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UNIFORM CONVERGENCE OF THE SPECTRAL EXPANSIONS IN TERMS OF ROOT FUNCTIONS FOR A SPECTRAL PROBLEM

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Abstract. In this article, we consider the spectral problem

$$-y'' + q(x)y = \lambda y, \quad 0 < x < 1,$$
$$y'(0)\sin\beta = y(0)\cos\beta, \quad 0 \le \beta < \pi; \quad y'(1) = (a\lambda + b)y(1)$$

where λ is a spectral parameter, a and b are real constants and $a < 0$, $q(x)$ is a real-valued continuous function on the interval $[0, 1]$. The root function system of this problem can also consist of associated functions. We investigate the uniform convergence of the spectral expansions in terms of root functions.

1. Introduction

Consider the spectral problem

$$-y'' + q(x)y = \lambda y, \quad 0 < x < 1, \tag{1.1}$$
$$y'(0)\sin\beta = y(0)\cos\beta, \quad 0 \le \beta < \pi, \tag{1.2}$$
$$y'(1) = (a\lambda + b)y(1), \tag{1.3}$$

where λ is a spectral parameter, a and b are real constants and $a < 0$, $q(x)$ is real-valued continuous function on the interval $[0, 1]$.

In this article, we study the uniform convergence of the expansions in terms of root functions of the boundary value problem (1.1)–(1.3) for the functions which belong to $C[0, 1]$. There are many articles which investigate the uniform convergence of the expansions for the functions in terms of root functions of some differential operators with a spectral parameter in the boundary conditions (see, for example, [4, 5, 7, 8, 9, 10, 11, 12, 13, 14]).

Especially, the spectral problems which investigated the uniform convergence of the spectral expansions underlie an important class of the mathematical physics problems. For example, the problem

$$u''(x) + \lambda u(x) = 0 \quad (0 < x < 1),$$
$$u(1) = 0, \quad (a - \lambda)u'(0) + b\lambda u(0) = 0, \quad a, b > 0$$

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appears in a model of transrelaxation heat process and in the mathematical description of vibrations of a loaded string (see [8]), and the problems on vibrations of a homogeneous loaded string, torsional vibrations of a rod with a pulley at one end, heat propagation in a rod with lumped heat capacity at one end, the current in a cable ground at one end through a concentrated capacitance or inductance lead to the spectral problem

$$u''(x) + \lambda u(x) = 0\,(0 < x < 1),$$
$$u(0) = 0, \quad u'(1) = d\lambda u(1), \quad d > 0$$

(see [8, 9]).

In [13], it has been investigated the uniform convergence of the Fourier series expansions in terms of eigenfunctions for the spectral problem

$$-y'' + q(x)y = \lambda y, \quad 0 < x < 1, \tag{1.4}$$
$$b_0 y(0) = d_0 y'(0), \quad (a_1\lambda + b_1)y(1) = (c_1\lambda + d_1)y'(1), \tag{1.5}$$

where λ is a spectral parameter, $q(x)$ is a real-valued continuous function on the interval $[0, 1]$, and a_1, b_0, b_1, c_1, d_0 and d_1 are real constants that satisfy the conditions

$$|b_0| + |d_0| \neq 0, \quad \sigma = a_1 d_1 - b_1 c_1 > 0. \tag{1.6}$$

Note that all the eigenvalues of problem (1.4), (1.5) are real and simple, hence the root functions system of this problem consists of only eigenfunctions. Problem (1.1)–(1.3) does not satisfy the condition (1.6), because $\sigma = a < 0$.

It was proved [3] that the eigenvalues of (1.1)–(1.3) form an infinite sequence λ_n, $(n = 0, 1, 2, \dots)$ without finite limit points and only the following cases are possible:

(i) all the eigenvalues are real and simple.
(ii) all the eigenvalues are real and all, except one double, are simple.
(iii) all the eigenvalues are real and all, except one triple, are simple.
(iv) all the eigenvalues are simple and all, except a conjugate pair of non-real, are real.

Note that the eigenvalues λ_n $(n = 0, 1, 2, \dots)$ were considered to be listed according to non-decreasing real part and repeated according to algebraic multiplicity. Therefore, the results of the article [13] cannot be applied directly to the problem (1.1)–(1.3).

We need some properties of eigenvalues, eigenfunctions and associated functions of problem (1.1)–(1.3), for the uniform convergence of the spectral expansions in terms of root functions of this problem.

Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ denote the solutions of (1.1) which satisfy the initial conditions

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h, \tag{1.7}$$
$$\psi(0, \lambda) = 0, \quad \psi'(0, \lambda) = 1, \tag{1.8}$$

where $h = \cot\beta$, $(0 < \beta < \pi)$.

It is easy to see by the same method as in [13, theorem 2.1] that the following asymptotic formulae are valid for sufficiently large n:

(i) If $\beta = 0$ and $\lambda_n = \mu_n^2$, ($\operatorname{Re}\mu_n \geq 0$), then

$$\mu_n = n\pi + \frac{A_1}{n\pi} + O\big(\frac{\delta_n^{(1)}}{n}\big), \tag{1.9}$$

$$\begin{aligned} y_n(x) &= \psi(x,\lambda_n) \\ &= \frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{(n\pi)^2}\big[A_1 x - \frac{1}{2}\int_0^x q(\tau)d\tau + \frac{1}{2}\int_0^x q(\tau)\cos 2n\pi\tau d\tau\big] \\ &\quad + \frac{\sin n\pi x}{2(n\pi)^2}\int_0^x q(\tau)\sin(2n\pi\tau)d\tau + O\big(\frac{\delta_n^{(1)}}{n^2}\big), \end{aligned} \tag{1.10}$$

where

$$A_1 = \frac{1}{a} + \frac{1}{2}\int_0^1 q(\tau)d\tau, \quad \delta_n^{(1)} = \Big|\int_0^1 q(\tau)\cos(2n\pi\tau)d\tau\Big| + \frac{1}{n}.$$

(ii) If $0 < \beta < \pi$ and $\lambda_n = \mu_n^2(\operatorname{Re}\mu_n \geq 0)$ then

$$\mu_n = (n - \frac{1}{2})\pi + \frac{A_2}{(n-\frac{1}{2})\pi} + O\big(\frac{\delta_n^{(2)}}{n}\big), \tag{1.11}$$

$$\begin{aligned} y_n(x) &= \varphi(x,\lambda_n) \\ &= \cos(n-\frac{1}{2})\pi x + \frac{\sin(n-\frac{1}{2})\pi x}{(n-\frac{1}{2})\pi}\Big[h - A_2 x + \frac{1}{2}\int_0^x q(\tau)d\tau \\ &\quad + \frac{1}{2}\int_0^x q(\tau)\cos(2n-1)\pi\tau d\tau\Big] \\ &\quad - \frac{\cos(n-\frac{1}{2})\pi x}{(2n-1)\pi}\int_0^x q(\tau)\sin(2n-1)\pi\tau d\tau + O\big(\frac{\delta_n^{(2)}}{n}\big), \end{aligned} \tag{1.12}$$

where

$$A_2 = h + \frac{1}{a} + \frac{1}{2}\int_0^1 q(\tau)d\tau, \quad \delta_n^{(2)} = \Big|\int_0^1 q(\tau)\cos(2n-1)\pi\tau d\tau\Big| + \frac{1}{n}.$$

Let λ_k be a multiple eigenvalue ($\lambda_k = \lambda_{k+1}$). Then for the first order associated function y_{k+1} corresponding to the eigenfunction y_k, the following relations hold [15, p. 28]

$$\begin{gathered} -y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k, \\ y'_{k+1}(0)\sin\beta = y_{k+1}(0)\cos\beta, \\ y'_{k+1}(1) = (a\lambda_k + b)y_{k+1}(1) + a y_k(1). \end{gathered}$$

Let λ_k be a triple eigenvalue ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$). Then for the first order associated function y_{k+1} there exist the second order associated function y_{k+2} for which the following relations hold

$$\begin{gathered} -y''_{k+2} + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1}, \\ y'_{k+2}(0)\sin\beta = y_{k+2}(0)\cos\beta, \\ y'_{k+2}(1) = (a\lambda_k + b)y_{k+2}(1) + a y_{k+1}(1). \end{gathered}$$

Note that the functions $y_{k+1} + c y_k$ and $y_{k+2} + d y_k$, where c and d are arbitrary constants, are also associated functions of the first and second order respectively.

Let $y(x,\lambda)$ denote the solution of the equation (1.1) which satisfy the initial condition (1.7) if $0<\beta<\pi$ or (1.8) if $\beta=0$. Then, the eigenvalues of (1.1)–(1.3) are the roots of the characteristic equation

$$\omega(\lambda)=y'(1,\lambda)-(a\lambda+b)y(1,\lambda). \tag{1.13}$$

It was proven in [1] that if λ_k is a multiple (double or triple) eigenvalue of (1.1)–(1.3), then

$$\begin{gathered} y(x,\lambda)\to y_k(x),\quad y'(x,\lambda)\to {y'}_k(x),\\ y_\lambda(x,\lambda)\to \tilde{y}_{k+1}(x),\quad {y'}_\lambda(x,\lambda)\to \tilde{y}'_{k+1}(x) \end{gathered} \tag{1.14}$$

uniformly according to $x\in[0,1]$, as $\lambda\to\lambda_k$ (see also [6]), where $\tilde{y}_{k+1}$ is one of the associated functions of the first order. It is obvious that $\tilde{y}_{k+1}=y_{k+1}+\tilde{c}y_k$.

Furthermore, if λ_k is a triple eigenvalue of (1.1)–(1.3), then

$$y_{\lambda\lambda}(x,\lambda)\to 2\tilde{y}_{k+2}(x),\quad {y'}_{\lambda\lambda}(x,\lambda)\to 2\tilde{y}'_{k+2}(x) \tag{1.15}$$

uniformly according to $x\in[0,1]$, as $\lambda\to\lambda_k$, where $\tilde{y}_{k+2}$ is one of the associated functions of the second order corresponding to the first associated function $\tilde{y}_{k+1}$. It is obvious that $\tilde{y}_{k+2}=y_{k+2}+\tilde{c}y_{k+1}+\tilde{d}y_k$ [1, 6].

It is easily seen from (1.14) and (1.15) that

$$\tilde{c}=\begin{cases}-{y'}_{k+1}(0), & \text{if } \beta=0,\\ -y_{k+1}(0), & \text{if } 0<\beta<\pi,\end{cases} \tag{1.16}$$

$$\tilde{d}=\begin{cases}\left({y'}_{k+1}(0)\right)^2-{y'}_{k+2}(0), & \text{if } \beta=0,\\ y_{k+1}^2(0)-y_{k+2}(0), & \text{if } 0<\beta<\pi.\end{cases} \tag{1.17}$$

The following systems were investigated in [1]:

(a) $y_n(x)$ $(n=0,1,\dots;n\neq l)$, if all of eigenvalues of (1.1)–(1.3) are real and simple, where l is an arbitrary non-negative integer.

(b) $y_n(x)$ $(n=0,1,\dots;n\neq k+1)$, if λ_k is double eigenvalue $(\lambda_k=\lambda_{k+1})$ of the problem (1.1)–(1.3).

(c) $y_n(x)$ $(n=0,1,\dots;n\neq k)$, if λ_k is double eigenvalue $(\lambda_k=\lambda_{k+1})$ of (1.1)–(1.3) and
$$\omega'''(\lambda_k)\neq 3\tilde{c}\omega''(\lambda_k). \tag{1.18}$$

(d) $y_n(x)$ $(n=0,1,\dots;n\neq l)$, if λ_k is double eigenvalue $(\lambda_k=\lambda_{k+1})$ of (1.1)–(1.3), where $l\neq k,k+1$ is an arbitrary non-negative integer.

(e) $y_n(x)$ $(n=0,1,\dots;n\neq k+2)$, if λ_k is triple eigenvalues $(\lambda_k=\lambda_{k+1}=\lambda_{k+2})$ of (1.1)–(1.3).

(f) $y_n(x)$ $(n=0,1,\dots;n\neq k+1)$, if λ_k is triple eigenvalues $(\lambda_k=\lambda_{k+1}=\lambda_{k+2})$ of (1.1)–(1.3) and
$$\omega^{IV}(\lambda_k)\neq 4\tilde{c}\omega'''(\lambda_k). \tag{1.19}$$

(h) $y_n(x)$ $(n=0,1,\dots;n\neq k)$, if λ_k is triple eigenvalues $(\lambda_k=\lambda_{k+1}=\lambda_{k+2})$ of (1.1)–(1.3) and
$$\frac{\omega^{IV}(\lambda_k)}{4!}\Big(\frac{\omega^{IV}(\lambda_k)}{4!}-\tilde{c}\frac{\omega'''(\lambda_k)}{3!}\Big)\neq\frac{\omega'''(\lambda_k)}{3!}\big(\frac{\omega^{V}(\lambda_k)}{5!}-\tilde{d}\frac{\omega'''(\lambda_k)}{3!}\big). \tag{1.20}$$

(h) $y_n(x)$ $(n=0,1,\dots;n\neq l)$, if λ_k is triple eigenvalues $(\lambda_k=\lambda_{k+1}=\lambda_{k+2})$ of (1.1)–(1.3), where $l\neq k,k+1,k+2$ is an arbitrary non-negative integer.

(i) $y_n(x)$ $(n = 0, 1, \dots; n \neq r)$, if λ_r and λ_s are conjugate of non-real eigenvalues $(\lambda_s = \overline{\lambda}_r)$ of (1.1)–(1.3).

(j) $y_n(x)$ $(n = 0, 1, \dots; n \neq l)$, if λ_r and λ_s are conjugate of non-real eigenvalues $(\lambda_s = \overline{\lambda}_r)$ of (1.1)–(1.3), where $l \neq r, s$ is an arbitrary non-negative integer.

It was proven in [1] that each of the systems (a)-(j) is a basis of $L_p(0, 1), 1 < p < \infty$; moreover, if $p = 2$, then this basis is unconditional.

We denote by $\{u_n(x)\}$ corresponding biorhogonally conjugate to each of the systems (a)-(j). For example, the system $u_n(x)\,(n = 0, 1, \dots; n \neq k)$ is biorhogonally conjugate to system (c).

The following auxiliary associated functions were considered in [1]:

$$y^*_{k+1} = y_{k+1} + c_1 y_k, \tag{1.21}$$

$$y^{**}_{k+1} = y_{k+1} + c_2 y_k, \tag{1.22}$$

$$y^{\#\#}_{k+2} = y_{k+2} + c_2 y_{k+1} + d_2 y_k \tag{1.23}$$

where

$$c_1 = -\frac{\omega'''(\lambda_k)}{3\omega''(\lambda_k)} - \frac{y_{k+1}(1)}{y_k(1)} + \tilde{c}, \tag{1.24}$$

$$c_2 = -\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} - \frac{y_{k+1}(1)}{y_k(1)} + \tilde{c}, \tag{1.25}$$

$$\begin{aligned} d_2 = &-\frac{\omega^{V}(\lambda_k)}{20\omega'''(\lambda_k)} + \Big(\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)}\Big)^2 \\ &+ \Big(\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \frac{y_{k+1}(1)}{y_k(1)}\Big)\Big(\frac{y_{k+1}(1)}{y_k(1)} - \tilde{c}\Big) - \frac{y_{k+2}(1)}{y_k(1)} + \tilde{d}. \end{aligned} \tag{1.26}$$

These auxiliary associated functions were studied for the basis properties of systems (c), (f) and (g) respectively. We will use them for the uniform convergence of the spectral expansions in systems (c), (f) and (g).

It is verified in [1] that if λ_k is double eigenvalue of the problem (1.1)–(1.3), the condition (1.18) is equivalent to the condition $y^*_{k+1}(1) \neq 0$; if λ_k is triple eigenvalue of (1.1)–(1.3), the conditions (1.19) and (1.20) are equivalent to the conditions $y^{**}_{k+1}(1) \neq 0$ and $y^{\#\#}_{k+2}(1) \neq 0$ respectively.

2. Uniform convergence of the spectral expansions for the boundary value problem (1.1)–(1.3)

In this section, we give uniformly convergent spectral expansions in terms of root functions of the problem (1.1)–(1.3). We define the trigonometric system $\{\theta_n(x)\}_{n=1}^{\infty}$ as follows:

$$\theta_n(x) = \begin{cases} \sqrt{2}\sin n\pi x, & \text{if } \beta = 0, \\ \sqrt{2}\cos(n - \frac{1}{2})\pi x, & \text{if } 0 < \beta < \pi. \end{cases}$$

Theorem 2.1. *Suppose that $f \in C[0, 1]$ and $f(x)$ has a uniformly convergent Fourier expansions in the system $\{\theta_n(x)\}_{n=1}^{\infty}$ on the interval $[0, 1]$. Then, the function $f(x)$ can be expanded in Fourier series in each of the systems (a)-(j) and these expansions are uniformly convergent on every interval $[0, b], 0 < b < 1$. Moreover, the Fourier series of $f(x)$ in systems (a)-(j) are uniformly convergent on $[0, 1]$ if and only if $(f, y_l) = 0$ for systems (a), (d), (h) and (j); $(f, y_k) = 0$ for the systems* *(b) and (e); $(f, y_{k+1}^*) = 0$ for system (c); $(f, y_{k+1}^{**}) = 0$ for system (f); $(f, y_{k+2}^{\#\#}) = 0$ for system (g) and $(f, y_s) = 0$ for system (i).*

Proof. We only prove theorem 2.1 for system (c). The proof of the theorem for other systems is similar.

Let $\beta = 0$. Consider the Fourier series $f(x)$ on the interval $[0, 1]$ in system (c):

$$F(x) = \sum_{n=0, n\neq k}^{\infty} (f, u_n) y_n(x), \tag{2.1}$$

where the system $u_n(x)$ $(n = 0, 1, \dots; n \neq k)$ is defined by (see [1])

$$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y_{k+1}^*(1)} y_{k+1}^*(x)}{\|y_n\|^2 + a y_n^2(1)}, u_{k+1}(x) = \frac{y_k(x) - \frac{y_k(1)}{y_{k+1}^*(1)} y_{k+1}^*(x)}{-y_k(1)\frac{\omega''(\lambda_k)}{2}} \tag{2.2}$$

where y_{k+1}^* is defined by (1.21). Let

$$g_n = \Big(\|y_n\|^2 + a y_n^2(1)\Big)^{-1}. \tag{2.3}$$

Then according to (2.2), we obtain

$$u_n(x) = g_n\Big(y_n(x) - \frac{y_n(1)}{y_{k+1}^*(1)} y_{k+1}^*(x)\Big). \tag{2.4}$$

By (1.10), we have the estimates

$$y_n(1) = \frac{(-1)^n}{a(n\pi)^2} + O\big(\frac{\delta_n^{(1)}}{n^2}\big), \tag{2.5}$$

$$\|y_n\|^2 = \frac{1}{2(n\pi)^2} + O(n^{-3}). \tag{2.6}$$

By (2.5) and (2.6), equality (2.3) can be written as

$$g_n = 2(n\pi)^2 + O(n). \tag{2.7}$$

Note that the series (2.1) is uniformly convergent on $[0, 1]$ if and only if the series

$$F_1(x) = \sum_{n=k+2}^{\infty} (f, u_n) y_n(x) \tag{2.8}$$

is uniformly convergent on $[0, 1]$. Suppose that the sequence $\{S_m(x)\}_{m=k+2}^{\infty}$ is the partial sum of the series (2.8). By using (2.4), the equality

$$S_m(x) = S_{m,1}(x) + S_{m,2}(x)$$

holds, where

$$\begin{gathered} S_{m,1}(x) = \sum_{n=k+2}^{m} g_n (f, y_n) y_n(x), \\ S_{m,2}(x) = -\frac{(f, y_{k+1}^*)}{y_{k+1}^*(1)} \sum_{n=k+2}^{m} g_n y_n(1) y_n(x). \end{gathered} \tag{2.9}$$

Firstly, we analyze the uniform convergence of the first sequence in (2.9). Using (2.7), we obtain

$$g_n(f, y_n) y_n(x) = 2(f, n\pi y_n) n\pi y_n(x) + (f, n\pi y_n) y_n(x) O(1). \tag{2.10}$$

By (1.10), the estimate

$$
\begin{aligned}
n\pi y_n(x) = \sin n\pi x &+ \frac{\alpha(x)\cos n\pi x}{n\pi} + \frac{\alpha_n(x)\cos n\pi x}{2n\pi} \\
&+ \frac{\beta_n(x)\sin n\pi x}{2n\pi} + O\Big(\frac{\delta_n^{(1)}}{n}\Big)
\end{aligned} \tag{2.11}
$$

holds, where

$$
\alpha(x) = A_1 x - \frac{1}{2}\int_0^x q(\tau)d\tau, \tag{2.12}
$$

$$
\alpha_n(x) = \int_0^x q(\tau)\cos 2n\pi\tau d\tau, \tag{2.13}
$$

$$
\beta_n(x) = \int_0^x q(\tau)\sin n\pi\tau d\tau. \tag{2.14}
$$

Note that $\alpha(x) \in C[0,1]$ and the functional sequences $\{\alpha_n(x)\}_{n=k+2}^{\infty}$, $\{\beta_n(x)\}_{n=k+2}^{\infty}$ are uniformly bounded. Hence, by (2.11), the equality (2.10) can be written as

$$
g_n(f, y_n)y_n(x) = 2(f, \sin n\pi x)\sin n\pi x + B_n(x),
$$

where

$$
\begin{aligned}
B_n(x) &= (f, \sin n\pi x)O(\frac{1}{n}) + (\alpha(x)f(x), \cos n\pi x)O(\frac{1}{n}) \\
&\quad + (f, \alpha_n(x)\cos n\pi x)O(\frac{1}{n}) + (f, \beta_n(x)\sin n\pi x)O(\frac{1}{n}) + O(\frac{\delta_n^{(1)}}{n}).
\end{aligned} \tag{2.15}
$$

Therefore

$$
S_{m,1}(x) = \sum_{n=k+2}^{m} (f, \sqrt{2}\sin n\pi x)\sqrt{2}\sin n\pi x + \sum_{n=k+2}^{m} B_n(x)\,.
$$

The series

$$
\sum_{n=k+2}^{\infty} B_n(x) \tag{2.16}
$$

is absolutely and uniformly convergent on $[0,1]$. Indeed, by (2.15) we have

$$
\begin{aligned}
|B_n(x)| &\le \frac{C_1}{n}\{|(f, \sin n\pi x)| + |(\alpha(x)f(x), \cos n\pi x)| \\
&\quad + |(f, \alpha_n(x)\cos n\pi x)| + |(f, \beta_n(x)\sin n\pi x)| + \delta_n^{(1)}\} \\
&\le C_2\Big\{|(f, \sin n\pi x)|^2 + |(\alpha(x)f(x), \cos n\pi x)|^2 \\
&\quad + \Big(\int_0^1 |f(x)\alpha_n(x)|dx\Big)^2 + \Big(\int_0^1 |f(x)\beta_n(x)|dx\Big)^2 + \frac{\delta_n^{(1)}}{n}\Big\},
\end{aligned}
$$

where C_1 and C_2 are certain positive constants. By the Bessel inequality for the Fourier coefficients, the numerical series

$$
\sum_{n=k+2}^{\infty} |(f, \sin n\pi x)|^2, \quad \sum_{n=k+2}^{\infty} |(\alpha(x)f(x), \cos n\pi x)|^2, \quad \sum_{n=k+2}^{\infty} \frac{\delta_n^{(1)}}{n}
$$

are convergent. By using Bessel inequality again and by (2.13), we obtain

$$
\begin{aligned}
\sum_{n=k+2}^{\infty}\Big(\int_0^1 |f(x)\alpha_n(x)|dx\Big)^2 &\le \|f\|^2\sum_{n=k+2}^{\infty}\int_0^1 |\alpha_n(x)|^2dx\\
&\le \|f\|^2\int_0^1\Big[\sum_{n=k+2}^{\infty}\Big|\int_0^x q(\tau)\cos 2n\pi\tau d\tau\Big|^2\Big]dx\\
&\le C_3\|f\|^2\int_0^1\int_0^x |q(\tau)|^2d\tau dx \le C_3\|f\|^2\|q\|^2,
\end{aligned}
$$

where C_3 is a certain positive constant. Similarly, by (2.14) we obtain the estimate

$$
\sum_{n=k+2}^{\infty}\Big(\int_0^1 |f(x)\beta_n(x)|dx\Big)^2 \le C_4\|f\|^2\|q\|^2,
$$

where C_4 is a certain positive constant. Thus, the functional series (2.16) is absolutely and uniformly convergent. Since the series

$$
\sum_{n=k+2}^{\infty}(f,\sqrt{2}\sin n\pi x)\sqrt{2}\sin n\pi x
$$

is uniformly convergent on the interval $[0,1]$. The sequence $\{S_{m,1}(x)\}_{m=k+2}^{\infty}$ is also uniformly convergent on this interval.

If $(f,y_{k+1}^*)=0$, then the equality $S_m(x)=S_{m,1}(x)$ holds. Hence, the functional sequence $\{S_m(x)\}_{m=k+2}^{\infty}$ is uniformly convergent on the interval $[0,1]$. Consequently, in the case $\beta=0$, the second part of the Theorem 2.1 is proven.

Suppose that $(f,y_{k+1}^*)\neq 0$. We now analyze the uniform convergence of the second functional sequence in (2.9). By using (1.10), (2.5) and (2.7), we obtain

$$
\sum_{n=k+2}^{m} g_n y_n(1)y_n(x)=\frac{2}{a\pi}\sum_{n=k+2}^{m}\frac{\sin n\pi(1+x)}{n}+\sum_{n=k+2}^{m}O(n^{-2}).
$$

Note that the series

$$
\sum_{n=k+2}^{\infty}\frac{\sin nt}{n}
$$

is uniformly convergent on every closed interval $[\delta,2\pi-\delta]$, where $0<\delta<\pi$ [2, Chapter I, §30, Theorem I]. So, the series

$$
\sum_{n=k+2}^{\infty}\frac{\sin n\pi(1+x)}{n}
$$

is uniformly convergent on the interval $[0,b]$, $0<b<1$. Hence, the functional sequence $\{S_{m,2}(x)\}_{m=r+1}^{\infty}$ is uniformly convergent on $[0,b]$, $0<b<1$.

Let $0<\beta<\pi$. Consider the Fourier series $f(x)$ on the interval $[0,1]$ in system (c):

$$
G(x)=\sum_{n=0,n\neq k}^{\infty}(f,u_n)y_n(x), \tag{2.17}
$$

where the system $u_n(x)\,(n=0,1,\dots;n\neq k)$ is defined by (2.2).

Note that the series (2.17) is uniformly convergent on $[0,1]$ if and only if the series

$$G_1(x) = \sum_{n=k+2}^{\infty} (f, u_n) y_n(x), \tag{2.18}$$

is uniformly convergent on $[0,1]$.

Suppose that the sequence $\{G_m(x)\}_{m=k+2}^{\infty}$ is the partial sum of the series (2.18). By using (2.2), the equality

$$G_m(x) = G_{m,1}(x) + G_{m,2}(x)$$

holds, where

$$\begin{gathered} G_{m,1}(x) = \sum_{n=k+2}^{m} h_n (f, y_n) y_n(x), \\ G_{m,2}(x) = -\frac{(f, y^*_{k+1})}{y^*_{k+1}(1)} \sum_{n=k+2}^{m} h_n y_n(1) y_n(x), \\ h_n = \left(\|y_n\|^2 + a y_n^2(1)\right)^{-1}. \end{gathered}$$

By using (1.12), we obtain the estimates

$$y_n(1) = \frac{2(-1)^n}{a(2n-1)\pi} + O\Big(\frac{\delta_n^{(2)}}{n}\Big), \tag{2.19}$$

$$h_n = 2 + O(n^{-1}). \tag{2.20}$$

From (1.12), (2.19) and (2.20),

$$h_n y_n(1) y_n(x) = -\frac{4}{a\pi} \frac{\sin(n-\frac{1}{2})\pi(1+x)}{2n-1} + O\Big(\frac{\delta_n^{(2)}}{n}\Big)$$

Since

$$\begin{aligned} \Big|\sum_{n=k+2}^{m} \sin(n-\frac{1}{2})\pi(1+x)\Big| &= \frac{|\cos(k+1)\pi(1+x) - \cos m\pi(1+x)|}{2\sin\frac{\pi(1+x)}{2}} \\ &\le \frac{1}{\sin\frac{\pi(1+x)}{2}} \le \frac{1}{\sin\frac{\pi(1+b)}{2}}, \end{aligned}$$

for $0 \le x \le b < 1$ and the numerical series $\sum_{n=k+2}^{\infty} \delta_n^{(2)}/n$ is convergent, then the sequence $\{G_{m,2}(x)\}_{m=k+2}^{\infty}$ is absolutely and uniformly convergent on the interval $[0,b]$, $0 < b < 1$ [2, Introductory material, §1 , Abel's Lemma].

Note that the sequence $\{G_{m,1}(x)\}_{m=k+2}^{\infty}$ is uniformly convergent on the interval $[0,1]$. This can be seen by the method of the case $\beta = 0$. The proof of the theorem 2.1 is complete. □

Theorem 2.2. *Suppose that $f \in C[0,1]$ and $f(x)$ has a uniformly convergent Fourier expansions in the system $\{\theta_n(x)\}_{n=1}^{\infty}$ on the interval $[0,1]$, then this function can be expanded in Fourier series in each of the systems $\{u_n(x)\}$ which are biorthogonally conjugates to systems* (a)-(j) *and these expansions are uniformly convergent on the interval $[0,1]$.*

Proof. We only prove theorem 2.2 for system (2.2) which is biorthogonally conjugate to system (c). The proof of the theorem for other systems is similar.

Let $\beta = 0$. Consider the Fourier series $f(x)$ on the interval $[0,1]$ in (2.2):

$$T(x) = \sum_{n=0, n\neq k}^{\infty} (f, y_n) u_n(x). \tag{2.21}$$

Note that the series (2.21) is uniformly convergent on $[0,1]$ if and only if the series

$$T_1(x) = \sum_{n=k+2}^{\infty} (f, y_n) u_n(x) \tag{2.22}$$

is uniformly convergent on $[0,1]$.

Suppose that the sequence $\{T_m(x)\}_{m=k+2}^{\infty}$ is the partial sum of the series (2.22). By using (2.2), the equality

$$T_m(x) = T_{m,1}(x) + T_{m,2}(x)$$

holds, where

$$T_{m,1}(x) = \sum_{n=k+2}^{m} g_n (f, y_n) y_n(x),$$

$$T_{m,2}(x) = -\frac{y_{k+1}^*(x)}{y_{k+1}^*(1)} \sum_{n=k+2}^{m} g_n y_n(1)(f, y_n).$$

The sequences $\{S_{m,1}(x)\}_{m=k+2}^{\infty}$ and $\{T_{m,1}(x)\}_{m=k+2}^{\infty}$ are the same. Therefore, the sequence $\{T_{m,1}(x)\}_{m=k+2}^{\infty}$ is uniformly convergent on the interval $[0,1]$.

Using (1.10), (2.5) and (2.7) we obtain

$$g_n y_n(1)(f, y_n) = \frac{2(-1)^n}{an\pi}(f, \sin n\pi x) + O\Big(\frac{\delta_n^{(1)}}{n}\Big).$$

From here, the estimate

$$|g_n y_n(1)(f, y_n)| \leq \frac{C_5}{n}\big\{|(f, \sin n\pi x)| + \delta_n^{(1)}\big\} \leq C_6\big\{|(f, \sin n\pi x)|^2 + \frac{\delta_n^{(1)}}{n}\big\}$$

holds, where C_5 and C_6 are certain positive number. The numerical series

$$\sum_{n=k+2}^{\infty} |(f, \sin n\pi x)|^2, \quad \sum_{n=k+2}^{\infty} \frac{\delta_n^{(1)}}{n}$$

are convergent. Consequently, the sequence $\{T_{m,2}(x)\}_{m=k+2}^{\infty}$ is absolutely and uniformly convergent on $[0,1]$.

In the case $0 < \beta < \pi$, the proof is similar. Theorem 2.2 is proven. □

3. Examples

Example 3.1. Consider the spectral problem

$$-y'' = \lambda y, \quad 0 < x < 1, \tag{3.1}$$

$$y(0) = 0,\ y'(1) = (-\frac{\lambda}{3} + 1) y(1) \tag{3.2}$$

where λ is a spectral parameter.

The eigenvalues of problem (3.1)–(3.2) are the root the equation $\omega(\lambda)=0$, where $\omega(\lambda)=(\frac{\lambda}{3}-1)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}+\cos\sqrt{\lambda}$ and $\operatorname{Re}\sqrt{\lambda}\geq 0$. It is easy to see that

$$\omega(\lambda)=-\lambda^2\sum_{n=0}^{\infty}\frac{4(-1)^n(n+1)(n+2)}{(2n+5)!}\lambda^n. \tag{3.3}$$

Therefore, $\lambda=0$ is double eigenvalue of (3.1)-(3.2). Hence, all the eigenvalues of (3.1)-(3.2) are real and all, except one double, are simple. Further, by (3.3), if $\lambda<0$, then $w(\lambda)<0$. Then, $\lambda=0$ is the first eigenvalue of (3.1)–(3.2) and $\lambda_0=\lambda_1=0$.

From (3.3), $\omega(0)=\omega'(0)=0$, $\omega''(0)=-\frac{2}{45}$ and $\omega'''(0)=\frac{1}{105}$. Eigenfunctions corresponding to $\lambda_n(0,2,3,\dots)$ are $y_0(x)=x$ and $y_n(x)=\frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}$ $(n\geq 2)$, associated function corresponding to eigenfunction y_0 is $y_1(x)=-\frac{x^3}{6}+cx$, where c is an arbitrary constant. From (1.16), $\tilde{c}=-c$. By (1.24),

$$c_1=-\frac{\omega'''(0)}{3\omega''(0)}-\frac{y_{k+1}(1)}{y_k(1)}+\tilde{c}=\frac{5}{21}-2c.$$

Note that $y_1^*=y_1+c_1y_0$ and $y_1^*(1)\neq 0$(or $\omega'''(\lambda_0)\neq 3\tilde{c}\omega''(\lambda_0)$), hence $c\neq 1/14$. Therefore, if $c\neq 1/14$, then the system $y_n(x)$ $(n=1,2,\dots)$ is a basis in $L_p(0,1), 1<p<\infty$ (see, [1]).

Let $f(x)=x^2-x$. Since

$$(f,\sin n\pi x)=\begin{cases}0, & \text{if } n \text{ is even}\\ -\frac{4}{n^3\pi^3}, & \text{if } n \text{ is odd},\end{cases}$$

the function $f(x)$ can be expanded uniformly convergent Fourier series in the system $\{\sqrt{2}\sin n\pi x\}_{n=1}^{\infty}$. Further, $(f,y_1{}^*)=\frac{8}{315}-\frac{c}{3}$. Consequently, if $c=\frac{8}{105}$, then the Fourier series of $f(x)$ in the system $y_n(x)$ $(n=1,2,\dots)$ is uniformly convergent on $[0,1]$; if $c\neq\frac{8}{105},\frac{1}{4}$, then the Fourier series of $f(x)$ in the system $y_n(x)$ $(n=1,2,\dots)$ is uniformly convergent on $[0,b]$, $0<b<1$.

Example 3.2. Consider the spectral problem

$$-y''=\lambda y,\quad 0<x<1, \tag{3.4}$$

$$y'(0)=\alpha y(0),\quad y'(1)=(a\lambda+b)y(1) \tag{3.5}$$

where λ is a spectral parameter, α is unique real root of the equation

$$\alpha^3+6\alpha^2+15\alpha+15=0 \tag{3.6}$$

(verify that $\alpha=\sqrt[3]{\frac{2}{1+\sqrt{5}}}-\sqrt[3]{\frac{1+\sqrt{5}}{2}}-2$) and

$$a=-\frac{\alpha^2+3\alpha+3}{3(\alpha+1)^2},\quad b=\frac{\alpha}{\alpha+1}. \tag{3.7}$$

The eigenvalues of (3.4)–(3.5) are the roots of the function

$$\omega(\lambda)=(-a\lambda+\alpha-b)\cos\sqrt{\lambda}-((\alpha a+1)\lambda+\alpha b)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \tag{3.8}$$

where $\operatorname{Re}\sqrt{\lambda}\geq 0$.

Note that by (3.6) and (3.7), the equalities $\alpha - b = \alpha b = 15a$, $\alpha a + 1 = -6a$ hold. Therefore, the equality (3.8) can be written as $\omega(\lambda) = (15a - a\lambda)\cos\sqrt{\lambda} + (6a\lambda - 15a)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}$. Hence, the Maclaurin series of $\omega(\lambda)$ forms

$$\omega(\lambda) = -a\lambda^3 \sum_{n=0}^{\infty} \frac{2(-1)^n(n+2)(n+3)(4n+19)}{(2n+7)!}\lambda^n. \tag{3.9}$$

Therefore, $\lambda = 0$ is triple eigenvalue of (3.4)–(3.5). Hence, all the eigenvalues of (3.4)–(3.5) are real and all, except one triple, are simple. Further, by (3.9), if $\lambda < 0$, then $w(\lambda) < 0$. Then, $\lambda = 0$ is the first eigenvalue of (3.4)–(3.5) and $\lambda_0 = \lambda_1 = \lambda_2 = 0$.

From (3.9), we obtain $\omega(0) = \omega'(0) = \omega''(0) = 0$, $\omega'''(0) = -\frac{288a}{7!}$, $\omega^{IV}(0) = \frac{4608a}{9!}$ and $\omega^{V}(0) = -\frac{57600a}{11!}$. Eigenfunctions corresponding to λ_n $(n = 0, 3, 4, \dots)$ are $y_0(x) = \alpha x + 1$ and $y_n(x) = \cos\sqrt{\lambda_n}x + \alpha\frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}$ $(n \geq 3)$. The first and the second order associated functions corresponding to y_0 are $y_1(x) = -\frac{\alpha x^3}{3!} - \frac{x^2}{2!} + \alpha A x + A$ and $y_2(x) = \frac{\alpha x^5}{5!} + \frac{x^4}{4!} - \frac{\alpha A x^3}{3!} - \frac{Ax^2}{2!} + \alpha B x + B$ respectively, where A and B are arbitrary constants.

Note that $0 < \beta < \pi$ for problem (3.4)–(3.5). From here, (1.16) and (1.17), $\tilde{c} = -A$ and $\tilde{d} = A^2 - B$. According to above calculations, the condition (1.20) can be written as

$$B \neq A^2 - \frac{A}{18} + \frac{13}{7128}. \tag{3.10}$$

Therefore, if condition (3.10) is satisfied, then the system $y_n(x)$ $(n = 1, 2, \dots)$ is a basis in $L_p(0,1)$ $(1 < p < \infty)$.

Let $F_s(x) = \mathrm{P}_s(2x-1)(x^2 - x)$, where $\mathrm{P}_s(t)(s = 0, 1, 2, \dots)$ are Legendre polynomials [16, p.162]:

$$\mathrm{P}_s(t) = \frac{1}{2^s s!}\frac{d^s}{x^s}[(x^2-1)^s].$$

Since $F_s(0) = F_s(1) = 0$, $(F_s, \cos(n - \frac{1}{2})\pi x) = O(n^{-2})$. It means that this function can be expanded uniformly convergent Fourier series in the system $\{\sqrt{2}\cos(n - \frac{1}{2})\pi x\}_{n=1}^{\infty}$ on the interval $[0, 1]$.

Note that the equalities

$$\int_{-1}^{1} t^k \mathrm{P}_s(t)dt = 0, \int_{-1}^{1} t^7 \mathrm{P}_7(t)dt = \frac{2^8(7!)^2}{15!}$$

hold, where $k = 0, 1, \dots, s-1$ [16, p.174 and 175]. From here, since the functions $(t^2 - 1)y_j(\frac{t+1}{2})$ $(j = 0, 1, 2)$ are polynomials of degree seven or less than seven, we obtain

$$\begin{aligned}
\int_0^1 F_s(x)y_j(x)dx &= \int_0^1 P_s(2x-1)(x^2-x)y_j(x)dx \\
&= \frac{1}{8}\int_{-1}^{1} P_s(t)(t^2-1)y_j(\frac{t+1}{2})dt \\
&= \begin{cases} 0, & \text{if } s \geq 8 \text{ and } j = 0, 1, 2, \\ \frac{\alpha(7!)^2}{5!15!}, & \text{if } s = 7 \text{ and } j = 2. \end{cases}
\end{aligned}$$

Hence, the condition

$$(F_s, y_2^{\#\#}) = \begin{cases} 0, & \text{if } s \geq 8 \\ \frac{\alpha(7!)^2}{5!15!}, & \text{if } s = 7 \end{cases}$$

is satisfied. Consequently, from theorem 2.1, the function $F_s(x)$ can be expanded uniformly convergent Fourier series in the system $y_n(x)$ $(n = 1, 2, \dots)$ on $[0, 1]$ for $s \geq 8$, on $[0, b]$ $(0 < b < 1)$ for $s = 7$.

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