large enough. By (3.17), for n and l large enough, we have

$$\begin{aligned}
&\int_{\mathbb{Q}\times\mathbb{Q}} H_0(u_1-u_2)\big\{(h_1-h_2)(\xi_x+\xi_y)+(b(u_2)-b(u_1))(\xi_s+\xi_t)\\
&+(f_2-f_1)\xi\big\}\rho_n\rho_l\,dy\,ds\,dx\,dt-\int_{\mathbb{Q}\times(\{0\}\times\mathbb{R})}(v_{0_1}-b(u_2))^+\xi\rho_n\rho_l dy\,dx\,dt \qquad (3.18)\\
&-\int_{(\{0\}\times\mathbb{R})\times\mathbb{Q}}(b(u_1)-v_{0_2})^+\xi\rho_n\rho_l\,dy\,ds\,dx\le 0.
\end{aligned}$$

Set

$$\varphi^l(t,x,y)=\int_t^T \xi(\frac{r}{2},\frac{x+y}{2})\rho_l(\frac{r}{2})dr=\int_{\min(t,\frac{1}{l})}^{\frac{1}{l}} \xi(\frac{r}{2},\frac{x+y}{2})\rho_l(\frac{r}{2})dr.$$

Since u_2 is entropy solution, we replace u by u_2, k by u_{0_1} and ξ by $\rho_n\varphi^l$ in (2.3) and integrate over $\mathbb{R}$ to obtain

$$\begin{aligned}
&-\int_{(\{0\}\times\mathbb{R})\times\mathbb{Q}}(v_{0_1}-b(u_2))^+\xi\rho_n\rho_l dy\,dx\,dt\\
&=\int_{(\{0\}\times\mathbb{R})\times\mathbb{Q}}(v_{0_1}-b(u_2))^+\rho_n\varphi^l_t dy\,dx\,dt\\
&\ge-\int_{(\{0\}\times\mathbb{R})\times\mathbb{Q}} H_0(u_{0_1}-u_2)\big\{(a(u_2,\varphi(u_2)_x)-a(u_{0_1},0))(\rho_n\varphi^l)_x\\
&\quad-f_2\rho_n\varphi^l\big\}dy\,dx\,dt-\int_{(\{0\}\times\mathbb{R})\times(\{0\}\times\mathbb{R})}(v_{0_1}-v_{0_2})^+\rho_n\varphi^l dy\,dx.
\end{aligned}$$

Since $\varphi^l=0$, when $t\ge\frac{1}{l}$, we have

$$\begin{aligned}
&-\int_{(\{0\}\times\mathbb{R})\times\mathbb{Q}}(v_{0_1}-b(u_2))^+\xi\rho_n\rho_l dy\,dx\,dt\\
&\ge-\int_{(\{0\}\times\mathbb{R})\times((0,\frac{1}{l})\times\mathbb{R})} H_0(u_{0_1}-u_2)\big\{(a(u_2,\varphi(u_2)_x)-a(u_{0_1},0))(\rho_n\varphi^l)_x \qquad (3.19)\\
&\quad-f_2\rho_n\varphi^l\big\}dy\,dx\,dt-\int_{(\{0\}\times\mathbb{R})\times(\{0\}\times\mathbb{R})}(v_{0_1}-v_{0_2})^+\rho_n\varphi^l dy\,dx.
\end{aligned}$$

It is easy to see that the first integral on the right side of inequality (3.19) converges to 0 when $l\to+\infty$. Moreover, without loss of generality, we can assume that $\rho_l(s)=\rho_l(-s)$ for any $s\in\mathbb{R}$; then

$$\lim_{l\to+\infty}\varphi^l(0,x,y)=\xi(0,\frac{x+y}{2})\lim_{l\to+\infty}\int_0^T\rho_l(r)dr=\frac{\xi(0,\frac{x+y}{2})}{2}$$

for any $(x,y)\in\mathbb{R}\times\mathbb{R}$. Since $\varphi^l(0,x,y)$ is uniformly bounded in $L^\infty(\mathbb{R})\times L^\infty(\mathbb{R})$, we deduce that the second integral on the right side of inequality (3.19) converges to:

$$\frac{1}{2}\int_{(\{0\}\times\mathbb{R})\times(\{0\}\times\mathbb{R})}(v_{0_1}-v_{0_2})^+\rho_n\xi dy\,dx.$$

Then we conclude that

$$\begin{aligned}
&-\limsup_{n\to+\infty}\limsup_{l\to+\infty}\int_{(\{0\}\times\mathbb{R})\times\mathbb{Q}}(v_{0_1}-b(u_2))^+\xi\rho_n\rho_l dy\,dx\,dt\\
&\geq -\lim_{n\to+\infty}\frac{1}{2}\int_{(\{0\}\times\mathbb{R})\times(\{0\}\times\mathbb{R})}(v_{0_1}-v_{0_2})^+\rho_n\xi dy\,dx \\
&=-\frac{1}{2}\int_{(\{0\}\times\mathbb{R})}(v_{0_1}-v_{0_2})^+\xi\,dx.
\end{aligned}\tag{3.20}$$

Similarly, by considering the function:

$$\widetilde{\varphi}^l(s,x,y)=\int_s^T\xi(\frac{r}{2},\frac{x+y}{2})\rho_l(\frac{-r}{2})dr=\int_{min(s,\frac{1}{l})}^{\frac{1}{l}}\xi(\frac{r}{2},\frac{x+y}{2})\rho_l(\frac{-r}{2})dr$$

and the fact that u_1 is an entropy solution and letting $u=u_1$, $k=u_{0_2}$, $\xi=\rho_n\widetilde{\varphi}^l$ in (2.2); we deduce that

$$\begin{aligned}
&-\limsup_{n\to+\infty}\limsup_{l\to+\infty}\int_{\mathbb{Q}\times(\{0\}\times\mathbb{R})}(b(u_1)-v_{0_2})^+\xi\rho_n\rho_l dy\,dx\,ds\\
&\geq -\lim_{n\to+\infty}\frac{1}{2}\int_{(\{0\}\times\mathbb{R})\times(\{0\}\times\mathbb{R})}(v_{0_1}-v_{0_2})^+\rho_n\xi dy\,dx \\
&=-\frac{1}{2}\int_{(\{0\}\times\mathbb{R})}(v_{0_1}-v_{0_2})^+\xi dx.
\end{aligned}\tag{3.21}$$

Finally, taking limit as $n\to+\infty$ and $l\to+\infty$ in (3.18), and using (3.20) and (3.21), we get (3.1). This completes the proof. □

Corollary 3.4. *Let (H1), (H2), (H3) and (H4) hold: Let u_i be an entropy solution of (1.1) with respect to the data (f_i,v_{0_i}) for $i=1,2$. Then*

$$\int_{\mathbb{R}}(b(u_1(t))-b(u_2(t)))^+dx\leq\int_{\mathbb{R}}(v_{0_1}-v_{0_2})^+dx+\int_0^t\int_{\mathbb{R}}H_0(u_1-u_2)(f_1-f_2)dxds,$$

and, therefore,

$$\|b(u_1(t))-b(u_2(t))\|_{L^1(\mathbb{R})}\leq\|v_{0_1}-v_{0_2}\|_{L^1(\mathbb{R})}+\int_0^t\|f_1-f_2\|_{L^1(\mathbb{R})}ds.$$

In particular, if $v_{0_1}\leq v_{0_2}$ almost everywhere in $\mathbb{R}$ and $f_1\leq f_2$ almost everywhere in $\mathbb{Q}$, then

$$b(u_1)\leq b(u_2)\quad a.e\ in\ \mathbb{Q}.$$

Moreover, if $f_1=f_2$ and $v_{0_1}=v_{0_2}$, then $b(u_1)=b(u_2)$.

Proof. Let $\psi\in D^+([0,T))$. Let $\xi\in D^+(\mathbb{R})$ be such that $0\leq\xi\leq1$, $\xi\equiv1$ on $[-1,1]$ and $\operatorname{supp}\xi\subset[-2,2]$. Set $\xi_n(x)=\xi(\frac{x}{n})$; then $\xi_n\in D^+(\mathbb{R})$, $0\leq\xi_n\leq1$, $\xi_n'\equiv0$ on $\mathbb{R}\setminus\{x\in\mathbb{R}/n\leq|x|\leq2n\}$. Then we have $\widetilde{\xi_n}(t,x)=\psi(t)\xi_n(x)\in D^+([0,T)\times\mathbb{R})$.

Taking $\widetilde{\xi_n}$ as a test function in inequality (3.1), we have

$$\begin{aligned}
&-\int_{\mathbb{Q}}(b(u_1)-b(u_2))^+\psi_t\xi_n\,dx\,dt+\int_{\mathbb{Q}}H_0(u_1-u_2)(h_1-h_2)\psi(\xi_n)_x\,dx\,dt\\
&-\int_{\mathbb{R}}(v_{0_1}-v_{0_2})^+\psi(0)\xi_n dx\\
&\leq\int_{\mathbb{Q}}H_0(u_1-u_2)(f_1-f_2)\psi\xi_n\,dx\,dt.
\end{aligned}$$

The above inequality gives

$$
\begin{aligned}
&-\int_{\mathbb{Q}}(b(u_1)-b(u_2))^+\psi_t\xi_n\,dx\,dt+\frac{1}{n}\int_{\{n\le|x|\le 2n\}}H_0(u_1-u_2)(h_1-h_2)\psi\xi'(\frac{x}{n})\,dx\,dt\\
&-\int_{\mathbb{R}}(v_{0_1}-v_{0_2})^+\psi(0)\xi_n dx\\
&\le\int_{\mathbb{Q}}H_0(u_1-u_2)(f_1-f_2)\psi\xi_n\,dx\,dt.
\end{aligned}
$$

Note that h_1, $h_2\in L^1(\mathbb{Q})$ since u_1 and u_2 are weak solutions; thus applying the Lebesgue Theorem, we deduce that as $n\to+\infty$,

$$
\begin{aligned}
&-\int_{\mathbb{Q}}(b(u_1)-b(u_2))^+\psi_t\,dx\,dt-\int_{\mathbb{R}}(v_{0_1}-v_{0_2})^+\psi(0)dx\\
&\le\int_0^T(\int_{\mathbb{R}}H_0(u_1-u_2)(f_1-f_2)dx)\psi dt.
\end{aligned}
$$

This inequality gives

$$
\begin{aligned}
&-\int_0^T(\int_{\mathbb{R}}(b(u_1)-b(u_2))^+dx)\psi_t dt-\int_{\mathbb{R}}(v_{0_1}-v_{0_2})^+\psi(0)dx\\
&\le\int_0^T(\int_{\mathbb{R}}H_0(u_1-u_2)(f_1-f_2)dx)\psi dt.
\end{aligned}
$$

From the above inequality, we deduce that

$$
\begin{aligned}
&-\int_0^T(\int_{\mathbb{R}}[(b(u_1)-b(u_2))^+-(v_{0_1}-v_{0_2})^+]dx)\psi_t dt\\
&\le\int_0^T(\int_{\mathbb{R}}H_0(u_1-u_2)(f_1-f_2)dx)\psi dt.
\end{aligned}
\tag{3.22}
$$

Now, put

$$
G(t)=\begin{cases}\int_{\mathbb{R}}[(b(u_1(t))-b(u_2(t)))^+-(v_{0_1}-v_{0_2})^+]dx & \text{for } t\in(0,T)\\ 0 & \text{for } t\in(-T,0),\end{cases}
$$

$$
F(t)=\begin{cases}\int_{\mathbb{R}}H_0(u_1(t)-u_2(t))(f_1(t)-f_2(t))dx & \text{for } t\in(0,T)\\ 0 & \text{for } t\in(-T,0).\end{cases}
$$

Then from (3.22), we deduce that

$$
\frac{dG}{dt}\le F\quad\text{in } D'(-T,T),
$$

and therefore, since G and F vanish for $t<0$, we have that

$$
G(t)\le\int_0^t F(s)ds.
$$

Hence, we easily deduce

$$
\int_{\mathbb{R}}(b(u_1(t))-b(u_2(t)))^+dx\le\int_{\mathbb{R}}(v_{0_1}-v_{0_2})^+dx+\int_0^t\int_{\mathbb{R}}H_0(u_1-u_2)(f_1-f_2)dx\,ds.
$$

$\square$

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