*Electronic Journal of Differential Equations*, Vol. 2021 (2021), No. 92, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# LOWER ORDER FOR MEROMORPHIC SOLUTIONS TO LINEAR DELAY-DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the order of growth for solutions of the non-homogeneous linear delay-differential equation

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij} f^{(j)}(z+c_i) = F(z),$$

where $A_{ij}(z)$ $(i = 0, \dots, n; j = 0, \dots, m)$, $F(z)$ are entire or meromorphic functions and $c_i$ $(0, 1, \dots, n)$ are non-zero distinct complex numbers. Under the condition that there exists one coefficient having the maximal lower order, or having the maximal lower type, strictly greater than the order, or the type, of the other coefficients, we obtain estimates of the lower bound of the order of meromorphic solutions of the above equation.

## 1. Introduction and statement of main results

Throughout this article, a meromorphic function means a function that is meromorphic in the whole complex plane $\mathbb{C}$. We use the basic notations such as $m(r, f)$, $N(r, f)$, $T(r, f)$ and fundamental results of Nevanlinna's value distribution theory [6, 10, 12, 23]. Further, we denote respectively by $\rho(f), \mu(f), \tau(f), \underline{\tau}(f)$, the order, the lower order, the type, and the lower type of a meromorphic function $f$. Also when $f$ is an entire function, we use $\tau_M(f)$, $\underline{\tau}_M(f)$ respectively for the type and lower type of $f$.

Recently, a lot of results have been obtained for complex difference and complex difference equations [3, 5, 8, 9, 15]. The back-ground for these studies lies in the recent difference counterparts of Nevanlinna theory. The key result here is the difference analogue of the lemma on the logarithmic derivative obtained by Halburd-Korhonen [8, 9] and Chiang-Feng [5], independently. Properties of meromorphic solutions of complex linear difference equations of type

$$A_n f(z+c_n) + A_{n-1} f(z+c_{n-1}) + \cdots + A_1 f(z+c_1) + A_0 f(z) = A_{n+1}, \qquad (1.1)$$

where $A_j(z)$ $(j = 0, \dots, n+1)$ are entire or meromorphic functions and $c_i$ $(1, \dots, n)$ are non-zero distinct complex numbers, have been made where one of the coefficients is dominating in comparison with the other coefficients, see e.g. [5, 13]. The following two theorems have been obtained in [1].

2010 *Mathematics Subject Classification.* 30D35, 39B32, 39A10.
*Key words and phrases.* Linear difference equation; linear delay-differential equation; meromorphic solution; order; type; lower order, lower type.

Submitted June 25, 2021. Published November 18, 2021.

**Theorem 1.1** ([1]). *Let $A_j(z)$ $(j=0,\dots,n+1)$ be entire functions, and let $k,l\in\{0,1,\dots,n+1\}$. If the following three assumptions hold simultaneously:*

(1) $\max\{\mu(A_k),\rho(A_j),j\neq k,l\}=\rho\leq\mu(A_l)<\infty$, $\mu(A_l)>0$;
(2) $\underline{\tau}_M(A_l)>\underline{\tau}_M(A_k)$, *when* $\mu(A_l)=\mu(A_k)$;
(3) $\max\{\tau_M(A_j):\rho(A_j)=\mu(A_l),j\neq k,l\}=\tau_1<\underline{\tau}_M(A_l)$, *when* $\mu(A_l)=\max\{\rho(A_j),j\neq k,l\}$.

*Then every meromorphic solution $f$ of* (1.1) *satisfies $\rho(f)\geq\mu(A_l)$ if $A_{n+1}\not\equiv 0$. Furthermore, if $A_{n+1}(z)\equiv 0$, then every meromorphic solution $f\not\equiv 0$ of* (1.1) *satisfies $\rho(f)\geq\mu(A_l)+1$.*

**Theorem 1.2** ([1]). *Let $A_j(z)$ $(j=0,\dots,n+1)$ be meromorphic functions, and let $k,l\in\{0,1,\dots,n+1\}$. If the following five assumptions hold simultaneously.*

(1) $\max\{\mu(A_k),\rho(A_j),j\neq k,l\}=\rho\leq\mu(A_l)<\infty$;
(2) $\underline{\tau}(A_l)>\underline{\tau}(A_k)$, *when* $\mu(A_l)=\mu(A_k)$;
(3)
$$\tau_1=\sum_{\rho(A_j)=\mu(A_l),\, j\neq l,k}\tau(A_j)<\underline{\tau}(A_l)<+\infty$$
*when* $\mu(A_l)=\max\{\rho(A_j),j\neq l,k\}$;
(4) $\tau_1+\underline{\tau}(A_k)<\underline{\tau}(A_l)<+\infty$ *when* $\mu(A_l)=\mu(A_k)=\max\{\rho(A_j),j\neq k,l\}$;
(5) $\lambda(\frac{1}{A_l})<\mu(A_l)<\infty$.

*Then every meromorphic solution $f$ of* (1.1) *satisfies $\rho(f)\geq\mu(A_l)$ if $A_{n+1}\not\equiv 0$. Furthermore, if $A_{n+1}(z)\equiv 0$, then every meromorphic solution $f\not\equiv 0$ of* (1.1) *satisfies $\rho(f)\geq\mu(A_l)+1$.*

Historically, the study of complex delay-differential equations can be traced back to Naftalevich's research. By using operator theory and iteration method, Naftalevich [18] considered the meromorphic solutions on complex delay-differential equations. Also there are few investigations on complex delay-differential equation field using Nevanlinna theory. Recently Liu, Laine and Yang [15] presented developments and new results on complex delay-differential equations, an area with important and interesting applications, which also gathers increasing attention (see, [14, 17, 19, 20, 21]). Chen and Zheng [4] investigated the growth of solutions of the homogeneous linear delay-differential equation

$$\sum_{i=0}^{n}\sum_{j=0}^{m}A_{ij}f^{(j)}(z+c_i)=0 \tag{1.2}$$

and have obtained the following results.

**Theorem 1.3** ([4]). *Let $A_{ij}(z)$ $(i=0,\dots,n;j=0,\dots,m)$ be entire functions, and $a,l\in\{0,1,\dots,n\}$, $b\in\{0,1,\dots,m\}$ such that $(a,b)\neq(l,0)$. If the following three assumptions hold simultaneously:*

(1) $\max\{\mu(A_{ab}),\rho(A_{ij}),(i,j)\neq(a,b),(l,0)\}=\rho\leq\mu(A_{l0})<\infty,\mu(A_{l0})>0$;
(2) $\underline{\tau}_M(A_{l0})>\underline{\tau}_M(A_{ab})$, *when* $\mu(A_{l0})=\mu(A_{ab})$;
(3) $\underline{\tau}_M(A_{l0})>\max\{\tau_M(A_{ij}):\rho(A_{ij})=\mu(A_{l0}),(i,j)\neq(a,b),(l,0)\}$, *when* $\mu(A_{l0})=\max\{\rho(A_{ij}):(i,j)\neq(a,b),(l,0)\}$.

*Then any non zero meromorphic solution $f$ of* (1.2) *satisfies $\rho(f)\geq\mu(A_{l0})+1$.*

**Theorem 1.4** ([4]). *Let $A_{ij}(z)$ $(i=0,\dots,n; j=0,\dots,m)$ be meromorphic functions, and $a,l \in \{0,1,\dots,n\}, b \in \{0,1,\dots,m\}$ such that $(a,b) \neq (l,0)$. If the following four assumptions hold simultaneously:*

(1) $\delta(\infty, A_{l0}) = \liminf_{r\to+\infty} \frac{m(r,A_{l0})}{T(r,A_{l0})} = \delta > 0$;
(2) $\max\{\mu(A_{ab}), \rho(A_{ij}), (i,j) \neq (a,b),(l,0)\} = \rho \leq \mu(A_{l0}) < \infty$, $\mu(A_{l0}) > 0$;
(3) $\delta\underline{\tau}(A_{l0}) > \underline{\tau}(A_{ab})$, *when* $\mu(A_{l0}) = \mu(A_{ab})$;
(4) $\delta\underline{\tau}(A_{l0}) > \max\{\tau(A_{ij}) : \rho(A_{ij}) = \mu(A_{l0}), (i,j) \neq (a,b),(l,0)\}$ *when* $\mu(A_{l0}) = \max\{\rho(A_{ij}) : (i,j) \neq (a,b),(l,0)\}$.

*Then any non zero meromorphic solution $f$ of* (1.2) *satisfies $\rho(f) \geq \mu(A_{l0}) + 1$.*

In this article, by combining complex differential and difference equations, we extend the results of Theorems 1.3 and 1.4 for the complex non-homogeneous linear delay-differential equation

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij} f^{(j)}(z+c_i) = F. \tag{1.3}$$

Let us define

$$S := \{F, A_{ij} : (i,j) \neq (l,0),(k,p)\}, \quad \rho(S) := \max\{\rho(g) : g \in S\}.$$

The main results of this paper reads as follows.

**Theorem 1.5.** *Consider a delay-differential equation* (1.3) *with entire coefficients. Suppose that one of the coefficients, say $A_{l0}$ with $\mu(A_{l0}) > 0$, is dominante in the sense that:*

(1) $\rho := \max\{\mu(A_{kp}), \rho(S)\} \leq \mu(A_{l0}) < \infty$;
(2) $\underline{\tau}_M(A_{l0}) > \underline{\tau}_M(A_{kp})$, *whenever* $\mu(A_{l0}) = \mu(A_{kp})$;
(3) $\tau_1 := \max\{\tau_M(g) : \rho(g) = \mu(A_{l0}),\ g \in S\} < \underline{\tau}_M(A_{l0})$, *whenever* $\mu(A_{l0}) = \rho(S)$.

*Then every meromorphic solution $f$ of* (1.3) *satisfies $\rho(f) \geq \mu(A_{l0})$ if $F(z) \not\equiv 0$. Further, if $F(z) \equiv 0$, then every meromorphic solution $f \not\equiv 0$ of* (1.2) *satisfies $\rho(f) \geq \mu(A_{l0}) + 1$.*

**Theorem 1.6.** *Consider a delay-differential equation of type* (1.3) *with meromorphic coefficients. Suppose that one of the coefficients, say $A_{l0}$, is dominate in the sense that*

(1) $\rho := \max\{\mu(A_{kp}), \rho(S)\} \leq \mu(A_{l0}) < \infty$;
(2) $\underline{\tau}(A_{l0}) > \underline{\tau}(A_{kp})$, *whenever* $\mu(A_{l0}) = \mu(A_{kp})$;
(3)
$$\tau_1 = \sum_{\substack{\rho(A_{ij})=\mu(A_{l0}),\\ (i,j)\neq(l,0),(k,p)}} \tau(A_{ij}) + \tau(F) < \underline{\tau}(A_{l0}) < +\infty$$
*whenever* $\mu(A_{l0}) = \rho(S)$;
(4) $\tau_1 + \underline{\tau}(A_{kp}) < \underline{\tau}(A_{l0}) < +\infty$ *whenever* $\mu(A_{l0}) = \mu(A_{kp}) = \rho(S)$;
(5) $\lambda(\frac{1}{A_{l0}}) < \mu(A_{l0}) < \infty$.

*Then every meromorphic solution $f$ of* (1.3) *satisfies $\rho(f) \geq \mu(A_{l0})$ if $F(z) \not\equiv 0$. Further, if $F(z) \equiv 0$, then every meromorphic solution $f \not\equiv 0$ of* (1.2) *satisfies $\rho(f) \geq \mu(A_{l0}) + 1$.*

## 2. Some preliminary lemmas

**Lemma 2.1** ([7]). *Let $f$ be a transcendental meromorphic function of finite order $\rho(f)$, and let $k$ and $j$ be integers satisfying $k > j \geq 0$. Then for every $\varepsilon(> 0)$, there exists a subset $E_1 \subset (1, +\infty)$ which has finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}.$$

**Lemma 2.2** ([5]). *Let $f$ be a meromorphic function of finite order $\rho$, and let $c_1, c_2 (c_1 \neq c_2)$ be two arbitrary complex numbers. Let $\varepsilon > 0$ be given, then there exists a subset $E_2 \subset (1, +\infty)$ with finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_2$, we have*

$$\exp\{-r^{\rho-1+\varepsilon}\} \leq \Big|\frac{f(z + c_1)}{f(z + c_2)}\Big| \leq \exp\{r^{\rho-1+\varepsilon}\}.$$

**Lemma 2.3** ([6]). *Let $f$ be a meromorphic function, $c$ be a non-zero complex constant. Then we have that as $r \to +\infty$*

$$(1 + o(1))T(r - |c|, f(z)) \leq T(r, f(z + c)) \leq (1 + o(1))T(r + |c|, f(z)).$$

*Consequently*

$$\rho(f(z + c)) = \rho(f), \quad \mu(f(z + c)) = \mu(f).$$

**Lemma 2.4** ([2]). *Let $f$ be a meromorphic function of finite order $\rho$. Then for any given $\varepsilon > 0$, there exists a set $E_3 \subset (1, +\infty)$ having finite linear measure and finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_3$ and sufficiently large $r$, we have*

$$\exp\{-r^{\rho+\varepsilon}\} \leq \big|f(z)\big| \leq \exp\{r^{\rho+\varepsilon}\}.$$

**Lemma 2.5** ([11]). *Let $f$ be an entire function with $\mu(f) < \infty$. Then for any given $\varepsilon(> 0)$, there exists a subset $E_4 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_4$, we have*

$$\mu(f) = \lim_{r\to+\infty,\, r\in E_4} \frac{\log\log M(r, f)}{\log r},$$
$$M(r, f) < \exp\{r^{\mu(f)+\varepsilon}\}.$$

**Lemma 2.6** ([22]). *Let $f$ be an entire function with $0 < \mu(f) < \infty$. Then for any given $\varepsilon(> 0)$, there exists a subset $E_5 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_5$, we have*

$$\underline{\tau}_M(f) = \lim_{r\to+\infty,\, r\in E_5} \frac{\log M(r, f)}{\log r},$$
$$M(r, f) < \exp\{(\underline{\tau}_M(f) + \varepsilon)r^{\mu(f)}\}.$$

**Lemma 2.7** ([5]). *Let $f$ be a meromorphic function of finite order $\rho(f) < \infty$, and let $c_1, c_2$ be two distinct complex numbers. Then for each $\varepsilon > 0$, we have*

$$m\Big(r, \frac{f(z + c_1)}{f(z + c_2)}\Big) = O(r^{\rho(f)-1+\varepsilon}).$$

**Lemma 2.8** ([24]). *Let $f$ be a meromorphic function with $\mu(f) < \infty$. Then for any given $\varepsilon(>0)$, there exists a subset $E_6 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_6$, we have*

$$T(r, f) < r^{\mu(f)+\varepsilon}.$$

**Lemma 2.9** ([16]). *Let $f$ be a meromorphic function with $0 < \mu(f) < \infty$. Then for any given $\varepsilon(>0)$, there exists a subset $E_7 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_7$, we have*

$$T(r, f) < (\underline{\tau}(f) + \varepsilon) r^{\mu(f)}.$$

## 3. Proof of main results

*Proof of Theorem 1.5.* If $f$ has infinite order, then the result holds. Now, we suppose that $\rho(f) < \infty$. We divide (1.3) by $f(z + c_l)$ to obtain

$$\begin{aligned} &- A_{l0}(z) \\ &= \sum_{i=0,i\neq l,k}^{n} \sum_{j=0}^{m} A_{ij} \frac{f^{(j)}(z+c_i)}{f(z+c_i)} \frac{f(z+c_i)}{f(z+c_l)} + \sum_{j=0,j\neq p}^{m} A_{kj} \frac{f^{(j)}(z+c_k)}{f(z+c_k)} \frac{f(z+c_k)}{f(z+c_l)} \\ &\quad + A_{kp} \frac{f^{(p)}(z+c_k)}{f(z+c_k)} \frac{f(z+c_k)}{f(z+c_l)} + \sum_{j=1}^{m} A_{lj} \frac{f^{(j)}(z+c_l)}{f(z+c_l)} - \frac{F(z)}{f(z+c_l)}. \end{aligned} \tag{3.1}$$

Therefore

$$\begin{aligned} |A_{l0}(z)| \leq & \sum_{i=0,i\neq l,k}^{n} \sum_{j=0}^{m} |A_{ij}| \Big| \frac{f^{(j)}(z+c_i)}{f(z+c_i)} \Big| \Big| \frac{f(z+c_i)}{f(z+c_l)} \Big| \\ &+ \sum_{j=0,j\neq p}^{m} |A_{kj}| \Big| \frac{f^{(j)}(z+c_k)}{f(z+c_k)} \Big| \Big| \frac{f(z+c_k)}{f(z+c_l)} \Big| \\ &+ |A_{kp}| \Big| \frac{f^{(p)}(z+c_k)}{f(z+c_k)} \Big| \Big| \frac{f(z+c_k)}{f(z+c_l)} \Big| \\ &+ \sum_{j=1}^{m} |A_{lj}| \Big| \frac{f^{(j)}(z+c_l)}{f(z+c_l)} \Big| + \Big| \frac{F(z)}{f(z+c_l)} \Big|. \end{aligned} \tag{3.2}$$

From Lemmas 2.1 and 2.3, for any given $\varepsilon(>0)$, there exists a subset $E_1 \subset (1, +\infty)$ which has finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\Big| \frac{f^{(j)}(z+c_i)}{f(z+c_i)} \Big| \leq |z|^{j(\rho(f+c_i)-1+\varepsilon)} = |z|^{j(\rho(f)-1+\varepsilon)}, \quad (i, j) \neq (l, 0). \tag{3.3}$$

It follows by Lemma 2.2 that for any $\varepsilon(>0)$, there exists a subset $E_2 \subset (1, +\infty)$ with finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_2$, we have

$$\Big| \frac{f(z+c_i)}{f(z+c_l)} \Big| \leq \exp\{r^{\rho(f)-1+\varepsilon}\}, \quad i \neq l. \tag{3.4}$$

From Lemma 2.3, we obtain

$$\rho(f(z+c_l)) = \rho\Big(\frac{1}{f(z+c_l)}\Big) = \rho(f).$$

So, by Lemma 2.4, for any given $\varepsilon > 0$, there exists a subset $E_3 \subset (1, +\infty)$ having finite linear measure and finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_3$ sufficiently large, we have

$$\Big|\frac{1}{f(z + c_l)}\Big| \le \exp\{r^{\rho(f)+\varepsilon}\}. \tag{3.5}$$

We divide the rest of the proof into four cases. **Case 1:** $\rho < \mu(A_{l0})$. For $g \in S$, by the definition of $\rho(S)$, for any given $\varepsilon > 0$ and sufficiently large $r$, we have

$$|g(z)| \le \exp\{r^{\rho(S)+\varepsilon}\} \le \exp\{r^{\rho+\varepsilon}\}. \tag{3.6}$$

From the definition of $\mu(A_{l0})$, for sufficiently small $\varepsilon > 0$ and sufficiently large $r$, we have

$$|A_{l0}(z)| \ge \exp\{r^{\mu(A_{l0})-\varepsilon}\}. \tag{3.7}$$

It also follows by the definition of $\mu(A_{kp})$ and Lemma 2.5, that for any $\varepsilon(> 0)$, there exists a subset $E_4 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_4$, we have

$$|A_{kp}(z)| \le \exp\{r^{\mu(A_{kp})+\varepsilon}\}. \tag{3.8}$$

By substituting (3.3)–(3.8) into (3.2), for all $z$ satisfying $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$, we obtain

$$\begin{aligned}
&\exp\{r^{\mu(A_{l0})-\varepsilon}\}\\
&\le \sum_{i=0, i\ne l,k}^{n} \sum_{j=0}^{m} \exp\{r^{\rho+\varepsilon}\}|z|^{j(\rho(f)-1+\varepsilon)} \exp\{r^{\rho(f)-1+\varepsilon}\}\\
&\quad + \sum_{j=0, j\ne p}^{m} \exp\{r^{\rho+\varepsilon}\}|z|^{j(\rho(f)-1+\varepsilon)} \exp\{r^{\rho(f)-1+\varepsilon}\}\\
&\quad + |z|^{p(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{kp})+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\\
&+ \quad \sum_{j=1}^{m} \exp\{r^{\rho+\varepsilon}\}|z|^{j(\rho(f)-1+\varepsilon)} + \exp\{r^{\rho+\varepsilon}\} \exp\{r^{\rho(f)+\varepsilon}\}\\
&\le ((n-1)(m+1) + 2m) r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\rho+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\\
&\quad + r^{p(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{kp})+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\\
&\quad + \exp\{r^{\rho+\varepsilon}\} \exp\{r^{\rho(f)+\varepsilon}\}.
\end{aligned} \tag{3.9}$$

Now, we choose $\varepsilon$ sufficiently small to satisfy $0 < 3\varepsilon < \mu(A_{l0}) - \rho$. We deduce from (3.9) that for $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$, $r \to +\infty$,

$$\exp\{r^{\mu(A_{l0})-2\varepsilon}\} \le \exp\{r^{\rho(f)+\varepsilon}\}.$$

Therefore, $\mu(A_{l0}) \le \rho(f) + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_{l0})$.

Further, if $F \equiv 0$, then by substituting (3.3), (3.4) and (3.6)–(3.8) into (3.2), for all $z$ satisfying $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2)$, we obtain

$$\begin{aligned}
\exp\{r^{\mu(A_{l0})-\varepsilon}\} &\le (nm + n + m - 1) r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\rho+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\\
&\quad + r^{p(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{kp})+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}.
\end{aligned} \tag{3.10}$$

By choosing sufficiently small $\varepsilon$ satisfying $0 < 3\varepsilon < \mu(A_{l0}) - \rho$, we deduce from (3.10) that for $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \to +\infty$,

$$\exp\{r^{\mu(A_{l0})-2\varepsilon}\} \le \exp\{r^{\rho(f)-1+\varepsilon}\},$$

that is, $\mu(A_{l0}) \le \rho(f) - 1 + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_{l0}) + 1$.

**Case 2:** $\beta = \rho(S) < \mu(A_{l0}) = \mu(A_{kp})$ and $\underline{\tau}_M(A_{l0}) > \underline{\tau}_M(A_{kp})$. For $g \in S$, by the definition of $\rho(S)$, for any given $\varepsilon(> 0)$, and sufficiently large $r$, we have

$$|g(z)| \le \exp\{r^{\rho(S)+\varepsilon}\} \le \exp\{r^{\beta+\varepsilon}\}. \tag{3.11}$$

From the definition of $\underline{\tau}_M(A_{l0})$, for sufficiently small $\varepsilon > 0$ and sufficiently large $r$, we have

$$|A_{l0}(z)| \ge \exp\{(\underline{\tau}_M(A_{l0}) - \varepsilon) r^{\mu(A_{l0})}\}. \tag{3.12}$$

Also, from the definition of $\underline{\tau}_M(A_{kp})$ and Lemma 2.6, for any given $\varepsilon(> 0)$, there exists a subset $E_5 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_5$, we have

$$|A_{kp}(z)| \le \exp\{(\underline{\tau}_M(A_{kp}) + \varepsilon) r^{\mu(A_{kp})}\} = \exp\{(\underline{\tau}_M(A_{kp}) + \varepsilon) r^{\mu(A_{l0})}\}. \tag{3.13}$$

By substituting (3.3)–(3.5) and (3.11)–(3.13) into (3.2), for all $z$ satisfying $|z| = r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$, we obtain

$$\begin{aligned} &\exp\{(\underline{\tau}_M(A_{l0}) - \varepsilon) r^{\mu(A_{l0})}\} \\ &\le (nm + n + m - 1) r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\beta+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + r^{p(\rho(f)-1+\varepsilon)} \exp\{(\underline{\tau}_M(A_{kp}) + \varepsilon) r^{\mu(A_{l0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + \exp\{r^{\beta+\varepsilon}\} \exp\{r^{\rho(f)+\varepsilon}\}. \end{aligned} \tag{3.14}$$

Therefore, we may choose $\varepsilon$ sufficiently small, $0 < 2\varepsilon < \min\{\mu(A_{l0}) - \beta, \underline{\tau}_M(A_{l0}) - \underline{\tau}_M(A_{kp})\}$, then from (3.14) for $r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$ sufficiently large, we obtain

$$\exp\{(\underline{\tau}_M(A_{l0}) - \underline{\tau}_M(A_{kp}) - 2\varepsilon) r^{\mu(A_{l0})-\varepsilon}\} \le \exp\{r^{\rho(f)+\varepsilon}\}.$$

Then, $\mu(A_{l0}) \le \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, so $\rho(f) \ge \mu(A_{l0})$.

Further, if $F \equiv 0$, then by substituting (3.3), (3.4) and (3.11)–(3.13) into (3.2), for all $z$ satisfying $|z| = r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2)$, we have

$$\begin{aligned} &\exp\{(\underline{\tau}_M(A_{l0}) - \varepsilon) r^{\mu(A_{l0})}\} \\ &\le (nm + n + m - 1) r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\beta+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + r^{p(\rho(f)-1+\varepsilon)} \exp\{(\underline{\tau}_M(A_{kp}) + \varepsilon) r^{\mu(A_{l0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\}. \end{aligned} \tag{3.15}$$

Now, we choose $\varepsilon$ sufficiently small, $0 < 2\varepsilon < \min\{\mu(A_{l0}) - \beta, \underline{\tau}_M(A_{l0}) - \underline{\tau}_M(A_{kp})\}$. Then from (3.15) for $r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2)$ sufficiently large, we obtain

$$\exp\{(\underline{\tau}_M(A_{l0}) - \underline{\tau}_M(A_{kp}) - 2\varepsilon) r^{\mu(A_{l0})-\varepsilon}\} \le \exp\{r^{\rho(f)-1+\varepsilon}\},$$

that is, $\mu(A_{l0}) \le \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_{l0}) + 1$.

**Case 3:** $\mu(A_{l0}) = \rho(S) > \mu(A_{kp})$ and $\max\{\tau_M(g) : \rho(g) = \mu(A_{l0}), g \in S\} = \tau_1 < \underline{\tau}_M(A_{l0})$. For $g \in S$, by the definitions of $\rho(S)$ and $\tau_M(g)$, for any given $\varepsilon > 0$ and sufficiently large $r$, we have

$$|g(z)| \le \begin{cases} \exp\{r^{\rho(S)+\varepsilon}\} \le \exp\{r^{\mu(A_{l0})-\varepsilon}\}, & \text{if } \rho(S) < \mu(A_{l0}), \\ \exp\{(\tau_1 + \varepsilon) r^{\mu(A_{l0})}\}, & \text{if } \rho(S) = \mu(A_{l0}). \end{cases} \tag{3.16}$$

Then, by substituting (3.3)–(3.5), (3.8), (3.12) and (3.16) into (3.2), for all $z$ satisfying $|z| = r \in E_4 \setminus ([0,1] \cup E_1 \cup E_2 \cup E_3)$ sufficiently large, we obtain

$$\begin{aligned}
&\exp\{(\underline{\tau}_M(A_{l0}) - \varepsilon) r^{\mu(A_{l0})}\} \\
&\le O\Big(r^{m(\rho(f)-1+\varepsilon)} \exp\{(\tau_1+\varepsilon) r^{\mu(A_{l0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\Big) \\
&\quad + O\Big(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l0})-\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\Big) \\
&\quad + r^{p(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{kp})+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\
&\quad + O\Big(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l0})-\varepsilon}\}\Big) \\
&\quad + O\Big(r^{m(\rho(f)-1+\varepsilon)} \exp\Big\{(\tau_1+\varepsilon) r^{\mu(A_{l0})}\Big\}\Big) \\
&\quad + \exp\{(\tau_1+\varepsilon) r^{\mu(A_{l0})}\} \exp\{r^{\rho(f)+\varepsilon}\}.
\end{aligned} \tag{3.17}$$

Now, we choose $\varepsilon$ sufficiently small satisfying

$$0 < 2\varepsilon < \min\{\mu(A_{l0}) - \mu(A_{kp}), \underline{\tau}_M(A_{l0}) - \tau_1\},$$

then from (3.17) for sufficiently large $r \in E_4 \setminus ([0,1] \cup E_1 \cup E_2 \cup E_3)$, we obtain

$$\exp\{(\underline{\tau}_M(A_{l0}) - \tau_1 - 2\varepsilon) r^{\mu(A_{l0})-\varepsilon}\} \le \exp\{r^{\rho(f)+\varepsilon}\}.$$

That means, $\mu(A_{l0}) \le \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_{l0})$.

Further, if $F \equiv 0$, then by substituting (3.3), (3.4), (3.8), (3.12) and (3.16) into (3.2), for all $z$ satisfying $|z| = r \in E_4 \setminus ([0,1] \cup E_1 \cup E_2)$ sufficiently large, we have

$$\begin{aligned}
&\exp\{(\underline{\tau}_M(A_{l0}) - \varepsilon) r^{\mu(A_{l0})}\} \\
&\le O\Big(r^{m(\rho(f)-1+\varepsilon)} \exp\{(\tau_1+\varepsilon) r^{\mu(A_{l0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\Big) \\
&\quad + O\Big(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l0})-\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\Big) \\
&\quad + r^{p(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{kp})+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\
&\quad + O(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l0})-\varepsilon}\}) \\
&\quad + O(r^{m(\rho(f)-1+\varepsilon)} \exp\{(\tau_1+\varepsilon) r^{\mu(A_{l0})}\}).
\end{aligned} \tag{3.18}$$

Now, we choose $\varepsilon$ sufficiently small satisfying

$$0 < 2\varepsilon < \min\{\mu(A_{l0}) - \mu(A_{kp}), \underline{\tau}_M(A_{l0}) - \tau_1\}.$$

Then from (3.18) for sufficiently large $r \in E_4 \setminus ([0,1] \cup E_1 \cup E_2)$, we obtain

$$\exp\{(\underline{\tau}_M(A_{l0}) - \tau_1 - 2\varepsilon) r^{\mu(A_{l0})-\varepsilon}\} \le \exp\{r^{\rho(f)-1+\varepsilon}\}.$$

That means, $\mu(A_{l0}) \le \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_{l0}) + 1$.

**Case 4:** $\rho(S) = \mu(A_{kp}) = \mu(A_{l0})$ and $\max\{\underline{\tau}_M(A_{kp}), \tau_M(g) : \rho(g) = \mu(A_{l0}), g \in S\} = \tau_2 < \underline{\tau}_M(A_{l0})$. It follows by substituting (3.3)–(3.5), (3.12), (3.13) and (3.16) into (3.2), for all $z$ satisfying $|z| = r \in E_5 \setminus ([0,1] \cup E_1 \cup E_2 \cup E_3)$ sufficiently large, we have

$$\begin{aligned}
&\exp\{(\underline{\tau}_M(A_{l0})-\varepsilon)r^{\mu(A_{l0})}\}\\
&\le O\Big(r^{m(\rho(f)-1+\varepsilon)}\exp\{(\tau_2+\varepsilon)r^{\mu(A_{l0})}\}\exp\{r^{\rho(f)-1+\varepsilon}\}\Big)\\
&\quad+O\Big(r^{m(\rho(f)-1+\varepsilon)}\exp\{r^{\mu(A_{l0})-\varepsilon}\}\exp\big\{r^{\rho(f)-1+\varepsilon}\big\}\Big)\\
&\quad+r^{p(\rho(f)-1+\varepsilon)}\exp\{(\underline{\tau}_M(A_{kp})+\varepsilon)r^{\mu(A_{l0})}\}\exp\big\{r^{\rho(f)-1+\varepsilon}\big\}\\
&\quad+O\big(r^{m(\rho(f)-1+\varepsilon)}\exp\{r^{\mu(A_{l0})-\varepsilon}\}\big)\\
&\quad+O\Big(r^{m(\rho(f)-1+\varepsilon)}\exp\big\{(\tau_2+\varepsilon)r^{\mu(A_{l0})}\big\}\Big)\\
&\quad+\exp\{(\tau_2+\varepsilon)r^{\mu(A_{l0})}\}\exp\big\{r^{\rho(f)+\varepsilon}\big\}.
\end{aligned}\tag{3.19}$$

Now, we choose $\varepsilon$ sufficiently small satisfying

$$0<2\varepsilon<\underline{\tau}_M(A_{l0})-\tau_2,$$

from (3.19) for sufficiently large $r\in E_5\setminus([0,1]\cup E_1\cup E_2\cup E_3)$, we obtain

$$\exp\{(\underline{\tau}_M(A_{l0})-\tau_2-2\varepsilon)r^{\mu(A_{l0})-\varepsilon}\}\le\exp\big\{r^{\rho(f)+\varepsilon}\big\},$$

this means, $\mu(A_{l0})\le\rho(f)+2\varepsilon$, since $\varepsilon>0$ is arbitrary, it follows that $\rho(f)\ge\mu(A_{l0})$.

Further, if $F\equiv 0$, by substituting (3.3), (3.4), (3.12), (3.13) and (3.16) into (3.2), for all $z$ satisfying $|z|=r\in E_5\setminus([0,1]\cup E_1\cup E_2)$ sufficiently large, we have

$$\begin{aligned}
&\exp\{(\underline{\tau}_M(A_{l0})-\varepsilon)r^{\mu(A_{l0})}\}\\
&\le O\Big(r^{m(\rho(f)-1+\varepsilon)}\exp\{(\tau_2+\varepsilon)r^{\mu(A_{l0})}\}\exp\big\{r^{\rho(f)-1+\varepsilon}\big\}\Big)\\
&\quad+O\Big(r^{m(\rho(f)-1+\varepsilon)}\exp\{r^{\mu(A_{l0})-\varepsilon}\}\exp\big\{r^{\rho(f)-1+\varepsilon}\big\}\Big)\\
&\quad+r^{p(\rho(f)-1+\varepsilon)}\exp\{(\underline{\tau}_M(A_{kp})+\varepsilon)r^{\mu(A_{l0})}\}\exp\big\{r^{\rho(f)-1+\varepsilon}\big\}\\
&\quad+O\Big(r^{m(\rho(f)-1+\varepsilon)}\exp\{r^{\mu(A_{l0})-\varepsilon}\}\Big)\\
&\quad+O\Big(r^{m(\rho(f)-1+\varepsilon)}\exp\big\{(\tau_2+\varepsilon)r^{\mu(A_{l0})}\big\}\Big).
\end{aligned}\tag{3.20}$$

Now, we choose $\varepsilon$ sufficiently small satisfying

$$0<2\varepsilon<\underline{\tau}_M(A_{l0})-\tau_2,$$

from (3.20) for sufficiently large $r\in E_5\setminus([0,1]\cup E_1\cup E_2)$, we obtain

$$\exp\{(\underline{\tau}_M(A_{l0})-\tau_2-2\varepsilon)r^{\mu(A_{l0})-\varepsilon}\}\le\exp\big\{r^{\rho(f)-1+\varepsilon}\big\}.$$

That means, $\mu(A_{l0})\le\rho(f)-1+2\varepsilon$, since $\varepsilon>0$ is arbitrary, then $\rho(f)\ge\mu(A_{l0})+1$. The proof of Theorem 1.5 is complete. □

*Proof of the Theorem 1.6.* If $f$ has infinite order, then the result holds. Now, we suppose that $\rho(f)<\infty$. By (3.1), we have

$$\begin{aligned}
&T(r,A_{l0}(z))\\
&=m(r,A_{l0}(z))+N(r,A_{l0}(z))\\
&\le\sum_{i=0,i\ne l,k}^{n}\sum_{j=0}^{m}m(r,A_{ij}(z))+m(r,A_{kp}(z))
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0,j\neq p}^{m} m(r,A_{kj}(z))+\sum_{j=1}^{m} m(r,A_{lj}(z))+\sum_{i=0,i\neq l,k}^{n}\sum_{j=0}^{m} m\Big(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\Big)\\
&+\sum_{i=0,i\neq l,k}^{n} m\Big(r,\frac{f(z+c_i)}{f(z+c_l)}\Big)+\sum_{j=1}^{m} m\Big(r,\frac{f^{(j)}(z+c_k)}{f(z+c_k)}\Big)+2m\Big(r,\frac{f(z+c_k)}{f(z+c_l)}\Big)\\
&+\sum_{j=1}^{m} m\Big(r,\frac{f^{(j)}(z+c_l)}{f(z+c_l)}\Big)+m(r,F(z))+m\Big(r,\frac{1}{f(z+c_l)}\Big)\\
&+N(r,A_{l0}(z))+O(1)\\
\le&\sum_{i=0,i\neq l,k}^{n}\sum_{j=0}^{m} T(r,A_{ij}(z))+T(r,A_{kp}(z))+\sum_{j=0,j\neq p}^{m} T(r,A_{kj}(z))\\
&+\sum_{j=1}^{m} T(r,A_{lj}(z))+\sum_{i=0,i\neq l,k}^{n}\sum_{j=1}^{m} m\Big(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\Big)\\
&+\sum_{i=0,i\neq l,k}^{n} m(r,\frac{f(z+c_i)}{f(z+c_l)})+\sum_{j=1}^{m} m(r,\frac{f^{(j)}(z+c_k)}{f(z+c_k)})+2m(r,\frac{f(z+c_k)}{f(z+c_l)})\\
&+\sum_{j=1}^{m} m\Big(r,\frac{f^{(j)}(z+c_l)}{f(z+c_l)}\Big)+T(r,F(z))+T(r,\frac{1}{f(z+c_l)})\\
&+N(r,A_{l0}(z))+O(1).
\end{aligned}
$$

By Lemma 2.3 and the first main theorem of Nevanlinna, when $r$ sufficiently large, we have

$$
T\Big(r,\frac{1}{f(z+c_l)}\Big)=T(r,f(z+c_l))+O(1)\le(1+o(1))T(r+|c_l|,f)\le 2T(2r,f).
$$

So, for $r$ sufficiently large, we obtain

$$
\begin{aligned}
&T(r,A_{l0}(z))\\
&\le\sum_{i=0,i\neq l,k}^{n}\sum_{j=0}^{m} T(r,A_{ij}(z))+T(r,A_{kp}(z))+\sum_{j=0,j\neq p}^{m} T(r,A_{kj}(z))\\
&\quad+\sum_{j=1}^{m} T(r,A_{lj}(z))+\sum_{i=0,i\neq l,k}^{n}\sum_{j=1}^{m} m(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)})\\
&\quad+\sum_{i=0,i\neq l,k}^{n} m\Big(r,\frac{f(z+c_i)}{f(z+c_l)}\Big)+T(r,F(z))+2T(2r,f)\\
&\quad+\sum_{j=1}^{m} m\Big(r,\frac{f^{(j)}(z+c_l)}{f(z+c_l)}\Big)+\sum_{j=1}^{m} m(r,\frac{f^{(j)}(z+c_k)}{f(z+c_k)})\\
&\quad+2m\Big(r,\frac{f(z+c_k)}{f(z+c_l)}\Big)+N(r,A_{l0}(z))+O(1).
\end{aligned}
\tag{3.21}
$$

By Lemma 2.7, for any positive $\varepsilon$, we have

$$
m\Big(r,\frac{f(z)}{f(z+c_l)}\Big)=O(r^{\rho(f)-1+\varepsilon}),\quad m\Big(r,\frac{f(z+c_j)}{f(z+c_l)}\Big)=O(r^{\rho(f)-1+\varepsilon}),\tag{3.22}
$$

for $j \neq l$. By the lemma of logarithmic derivative [10], there exists a subset $E_8 \subset [0,+\infty[$ of a finite linear measure such that for all $r \notin E_8$ sufficiently large, we have

$$m\Big(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}\Big) = O(\log r) \quad (i = 0,\dots,n; j = 1,\dots,m). \tag{3.23}$$

From the definition of $\lambda(\frac{1}{A_{l0}})$, for any $\varepsilon > 0$ and sufficiently large $r$, we have

$$N(r, A_{l0}) \leq r^{\lambda(\frac{1}{A_{l0}})+\varepsilon}. \tag{3.24}$$

We divide the rest of the proof into four cases.

**Case 1:** $\rho < \mu(A_{l0})$. For $g \in S$, from the definition of $\rho(S)$ and $\rho(f)$ for any given $\varepsilon > 0$ and sufficiently large $r$, we have

$$T(r, g) \leq r^{\rho(S)+\varepsilon} \leq r^{\rho+\varepsilon}, \tag{3.25}$$

$$T(r, f) \leq r^{\rho(f)+\varepsilon}. \tag{3.26}$$

It follows from the definition of $\mu(A_{l0})$, for sufficiently small $\varepsilon > 0$ and sufficiently large $r$, we have

$$T(r, A_{l0}) \geq r^{\mu(A_{l0})-\varepsilon}. \tag{3.27}$$

It follows from the definition of $\mu(A_{kp})$ and Lemma 2.8, for any $\varepsilon(> 0)$, there exists a subset $E_6 \subset (1,+\infty)$ with infinite logarithmic measure such that for all $r \in E_6$, we have

$$T(r, A_{kp}) \leq r^{\mu(A_{kp})+\varepsilon}. \tag{3.28}$$

By substituting (3.22)–(3.28) into (3.21) for sufficiently large $r \in E_6 \setminus E_8$, we obtain

$$\begin{aligned} r^{\mu(A_{l0})-\varepsilon} &\leq ((n-1)(m+1)+2m)r^{\rho+\varepsilon} + r^{\mu(A_{kp})+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) \\ &\quad + 2(2r)^{\rho(f)+\varepsilon} + r^{\rho+\varepsilon} + r^{\lambda(\frac{1}{A_{l0}})+\varepsilon} + O(\log r). \end{aligned} \tag{3.29}$$

We may choose $\varepsilon$ sufficiently small satisfying

$$0 < 3\varepsilon < \min\big\{\mu(A_{l0}) - \rho, \mu(A_{l0}) - \lambda(\frac{1}{A_{l0}})\big\},$$

it follows from (3.29) that for $r \in E_6 \setminus E_8$, $r \to +\infty$,

$$r^{\mu(A_{l0})-2\varepsilon} \leq r^{\rho(f)+\varepsilon},$$

this means, $\mu(A_{l0}) \leq \rho(f) + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l0})$.

Further, if $F \equiv 0$, then by substituting (3.22)–(3.25), (3.27) and (3.28) into (3.21) for sufficiently large $r \in E_6 \setminus E_8$, we obtain

$$\begin{aligned} r^{\mu(A_{l0})-\varepsilon} &\leq ((n-1)(m+1)+2m)r^{\rho+\varepsilon} + r^{\mu(A_{kp})+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) \\ &\quad + r^{\lambda(\frac{1}{A_{l0}})+\varepsilon} + O(\log r). \end{aligned} \tag{3.30}$$

We choose $\varepsilon$ sufficiently small satisfying

$$0 < 3\varepsilon < \min\big\{\mu(A_{l0}) - \rho, \mu(A_{l0}) - \lambda(\frac{1}{A_{l0}})\big\},$$

from (3.30) that for $r \in E_6 \setminus E_8$, $r \to +\infty$,

$$r^{\mu(A_{l0})-2\varepsilon} \leq r^{\rho(f)-1+\varepsilon},$$

this means, $\mu(A_{l0}) \leq \rho(f) - 1 + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l0}) + 1$.

**Case 2:** $\beta = \rho(S) < \mu(A_{l0}) = \mu(A_{kp})$, and $\underline{\tau}(A_{l0}) > \underline{\tau}(A_{kp})$. For $g \in S$, by the definition of $\rho(S)$, for any given $\varepsilon(>0)$ and sufficiently large $r$, we obtain

$$T(r, g) \le r^{\rho(S)+\varepsilon} \le r^{\beta+\varepsilon}. \tag{3.31}$$

From the definition of $\underline{\tau}(A_{l0})$, for sufficiently small $\varepsilon > 0$ and sufficiently large $r$, we have

$$T(r, A_{l0}) \ge (\underline{\tau}(A_{l0}) - \varepsilon) r^{\mu(A_{l0})}. \tag{3.32}$$

It follows from the definition of $\underline{\tau}(A_{kp})$ and Lemma 2.9, that for any positive $\varepsilon$, there exists a subset $E_7 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_7$, we have

$$T(r, A_{kp}) \le (\underline{\tau}(A_{kp}) + \varepsilon) r^{\mu(A_{kp})} \le (\underline{\tau}(A_{kp}) + \varepsilon) r^{\mu(A_{l0})}. \tag{3.33}$$

By substituting (3.22)–(3.24), (3.26) and (3.31)–(3.33) into (3.21), for sufficiently large $r \in E_7 \setminus E_8$, we obtain

$$\begin{aligned}
&(\underline{\tau}(A_{l0}) - \varepsilon) r^{\mu(A_{l0})} \\
&\le ((n-1)(m+1) + 2m) r^{\beta+\varepsilon} + (\underline{\tau}(A_{kp}) + \varepsilon) r^{\mu(A_{l0})} \\
&\quad + O(r^{\rho(f)-1+\varepsilon}) + 2(2r)^{\rho(f)+\varepsilon} + r^{\beta+\varepsilon} + r^{\lambda(\frac{1}{A_{l0}})+\varepsilon} + O(\log r).
\end{aligned} \tag{3.34}$$

Now, we choose $\varepsilon$ sufficiently small satisfying

$$0 < 2\varepsilon < \min\big\{\mu(A_{l0}) - \beta, \underline{\tau}(A_{l0}) - \underline{\tau}(A_{kp}), \mu(A_{l0}) - \lambda\big(\frac{1}{A_{l0}}\big)\big\},$$

so from (3.34) for sufficiently large $r \in E_7 \setminus E_8$, we have

$$(\underline{\tau}(A_{l0}) - \underline{\tau}(A_{kp}) - 2\varepsilon) r^{\mu(A_{l0})-\varepsilon} \le r^{\rho(f)+\varepsilon},$$

this means, $\mu(A_{l0}) \le \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_{l0})$.

Further, if $F \equiv 0$, then by substituting (3.22)–(3.24) and (3.31)–(3.33) into (3.21), for sufficiently large $r \in E_7 \setminus E_8$, we obtain

$$\begin{aligned}
(\underline{\tau}(A_{l0}) - \varepsilon) r^{\mu(A_{l0})} &\le ((n-1)(m+1) + 2m) r^{\beta+\varepsilon} + (\underline{\tau}(A_{kp}) + \varepsilon) r^{\mu(A_{l0})} \\
&\quad + O(r^{\rho(f)-1+\varepsilon}) + r^{\lambda(\frac{1}{A_{l0}})+\varepsilon} + O(\log r).
\end{aligned} \tag{3.35}$$

Now, we choose $\varepsilon$ sufficiently small satisfying

$$0 < 2\varepsilon < \min\big\{\mu(A_{l0}) - \beta, \underline{\tau}(A_{l0}) - \underline{\tau}(A_{kp}), \mu(A_{l0}) - \lambda\big(\frac{1}{A_{l0}}\big)\big\}.$$

From (3.35) for sufficiently large $r \in E_7 \setminus E_8$, we obtain

$$(\underline{\tau}(A_{l0}) - \underline{\tau}(A_{kp}) - 2\varepsilon) r^{\mu(A_{l0})-\varepsilon} \le r^{\rho(f)-1+\varepsilon},$$

this means, $\mu(A_{l0}) \le \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_{l0}) + 1$.

**Case 3:** $\mu(A_{l0}) = \rho(S) > \mu(A_{kp})$ and

$$\tau_1 = \sum_{\substack{\rho(A_{ij})=\mu(A_{l0}),\\ (i,j)\neq(l,0),(k,p)}} \tau(A_{ij}) + \tau(F) < \underline{\tau}(A_{l0}).$$

Then there exists a subset $J \subseteq \{0, 1, \dots, n\} \times \{0, 1, \dots, m\} \setminus \{(l, 0), (k, p)\}$ such that for all $(i, j) \in J$, when $\rho(A_{ij}) = \mu(A_{l0})$, we have $\sum_{(i,j)\in J} \tau(A_{ij}) < \underline{\tau}(A_{l0}) - \tau(F)$, and for $(i,j) \in \Pi = \{0,1,\dots,n\} \times \{0,1,\dots,m\} \setminus (J \cup \{(l,0),(k,p)\})$ we have $\rho(A_{ij}) < \mu(A_{l0})$. Hence, for any $\varepsilon > 0$ and sufficiently large $r$, we obtain

$$T(r, A_{ij}) \le \begin{cases} (\tau(A_{ij}) + \varepsilon) r^{\mu(A_{l0})}, & \text{if } (i,j) \in J, \\ r^{\rho(A_{ij})+\varepsilon} \le r^{\mu(A_{l0})-\varepsilon}, & \text{if } (i,j) \in \Pi \end{cases} \tag{3.36}$$

and

$$T(r, F) \le \begin{cases} (\tau(F) + \varepsilon) r^{\mu(A_{l0})}, & \text{if } \rho(F) = \mu(A_{l0}), \\ r^{\rho(F)+\varepsilon} \le r^{\mu(A_{l0})-\varepsilon}, & \text{if } \rho(F) < \mu(A_{l0}). \end{cases} \tag{3.37}$$

Then, by substituting (3.22)–(3.24), (3.26), (3.28), (3.32), (3.36) and (3.37) into (3.21), for all $z$ satisfying $|z| = r \in E_6 \setminus E_8$ sufficiently large $r$, we obtain

$$\begin{aligned} &(\underline{\tau}(A_{l0}) - \varepsilon) r^{\mu(A_{l0})} \\ &\le \sum_{(i,j)\in J} (\tau(A_{ij}) + \varepsilon) r^{\mu(A_{l0})} + \sum_{(i,j)\in\Pi} r^{\mu(A_{l0})-\varepsilon} + r^{\mu(A_{kp})+\varepsilon} \\ &\quad + (\tau(F) + \varepsilon) r^{\mu(A_{l0})} + r^{\lambda(\frac{1}{A_{l0}})+\varepsilon} + 2(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) + O(\ln r) \\ &\le (\tau_1 + (nm+n+m)\varepsilon) r^{\mu(A_{l0})} + O(r^{\mu(A_{l0})-\varepsilon}) + r^{\mu(A_{kp})+\varepsilon} \\ &\quad + r^{\lambda(\frac{1}{A_{l0}})+\varepsilon} + 2(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) + O(\log r). \end{aligned} \tag{3.38}$$

Now, we choose $\varepsilon$ sufficiently small satisfying

$$0 < \varepsilon < \min\Big\{ \frac{\mu(A_{l0}) - \mu(A_{kp})}{2}, \frac{\underline{\tau}(A_{l0}) - \tau_1}{nm+n+m+1}, \frac{\mu(A_{l0}) - \lambda(\frac{1}{A_{l0}})}{2} \Big\},$$

then from (3.38) for sufficiently large $r \in E_6 \setminus E_8$, we obtain

$$(\underline{\tau}(A_{l0}) - \tau_1 - (nm+n+m+1)\varepsilon) r^{\mu(A_{l0})-\varepsilon} \le r^{\rho(f)+\varepsilon},$$

this means, $\mu(A_{l0}) \le \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_{l0})$.

Further, if $F \equiv 0$, then by substituting (3.22)–(3.24), (3.28), (3.32) and (3.36) into (3.21), for all $z$ satisfying $|z| = r \in E_6 \setminus E_8$ sufficiently large $r$, we obtain

$$\begin{aligned} (\underline{\tau}(A_{l0}) - \varepsilon) r^{\mu(A_{l0})} &\le (\tau_1 + (nm+n+m-1)\varepsilon) r^{\mu(A_{l0})} + O\big(r^{\mu(A_{l0})-\varepsilon}\big) \\ &\quad + r^{\mu(A_{kp})+\varepsilon} + r^{\lambda(\frac{1}{A_{l0}})+\varepsilon} + O\big(r^{\rho(f)-1+\varepsilon}\big) + O(\log r). \end{aligned} \tag{3.39}$$

Now, we choose $\varepsilon$ sufficiently small satisfying

$$0 < \varepsilon < \min\Big\{ \frac{\mu(A_{l0}) - \mu(A_{kp})}{2}, \frac{\underline{\tau}(A_{l0}) - \tau_1}{nm+n+m}, \frac{\mu(A_{l0}) - \lambda(\frac{1}{A_{l0}})}{2} \Big\},$$

then from (3.39) for sufficiently large $r \in E_6 \setminus E_8$, we obtain

$$(\underline{\tau}(A_{l0}) - \tau_1 - (nm+n+m)\varepsilon) r^{\mu(A_{l0})-\varepsilon} \le r^{\rho(f)-1+\varepsilon},$$

this means, $\mu(A_{l0}) \le \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_{l0}) + 1$.

**Case 4:** $\rho(S) = \mu(A_{l0}) = \mu(A_{kp})$ with $\tau_1 + \underline{\tau}(A_{kp}) < \underline{\tau}(A_{l0})$. It follows by substituting (3.22)–(3.24), (3.26), (3.32), (3.33), (3.36) and (3.37) into (3.21), for all sufficiently large $r \in E_7 \setminus E_8$, we have

$$\begin{aligned} (\underline{\tau}(A_{l0}) - \varepsilon) r^{\mu(A_{l0})} &\le (\tau_1 + (nm+n+m)\varepsilon) r^{\mu(A_{l0})} + O\big(r^{\mu(A_{l0})-\varepsilon}\big) \\ &\quad + (\underline{\tau}(A_{kp}) + \varepsilon) r^{\mu(A_{l0})} + r^{\lambda(\frac{1}{A_{l0}})+\varepsilon} + 2(2r)^{\rho(f)+\varepsilon} \\ &\quad + O(r^{\rho(f)-1+\varepsilon}) + O(\log r). \end{aligned} \tag{3.40}$$

Now, we choose $\varepsilon$ sufficiently small satisfying

$$0<\varepsilon<\min\Big\{\frac{\underline{\tau}(A_{l0})-\tau_1-\underline{\tau}(A_{kp})}{nm+n+m+2},\frac{\mu(A_{l0})-\lambda(\frac{1}{A_{l0}})}{2}\Big\},$$

then from (3.40) for sufficiently large $r\in E_7\setminus E_8$, we obtain

$$(\underline{\tau}(A_{l0})-\tau_1-\underline{\tau}(A_{kp})-(nm+n+m+2)\varepsilon)r^{\mu(A_{l0})-\varepsilon}\le r^{\rho(f)+\varepsilon},$$

this means, $\mu(A_{l0})\le\rho(f)+2\varepsilon$, since $\varepsilon>0$ is arbitrary, then $\rho(f)\ge\mu(A_{l0})$.

Further, if $F\equiv 0$, then by substituting (3.22)–(3.24), (3.32), (3.33) and (3.36) into (3.21), for all sufficiently large $r\in E_7\setminus E_8$, we have

$$\begin{aligned}(\underline{\tau}(A_{l0})-\varepsilon)r^{\mu(A_{l0})}&\le(\tau_1+(nm+n+m-1)\varepsilon)r^{\mu(A_{l0})}+O\big(r^{\mu(A_{l0})-\varepsilon}\big)\\&\quad+(\underline{\tau}(A_{kp})+\varepsilon)r^{\mu(A_{l0})}+r^{\lambda(\frac{1}{A_{l0}})+\varepsilon}\\&\quad+O(r^{\rho(f)-1+\varepsilon})+O(\log r).\end{aligned}\tag{3.41}$$

Now, we choose $\varepsilon$ sufficiently small satisfying

$$0<\varepsilon<\min\Big\{\frac{\underline{\tau}(A_{l0})-\tau_1-\underline{\tau}(A_{kp})}{nm+n+m+1},\frac{\mu(A_{l0})-\lambda(\frac{1}{A_{l0}})}{2}\Big\},$$

then from (3.41) for sufficiently large $r\in E_7\setminus E_8$, we obtain

$$(\underline{\tau}(A_{l0})-\tau_1-\underline{\tau}(A_{kp})-(nm+n+m+1)\varepsilon)r^{\mu(A_{l0})-\varepsilon}\le r^{\rho(f)-1+\varepsilon},$$

so this means, $\mu(A_{l0})\le\rho(f)-1+2\varepsilon$, since $\varepsilon>0$ is arbitrary, then $\rho(f)\ge\mu(A_{l0})+1$. The proof of Theorem 1.6 is complete. $\square$

## 4. Examples

**Example 4.1.** We consider the non-homogeneous linear delay-differential equation with entire coefficients

$$\begin{aligned}&A_{02}(z)f''(z)+A_{11}(z)f'(z+1)+A_{01}(z)f'(z)+A_{10}(z)f(z+1)\\&+A_{00}(z)f(z)=F(z).\end{aligned}\tag{4.1}$$

**Case 1:** $\rho(S)<\mu(A_{l0})$. In (4.1), for

$$A_{00}(z)=\pi^2+2\pi^4z^2,\quad A_{10}(z)=e^{-\pi^2z^2-\pi^2},\quad A_{01}(z)=2\pi^2(z+1)e^{2\pi^2z+\pi^2},$$
$$A_{11}(z)=-2\pi^2z,\quad A_{02}(z)=-\frac{1}{2},\quad F(z)=e^{2\pi^2z},$$

we have

$$\max\{\mu(A_{11}),\rho(F),\rho(A_{ij}):(i,j)\ne(1,0),(1,1)\}=1<\mu(A_{10})=2.$$

We see that the conditions of Theorem 1.5 are satisfied. The function $f(z)=e^{\pi^2z^2}$ is a solution of (4.1) and satisfies $\rho(f)=2\ge\mu(A_{10})=2$.

**Case 2:** $\rho(S)<\mu(A_{l0})=\mu(A_{kp})$ with $\underline{\tau}_M(A_{l0})>\underline{\tau}_M(A_{kp})$. In (4.1), for

$$A_{00}(z)=2\pi^2,\quad A_{10}(z)=2\pi^2(z+1)e^{z^2}+e^{-\pi^2z^2-\pi^2},\quad A_{01}(z)=2\pi^2z,$$
$$A_{11}(z)=-e^{z^2},\quad A_{02}(z)=-1,\quad F(z)=e^{2\pi^2z},$$

we obtain $\max\{\rho(F),\rho(A_{ij}):(i,j)\ne(1,0),(1,1)\}=1<\mu(A_{10})=\mu(A_{11})=2$ and $\underline{\tau}_M(A_{10})=\pi^2>\underline{\tau}_M(A_{11})=1$. Hence, the conditions of Theorem 1.5 are satisfied. The function $f(z)=e^{\pi^2z^2}$ is a solution of (4.1) and satisfies $\rho(f)=2\ge\mu(A_{10})=2$.

**Case 3:** $\mu(A_{l0}) = \rho(S) > \mu(A_{kp})$ with $\underline{\tau}_M(A_{l0}) > \tau_1 = \max\{\tau_M(g) : \rho(g) = \mu(A_{l0}), g \in S\}$. In (4.1), for

$$A_{00}(z) = \pi^2 + 2\pi^4 z^2, \quad A_{10}(z) = e^{-\frac{4}{5}\pi^2 z^2 - \pi^2}, \quad A_{01}(z) = 2\pi^2(z+1)e^{2\pi^2 z + \pi^2},$$
$$A_{11}(z) = -2\pi^2 z, \quad A_{02}(z) = -\frac{1}{2}, \quad F(z) = e^{\frac{1}{5}\pi^2 z^2 + 2\pi^2 z},$$

we have $\mu(A_{10}) = \max\{\rho(F), \rho(A_{ij}) : (i,j) \neq (1,0),(0,1)\} = 2 > \mu(A_{01}) = 1$ and $\underline{\tau}_M(A_{10}) = \frac{4\pi^2}{5} > \tau_1 = \tau_M(F) = \frac{\pi^2}{5}$. Obviously, the conditions of Theorem 1.5 are satisfied. The function $f(z) = e^{\pi^2 z^2}$ is a solution of (4.1) and satisfies $\rho(f) = 2 \geq \mu(A_{10}) = 2$.

**Case 4:** $\mu(A_{l0}) = \mu(A_{kp}) = \rho(S)$ and $\underline{\tau}_M(A_{10}) > \max\{\tau_1, \underline{\tau}_M(A_{kp})\}$. In (4.1), for

$$A_{00}(z) = 2\pi^2, \quad A_{10}(z) = 2\pi^2(z+1)e^{z^2} + e^{-\frac{4}{5}\pi^2 z^2 - \pi^2}, \quad A_{01}(z) = 2\pi^2 z,$$
$$A_{11}(z) = -e^{z^2}, \quad A_{02}(z) = -1, \quad F(z) = e^{\frac{1}{5}\pi^2 z^2 + 2\pi^2 z},$$

we obtain $\mu(A_{10}) = \mu(A_{11}) = \max\{\rho(F), \rho(A_{ij}) : (i,j) \neq (1,0),(1,1)\} = 2$ and $\underline{\tau}_M(A_{10}) = \frac{4\pi^2}{5} > \max\{\tau_1, \underline{\tau}_M(A_{11})\} = \max\{\tau_M(F), \underline{\tau}_M(A_{11})\} = \frac{\pi^2}{5}$. We see that the conditions of Theorem 1.5 are satisfied. The function $f(z) = e^{\pi^2 z^2}$ is a solution of equation (4.1) and satisfies $\rho(f) = 2 \geq \mu(A_{10}) = 2$.

**Example 4.2.** We consider the homogeneous linear delay-differential equation with entire coefficients

$$A_{11}(z)g'(z-1) + A_{20}(z)g(z+3) + A_{00}(z)g(z) = 0. \tag{4.2}$$

**Case 1:** $\max\{\mu(A_{kp}), \rho(A_{ij}) : (i,j) \neq (l,0),(k,p)\} < \mu(A_{l0})$. In (4.2), for

$$A_{00}(z) = 1, \quad A_{20}(z) = (4\pi i(1-z) - e^{4\pi iz})e^{-16\pi iz}, \quad A_{11}(z) = 1,$$

we have

$$\max\{\mu(A_{11}), \rho(A_{ij}) : (i,j) \neq (2,0),(1,1)\} = 0 < \mu(A_{20}) = 1.$$

So, the conditions of Theorem 1.5 are satisfied. The function $g(z) = e^{2\pi iz^2}$ is a solution of (4.2) and $g$ satisfies $\rho(g) = 2 \geq \mu(A_{20}) + 1 = 2$.

**Case 2:** $\max\{\rho(A_{ij}) : (i,j) \neq (l,0),(k,p)\} < \mu(A_{l0}) = \mu(A_{kp})$ with $\underline{\tau}_M(A_{l0}) > \underline{\tau}_M(A_{kp})$. In (4.2), for

$$A_{00}(z) = 1, \quad A_{20}(z) = (4\pi i(1-z) - e^{2\pi iz})e^{-14\pi iz}, \quad A_{11}(z) = e^{2\pi iz},$$

we obtain $\mu(A_{20}) = \mu(A_{11}) = 1 > \max\{\rho(A_{ij}) : (i,j) \neq (2,0),(1,1)\} = 0$ and $\underline{\tau}_M(A_{20}) = 14\pi > \underline{\tau}_M(A_{11}) = 2\pi$. Obviously, the conditions of Theorem 1.5 are satisfied. The function $g(z) = e^{2\pi iz^2}$ is a solution of (4.2) and $g$ satisfies $\rho(g) = 2 \geq \mu(A_{20}) + 1 = 2$.

**Case 3:** $\mu(A_{l0}) = \max\{\rho(A_{ij}) : (i,j) \neq (l,0),(k,p)\} > \mu(A_{kp})$ with $\underline{\tau}_M(A_{l0}) > \tau_1 = \max\{\tau_M(A_{ij}) : \rho(A_{ij}) = \mu(A_{l0}), (i,j) \neq (l,0),(k,p)\}$. In (4.2), for

$$A_{00}(z) = e^{2\pi iz}, \quad A_{20}(z) = (4\pi i(1-z) - e^{6\pi iz})e^{-16\pi iz}, \quad A_{11}(z) = 1,$$

we obtain $\mu(A_{20}) = \max\{\rho(A_{ij}) : (i,j) \neq (2,0),(1,1)\} = 1 > \mu(A_{11}) = 0$ and $\underline{\tau}_M(A_{20}) = 16\pi > \tau_1 = \tau_M(A_{00}) = 2\pi$. Obviously, the conditions of Theorem 1.5 are satisfied. The function $g(z) = e^{2\pi iz^2}$ is a solution of (4.2) and $g$ satisfies $\rho(g) = 2 \geq \mu(A_{20}) + 1 = 2$.

**Case 4.** $\mu(A_{l0}) = \mu(A_{kp}) = \max\{\rho(A_{ij}) : (i,j) \neq (l,0),(k,p)\}$ with $\underline{\tau}_M(A_{l0}) > \tau_2 = \max\{\underline{\tau}_M(A_{kp}), \tau_M(A_{ij}) : \rho(A_{ij}) = \mu(A_{l0}), (i,j) \neq (l,0),(k,p)\}$. In (4.2), for

$$A_{00}(z) = e^{-2\pi iz}, \quad A_{20}(z) = (4\pi i(1-z)-1)e^{-14\pi iz}, \quad A_{11}(z) = e^{2\pi iz},$$

we have $\mu(A_{20}) = \mu(A_{11}) = \max\{\rho(A_{ij}) : (i,j) \neq (2,0),(1,1)\} = 1$ and $\underline{\tau}_M(A_{20}) = 14\pi > \max\{\tau_M(A_{00}), \underline{\tau}_M(A_{11})\} = 2\pi$. It is clear that the conditions of Theorem 1.5 are satisfied. The function $g(z) = e^{2\pi iz^2}$ is a solution of (4.2) and $g$ satisfies $\rho(g) = 2 \geq \mu(A_{20}) + 1 = 2$.

**Example 4.3.** We consider the non-homogeneous linear delay-differential equation with meromorphic coefficients

$$A_{11}(z)f'(z-1) + A_{01}(z)f'(z) + A_{20}(z)f(z+1) + A_{10}(z)f(z-1) = F(z). \quad (4.3)$$

**Case 1:** $\max\{\mu(A_{kp}), \rho(S)\} < \mu(A_{l0})$. In (4.3), for

$$A_{10}(z) = e^{-\pi^3 z^3 + 3\pi^3 z^2 - 3\pi^3 z + \pi^3}, \quad A_{20}(z) = 3\pi^3(2z-1)e^{-3\pi^3 z^2 - 3\pi^3 z - \pi^3},$$
$$A_{01}(z) = -1, \quad A_{11}(z) = e^{3\pi^3 z^2 - 3\pi^3 z + \pi^3}, \quad F(z) = \tan(\pi z),$$

we have

$$\max\{\mu(A_{11}), \rho(F), \rho(A_{ij}) : (i,j) \neq (1,0),(1,1)\} = 2 < \mu(A_{10}) = 3,$$
$$\lambda(\frac{1}{A_{10}}) = 0 < \mu(A_{10}) = 3.$$

It is easy to see that the conditions of Theorem 1.6 are satisfied. The meromorphic function

$$f(z) = e^{\pi^3 z^3}\tan(\pi z)$$

is a solution of (4.3) and satisfies $\rho(f) = 3 \geq \mu(A_{10}) = 3$.

**Case 2:** $\rho(S) < \mu(A_{l0}) = \mu(A_{kp})$ with $\underline{\tau}(A_{l0}) > \underline{\tau}(A_{kp})$. In (4.3), for

$$\begin{aligned} A_{10}(z) &= e^{-\pi^3 z^3 + 3\pi^3 z^2 - 3\pi^3 z + \pi^3} \\ &\quad + \Big(3\pi^3(2z-1) - 3z^2\pi^3 - \pi\tan(\pi z) + \frac{\pi}{\tan(\pi z)}\Big)e^{-z^3}, \end{aligned}$$
$$A_{20}(z) = 3\pi^3 z^2 + \pi\tan(\pi z) + \frac{\pi}{\tan(\pi z)}, \quad A_{01}(z) = -e^{3\pi^3 z^2 + 3\pi^3 z + \pi^3},$$
$$A_{11}(z) = e^{-z^3}, \quad F(z) = \tan(\pi z),$$

we obtain

$$\max\{\rho(F), \rho(A_{ij}) : (i,j) \neq (1,0),(1,1)\} = 2 < \mu(A_{10}) = \mu(A_{11}) = 3,$$
$$\lambda(\frac{1}{A_{10}}) = 1 < \mu(A_{10}) = 3,$$
$$\underline{\tau}(A_{10}) = \pi^2 > \underline{\tau}(A_{11}) = \frac{1}{\pi}.$$

Hence, the conditions of Theorem 1.6 are satisfied. The function $f(z) = e^{\pi^3 z^3}$ is a solution of (4.3) and $f$ satisfies $\rho(f) = 3 \geq \mu(A_{10}) = 3$.

**Case 3:** $\mu(A_{l0}) = \rho(S) > \mu(A_{kp})$ with

$$\underline{\tau}(A_{l0}) > \tau_1 = \sum_{\substack{\rho(A_{ij}) = \mu(A_{l0}),\\ (i,j) \neq (l,0),(k,p)}} \tau(A_{ij}) + \tau(F).$$

In (4.3), for

$$A_{10}(z)=e^{-2\pi^3z^3+3\pi^3z^2-3\pi^3z+\pi^3},\quad A_{20}(z)=3\pi^3(2z-1)e^{-3\pi^3z^2-3\pi^3z-\pi^3},$$
$$A_{01}(z)=-1,\quad A_{11}(z)=e^{3\pi^3z^2-3\pi^3z+\pi^3},\quad F(z)=\frac{\tan(\pi z)}{e^{\pi^3z^3}},$$

we have

$$\mu(A_{10})=\max\{\rho(F),\rho(A_{ij}):(i,j)\neq(1,0),(1,1)\}=3>\mu(A_{11})=2,$$
$$\lambda(\frac{1}{A_{10}})=0<\mu(A_{10})=3$$

and $\underline{\tau}(A_{10})=2\pi^2>\tau_1=\tau(F)=\pi^2$. We can see that the conditions of Theorem 1.6 are satisfied. The meromorphic function

$$f(z)=e^{\pi^3z^3}\tan(\pi z)$$

is a solution of (4.3) and satisfies $\rho(f)=3\geq\mu(A_{10})=3$.

**Case 4:** $\mu(A_{l0})=\mu(A_{k0})=\rho(S)$ and $\underline{\tau}(A_{10})>\tau_1+\underline{\tau}(A_{kp})$. In (4.3), for

$$A_{10}(z)=e^{-2\pi^3z^3+3\pi^3z^2-3\pi^3z+\pi^3},\quad A_{20}(z)=3\pi^3(2z-1)e^{(\frac{\pi}{4}z)^3-3\pi^3z^2-3\pi^3z-\pi^3},$$
$$A_{01}(z)=-e^{(\frac{\pi}{4}z)^3},\quad A_{11}(z)=e^{(\frac{\pi}{4}z)^3+3\pi^3z^2-3\pi^3z+\pi^3},\quad F(z)=\frac{\tan(\pi z)}{e^{\pi^3z^3}},$$

we obtain

$$\mu(A_{10})=\mu(A_{11})=\max\{\rho(F),\rho(A_{ij}):(i,j)\neq(1,0),(1,1)\}=3,$$
$$\lambda(\frac{1}{A_{10}})=0<\mu(A_{10})=3,$$
$$\begin{aligned}\tau_1+\underline{\tau}(A_{11})&=\tau(A_{01})+\tau(A_{20})+\tau(F)+\underline{\tau}(A_{11})\\&=(\frac{2}{4^3}+1)\pi^2+\frac{\pi^2}{4^3}=\frac{67}{64}\pi^2<\underline{\tau}(A_{10})=2\pi^2.\end{aligned}$$

Obviously, the conditions of Theorem 1.6 are satisfied. The meromorphic function

$$f(z)=e^{\pi^3z^3}\tan(\pi z)$$

is a solution of (4.3) and satisfies $\rho(f)=3\geq\mu(A_{10})=3$.

**Example 4.4.** We consider the homogeneous linear delay-differential equation with meromorphic coefficients

$$A_{11}(z)h'(z+i\pi)+A_{20}(z)h(z+2i\pi)+A_{00}(z)h(z)=0. \tag{4.4}$$

**Case 1:** $\max\{\mu(A_{kp}),\rho(A_{ij}):(i,j)\neq(l,0),(k,p)\}<\mu(A_{l0})$. In (4.4), for

$$A_{00}(z)=-1,\quad A_{20}(z)=e^{12\pi z^2+24\pi^2iz-16\pi^3}-e^{6\pi z^2+18\pi^2iz-14\pi^3},$$
$$A_{11}(z)=\frac{\cos(2iz)}{6i(z+i\pi)^2\cos(2iz)+2i\sin(2iz)},$$

we have $\max\{\mu(A_{11}),\rho(A_{ij}):(i,j)\neq(2,0),(1,1)\}=1<\mu(A_{20})=2$ and

$$\lambda(\frac{1}{A_{20}})=0<\mu(A_{20})=2.$$

Obviously, the conditions of Theorem 1.6 are satisfied. The meromorphic function

$$h(z)=\frac{e^{2iz^3}}{\cos(2iz)}$$

is a solution of (4.4) and satisfies $\rho(h) = 3 \geq \mu(A_{20}) + 1 = 3$.

**Case 2:** $\max\{\rho(A_{ij}) : (i,j) \neq (l,0),(k,p)\} < \mu(A_{l0}) = \mu(A_{kp})$ with $\underline{\tau}(A_{l0}) > \underline{\tau}(A_{kp})$. In (4.4), for

$$A_{00}(z) = 1, \quad A_{20}(z) = -2e^{12\pi z^2 + 24\pi^2 iz - 16\pi^3},$$
$$A_{11}(z) = \frac{e^{6\pi z^2 + 6\pi^2 iz - 2\pi^3}\cos(2iz)}{6i(z+i\pi)^2\cos(2iz) + 2i\sin(2iz)},$$

we obtain

$$\mu(A_{20}) = \mu(A_{11}) = 2 > \max\{\rho(A_{ij}) : (i,j) \neq (2,0),(1,1)\} = \rho(A_{00}) = 0,$$
$$\lambda(\frac{1}{A_{20}}) = 0 < \mu(A_{20}) = 2,$$
$$\underline{\tau}(A_{20}) = 12 > \underline{\tau}(A_{11}) = 6.$$

It is clear that the conditions of Theorem 1.6 are satisfied. The meromorphic function

$$h(z) = \frac{e^{2iz^3}}{\cos(2iz)}$$

is a solution of equation (4.4) and satisfies $\rho(h) = 3 \geq \mu(A_{20}) + 1 = 3$.

**Case 3:** $\mu(A_{l0}) = \max\{\rho(A_{ij}) : (i,j) \neq (l,0),(k,p)\} > \mu(A_{kp})$ with $\underline{\tau}(A_{l0}) > \sum_{\rho(A_{ij})=\mu(A_{l0}),\,(i,j)\neq(l,0),(k,p)} \tau(A_{ij})$. In (4.4), for

$$A_{00}(z) = -e^{\pi z^2}, \quad A_{20}(z) = e^{13\pi z^2 + 24\pi^2 iz - 16\pi^3} - e^{6\pi z^2 + 18\pi^2 iz - 14\pi^3},$$
$$A_{11}(z) = \frac{\cos(2iz)}{6i(z+i\pi)^2\cos(2iz) + 2i\sin(2iz)},$$

we obtain

$$\mu(A_{20}) = \max\{\rho(A_{ij}) : (i,j) \neq (2,0),(1,1)\} = \rho(A_{00}) = 2 > \mu(A_{11}) = 1,$$
$$\lambda(\frac{1}{A_{20}}) = 0 < \mu(A_{20}) = 2,$$
$$\underline{\tau}(A_{20}) = 13 > \sum_{\rho(A_{ij})=\mu(A_{l0}),\,(i,j)\neq(l,0),(k,p)} \tau(A_{ij}) = \tau(A_{00}) = 1.$$

It is clear that the conditions of Theorem 1.6 are satisfied. The meromorphic function

$$h(z) = \frac{e^{2iz^3}}{\cos(2iz)}$$

is a solution of (4.4) and satisfies $\rho(h) = 3 \geq \mu(A_{20}) + 1 = 3$.

**Case 4:** $\mu(A_{l0}) = \mu(A_{kp}) = \max\{\rho(A_{ij}) : (i,j) \neq (l,0),(k,p)\}$ with $\underline{\tau}(A_{l0}) > \sum_{\rho(A_{ij})=\mu(A_{l0}),\,(i,j)\neq(l,0),(k,p)} \tau(A_{ij}) + \underline{\tau}(A_{kp})$. In (4.4), for

$$A_{00}(z) = e^{\pi z^2}, \quad A_{20}(z) = -2e^{13\pi z^2 + 24\pi^2 iz - 16\pi^3},$$
$$A_{11}(z) = \frac{e^{7\pi z^2 + 6\pi^2 iz - 2\pi^3}\cos(2iz)}{6i(z+i\pi)^2\cos(2iz) + 2i\sin(2iz)},$$

we have

$$\mu(A_{20}) = \mu(A_{11}) = \max\{\rho(A_{ij}) : (i,j) \neq (2,0),(1,1)\} = \rho(A_{00}) = 2,$$

$$\lambda(\frac{1}{A_{20}}) = 0 < \mu(A_{20}) = 2,$$
$$\underline{\tau}(A_{20}) = 13 > \tau(A_{00}) + \underline{\tau}(A_{11}) = 1 + 7 = 8.$$

It is easy to see that the conditions of Theorem 1.6 are satisfied. The meromorphic function

$$h(z) = \frac{e^{2iz^3}}{\cos(2iz)}$$

is a solution of equation (4.4) and satisfies $\rho(h) = 3 \geq \mu(A_{20}) + 1 = 3$.

**Acknowledgements.** The authors would like to thank the anonymous referees for their insightful comments, which allow us to improve our original version. This work was supported by the Directorate-General for Scientific Research and Technological Development (DGRSDT).

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