

A CHARACTERIZATION OF ORIENTED HYPERGRAPHIC LAPLACIAN
AND ADJACENCY COEFFICIENTS AND MINORS

by

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TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENTS	iv
LIST OF FIGURES	vii
CHAPTER	
1. INTRODUCTION	1
1.1 Oriented Hypergraphs	2
1.2 Weak Walks	3
1.3 Oriented Hypergraphic Matrices	3
1.4 Sachs' Theorem	4
2. BACKGROUND	5
2.1 Oriented Hypergraphs	5
2.2 Weak Walks	7
2.3 Oriented Hypergraphic Matrices	8
2.4 Sachs' Theorem	10
2.5 The Matrix Tree Theorem	10
3. PARTITIONS OF THE SET OF CONTRIBUTORS	13
3.1 Contributors	13
3.2 Permutomorphic Contributors	15
3.3 Generalized Basic Figures and Cycle Covers	17
4. PERMANENTS AND DETERMINANTS	19
4.1 Permanents and Determinants of the Laplacian and Adjacency Matrices	20
4.2 Optimizing	25
5. CHARACTERISTIC POLYNOMIALS	26
5.1 Coefficient Theorems	26

5.2 Alternative Proof of Sachs' Theorem	30
6. THE ALL MINORS MATRIX TREE THEOREM	32
6.1 The All Minors Matrix Tree Theorem for Oriented Hypergraphs ...	32
REFERENCES	35

LIST OF FIGURES

Figure	Page
1. An oriented hypergraph, G_1	2
2. Positive and Negative Signed Adjacencies	6
3. An oriented simple graph G_2	11
4. All 8 spanning trees of G_2 (the removed edges are left as dashed lines)	11
5. Examples of contributors of G_1 and G_2	15
6. Examples of π -permutomorphic contributors of G_1 and G_2	16
7. A contributor from the graph G_1 with permutation $\pi = (12)$	24
8. All of the contributors of G_2	29

1. INTRODUCTION

In this thesis we use the Weak Walk Theorem for oriented hypergraphs [3, 6] to unify and generalize Sachs' Coefficient Theorem [4] and the All Minors Matrix Tree Theorem [2] for graphs and signed graphs. Oriented hypergraphs are an incidence-centric generalization of graphs and signed graphs; Sachs' Coefficient Theorem provides expressions for the coefficients of the characteristic polynomial of the adjacency matrix for graphs; the All Minors Matrix Tree Theorem relates the number of forests in graphs to a specific minor of the Laplacian matrix.

Applying the Weak Walk Theorem for oriented hypergraphs from [3, 6], we relate the entries of the Laplacian and adjacency matrices. This allows for a single combinatorial approach to examine both determinant and permanent versions of Sachs' Coefficient Theorem and the All Minors Matrix Tree Theorem when applied to the Laplacian and adjacency matrices. To unify these theorems, we construct incidence preserving maps from a disjoint union of paths of length 1 into a given oriented hypergraph and examine the signed counts of these functions with varying parameters. These functions, called contributors, are in one-to-one correspondence with an oriented version of the basic figures in [4] which allows us to provide a meaningful explanation of the coefficients of the characteristic polynomial of the Laplacian matrix; the contributors that represent cycle covers are counted in the characteristic polynomial of the adjacency matrix. We then use these contributors to create and prove a general All Minors Matrix Tree Theorem by classifying which objects are to be counted in correspondence with particular minors of the Laplacian or adjacency matrix.

1.1 Oriented Hypergraphs

The definition of oriented hypergraph used for this paper is an adaptation of the definition introduced in [6] and expanded on in [7]. An oriented hypergraph is a graphical structure that includes orientations on each incidence (where an edge meets a vertex) and allows for loops, multiple edges between a single pair of vertices, and hyperedges, or edges that are incident to more than one vertex. Exact definitions for these terms will be given in section 2.1, but an example is given in figure 1 where the oriented hypergraph G_1 has three vertices indicated by points in the plane and three edges where edges connecting two vertices are lines and edges connecting three vertices are indicated by shaded regions. There are multiple edges between v_1 and v_2 , namely e_1 and e_2 . Also, e_2 is a hyperedge connecting all three vertices and e_3 is a loop. The arrows represent the orientation of each incidence: an arrow entrant to a vertex has sign $+1$ and an arrow salient to a vertex has sign -1 .

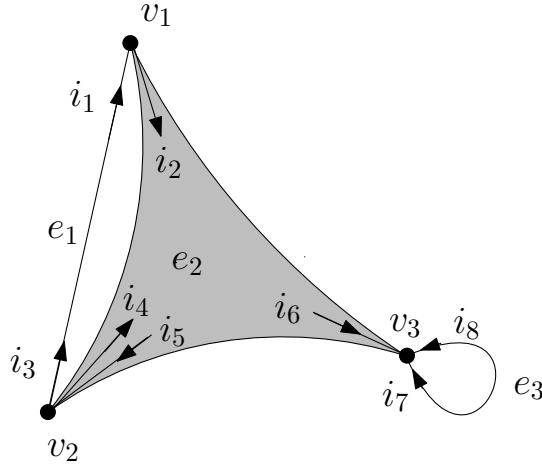


Figure 1: An oriented hypergraph, G_1 .

We work with oriented hypergraphs in order to allow the most generality in our conclusions. Thus, all of the theorems in this paper can easily be applied to both graphs, and signed graphs. Minor modifications to the theorems will even allow applications to directed graphs.

1.2 Weak Walks

A large emphasis of this paper is on the reinterpretation of theorems using weak walks to combinatorially describe what is being counted. A weak walk is an incidence sequence that alternates between vertices/edges and incidences in a graph. The main focus of this paper is on weak walks of length 1 that are of the form:

vertex, incidence, edge, incidence, vertex

This term is related to the more strict and common term, walk, where an incidence cannot be repeated before reaching a new vertex. An example of a length 1 weak walk in the oriented hypergraph G_1 is v_1, i_1, e_2, i_2, v_2 where i_1 is the incidence between v_1 and e_2 , and i_2 is the incidence between e_2 and v_2 . Notice that v_1, i_1, e_2, i_1, v_1 is also a weak walk and, in fact, our first example is also a walk.

1.3 Oriented Hypergraphic Matrices

We organize information from our graphs into matrices that are defined in section 2.3. The four major matrices we will use are the degree matrix, the adjacency matrix, the incidence matrix, and the Laplacian matrix, the last of which is formed by either subtracting the adjacency matrix from the degree matrix or multiplying the incidence matrix by its transpose.

We also discuss an important theorem introduced in [7] and refined in [3] that states that the ij -entry of the Laplacian matrix for an oriented hypergraph is the negative of the sum of the signs of the weak walks from vertex i to vertex j . This connection between weak walks and the entries in the Laplacian matrix is the driving force behind the creation of the single combinatorial approach that universally proves both the All Minors Matrix Tree Theorem and Sachs' Coefficient Theorem for oriented hypergraphs.

1.4 Sachs' Theorem

Sachs' Theorem, sometimes called the “coefficients theorem for digraphs,” appears in [4] and is one of the major building blocks for the research in this thesis. Sachs' Theorem enumerates the coefficients of the characteristic polynomial of the adjacency matrix of arbitrary directed multigraphs. Although Sachs specifically applies this theorem to directed and undirected multigraphs, our version is easily applied to oriented hypergraphs. In [4], two versions of this theorem are introduced: one by using a determinant calculation and the other using a permanent calculation. These theorems along with extensions to the Laplacian matrix are proved in section 5.1. In section 5.2, we show that Sachs' Theorem is a direct result of the theorem we prove in the previous section.

2. BACKGROUND

The majority of the following definitions come from [3, 6] and [7] and have been modified to fit this particular paper. In section 2.1 we formally define the graphical structures that were introduced in section 1.1. In section 2.2 we introduce weak walks which enumerate the entries of the oriented hypergraphic matrices in section 2.3. We introduce Sachs' Theorem in section 2.4, and introduce the Matrix Tree Theorem for simple graphs in section 2.5.

2.1 Oriented Hypergraphs

Let V , E , and $\mathcal{I}$ denote disjoint sets of *vertices*, *edges*, and *incidences*, respectively. Consider $\mathcal{I} \xrightarrow{\iota} V \times E$, where ι is the incidence function, and we say v and e are *incident along i* if $\iota(i) = (v, e)$ (alternatively, i is *between v and e*). Two incidences i and j are said to be *parallel* if $\iota(i) = \iota(j)$. This provides an equivalence class of parallel incidences, and the size of each equivalence class is called the *multiplicity of incidence i* (for some i in the class). An *incidence orientation function* is a function $\sigma : \mathcal{I} \rightarrow \{+1, -1\}$, we say $\sigma(i) = +1$ for an incidence i entrant to a vertex and $\sigma(j) = -1$ for an incidence j salient to a vertex. An *oriented hypergraph* is a quintuple $(V, E, \mathcal{I}, \iota, \sigma)$.

The *degree of vertex v* is $\deg(v) := |\{i \in \mathcal{I} \mid (\pi_V \circ \iota)(i) = v\}|$, while the *size of an edge e* is $\text{size}(e) := |\{i \in \mathcal{I} \mid (\pi_E \circ \iota)(i) = e\}|$. Vertices v and w are said to be *adjacent with respect to edge e* if there are incidences $i \neq j$ such that $\iota(i) = (v, e)$, and $\iota(j) = (w, e)$. A *directed adjacency* is a quintuple (v, i, e, j, w) where v and w are adjacent with respect to edge e using incidences i and j and $i \neq j$. Observe that if the directed adjacency (v, i, e, j, w) exists, then the opposite directed adjacency (w, j, e, i, v) also exists. An *adjacency* is the set associated to a directed adjacency. The directedness condition can easily be modified to “directed” oriented

hypergraphs. The *sign of the adjacency* (v, i, e, j, w) is

$$\text{sgn}(v, i, e, j, w) = -\sigma(i)\sigma(j),$$

and $\text{sgn}(v, i, e, j, w) = 0$ if v and w are not adjacent. Figure 2 depicts the three possibilities for the signing of adjacencies.

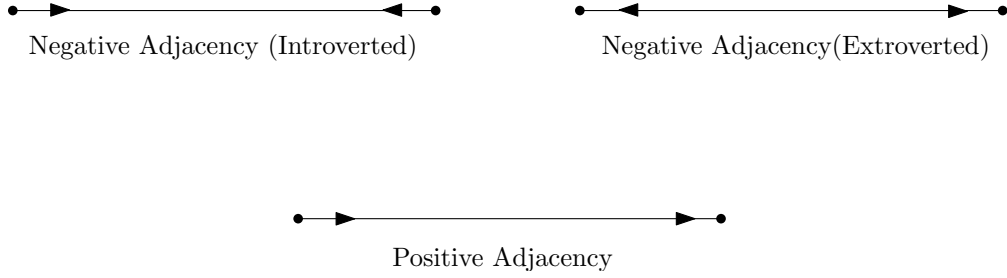


Figure 2: Positive and Negative Signed Adjacencies

Example 1. We now refer back to the oriented hypergraph G_1 in figure 1 which we will continue to do for the majority of this paper. This example is intended to increase understanding of the definitions in this section.

For the oriented hypergraph G_1 in figure 1, we have:

$$V = \{v_1, v_2, v_3\}, \quad E = \{e_1, e_2, e_3\}, \quad \mathcal{I} = \{i_1, i_2, \dots, i_8\}$$

Notice that $\iota(i_4) = \iota(i_5) = (v_2, e_2)$, and $\iota(i_7) = \iota(i_8) = (v_3, e_3)$. Also $\deg(v_2) = 3$, and $\text{size}(e_2) = 4$. Notice that v_1 is adjacent to v_2 by the three adjacencies $(v_1, i_2, e_2, i_4, v_2)$, $(v_1, i_2, e_2, i_5, v_2)$, and $(v_1, i_1, e_1, i_3, v_2)$ of which the first is negative and the other two are positive. In addition, v_3 is adjacent to v_3 by the negative adjacency $(v_3, i_7, e_3, i_8, v_3)$.

2.2 Weak Walks

A (*directed*) *weak walk* is a sequence $W = (a_0, i_1, a_1, i_2, a_2, i_3, a_3, \dots, a_{n-1}, i_n, a_n)$ of vertices, edges and incidences, where $\{a_\ell\}$ is an alternating sequence of vertices and edges, and i_h is an incidence between a_{h-1} and a_h . Specifically we have:

$$\{(proj_V \circ \iota)(i_h), (proj_E \circ \iota)(i_h)\} = \{a_{h-1}, a_h\}.$$

As with directed adjacencies, a *weak walk* is the set associated to a directed weak walk. The prefix *vertex/edge/cross* is used when the end points of a weak walk are vertices/edges/one edge and one vertex. The *length of a weak walk* is half the number of incidences in the weak walk.

A *vertex walk* is a weak walk where $a_0, a_n \in V$, and $i_{2h-1} \neq i_{2h}$ — this forbids the weak walk from entering an edge and immediately returning to the same vertex along the same incidence. Clearly, an adjacency is a vertex walk of length 1. A *vertex backstep* is a weak walk of length 1 of the form (v, i, e, i, v) , while a *loop* is a vertex walk of the form (v, i, e, j, v) where $i \neq j$. A *vertex path* is a vertex walk where no vertex or edge is repeated, while a *circle* is a vertex-path except $a_0 = a_n$. Analogous edge-centric definitions exist for the incidence dual and the results are inherited. A *connected graph* is a graph in which there exists a path from each vertex to every other vertex. Throughout this paper, all graphs are connected graphs.

The *sign of a weak walk* W is

$$sgn(W) = (-1)^{\lfloor n/2 \rfloor} \prod_{h=1}^n \sigma(i_h),$$

which is equivalent to taking the product of the signed adjacencies if W is a vertex-walk. Observe that vertex-backsteps are always negative. Throughout this paper, all walks and weak walks begin and end at a vertex and are of length 1. For this

reason, we can refer to them as simply either backsteps, or adjacencies (some of which are loops).

Example 2. *The walks below are from the oriented hypergraph G_1 in figure 1.*

- *Vertex Weak Walk of length 2:* $W = v_1, i_2, e_2, i_2, v_1, i_1, e_1, i_3, v_2, \text{sgn}(W) = -1$.
- *Edge Walk of length 2:* $e_2, i_6, i_7, e_3, i_8, v_3, i_6, e_2$.
- *Backstep:* v_2, i_4, e_2, i_4, v_2 .
- *Loop:* v_3, i_8, e_3, i_7, v_3 .
- *Adjacency:* v_2, i_5, e_2, i_6, v_3 .

2.3 Oriented Hypergraphic Matrices

In this section, we define five graphic matrices and provide examples of the matrices corresponding to the oriented hypergraph G_1 in Figure 1. We also discuss what the entries in certain matrices represent in the graph. This is important because the link between weak walks and the Laplacian matrix is the key idea to the following proof of what is being counted in the coefficients of the characteristic polynomial. The following definitions and theorems are adapted from [6].

The *incidence matrix* H_G of an oriented hypergraph G is the $V \times E$ matrix in which the (v, e) entry is $\sum \sigma(i)$, where the sum is taken over all incidences i such that $\iota(i) = (v, e)$. From [6] we know that the matrix $H_G^T = H_{G^*}$ where G^* is the incidence-dual of G .

The *adjacency matrix* A_G of an oriented hypergraph G is the $V \times V$ matrix whose (v, w) -entry is the sum of all signed adjacencies of the form (v, i, e, j, w) . Clearly, $\text{sgn}(v, i, e, j, w) = \text{sgn}(w, j, e, i, v)$ so A_G is a symmetric matrix.

The *degree matrix* of an oriented hypergraph G is the $V \times V$ diagonal matrix $D_G := \text{diag}(\deg(v_1), \dots, \deg(v_n))$. The *Laplacian matrix* of G is defined as $L_G :=$

$H_G H_G^T = D_G - A_G$ for all oriented hypergraphs.

Example 3. For the graph G_1 in Figure 1, we have

$$D_{G_1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad A_{G_1} = \begin{bmatrix} 0 & 1+1-1 & 1 \\ 1+1-1 & 1+1 & 1-1 \\ 1 & 1-1 & -1-1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$H_{G_1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1-1 & 0 \\ 0 & 1 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$L_{G_1} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

The (vertex) weak walk matrix of length k of an oriented hypergraph G is the $V \times V$ matrix $W_{G,k}$ where the ij -entry is the number of positive weak walks from v_i to v_j minus the number of negative weak walks from v_i to v_j . It was shown in [6], and improved in [3], that the entries of the Laplacian are the 1-weak-walk counts, and we collect the necessary relevant results.

Theorem 2.3.1 (See [6]). *Let G be an oriented hypergraph.*

1. *The ij -entry of D_G is negative the sum of the signs of the strictly weak walks of length 1 from v_i to v_j . That is, the number of backsteps from v_i to v_j .*
2. *The ij -entry of A_G is the sum of the signs of the (non-weak) walks of length 1 from v_i to v_j .*
3. *The ij -entry of L_G is negative the sum of the signs of the weak walks of length 1 from v_i to v_j . That is, $L_G = -W_{G,1}$.*

Part 3 of theorem 2.3.1 is referred to as the Weak Walk Theorem for oriented hypergraphs. In section 3.1 we find an equivalent statement to the weak walk theorem that will motivate the major proofs in this paper. It is important to note that there are corresponding matrices and theorems for edge walks and cross walks.

2.4 Sachs' Theorem

The following definitions are from [4] and Theorem 2.4.1 was first stated in 1963. The objects described below are similar to the objects called contributors defined in this thesis.

An *elementary figure* is either a K_2 (link graph) or a C_n (a cycle on n vertices) where $n \geq 1$. A *basic figure* U is a graph that is the disjoint union of elementary figures. Let $\mathcal{U}_i$ denote the set of all basic figures that are contained in G and have exactly i isolated vertices, let $p(U)$ be the number of elementary figures that make up U and let $c(U)$ be the number of circuits in U .

Theorem 2.4.1 (Sachs' Theorem). *For a multigraph G with $n = |V(G)|$,*

$$\chi(A_G, x) = \sum_{i=0}^n \left(\sum_{U \in \mathcal{U}_i} (-1)^{p(U)} (2)^{c(U)} \right) x^i.$$

2.5 The Matrix Tree Theorem

This section is a special case of a theorem for simple graphs. For more on a Matrix Tree Theorem for oriented simple graphs, see [8]. The following definitions only hold for graphs and have not been interpreted to include signed graphs and oriented hypergraphs.

A *tree* is a connected circle-free graph. A *spanning tree* of G is a tree that contains all vertices of the graph. It is known from [5] that every connected graph is a tree if and only if it contains $|V| - 1$ edges. Thus, a spanning tree of G contains

$|V|$ vertices, $|V| - 1$ edges, and no circles. We give an example of the spanning trees of a simple graph G_2 in figures 3, and 4.

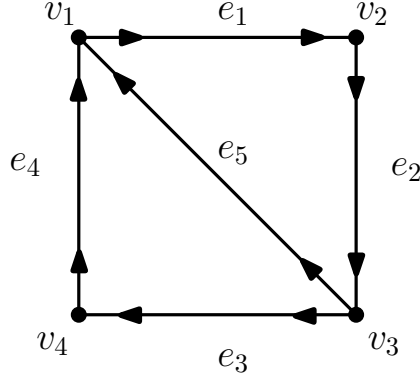


Figure 3: An oriented simple graph G_2

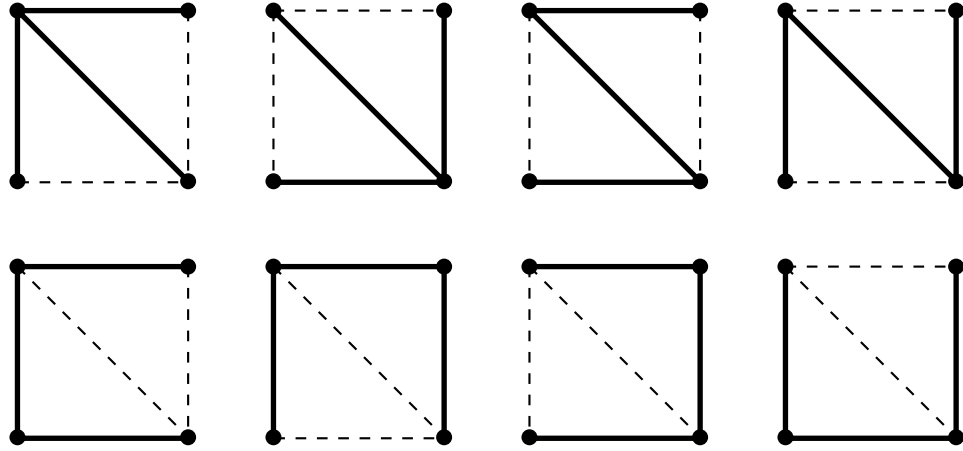


Figure 4: All 8 spanning trees of G_2 (the removed edges are left as dashed lines).

The following theorem is only for graphs.

Theorem 2.5.1 (Matrix Tree Theorem). *The number of spanning trees of a graph G is the value of any cofactor of the Laplacian matrix.*

One proof of this theorem is found in [8] where Tutte uses spanning aborescences, darts and the conductances of link-darts to prove this theorem. Another proof can be found in [2] where Chaiken uses directed arcs and also provides a count of objects when multiple rows and columns are deleted. This is called the

All Minors Matrix Tree Theorem and a version of this theorem is proved in section 6. We will provide an alternate proof method to show that this theorem is true and extend the results to include oriented hypergraphs.

Example 4. *For the graph G_2 in Figure 3, we have*

$$D_{G_2} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad A_{G_2} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad H_{G_2} = \begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$$L_{G_2} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \widetilde{W}_{G_2} = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}.$$

We now take the 1,1 cofactor of the Laplacian matrix and see that we get the number of spanning trees $T(G_2)$ pictured in figure 4 as follows:

$$T(G_2) = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 8.$$

3. PARTITIONS OF THE SET OF CONTRIBUTORS

In this section, we introduce incidence preserving maps called contributors which we associate with their image. We sum over these objects to prove the coefficient theorem and Matrix Tree Theorem that follow. In section 3.1 we are able to restate the Weak Walk Theorem in terms of incidence preserving maps that help to build these contributors. In section 3.2 we create equivalence classes of these contributors. In section 3.3 we consider partitions of contributors and relate them to the basic figures introduced in section 2.4 in order to reclaim Sachs' coefficient theorem.

3.1 Contributors

Given hypergraphs $H = (V_H, E_H, \mathcal{I}_H, \iota_H)$, and $G = (V_G, E_G, \mathcal{I}_G, \iota_G)$, an *incidence preserving map* is a function $\alpha : H \rightarrow G$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{I}_H & \xrightarrow{\alpha_{\mathcal{I}}} & \mathcal{I}_G \\ \downarrow \iota_H & & \downarrow \iota_G \\ V_H \times E_H & \xrightarrow{(\alpha_V \times \alpha_E)} & V_G \times E_G \end{array}$$

Let $\vec{P}_k$ be a directed vertex path graph of length k .

Lemma 3.1.1. *W is a weak walk of length k in G if, and only if, there is an incidence preserving map $\omega : \vec{P}_k \rightarrow G$ such that $\omega(\vec{P}_k) = W$.*

Proof. Let W be a vertex weak walk of length k and $\vec{P}_k$ be a directed vertex path of length k as follows:

$$W = (a_0, i_1, a_1, i_2, a_2, \dots, a_{2k-1}, i_{2k}, a_{2k})$$

$$\vec{P}_k = (v_0, j_1, e_1, j_2, v_1, j_3, e_2, \dots, e_k, j_{2k}, v_k).$$

We then have that $\omega : \vec{P}_k \rightarrow G$ where $\omega(v_b) = a_{2b}$, $\omega(e_b) = a_{2b-1}$, and $\omega(j_b) = i_b$ for all $b \in \{0, 1, \dots, k\}$ is an incidence preserving map with $\omega(\vec{P}_k) = W$.

On the other hand, if $\vec{P}_k$ is mapped into G via an incidence preserving map ω , then $\omega(\vec{P}_k)$ is determined by a sequence of possibly repeating incidences in G where the incidence nature of G is preserved. Thus $\omega(\vec{P}_k) = W$ for some weak walk W of length k in G . $\square$

Given $\omega : \vec{P}_k \rightarrow G$ such that $\omega(\vec{P}_k) = W$, we can redefine the sign of a weak walk as

$$\text{sgn}(W) = (-1)^{\lfloor k/2 \rfloor} \prod_{n=1}^k \sigma(\omega(i_n)) \sigma(\omega(j_n)).$$

From here we are able to restate the Weak Walk Theorem for $\mathbf{L}_G$ from [6, 3] (for weak walks of length 1) in terms of incidence preserving maps.

Theorem 3.1.2 (Weak Walk Theorem). *Let $\vec{P}_1 = (t, i, e, j, h)$ and $\omega : \vec{P}_1 \rightarrow G$. The vw -entry of $\mathbf{L}_G$ is $\sum_{\omega} -\text{sgn}(\omega(\vec{P}_1))$. Where the sum is over all incidence preserving maps ω such that $\omega(t) = v$ and $\omega(h) = w$.*

Let $\vec{P}_1 = (t, i, e, j, h)$ be the directed path of length 1, a *contributor* of G is an incidence preserving map $c : \prod_{v \in V} \vec{P}_1 \rightarrow G$ such that $c(t_v) = v$ and $\{c(h_v) \mid v \in V\} = V$. Let $\mathfrak{C}(G)$ denote the set of contributors. For a contributor $c \in \mathfrak{C}(G)$ let $ec(c)$ and $oc(c)$ be the number of even and odd circles in c , respectively. Similarly, let $pc(c)$ and $nc(c)$ be the number of positive and negative circles in c . Also, let $bs(c)$ be the number of backsteps in the contributor c .

By definition, each contributor creates a natural bijection from the vertex set to itself. Note that if for some $\vec{P}_1$, $c(i_v) = c(j_v)$, then this path is mapped to a backstep. If this condition does not hold, the the path is mapped to an adjacency. We associate each contributor, c , with its image in the codomain. In this way, we

will simply refer to these images as contributors. Figure 5 shows an example of four contributors from each of the two graphs G_1 and G_2 in figures 1 and 3 respectively. Notice that a 2 in the figure is an indication of the multiplicity of that incidence and therefore indicates a backstep or a repeated adjacency. Also, the underlying graphical structure is shown in gray behind every contributor.

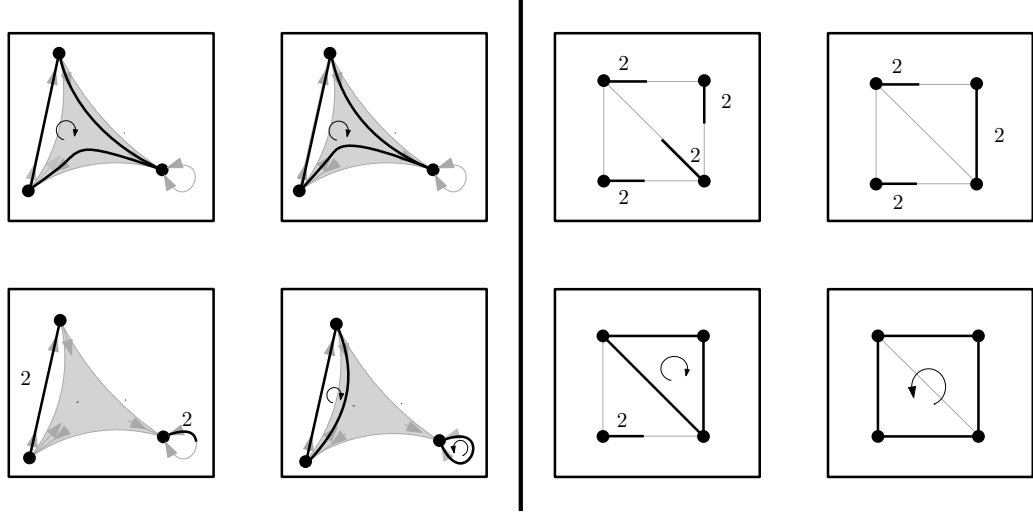


Figure 5: Examples of contributors of G_1 and G_2 .

3.2 Permutomorphic Contributors

Lemma 3.2.1. *Every contributor c is associated to a single permutation $\pi \in S_V$.*

Proof. We consider the permutations on the set S_V . Given a contributor of the graph G , we let the backsteps and loops in the contributor be fixed elements in the permutation and the directed circles of size 2 or larger and the degenerate 2-circles be cycles in the permutation. Notice that while loops in a contributor are considered circles, they will appear as fixed elements in the associated permutation simply because they are cycles of length 1. A contributor cannot be associated with two different permutations because then it would have to contain 2 different sets of circles of length greater than 2. Therefore, every contributor c is associated to a single permutation $\pi \in S_V$. $\square$

Two contributors that are associated to the same permutation π are said to be π -permutomorphic, let $\mathfrak{C}_\pi(G)$ denote the set of π -permutomorphic contributors. Figure 6 depicts the contributors in Figure 5 sorted by those that are π -permutomorphic and their corresponding permutations are listed below the contributors. Notice that for some permutations, $\mathfrak{C}_\pi(G)$ may be empty. This will occur if the circle associated to some cycle in π does not exist in the graph G .

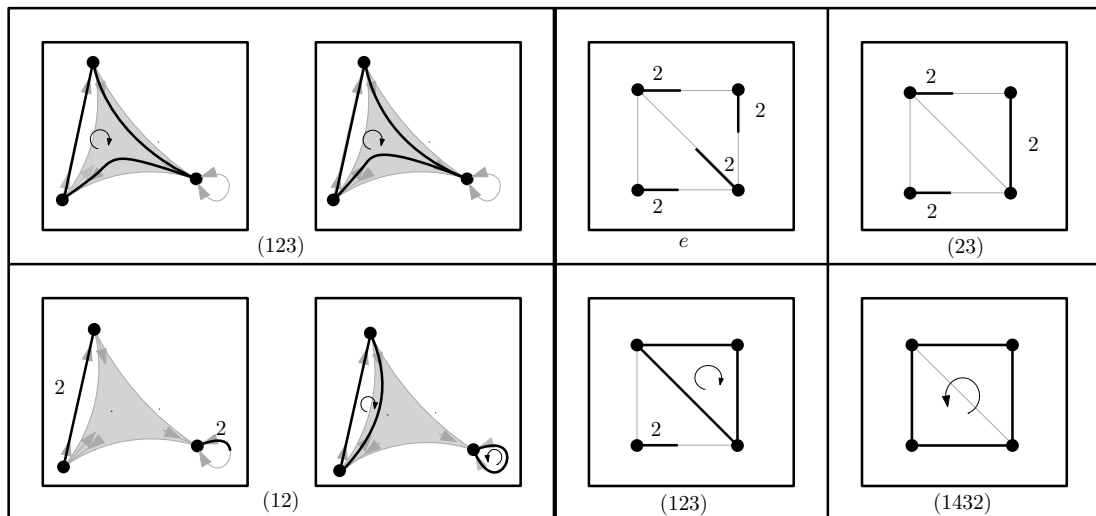


Figure 6: Examples of π -permutomorphic contributors of G_1 and G_2 .

Lemma 3.2.2. *The identification of contributors into permutomorphic sets is an equivalence relation. Furthermore, the sets $\mathfrak{C}_\pi(G)$ form the equivalence classes of $\mathfrak{C}(G)$.*

Observe that π -permutomorphic contributors need not be isomorphic for if $c(h_v) = v$ the associated algebraic 1-cycle may be a result of either a loop or a back-step. Similarly, an algebraic 2-cycle may arise from a repeated adjacency or two distinct adjacencies. However, these are the only obstructions to being isomorphic, as seen in the following lemma.

When an algebraic 2-cycle in the permutation corresponding contributor arises from a repeated adjacency, we call this a *degenerate 2-circle*. This is an important distinction because while a 2-circle has two possible cycle orientations, a

degenerate 2-circle has only one orientation.

Lemma 3.2.3. *Permutomorphic contributors are graphically-isomorphic on circles of size 3 or greater. However, they are not isomorphic on cycles of length 1 or 2 because loops and backsteps are indistinguishable in the associated permutation as are 2-circles and degenerate 2-circles.*

An example of Lemma 3.2.3 is seen in figure 6 where the contributors corresponding to the permutation (12) are not isomorphic due to two obstructions: the loop and backstep at v_3 , and the 2-circle and degenerate 2-circle between v_1 and v_2 .

3.3 Generalized Basic Figures and Cycle Covers

Let $\mathfrak{C}_{=k}(G)$ be the set of contributors of G with exactly k backsteps, $\mathfrak{C}_{\geq k}(G)$ be the set of contributors of G with at least k backsteps. Then define $\widehat{\mathfrak{C}}_{=k}(G)$ as the collection of sub-contributors of G formed by the contributors in $\mathfrak{C}_{=k}(G)$ after removing exactly k backsteps. Similarly, we define $\widehat{\mathfrak{C}}_{\geq k}(G)$ as the collection of sub-contributors of G formed by the contributors from $\mathfrak{C}_{\geq k}(G)$ after removing exactly k backsteps. Notice that more than one subcontributor in $\widehat{\mathfrak{C}}_{\geq k}(G)$ may come from the same contributor in $\mathfrak{C}_{\geq k}(G)$.

The following lemmas only hold for multigraphs (simple graphs with multiple adjacencies allowed). Recall that these are the structures on which Sachs' basic figures are defined. We will use these lemmas to reclaim Sachs' theorem in section 5.2.

Lemma 3.3.1. *For a graph G , contributors are disjoint unions of backsteps and adjacencies and these adjacencies are either contained in a degenerate 2-cycle, or a C_n .*

Proof. By definition contributors are disjoint unions of backsteps and adjacencies. Now assume there is an adjacency $a = (v, i, e, j, w)$ that is not contained in a degen-

erate 2-cycle. Thus to, fulfill the conditions of a contributor, there must be another adjacency from each of the 2 vertices contained in the first adjacency. Since $|V|$ is finite, at some point, the adjacencies must form a cycle. $\square$

Lemma 3.3.2. *For a multigraph G , $\widehat{\mathfrak{C}}_k(G)$ is the set of oriented basic figures on $|V| - k$ vertices.*

Proof. The set $\widehat{\mathfrak{C}}_k(G)$ contains those contributors with exactly k backsteps and k backsteps deleted. Thus, these contributors are disjoint unions of degenerate 2-cycles and oriented C_n 's that have exactly $|V| - k$ vertices. We see that degenerate 2-cycles are in one-to-one correspondence with K_2 's. Thus, the contributors in $\widehat{\mathfrak{C}}_k(G)$ are those that are disjoint unions of K_2 's and C_n 's. Therefore these contributors are the oriented basic figures on $|V| - k$ vertices. $\square$

A *cycle cover*, sometimes called a vertex cycle cover, of a graph G is a union of disjoint cycles which are subgraphs of G and contain all of the vertices of G . Notice that the cycle covers of a graph are simply the contributors that do not contain any backsteps. This fact is proved in the following Lemma.

Lemma 3.3.3. *For a multigraph G , $\widehat{\mathfrak{C}}_0(G)$ is the set of oriented cycle covers.*

Proof. By Lemma 3.3.2 with $k = 0$, we see that $\widehat{\mathfrak{C}}_0(G)$ is the set of oriented basic figures on $|V|$ vertices. Basic figures are disjoint unions of cycles, thus $\widehat{\mathfrak{C}}_0(G)$ is the set of cycle covers of G . $\square$

4. PERMANENTS AND DETERMINANTS

In this section we will see that when we take the permanent and determinant of the Laplacian and adjacency matrices, our calculations are simplified by using the fact that the entries in these matrices are characterized through weak walks. The following subsections show us that these calculations are simply the sums over contributors with specific attributes given a particular signing scheme for the contributors. Notice that the signing scheme in Sachs' Theorem in section 2.4.1 is $(-1)^{p(U)}$ where $p(U)$ is the number of basic figures in U . We will see that $p(U)$ in Sachs' Theorem is equivalent to the total number of circles in a given contributor. However, Sachs' Theorem is specifically for multigraphs, not signed graphs. Thus we must also discuss the occurrence of negative circles in a contributor. It turns out that our signing scheme depends on the number of positive circles, negative circles, even circles, and odd circles in a contributor.

We will now discuss why the number of odd circles, negative circles, and positive circles in a contributor might be relevant to how that contributor is counted. A simple graph may be regarded as a signed graph where all edges are positive, thus making all circles in the graph positive. This has lead to many signed graphic theorems based on signed graphs where all circles are positive and complementary theorems where all circles are negative (see [9, 10]). Thus, the number of negative circles in a graph can be seen as a way to measure how “far away” a signed graph is from being a graph.

The classical development of hypergraphs in [1] uses a $\{0, 1\}$ -incidence matrix which corresponds to the all-extroverted orientation of the associated hypergraph. This orientation causes all adjacencies in the hypergraph to be negative. This “trick” is used because it is very often impossible to make all of the adjacencies in an oriented hypergraph positive. When all of the adjacencies in the graph

have the same sign, it becomes much easier to talk about the signs of the circles in the graph. For instance, when using the classical hypergraphic incidence matrix, a circle is negative if, and only if, it is odd and a circle is positive if, and only if, it is even. Thus the parity of the circle size can often be used in place of the sign of the circle. When referring to the classical hypergraph, we will call its Laplacian matrix the *signless Laplacian matrix*

Two other important terms that are used when characterizing signed graphs and oriented hypergraphs are balance and balanceability. An oriented hypergraph is *balanced* if all circles are positive and is *balanceable* if there are incidences that can be negated such that the resulting oriented hypergraph is balanced [7]. Thus, the signs of the circles in an oriented hypergraph directly affect the balance of it and therefore play an important role in the characterization of the graph itself. While all graphs are clearly balanced, all signed graphs are balanceable. However, there are obstructions to oriented hypergraphs being balanceable which can be found in [7] and [3].

4.1 Permanents and Determinants of the Laplacian and Adjacency Matrices

The following theorem provides the correlation between the contributors discussed in section 3.1 and the permanent and determinant of the Laplacian matrices of oriented hypergraphs. It is important to note that these theorems not only apply to graphs, multigraphs and signed graphs but extend all the way to the generalized oriented hypergraph. Notice that part 3 and 4 of theorem 4.1.1 are stronger versions of Sachs' Theorem introduced in section 2.4.1.

Theorem 4.1.1. *Let G be an oriented hypergraph with adjacency matrix A_G and Laplacian matrix L_G , then*

$$1. \text{ perm}(L_G) = \sum_{c \in \mathfrak{C}(G)} (-1)^{oc(c)+nc(c)}$$

$$2. \det(L_G) = \sum_{c \in \mathfrak{C}(G)} (-1)^{pc(c)}$$

$$3. \text{perm}(A_G) = \sum_{c \in \mathfrak{C}_{=0}(G)} (-1)^{nc(c)}$$

$$4. \det(A_G) = \sum_{c \in \mathfrak{C}_{=0}(G)} (-1)^{ec(c)+nc(c)}$$

Proof. 1. From the definition of determinant and Theorem 3.1.2 we have

$$\text{perm}(L_G) = \sum_{\pi \in S_V} \prod_{v \in V} \sum_{\omega} -\text{sgn}(\omega(\vec{P}_1)),$$

Where the sum is over all incidence preserving maps ω such that $\omega(t) = v$ and $\omega(h) = \pi(v)$.

We now want to distribute the inner sums for all $v \in V$ and thus have a sum of products instead of a product of sums. In order to do this we must pass from the incidence preserving maps $\omega : \vec{P}_1 \rightarrow G$ with $\omega(t) = v$ and $\omega(h) = \pi(v)$ to the incidence preserving maps $c : \coprod_{v \in V} \vec{P}_1 \rightarrow G$ with $\omega(t_v) = v$ and $\omega(h_v) = \pi(v)$. Now we collect the permutomorphic contributors and get:

$$\text{perm}(L_G) = \sum_{\pi \in S_V} \sum_{c \in \mathfrak{C}_{\pi}(G)} \prod_{v \in V} \sigma(c(i_v))\sigma(c(j_v)).$$

For a fixed permutation π , we see that we are unable to distinguish the number and type of circles in permutomorphic contributors because the algebraic 1-cycles and 2-cycles in the permutation may or may not correspond to circles in the contributor as seen in Lemma 3.2.3. Thus we calculate the product $\prod_{v \in V} \sigma(c(i_v))\sigma(c(j_v))$ by factoring out -1 's from the adjacencies in the contributors. First we factor out a -1 for each adjacency in c which is equivalent to a net factor of $(-1)^{oc(c)}$ because the even cycles will have an even number of adjacencies. This now causes the negative adjacencies in G to appear as negative and the positive adjacencies to appear as positive because of the fact that

$L_G = D_G - A_G$. Then we factor out a -1 from every adjacency that is negative which is equivalent to a net factor of $(-1)^{nc(c)}$ because the positive circles must have an even number of negative adjacencies. Thus the total value factored out is $(-1)^{oc(c)+nc(c)}$ and no negative signs remain inside the product (now a product of positive 1's) so we get:

$$perm(L_G) = \sum_{\pi \in S_V} \sum_{c \in \mathfrak{C}_\pi(G)} (-1)^{oc(c)+nc(c)}$$

Finally from Lemma 3.2.2 we get:

$$perm(L_G) = \sum_{c \in \mathfrak{C}(G)} (-1)^{oc(c)+nc(c)}.$$

2. The proof of part 2 has a small adaptation from part 1 as now include the sign of the permutation in the calculation. Thus we have:

$$det(L_G) = \sum_{\pi \in S_V} (-1)^{ec(\pi)} \sum_{c \in \mathfrak{C}_\pi(G)} (-1)^{oc(c)+nc(c)}$$

$$det(L_G) = \sum_{\pi \in S_V} \sum_{c \in \mathfrak{C}_\pi(G)} (-1)^{ec(\pi)+oc(c)+nc(c)}$$

However, all of the even cycles in π correspond to even circles in all $c \in \mathfrak{C}(G)$. This is because the only obstruction is the 1-cycles in π which don't always correspond to 1-circles in all $c \in \mathfrak{C}(G)$ (see Lemma 3.2.3). Thus $(-1)^{ec(\pi)} = (-1)^{ec(c)}$, and we get:

$$det(L_G) = \sum_{\pi \in S_V} \sum_{c \in \mathfrak{C}_\pi(G)} (-1)^{ec(c)+oc(c)+nc(c)}$$

Finally, since $(-1)^{ec(c)+oc(c)+nc(c)} = (-1)^{tc(c)+nc(c)} = (-1)^{tc(c)-nc(c)} = (-1)^{pc(c)}$:

$$\det(L_G) = \sum_{\pi \in S_V} \sum_{c \in \mathfrak{C}_\pi(G)} (-1)^{pc(c)}$$

$$\det(L_G) = \sum_{c \in \mathfrak{C}(G)} (-1)^{pc(c)}$$

3. The proof of part 3 is also similar to part 1. One difference is that the sum is now over an adjacency preserving ω we call ω' . Also, when we factor out negative ones, we only need to take one out for every negative adjacency which is equivalent to a net factor of $(-1)^{nc(c)}$ as before. Thus we get:

$$\text{perm}(A_G) = \sum_{\pi \in S_V} \sum_{c \in \mathfrak{C}_{=0,\pi}(G)} (-1)^{nc(c)},$$

$$\text{perm}(A_G) = \sum_{c \in \mathfrak{C}_{=0}(G)} (-1)^{nc(c)}.$$

4. The proof of part 4 combines part 2 and 3 and we get:

$$\det(A_G) = \sum_{\pi \in S_V} \sum_{c \in \mathfrak{C}_{=0,\pi}(G)} (-1)^{ec(c)+nc(c)},$$

$$\det(A_G) = \sum_{c \in \mathfrak{C}_{=0}(G)} (-1)^{ec(c)+nc(c)}.$$

□

Example 5. We now provide an example of this proof for the graph G_1 in figure 1. We will only do this example on part 1 of the theorem, $\text{perm}(L_G)$. In order to see where the contributors are coming from, we need to write the Laplacian in an expanded form so that the signs of every adjacency are present. We do this by subtracting the expanded form of the adjacency matrix from the degree matrix. Referring

to the matrices in section 2.3, we have:

$$L(G_1) = \begin{bmatrix} (2) - (0) & (0) - (1 + 1 - 1) & (0) - (1) \\ (0) - (1 + 1 - 1) & (3) - (1 + 1) & (0) - (1 - 1) \\ (0) - (1) & (0) - (1 - 1) & (3) - (-1 - 1) \end{bmatrix}$$

Now consider the permutation $\pi = (12)$ and the contributor from figure 6 pictured below:

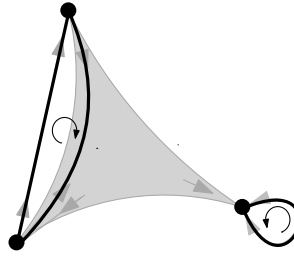


Figure 7: A contributor from the graph G_1 with permutation $\pi = (12)$

In order to find the sign associated to this contributor in the summation we take the entries of the Laplacian matrix corresponding to the permutation and we get:

$$[(0) - (1 + 1 - 1)] \cdot [(0) - (1 + 1 - 1)] \cdot [(3) - (-1 - 1)]$$

When these sums are distributed, our contributor comes from the numbers in braces below:

$$[(0) - (1 + 1 \underbrace{-1})] \cdot [(0) - (\underbrace{1} + 1 - 1)] \cdot [(3) - (\underbrace{-1} - 1)].$$

Thus, the contribution of our contributor is:

$$[-(-1)] \cdot [-(1)] \cdot [-(-1)].$$

Factoring out a negative for each adjacency: $(-1)^3[-1] \cdot [1] \cdot [-1]$.

Factoring out a negative from each negative adjacency: $(-1)^{3+2}[1] \cdot [1] \cdot [1]$.

The net sign on our contributor is $(-1)^5 = (-1)$.

Also $(-1)^{oc(c)+nc(c)} = (-1)^{1+2} = (-1)$.

4.2 Optimizing

The following theorem provides a way to maximize the value of the permanent of the Laplacian matrix for an oriented hypergraph.

Theorem 4.2.1. *For a fixed underlying oriented hypergraph G and varied orientation function σ , the following are equivalent:*

1. $perm(L_G)$ achieves its max for σ ,
2. σ is the all extroverted or all introverted orientation,
3. L_G is the signless Laplacian,
4. $perm(L_G) = |\mathfrak{C}(G)|$.

Proof. Part 2 and 3 are trivially equivalent. We will now show the equivalence of parts 1 and 3 using the fact that in the signless Laplacian, a circle in an associated contributor is negative if, and only if, it is odd. From Theorem 4.1.1, $perm(L_G)$ is maximal when for every contributor c , $oc(c)$ and $nc(c)$ have the same parity.

If $oc(c) \neq nc(c)$ then there exists a circle in a contributor which is either odd and not negative or negative and not odd. Refine the corresponding algebraic cycle into fixed elements thus causing each element in the circle to become a back-step, thus forming a new contributor c' . Now the parity of $oc(c')$ and $nc(c')$ are not equal, so it must be true that $oc(c) = nc(c)$ for all $c \in \mathfrak{C}(G)$.

It must also be true that when $oc(c) = nc(c)$ they are the same set of circles by the same argument as above. Thus $perm(L_G)$ is maximal if, and only if, L_G is the signless Laplacian.

The equivalence for part 4 is obvious because if $perm(L_G)$ is maximal then by Theorem 4.1.1, $perm(L_G)$ is the number of contributors. □

5. CHARACTERISTIC POLYNOMIALS

We now investigate the coefficients of the characteristic polynomial of the Laplacian and adjacency matrices. In this section we will also see the equivalence of Sachs' Theorem and the version in theorem 5.1.1

Let $\chi^D(M, x) = \det(xI - M)$ be the determinant-based characteristic polynomial and $\chi^P(M, x) = \text{perm}(xI - M)$ be the permanent-based characteristic polynomial. It is important to note that sometimes the characteristic polynomial is formed by the determinant of $M - xI$ which only differs from the version used above by a factor of $(-1)^{|V|}$. we chose the version above to parallel Sachs' approach to the following theorem.

5.1 Coefficient Theorems

Theorem 5.1.1. *Let G be an oriented hypergraph with adjacency matrix A_G and Laplacian matrix L_G , then the characteristic polynomials are as follows:*

$$\begin{aligned}
 1. \quad \chi^P(A_G, x) &= \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k}(G)} (-1)^{oc(c)+nc(c)} \right) x^k, \\
 2. \quad \chi^D(A_G, x) &= \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k}(G)} (-1)^{pc(c)} \right) x^k, \\
 3. \quad \chi^P(L_G, x) &= \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{\geq k}(G)} (-1)^{nc(c)+bs(c)} \right) x^k, \\
 4. \quad \chi^D(L_G, x) &= \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{\geq k}(G)} (-1)^{ec(c)+nc(c)+bs(c)} \right) x^k.
 \end{aligned}$$

Proof. We begin by proving part 1, as it parallels the techniques used in part 1 of Theorem 4.1.1. Notice that this means the proof of the permanent of the Laplacian matrix is extremely similar to the proof of the permanent-based characteristic polynomial of the adjacency matrix.

1. To prove part 1, we evaluate $\text{perm}(xI - A_G)$. In order to do this, we create a choice function for a given permutation π and a vertex v as follows:

$$\alpha : v \rightarrow \left\{ x \cdot \delta(v, \pi(v)), \sum_{\omega'} \text{sgn}(\omega'(\vec{P}_1)) \right\}.$$

Recall that ω' is an adjacency preserving and incidence preserving map from $\vec{P}_1 \rightarrow G$ such that $\omega'(t) = v$ and $\omega'(h) = \pi(v)$. Observe that if $\pi(v) = v$, then α maps v to either $\sum_{\omega'} \text{sgn}(\omega'(\vec{P}_1))$ or x . However, if $\pi(v) \neq v$, then α maps v to either $\sum_{\omega'} \text{sgn}(\omega'(\vec{P}_1))$ or 0. Thus, the determinant can be written as:

$$\begin{aligned} \chi^P(A_G, x) &= \text{perm}(xI - A_G) \\ &= \sum_{\pi \in S_V} \prod_{v \in V} \sum_{\alpha} \alpha(v) \end{aligned}$$

We now define another function β that will allow us to distribute and thus switch the order of the sum and product.

$$\beta : V \rightarrow \left\{ x \cdot \delta(v, \pi(v)), \sum_{\omega'} \text{sgn}(\omega'(\vec{P}_1)) \right\}.$$

After distributing, we get

$$\chi^P(A_G, x) = \sum_{\pi \in S_V} \sum_{\beta} \prod_{v \in V} \beta(v).$$

Notice that if a β maps any non-fixed point v to $x \cdot \delta(v, \pi(v))$, then $\prod_{v \in V} \beta(v) = 0$, so we may ignore them in the sum. We now sum the β 's with exactly k mapping to x . This produces a term of x^k . The remaining β 's are necessarily closed walk covers of the remaining $|V| - k$ vertices, or elements of $\widehat{\mathfrak{C}}_{=k}(G)$,

$$= \sum_{\pi \in S_V} \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k, \pi}(G)} \prod_{v \in c(V)} \sigma(c(i_v)) \cdot \sigma(c(j_v)) \right) x^k.$$

Now we reverse the order of the first two summations because they don't depend on each other. Proceeding as in the proof of part 1 of Theorem 4.1.1, we factor out a -1 for each adjacency and a -1 for each negative adjacency to get

$$\begin{aligned}
&= \sum_{k=0}^{|V|} \sum_{\pi \in S_V} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k, \pi}(G)} \prod_{v \in c(V)} \sigma(c(i_v)) \cdot \sigma(c(j_v)) \right) x^k, \\
&= \sum_{k=0}^{|V|} \sum_{\pi \in S_V} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k, \pi}(G)} (-1)^{oc(c) + nc(c)} \right) x^k, \\
&= \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k}(G)} (-1)^{oc(c) + nc(c)} \right) x^k.
\end{aligned}$$

2. The proof of part 2 mirrors the proof of part 2 in Theorem 4.1.1. Thus we repeat the proof above with the inclusion of the sign of the permutation and the last few steps now appear as

$$\begin{aligned}
\chi^D(A_G, x) &= \sum_{k=0}^{|V|} \sum_{\pi \in S_V} (-1)^{ec(\pi)} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k, \pi}(G)} (-1)^{oc(c) + nc(c)} \right) x^k, \\
&= \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k}(G)} (-1)^{ec(\pi) + oc(c) + nc(c)} \right) x^k, \\
&= \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k}(G)} (-1)^{ec(c) + oc(c) + nc(c)} \right) x^k, \\
&= \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k}(G)} (-1)^{pc(c)} \right) x^k
\end{aligned}$$

3. Part 3 is similar to part 1, but we use the incidence preserving map ω as used in part 1 and 2 of Theorem 4.1.1 so the contributors we are summing over are now elements of $\widehat{\mathfrak{C}}_{\geq k}(G)$. Also, there are changes when we begin to factor out the -1 's. Notice that $xI - L_G = xI - D_G + A_G$, so the signs of the adjacencies

are represented truthfully, while the signs of the backsteps are opposite. Thus, we factor out a -1 for every backstep in the contributor to produce $(-1)^{bs(c)}$. Next, as usual, we factor out a (-1) for every negative adjacency, producing $(-1)^{nc(c)}$.

4. Part 4 simply differs by $(-1)^{ec(c)}$ which comes from the sign of the permutation.

□

Example 6. To see the objects that Theorem 5.1.1 enumerates, consider G_2 from figure 3. In figure 8 below, all of the contributors of the graph G_2 are shown.

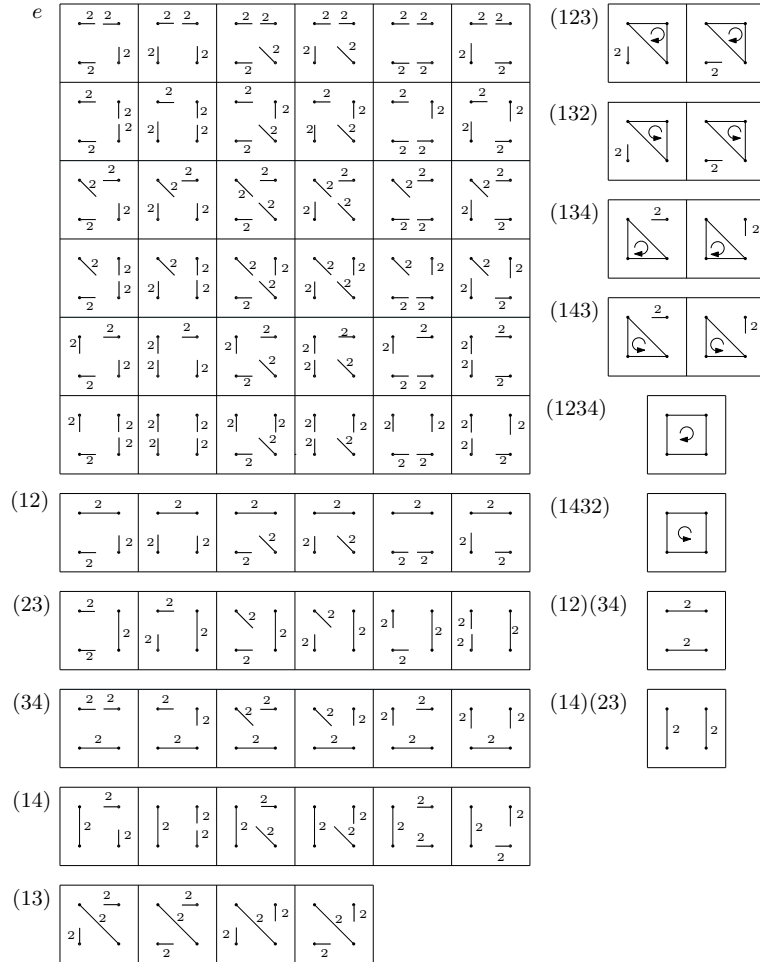


Figure 8: All of the contributors of G_2 .

Now, using the matrices in example 4 we calculate

$$\chi^D(A_{G_2}, x) = x^4 - 5x^2 - 4x.$$

By theorem 5.1.1, the coefficient on x^k is the signed count of sub-contributors formed from the contributors with exactly k backsteps after k backsteps have been removed, where the signing function is $(-1)^{pc(c)}$. The 1 in front of x^4 is the set of isolated vertices. There is a 0 in front of x^3 because there are no contributors with exactly 3 backsteps. The -5 in front of the x^2 is represented by one representative from each permutomorphic-grouping of contributors with exactly 2 backsteps. The sign of each of these is $(-1)^1 = -1$. The -4 in front of the x is a count of one representative from each permutomorphic grouping of contributors with exactly 1 backstep. The sign of each of these is $(-1)^1 = -1$. Finally, the constant is 0 because of the 4 contributors that have no backsteps, there are two of sign -1 and two of sign 1 so they add up to 0.

This process can be repeated for the characteristic polynomial

$$\chi^D(L_{G_2}, x) = x^4 - 10x^3 - 32x^2 - 34x.$$

5.2 Alternative Proof of Sachs' Theorem

Recall that Sachs' Theorem enumerates the coefficients for the determinant-based characteristic polynomial of the adjacency matrix for a directed multigraph. Theorem 5.1.1 part 2 can therefore be used to prove Sachs' Theorem.

Corollary 5.2.1. *Sachs' Theorem is equivalent to Theorem 5.1.1 part 2 if G is a multigraph and oriented circles are combined.*

Proof. We have

$$\chi^D(A_G, x) = \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k}(G)} (-1)^{pc(c)} \right) x^k.$$

Since G is not a signed graph, all circles are positive, so we can replace $(-1)^{pc(c)}$ with $(-1)^{tc(c)}$. Now since we are only summing over contributors in $\mathfrak{C}_{=k}(G)$, there are no backsteps, so every circle is an elementary figure. Thus we have $(-1)^{tc(c)} = (-1)^{\text{number of elementary figures in } c}$. Finally, since the circles in basic figures are unoriented, there are two contributors corresponding to each basic figure. Thus

$$\chi^D(A_G, x) = \sum_{i=0}^n \left(\sum_{U \in \mathcal{U}_i} (-1)^{p(U)} (2)^{c(U)} \right) x^i$$

□

6. THE ALL MINORS MATRIX TREE THEOREM

The All Minors Matrix Tree Theorem in [2] enumerates the minors of the Laplacian Matrix of a directed graph after deleting equal sized sets of rows and columns. In this section, we provide an alternative statement of this theorem that extends to oriented hypergraphs. In order to do this, we use similar proof methods to those in Theorem 4.1.1 and Theorem 5.1.1. Below is the statement of Chaiken's Theorem without the exact sum provided. See [2] for further explanation of this version of the All Minors Matrix Tree Theorem.

Theorem 6.0.1 (The All Minors Matrix Tree Theorem [2]). *The determinant of the matrix that results from deleting sets of k rows W and columns U from the Laplacian matrix of a digraph G is equal to the number of forests that have:*

1. k trees,
2. Each of the trees contains exactly one vertex in each U and W ,
3. Each arc is directed away from the vertex in U of the tree containing the arc.

6.1 The All Minors Matrix Tree Theorem for Oriented Hypergraphs

Before we can state a version of this theorem for oriented hypergraphs, we need to introduce another set of sub-contributors $\widehat{\mathfrak{C}}(U, W)$ as follows. Given $U, W \subseteq V$ with $|U| = |W|$,

$$\widehat{\mathfrak{C}}(U, W) = \left\{ c : \coprod_{v \in \bar{U}} \vec{P}_1 \rightarrow G \mid c(t_v) = v, \{c(h_v) \mid v \in (\bar{U})\} = \bar{W} \right\}.$$

This is the set of contributors formed by the incidence preserving maps from $|\bar{U}|$ disjoint unions of paths of length 1 to the oriented hypergraph G such that the tails of these paths are mapped to the vertices in $\bar{U}$ and the heads are mapped to

the vertices in $\bar{W}$. Then let

$$\widehat{\mathfrak{C}}_{\pi|_{\bar{U} \rightarrow \bar{W}}}(U, W) = \left\{ c : \coprod_{v \in \bar{U}} \vec{P}_1 \rightarrow G \mid c(t_v) = v, c(h_v) = \pi|_{\bar{U} \rightarrow \bar{W}}(v) \right\}.$$

We say that two contributors are *sub-permutomorphic* if they are associated to the same bijection $\pi|_{\bar{U} \rightarrow \bar{W}}$. Let $on(c)$ be the number of odd nontrivial components in the contributor c and let $nn(c)$ be the number of negative nontrivial components in the contributor c where a *trivial component* of a contributor is a component that does not contain an adjacency (namely a backstep).

Theorem 6.1.1. *Let G be an oriented hypergraph with Laplacian matrix L_G . Given $U, W \subseteq V$ with $|U| = |W|$, let $[L_G]_{UW}$ be the minor of L_G formed by deleting the rows corresponding to the vertices in U and deleting the columns corresponding to the vertices in W . Then we have,*

$$\text{perm}([L_G]_{UW}) = \sum_{c \in \widehat{\mathfrak{C}}(U, W)} (-1)^{on(c) + nn(c)}.$$

Proof. Define:

$$S_V|_{\bar{U} \rightarrow \bar{W}} = \{ \pi|_{\bar{U} \rightarrow \bar{W}} \mid \pi \in S_V \}$$

Now we see that the bijections in the set $S_V|_{\bar{U} \rightarrow \bar{W}}$ are not full permutations, but represent the permutations of the minors if the rows and columns were relabeled after deleting the rows corresponding to the vertices in U and the columns corresponding to the vertices in W . However, we will not relabel the rows and columns and will instead just call these functions bijections from $\bar{U}$ to $\bar{W}$. Thus we get,

$$\text{perm}([L_G]_{UW}) = \sum_{\pi|_{\bar{U} \rightarrow \bar{W}} \in S_V|_{\bar{U} \rightarrow \bar{W}}} \prod_{v \in \bar{U}} \sum_{\omega} -\text{sgn}(\omega(\vec{P}_1)).$$

Where the sum is over all incidence preserving maps ω such that $\omega(t) = v$

and $\omega(h) = \pi|_{\bar{U} \rightarrow \bar{W}}(v)$. Collecting the permutomorphic contributors, we get

$$\text{perm}([L_G]_{UW}) = \sum_{\pi|_{\bar{U} \rightarrow \bar{W}} \in S_V|_{\bar{U} \rightarrow \bar{W}}} \sum_{c \in \widehat{\mathfrak{C}}_{\pi|_{\bar{U} \rightarrow \bar{W}}}(U,W)} \prod_{v \in \bar{U}} \sigma(c(i_v)) \sigma(c(j_v)).$$

As before, we factor out a (-1) for every adjacency and then factor out a (-1) for every negative adjacency. It is clear that this is equivalent to a net factor of $(-1)^{on(c)+nn(c)}$.

$$\text{perm}([L_G]_{UW}) = \sum_{\pi|_{\bar{U} \rightarrow \bar{W}} \in S_V|_{\bar{U} \rightarrow \bar{W}}} \sum_{c \in \widehat{\mathfrak{C}}_{\pi|_{\bar{U} \rightarrow \bar{W}}}(U,W)} (-1)^{on(c)+nn(c)}.$$

$$\text{perm}([L_G]_{UW}) = \sum_{c \in \widehat{\mathfrak{C}}(U,W)} (-1)^{on(c)+nn(c)}$$

□

Theorem 6.1.2 (The All Minors Matrix Tree Theorem for Oriented Hypergraphs).

Let G be an oriented hypergraph with Laplacian matrix L_G . Given $U, W \subseteq V$ with $|U| = |V|$, let $[L_G]_{UW}$ be the minor of L_G formed by deleting the rows corresponding to the vertices in U and deleting the columns corresponding to the vertices in W . Then we have,

$$\det([L_G]_{UW}) = \sum_{c \in \widehat{\mathfrak{C}}(U,W)} \epsilon(\pi|_{\bar{U} \rightarrow \bar{W}}) (-1)^{on(c)+nn(c)}.$$

Where $\epsilon(\pi|_{\bar{U} \rightarrow \bar{W}}) = (-1)^{n(\pi|_{\bar{U} \rightarrow \bar{W}})}$, and $n(\pi|_{\bar{U} \rightarrow \bar{W}})$ is the number of inversions in $\pi|_{\bar{U} \rightarrow \bar{W}}$.

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