Electronic Journal of Differential Equations, Vol. 2005(2005), No. 99, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF SOLUTIONS FOR QUASISTATIC PROBLEMS OF UNILATERAL CONTACT WITH NONLOCAL FRICTION FOR NONLINEAR ELASTIC MATERIALS

TOUZALINE AREZKI, MIGNOT ALAIN

Abstract. This paper shows the existence of a solution of the quasi-static unilateral contact problem with nonlocal friction law for nonlinear elastic materials. We set up a variational incremental problem which admits a solution, when the friction coefficient is small enough, and then by passing to the limit with respect to time we obtain a solution.

1. Introduction

We consider a Signorini's quasistatic contact problem with nonlocal friction in nonlinear elasticity. In linear elasticity the quasistatic problem of unilateral contact using a normal compliance law has been solved in [1] by considering incremental problems and in [9] by an other method using a regularisation relative to time. The quasistatic contact problem with local or nonlocal friction has been solved respectively in [10] and in [4] by using a time-discretization method. In [2] the quasistatic contact problem with Coulomb friction was solved by the aid of an established shifting technique used to obtain increased regularity at the contact surface and by the aid of auxiliary problems involving regularized friction terms and a so-called normal compliance penalization technique. Signorini 's problem with friction for nonlinear elastic materials or viscoelastic materials has been solved in [5] by using the fixed point's method. In viscoelasticity, the quasistatic contact problem with a normal compliance law and friction has been solved in [11] by the same fixed point arguments. The book [8] introduces generally readers to a mathematical theory of contact problems involving deformable bodies. In carrying out the variational analysis, the authors systematically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators, and fixed points of operators.

In this paper we propose a variational formulation using a classical regularization [6] of the normal stress characterizing the notion of nonlocal friction as in the linear case. The variational formulation is written in the form of two variational inequalities as in [4]. By using an implicit scheme as in [4, 10], we are led to solve

2000 *Mathematics Subject Classification.* 35J85, 49J40, 47J20, 58E35.
Key words and phrases. Nonlinear elasticity; nonlocal friction; incremental; variational inequality; monotone.

Submitted June 11, 2004. Published September 19, 2005.

a sequence of incremental static problems and by passing to the limit, we show the existence of a solution of the quasistatic contact problem for a small enough friction coefficient.

2. Variational formulation

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, be the domain initially occupied by the nonlinear elastic body. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ. Γ is decomposed into three parts $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets and $\operatorname{meas}(\Gamma_1) > 0$. Let $T > 0$ and let $[0, T]$ denote the time interval of interest. The body is clamped on Γ_1 and thus the displacement field vanishes there. The body is acted upon by a volume force of density ϕ_1 on Ω and a surface traction of density ϕ_2 on Γ_2. On Γ_3 the body is in unilateral contact with a rigid support and the conditions of contact are supposed to be as in [4].

Under these conditions the classical formulation of the mechanical problem is the following.

Problem P1. . Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$ such that

$$\operatorname{div} \sigma(u) + \phi_1 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma(u) = F(\varepsilon(u)) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\mathbf{n}(u) = \phi_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.4}$$

$$\sigma_N(u) \leq 0, u_N \leq 0, \sigma_N(u).u_N = 0 \quad \text{on } \Gamma_3 \times (0, T), \tag{2.5}$$

$$\begin{cases} |\sigma_T| \leq \mu|R\sigma_N(u)| \\ |\sigma_T| < \mu|R\sigma_N(u)| \Longrightarrow \dot{u}_T = 0 \\ |\sigma_T| = \mu|R\sigma_N(u)| \Longrightarrow \sigma_T = -\lambda\dot{u}_T,\ \lambda \geq 0 \end{cases} \quad \text{on } \Gamma_3 \times (0, T), \tag{2.6}$$

$$u(0) = u_0 \quad \text{in } \Omega \tag{2.7}$$

We adopt the following notations as in [7]: Vector $\mathbf{n} = (n_i)$ is the outer unit normal vector to Γ; $u = (u_i)$ is the displacement field; $u_N = u_i n_i$ is the normal displacement on Γ; $u_T = u - u_N\mathbf{n}$ is the tangential displacement on Γ. The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = (\frac{1}{2}(u_{i,j} + u_{j,i})), i, j \in \{1, \dots, d\}\,;$$

the stress tensor is $\sigma = (\sigma_{ij})$; $\operatorname{div} \sigma = (\sigma_{ij,j})$ is the divergence of σ, $\sigma_N = (\sigma\mathbf{n}).\mathbf{n}$ is the normal stress; $\sigma_T = \sigma\mathbf{n} - \sigma_N\mathbf{n}$ is the tangential stress. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$.

To proceed with the variational formulation, we need the function spaces:

$$H = L^2(\Omega)^d, \quad H_1 = (H^1(\Omega))^d,$$

$$Q = \{\tau = (\tau_{ij}); \tau_{ij} = \tau_{ji} \in L^2(\Omega)\} = L^2(\Omega)_s^{d\times d},$$

$$H(\operatorname{div}; \Omega) = \{\sigma \in Q; \operatorname{div} \sigma \in H\}$$

Note that H and Q are Hilbert spaces equipped with the respective scalar products

$$(u, v)_H = \int_\Omega u_i v_i dx, \quad \langle\sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

We recall that Green's formula holds: for $\sigma \in H(\operatorname{div}; \Omega)$

$$\langle\sigma, \varepsilon(v)\rangle_Q + (\operatorname{div} \sigma, v)_H = \langle\sigma\mathbf{n}, v\rangle_{H^{-\frac{1}{2}}(\Gamma)^d \times H^{1/2}(\Gamma)^d} \forall v \in H_1.$$

Let V be the closed subspace of H_1 given by

$$V = \{v \in H_1; v = 0 \text{ on } \Gamma_1\}$$

and K be the set of admissible displacements given by

$$K = \{v \in V; v_N \leq 0 \text{ on } \Gamma_3\}.$$

Since $\operatorname{meas}\Gamma_1 > 0$, the following Korn's inequality holds (see [5]):

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V \tag{2.8}$$

where the constant c_Ω depends only on Ω and Γ_1. We equip V with the scalar product

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$$

and $\|\cdot\|_V$ is the associated norm. It follows from Korn's inequality (2.8) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V. Then $(V, \|\cdot\|_V)$ is a Hilbert space.

Moreover, by the Sobolev's trace theorem, there exists a positive constant d_Ω depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq d_\Omega \|v\|_V \quad \forall v \in V \tag{2.9}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to X; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

The forces and tractions are assumed to satisfy

$$\phi_1 \in W^{1,\infty}(0, T; H), \quad \phi_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2)^d) \tag{2.10}$$

Let $f : [0, T] \to V$ be given by

$$(f(t), v)_V = \int_\Omega \phi_1(t).v dx + \int_{\Gamma_2} \phi_2(t).v da \quad \forall v \in V, t \in [0, T].$$

We note that conditions (2.10) imply $f \in W^{1,\infty}(0, T; V)$. Let

$$H^{1/2}(\Gamma_3) = \{w\big|_{\Gamma_3} : w \in H^{1/2}(\Gamma) \text{ and } w = 0 \text{ on } \Gamma_1\}.$$

Let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $H^{-1/2}(\Gamma_3)$ and $H^{1/2}(\Gamma_3)$.

The normal stress $\sigma_N\ (u(t)) \in H^{-1/2}(\Gamma_3)$ associated to $u(t) \in V$ is defined by

$$\begin{gathered} \forall w \in H^{1/2}(\Gamma_3) : \\ \langle \sigma_N(u(t)), w\rangle = \langle F(\varepsilon(u(t))), \varepsilon(v)\rangle_Q - (f(t), v)_V \\ \forall v \in V : v_N = w,\ w_T = 0 \quad \text{on } \Gamma_3 \end{gathered} \tag{2.11}$$

$R : H^{-1/2}(\Gamma_3) \to L^2(\Gamma_3)$ is a linear compact mapping which respects the positivity (see [6]).

Hypotheses on the nonlinear elasticity operator. As in [5] we assume $\mathcal{F} : \Omega \times S_d \to S_d$ satisfies the following conditions:

(a) There exists $L_1 > 0$ such that $|\mathcal{F}(.,\varepsilon_1) - \mathcal{F}(.,\varepsilon_2)| \leq L_1|\varepsilon_1 - \varepsilon_2|$ for all $\varepsilon_1, \varepsilon_2$ in S_d, a.e in Ω.
(b) There exists $L_2 > 0$ such that $(\mathcal{F}(.,\varepsilon_1) - \mathcal{F}(.,\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \geq L_2|\varepsilon_1 - \varepsilon_2|^2$ for all $\varepsilon_1, \varepsilon_2 \in S_d$, a.e in Ω.
(c) For any $\varepsilon \in S_d$, the mapping $x \mapsto \mathcal{F}(x, \varepsilon)$ is measurable on Ω.
(d) $(x, 0_d) = 0$ for all x in Ω,

Remark 2.1. $\mathcal{F}(x, \tau(x)) \in Q$, for all $\tau \in Q$ and thus it is possible to consider $\mathcal{F}$ as an operator defined from Q to Q.

We assume that the friction coefficient satisfies

$$\mu \geq 0 \text{ a.e. on } \Gamma_3 \text{ and } \mu \in L^\infty(\Gamma_3) \tag{2.12}$$

Also we assume that the initial data $u_0 \in K$ satisfies

$$\langle \mathcal{F}(\varepsilon(u_0)), \varepsilon(v) - \varepsilon(u_0)\rangle_Q + j(u_0, v - u_0) \geq (f(0), v - u_0)_V \quad \forall v \in K. \tag{2.13}$$

Now assuming that the solution is sufficiently regular, we formally multiply the equilibrium equation (2.1) by $v - \dot{u}(t)$ and by using techniques similar to those exposed in [7] , we show that the problem (P1) has the following variational formulation.

Problem P2. Find a displacement field $u : [0, T] \to V$, verifying $u(0) = u_0$ in Ω and $u(t) \in K$ a.e. $t \in [0, T]$, and such that a.e. $t \in [0, T]$:

$$\begin{aligned} &\langle \mathcal{F}(\varepsilon(u(t))), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_Q + j(u(t), v) - j(u(t), \dot{u}(t)) \\ &\geq (f(t), v - \dot{u}(t))_V + \langle \sigma_N(u(t)), v_N - \dot{u}_N(t)\rangle \geq 0 \quad \forall v \in V \end{aligned} \tag{2.14}$$

$$\langle \sigma_N(u(t)), z_N - u_N(t)\rangle \geq 0, \quad \forall z \in K \tag{2.15}$$

where

$$j(u, v) = \int_{\Gamma_3} \mu |R\sigma_N(u)|\, |v_T| da.$$

The aim of this paper is to show the following result.

Theorem 2.2. *Let* (2.10)*,* (2.11)*,* (2.12) *and* (2.13) *hold. Then problem (P2) has at least a solution* $u \in W^{1,\infty}(0, T; V)$ *for a small enough friction coefficient* μ*. Moreover, there exists a constant* $C > 0$ *such that*

$$\|u\|_{W^{1,\infty}(0,T;V)} \leq C\|f\|_{W^{1,\infty}(0,T;V)}$$

For the proof of this theorem, we carry a time-discretization of problem (P2). For $n \in \mathbf{N}^*$, we set $\Delta t = \frac{T}{n}$, and $t_i = i\Delta t, i = 0, \dots, (n-1)$; denote by u^{t_i} the approached solution of the solution u at the time t_i and $\Delta u^{t_i} = u^{t_{i+1}} - u^{t_i}$. By using an implicit scheme, we obtain a sequence of incremental problems, for $u^0 \in K$, define as

Problem $(P_n^{t_i})$**.** Find $u^{t_{i+1}} \in K$ such that

$$\begin{aligned} &\langle \mathcal{F}(\varepsilon(u^{t_{i+1}})), \varepsilon(w) - \varepsilon(u^{t_{i+1}})\rangle_Q + j(u^{t_{i+1}}, w - u^{t_{i+1}}) - j(u^{t_{i+1}}, u^{t_{i+1}} - u^{t_i}) \\ &\geq (f^{t_{i+1}}, w - u^{t_{i+1}})_V + \langle \sigma_N(u^{t_{i+1}}), w_N - u_N^{t_{i+1}}\rangle, \quad \forall w \in V \end{aligned}$$

$$\langle \sigma_N(u^{t_{i+1}}), w_N - u_N^{t_{i+1}}\rangle \geq 0, \quad \forall w \in K$$

where $u^0 = u_0$, and $f^{t_{i+1}} = f(t_{i+1})$.

3. Existence of a solution of the incremental problem

Lemma 3.1. *Problem $(P_n^{t_i})$ is equivalent to the problem $(Q_n^{t_i})$ stated below.*

Problem $(Q_n^{t_i})$. Find $u^{t_{i+1}} \in K$ such that

$$\begin{aligned} &\langle \mathcal{F}(\varepsilon(u^{t_{i+1}})), \varepsilon(w) - \varepsilon(u^{t_{i+1}})\rangle_Q + j(u^{t_{i+1}}, w - u^{t_i}) - j(u^{t_{i+1}}, u^{t_{i+1}} - u^{t_i}) \\ &\geq (f^{t_{i+1}}, w - u^{t_{i+1}})_V \quad \forall w \in K \end{aligned} \tag{3.1}$$

For the proof of Lemma 3.1 in the linear case, see see [4].

Proposition 3.2. *There exists $\mu_0 > 0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, then problem $(Q_n^{t_i})$ admits a unique solution.*

To show proposition 3.2 we introduce an intermediate problem. We define the convex set

$$C_+^* = \{g \in L^2(\Gamma_3); g \geq 0 \text{ a.e. on } \Gamma_3\},$$

and the function

$$\varphi(w) = \int_{\Gamma_3} \mu g |w_T| da.$$

Then, we introduce an intermediate problem by replacing in $R\sigma_N(u^{t_{i+1}})$ in (3.1) by for $g \in C_+^*$ as follows:

Problem $(Q_{ng}^{t_i})$. Find u_g in K such that

$$\begin{aligned} &\langle \mathcal{F}(\varepsilon(u_g)), \varepsilon(w) - \varepsilon(u_g)\rangle_Q + \varphi(w - u^{t_i}) - \varphi(u_g - u^{t_i}) \\ &\geq (f^{t_{i+1}}, w - u_g)_V \quad \forall w \in K\,. \end{aligned} \tag{3.2}$$

Now, we have the following lemma.

Lemma 3.3. *For any $g \in C_+^*$ problem $(Q_{ng}^{t_i})$ has a unique solution u_g. Furthermore, there exists constants $c_i > 0$, $i = 1, 2$, such that*

$$\|u_g\|_V \leq c_1 \|\mu\|_{L^\infty(\Gamma_3)} \|g\|_{L^2(\Gamma_3)} + c_2 \|f^{t_{i+1}}\|_V \tag{3.3}$$

Proof. Using Riesz's representation theorem we define the nonlinear operator $A : V \to V$ by

$$(Av, w)_V = \langle \mathcal{F}(\varepsilon(v)), \varepsilon(w)\rangle_Q\,.$$

Then hypotheses (a) and and (b) on $\mathcal{F}$ imply that A is a strictly monotone, coercive and lipschitzian operator; on the other hand the functional φ is proper, convex and lower continuous. There results from the theory of elliptic variational inequalities [3] that the inequality (3.1) has an unique solution u_g. Setting $w = 0$ in the inequality (3.2) and using both the hypothesis (b) on $\mathcal{F}$ and the inequality

$$\big||(u_g - u^{t_i})_T| - |u_T^{t_i}|\big| \leq |u_{g\mathrm{T}}|$$

we see that there exist constants $c_i > 0$, $i = 1, 2$, such that

$$\|u_g\|_V^2 \leq c_1 \|\mu\|_{L^\infty(\Gamma_3)} \|g\|_{L^2(\Gamma_3)} \|u_g\|_V + c_2 \|f^{t_{i+1}}\|_V \|u_g\|_V.$$

Simplifying by the norm $\|u_g\|_V$ we have the inequality (3.3). □

Lemma 3.4. *Let $\Psi : C_+^* \to C_+^*$ be the mapping $\Psi(g) = -R\sigma_N(u_g)$ there exists $\mu_0 > 0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, then Ψ admits a fixed point g^* and u_{g^*} is a solution of problem $(Q_n^{t_i})$.*

Proof. Using (2.8), (2.11), the hypothesis (b) on $\mathcal{F}$ and the continuity of R, we deduce that there exists a constant $C > 0$ such that

$$\|\Psi(g_1) - \Psi(g_2)\|_{L^2(\Gamma_3)} \leq C\|u_{g_1} - u_{g_2}\|_V$$

On the other hand by setting $v = u_{g_1}$ in $(Q^{t_i}_{ng_2})$ and $v = u_{g_2}$ in $(Q^{t_i}_{ng_1})$ and then adding, we obtain by using the (b) on $\mathcal{F}$ and (2.8), that there exists a constant $C_1 > 0$ such that

$$\|u_{g_1} - u_{g_2}\|_V \leq C_1\|\mu\|_{L^\infty(\Gamma_3)}\|g_1 - g_2\|_{L^2(\Gamma_3)}$$

Hence there exists a constant $C_2 > 0$ such that

$$\|\Psi(g_1) - \Psi(g_2)\|_{L^2(\Gamma_3)} \leq C_2\|\mu\|_{L^\infty(\Gamma_3)}\|g_1 - g_2\|_{L^2(\Gamma_3)},$$

and when $\mu_0 = 1/C_2$, we have for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, the mapping Ψ is a contraction, thus it has a fixed point g^* and u_{g*} is the solution of problem $(Q^{t_i}_n)$. $\square$

4. Existence of a solution of the quasistatic problem

Lemma 4.1. *We have the following estimates: For a positive constant $\mu_1 > 0$, when $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$, there exists $d_i > 0$, $i = 1, 2$, such that*

$$\|u^{t_{i+1}}\|_V \leq d_1\|f^{t_{i+1}}\|_V \tag{4.1}$$

$$\|\Delta u^{t_i}\|_V \leq d_2\|\Delta f^{t_i}\|_V \tag{4.2}$$

Proof. By setting $w = 0$ in the inequality (3.1) and using hypothesis (b) on $\mathcal{F}$ and the properties of j, there exists $c_1 > 0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)} < c_1$, we deduce that there exists $d_1 > 0$ such that (4.1) is satisfied.

To show the inequality (4.2) we consider inequality of (3.1) translated at the time t_i that is:

$$\begin{aligned}&\langle \mathcal{F}(\varepsilon(u^{t_i})), \varepsilon(w) - \varepsilon(u^{t_i})\rangle_Q + j(u^{t_i}, v - u^{t_{i-1}}) - j(u^{t_i}, u^{t_i} - u^{t_{i-1}})\\ &\geq (f^{t_i}, w - u^{t_i})_V \ , \forall w \in K\end{aligned} \tag{4.3}$$

By setting $w = u^{t_i}$ in (3.1) and $w = u^{t_{i+1}}$ in (4.3) and add them up, we obtain the inequality

$$\begin{aligned}&- \langle \mathcal{F}(\varepsilon(u^{t_{i+1}})) - \mathcal{F}(\varepsilon(u^{t_i})), \varepsilon(\Delta u^{t_i})\rangle_Q - j(u^{t_{i+1}}, \Delta u^{t_i})\\ &+ j(u^{t_i}, u^{t_{i+1}} - u^{t_{i-1}}) - j(u^{t_i}, u^{t_i} - u^{t_{i-1}})\\ &\geq (-\Delta f^{t_i}, \Delta u^{t_i})_V\end{aligned}$$

furthermore using the inequality

$$\Big||u^{t_{i+1}}_T - u^{t_{i-1}}_T| - |u^{t_i}_T - u^{t_{i-1}}_T|\Big| \leq |u^{t_{i+1}}_T - u^{t_i}_T|$$

We have

$$j(u^{t_i}, u^{t_{i+1}} - u^{t_{i-1}}) - j(u^{t_i}, u^{t_i} - u^{t_{i-1}}) \leq \ j(u^{t_i}, \Delta u^{t_i})\,.$$

Therefore,

$$\begin{aligned}&- \langle \mathcal{F}(\varepsilon(u^{t_{i+1}})) - \mathcal{F}(\varepsilon(u^{t_i})), \varepsilon(\Delta u^{t_i})\rangle_Q + j(u^{t_i}, \Delta u^{t_i}) - j(u^{t_{i+1}}, \Delta u^{t_i})\\ &\geq (-\Delta f^{t_i}, \Delta u^{t_i})_V\,.\end{aligned}$$

Using the properties of j we have

$$-j(u^{t_i}, \Delta u^{t_i}) + j(u^{t_{i+1}}, \Delta u^{t_i}) \leq j(\Delta u^{t_i}, \Delta u^{t_i})\,.$$

As a consequence we obtain the inequality

$$\langle \digamma(\varepsilon(u^{t_{i+1}})) - \digamma(\varepsilon(u^{t_i})), \varepsilon(\Delta u^{t_i})\rangle_Q - j(\Delta u^{t_i}, \Delta u^{t_i}) - (\Delta f^{t_i}, \Delta u^{t_i})_V \leq 0. \quad (4.4)$$

Using the relation (2.11), there exists a constant $c_3 > 0$ such that

$$\|\sigma_N(\Delta u^{t_i})\|_{H^{-\frac{1}{2}}(\Gamma_3)} \leq c_3(\|\Delta u^{t_i}\|_V + \|\Delta f^{t_i}\|_V)\,.$$

Then using the hypothesis (b) on $\digamma$ and the properties of j we deduce that there exists $d_3 > 0$ such that

$$L_2\|\Delta u^{t_i}\|_V^2 \leq d_3\|k\|_{L^\infty(\Gamma_3)}\|\Delta u^{t_i}\|_V^2 + \|\Delta f^{t_i}\|_V\|\Delta u^{t_i}\|_V$$

Setting $c_2 = \frac{L_2}{2d_3}$, we deduce that if $\|\mu\|_{L^\infty(\Gamma_3)} < c_2$, there exists $d_4 > 0$ such that

$$\|\Delta u^{t_i}\|_V \leq d_4\|\Delta f^{t_i}\|_V$$

It suffices to take $\mu_1 = \min(c_1, c_2)$ and the lemma is proved. □

The proof of Theorem 2.2 is done as in [4], but in L^∞. For the next proposition, we define the continuous function $u^n : [0,T] \to V$ by

$$u^n(t) = u^{t_i} + \frac{(t-t_i)}{\Delta t}\Delta u^{t_i} \quad \text{on } [t_i, t_{i+1}],\ i = 0, \dots, n-1\,.$$

Proposition 4.2. *From the sequence (u^n) we can extract a subsequence still denoted (u^n) such that (u^n) converges weakly $*$ in $W^{1,\infty}(0,T;V)$ to a function u.*

Proof. From (4.1) we deduce that the sequence (u^n) is bounded in $C([0,T];V)$ and there exists a constant $c_3 > 0$ such that

$$\max_{0\leq t\leq T}\|u^n(t)\|_V \leq c_3\|f\|_{C([0,T];V)}$$

From (4.2) we deduce that the sequence $(\dot{u}^n)$ is bounded in $L^\infty(0,T;V)$ and there exists $c_4 > 0$ such that

$$\|\dot{u}^n\|_{L^\infty(0,T;V)} = \max_{0\leq i\leq n-1}\|\frac{\Delta u^{t_i}}{\Delta t}\|_V \leq c_4\|\dot{f}\|_{L^\infty(0,T;V)}$$

Then the sequence (u^n) is uniformly bounded in $W^{1,\infty}(0,T;V)$, and we thus can extract from it a subsequence still denoted (u^n) such that $u^n \to u$ in $W^{1,\infty}(0,T;V)$ weakly $*$ as $n \to \infty$ and satisfying

$$\|u\|_{W^{1,\infty}(0,T;V)} \leq C\|f\|_{W^{1,\infty}(0,T.V)}$$

with $C = \max(c_3, c_4)$. □

As in [10] let's introduce the piecewise constant functions $\widetilde{u}^n : [0,T] \to V$ and $\widetilde{f}^n : [0,T] \to V$ defined by

$$\widetilde{u}^n(t) = u^{t_{i+1}}, \quad \widetilde{f}^n(t) = f(t_{i+1}) \quad \forall t \in (t_i, t_{i+1}], i = 0, \dots, n-1\,.$$

Lemma 4.3. *From the sequence $(\tilde{u}^n)$ we can extract a subsequence still denoted $(\tilde{u}^n)$ which satisfies the convergence results:*

(i) $\widetilde{u}^n \to u$ *weak* $*$ *in* $L^\infty(0,T;V)$ *as* $n \to \infty$
(ii) $\widetilde{u}^n(t) \to u(t)$ *weakly in* V *a.e.* $t \in [0,T]$

Proof. From (4.1) we deduce that the sequence $(\widetilde{u}^n)$ is uniformly bounded in $L^\infty(0,T;V)$. Thus, we can extract from it a subsequence still denoted $(\widetilde{u}^n)$ which converges weakly $*$ in $L^\infty(0,T;V)$. On the other hand as in [8] we thus deduce for any $t \in (0,T)$ the inequality

$$\|\widetilde{u}^n(t) - u^n(t)\|_V \le \frac{T}{n}\|\dot{u}_n(t)\|_V,. \tag{4.5}$$

Since the sequence $(\dot{u}^n)$ is bounded in $L^\infty(0,T;V)$, we thus deduce from (4.5) that $\widetilde{u}^n \to u$ weak $*$ in $L^\infty(0,T;V)$ as $n \to \infty$, whence (i).

For the proof of (ii), since $W^{1,\infty}(0,T;V) \subset C([0,T];V)$, we have $u^n(t) \to u(t)$ weakly in V, for all $t \in [0,T]$, and from (4.5) we immediately we have the conclusion. □

Remark 4.4. Since $\widetilde{u}^n(t) \in K$ a.e. t in $[0,T]$ then $u(t) \in K$ a.e. t in $[0,T]$. On the other hand, since $f \in W^{1,\infty}(0,T;V)$, we deduce that

$$\widetilde{f}^n \to f \text{ strongly in } L^2(0,T;V) \tag{4.6}$$

Proposition 4.5. *The sequence $(\tilde{u}^n)$ converges strongly to u in $L^2(0,T;V)$ and u is a solution of problem (P2) for a small enough friction coefficient.*

Proof. From inequality (3.1) we deduce the inequality

$$\langle F(\varepsilon(u^{t_{i+1}})), \varepsilon(v) - \varepsilon(u^{t_{i+1}})\rangle_Q + j(u^{t_{i+1}}, v - u^{t_{i+1}}) \ge (f^{t_{i+1}}, v - u^{t_{i+1}})_V \quad \forall v \in K$$

whence

$$\begin{aligned} &\langle F(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v) - \varepsilon(\widetilde{u}^n(t))\rangle_Q + j(\widetilde{u}^n(t), v - \widetilde{u}^n(t)) \\ &\ge (\widetilde{f}^n(t), v - \tilde{u}^n(t))_V \quad \forall v \in K, \text{ a. e. } t \in [0,T], \end{aligned} \tag{4.7}$$

we also have the inequality

$$\begin{aligned} &\langle F(\varepsilon(\widetilde{u}^{n+m}(t))), \varepsilon(v) - \varepsilon(\widetilde{u}^{n+m}(t))\rangle_Q + j(\widetilde{u}^{n+m}(t), v - \widetilde{u}^{n+m}(t)) \\ &\ge (\widetilde{f}^{n+m}(t), v - \widetilde{u}^{n+m}(t))_V \forall v \in K, \text{ a.e. } t \in [0,T]\,. \end{aligned} \tag{4.8}$$

Setting $v = \widetilde{u}^n(t)$ in (4.7) and $v = \widetilde{u}^{n+m}(t)$ in (4.8) and adding them, we obtain the inequality

$$\begin{aligned} &\langle F(\varepsilon(\widetilde{u}^{n+m}(t))) - F(\varepsilon(\widetilde{u}^n(t))), \varepsilon(\widetilde{u}^n(t)) - \varepsilon(\widetilde{u}^{n+m}(t))\rangle_Q \\ &+ \int_{\Gamma_3} \mu(|R\sigma_N(\tilde{u}^{n+m}(t))| + |R\sigma_N(\tilde{u}^n(t))|)|\tilde{u}_T^{n+m}(t) - \tilde{u}_T^n(t)|da \\ &\ge -(\widetilde{f}^{n+m}(t) - \widetilde{f}^n(t), \widetilde{u}^{n+m}(t) - \widetilde{u}^n(t))_V \end{aligned}$$

Then using (2.10) and that the mapping R is compact, we deduce that

$$\begin{aligned} \|R\sigma_N(\tilde{u}^n(t))\|_{L^2(\Gamma_3)} &\le C\|\sigma_N(\tilde{u}^n(t))\|_{H^{-\frac{1}{2}}(\Gamma_3)} \\ &\le C_1(\sup_{t\in(0,T)}\|\tilde{u}^n(t)\|_V + \sup_{t\in(0,T)}\|\tilde{f}^n(t)\|_V) \end{aligned}$$

Since

$$\|\tilde{u}^n(t)\|_V \le \|u^n(t)\|_V + \frac{T}{n}\sup_{t\in(0,T)}\|\dot{u}^n(t)\|_V,$$

we have

$$\sup_{t\in(0,T)}\|\tilde{u}^n(t)\|_V \le \max_{t\in[0,T]}\|u^n(t)\|_V + T\sup_{t\in(0,T)}\|\dot{u}^n(t)\|_V$$

So we deduce that there exists a constant $C' > 0$ such that

$$\sup_{t\in(0,T)} \|\tilde{u}^n(t)\|_V \le C'\|f\|_{W^{1,\infty}(0,T;V)}$$

Whence we deduce also that there exits a constant $C_2 > 0$ such that

$$\begin{aligned}&\|\widetilde{u}^{n+m}(t) - \widetilde{u}^n(t)\|_V^2\\ &\le C_2(\|\mu\|_{L^\infty(\Gamma_3)}\|\tilde{u}_T^{n+m}(t) - \tilde{u}_{\rm T}^n(t)\|_{L^2(\Gamma_3)^d} + \|\widetilde{f}^{n+m}(t) - \widetilde{f}^n(t)\|_V^2).\end{aligned}$$

Having in mind that

$$\begin{aligned}\|\tilde{u}_T^{n+m}(t) - \tilde{u}_T^n(t)\|_{L^2(\Gamma_3)^d} &\le \|\tilde{u}_T^{n+m}(t) - u_T^{n+m}(t)\|_{L^2(\Gamma_3)^d}\\ &\quad + \|u_T^{n+m}(t) - u_T^n(t)\|_{L^2(\Gamma_3)^d} + \|u_{\rm T}^n(t) - \tilde{u}_T^n(t)\|_{L^2(\Gamma_3)^d},\end{aligned}$$

since (u^n) is bounded in $W^{1,\infty}(0,T;V)$, the sequence $(u^n|_{\Gamma_3})$ is relatively compact in $C([0,T];L^2(\Gamma_3)^d)$ and there exists a subsequence still denoted (u^n) such that for all $\eta > 0$ there exists $n_1 \in N$, so that for all $n \ge n_1$ and all $t \in [0,T]$,

$$\|u_T^{n+m}(t) - u_T^n(t)\|_{L^2(\Gamma_3)^d} \le \eta\,.$$

On the other hand we have

$$\|u_T^n(t) - \tilde{u}_T^n(t)\|_{L^2(\Gamma_3)^d} \le c\|u^n(t) - \tilde{u}^n(t)\|_V \le c\frac{T}{n}\|\dot{u}_n(t)\|_V,$$

where $(\dot{u}^n)$ is bounded in $L^\infty(0,T;V)$. Combining these results we obtain that there exists a positive constant C_3 such that

$$\int_0^T \|\widetilde{u}_T^{n+m}(t) - \widetilde{u}_T^n(t)\|_{L^2(\Gamma_3)^d}^2 dt \le C_3(\frac{1}{n^2} + \eta^2).$$

On the other hand from (4.6), we have: For all $\eta > 0$ there exists n_2 in $\mathbf{N}$ such that for all $n \ge n_2$ and all $m \in \mathbf{N}$,

$$\int_0^T \|\widetilde{f}^{n+m}(t) - \widetilde{f}^n(t)\|_V^2 dt \le \eta.$$

Then we obtain that there exists a constant $C_4 > 0$ such that for all $\eta > 0$ there exists n_3 in $\mathbf{N}$ such that for all $n \ge n_3 = \max(n_1,n_2)$ and all $m \in N$,

$$\int_0^T \|\widetilde{u}^{n+m}(t) - \widetilde{u}^n(t)\|_V^2 dt \le C_4(2\eta + \frac{1}{n})$$

On the other hand for all $\eta > 0$ there exists n_4 in $\mathbf{N}$ such that for all $n \ge n_4$, $\frac{1}{n} \le \eta$. We thus deduce that for all $\eta > 0$ there exists $n_5 = \max(n_4,n_3)$ such that for all $n \ge n_5$,

$$\int_0^T \|\widetilde{u}^{n+m}(t) - \widetilde{u}^n(t)\|_V^2 dt \le 3C_4\eta.$$

So we conclude that

$$\widetilde{u}_n \to u \text{ strongly in } L^2(0,T;V) \tag{4.9}$$

Now to prove that u is a solution of problem, in the inequality of problem $(P_n^{t_i})$, for $v \in V$ set $w = u^{t_i} + v\Delta t$ and divide by Δt; we obtain the inequality

$$\begin{aligned}&\langle \digamma(\varepsilon(u^{t_{i+1}})), \varepsilon(v) - \varepsilon(\frac{\Delta u^{t_t}}{\Delta t})\rangle_Q + j(u^{t_{i+1}}, v) - j(u^{t_{i+1}}, \frac{\Delta u^{t_i}}{\Delta t})\\ &\ge (f(t_{i+1}), v - \frac{\Delta u^{t_i}}{\Delta t})_V + \langle \sigma_N(u^{t_{i+1}}), v_N - \frac{\Delta u_N^{t_i}}{\Delta t}\rangle \quad \forall v \in V\end{aligned}$$

Whence for any $v \in L^2(0,T;V)$, we have

$$\begin{aligned}
&\langle \mathcal{F}(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v(t)) - \varepsilon\big(\frac{d}{dt}u^n(t)\big)\rangle_Q + j(\widetilde{u}^n(t), v(t)) - j\Big(\widetilde{u}^n(t), \frac{d}{dt}u^n(t)\Big) \\
&\geq (\widetilde{f}^n(t), v(t) - \frac{d}{dt}u^n(t))_V + \langle \sigma_N(\widetilde{u}^n(t)), v_N(t) - \frac{d}{dt}u^n_N(t)\rangle
\end{aligned}$$

Integrating both sides of the previous inequality on $(0,T)$, we obtain the inequality

$$\begin{aligned}
&\int_0^T \langle \mathcal{F}(\varepsilon\big(\widetilde{u}^n(t)\big)), \varepsilon(v(t)) - \varepsilon(\frac{d}{dt}u^n(t))\rangle_Q dt \\
&+ \int_0^T j(\widetilde{u}^n(t), v(t))dt - \int_0^T j(\widetilde{u}^n(t), \frac{d}{dt}u^n(t))dt \\
&\geq \int_0^T (\widetilde{f}^n(t), v - \frac{d}{dt}u^n(t))_V dt + \int_0^T \langle \sigma_N(\widetilde{u}^n(t)), v_N - \frac{d}{dt}u^n_N(t)\rangle dt
\end{aligned} \tag{4.10}$$

□

Lemma 4.6. *For any $v \in L^2(0,T;V)$ we have the following properties:*

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \langle \mathcal{F}(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v(t)) - \varepsilon(\frac{d}{dt}u^n(t))\rangle_Q dt \\
&= \int_0^T \langle \mathcal{F}(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}(t))\rangle_Q dt
\end{aligned} \tag{4.11}$$

$$\liminf_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), \frac{d}{dt}u^n(t))dt \geq \int_0^T j(u(t), \dot{u}(t))dt \tag{4.12}$$

$$\lim_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), v(t))dt = \int_0^T j(u(t), v(t))dt \tag{4.13}$$

$$\lim_{n\to\infty} \int_0^T (\widetilde{f}^n(t), v(t) - \frac{d}{dt}u^n(t))_V dt = \int_0^T (f(t), v(t) - \dot{u}(t))_V dt \tag{4.14}$$

Proof. For proving (4.11), we write

$$\begin{aligned}
&\int_0^T \langle \mathcal{F}(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v(t)) - \varepsilon(\frac{d}{dt}u^n(t))\rangle_Q dt \\
&= \int_0^T \langle \mathcal{F}(\varepsilon(\widetilde{u}^n(t))) - \mathcal{F}(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(\frac{d}{dt}u^n(t))\rangle_Q dt \\
&\quad + \int_0^T \langle \mathcal{F}(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(\frac{d}{dt}u^n(t))\rangle_Q dt
\end{aligned}$$

Using (4.9) and the hypothesis (a) on $\mathcal{F}$, we have

$$\begin{aligned}
&\Big| \int_0^T \langle \mathcal{F}(\varepsilon(\widetilde{u}^n(t))) - \mathcal{F}(\varepsilon(u(t)), \varepsilon(v(t)) - \varepsilon(\frac{d}{dt}u^n(t))\rangle_Q dt \Big| \\
&\leq c\|\widetilde{u}^n - u\|_{L^2(0,T;V)}(\|v\|_{L^2(0,T;V)} + \|\dot{u}^n\|_{L^2(0,T;V)}) \to 0
\end{aligned}$$

We deduce that

$$\lim_{n\to\infty} \int_0^T \langle \mathcal{F}(\varepsilon(\widetilde{u}^n(t))) - \mathcal{F}(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(\frac{d}{dt}u^n(t))\rangle_Q dt = 0.$$

On the other hand, we have

$$\int_0^T \langle \mathcal{F}(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(\frac{d}{dt}u^n(t))\rangle_Q dt = \int_0^T (Au(t), v(t) - \frac{d}{dt}u^n(t))_V dt$$

which approaches

$$\int_0^T (Au(t), v(t) - \frac{d}{dt}u(t))_V dt = \int_0^T \langle \mathcal{F}(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}(t))\rangle_Q dt.$$

To prove (4.12) we write

$$j(\widetilde{u}^n(t), \frac{d}{dt}u^n(t)) = j(\widetilde{u}^n(t) - u(t), \frac{d}{dt}u^n(t)) + j(u(t), \frac{d}{dt}u^n(t))$$

then we have

$$\begin{aligned} &\Big|\int_0^T j(\widetilde{u}^n(t) - u(t), \frac{d}{dt}u^n(t))dt\Big| \\ &\le c\|\mu\|_{L^\infty(\Gamma_3)}\|R\sigma_N(\widetilde{u}^n - u)\|_{L^2(0,T;L^2(\Gamma_3))}\|\dot{u}_T^n\|_{L^2(0,T;L^2(\Gamma_3)^d)}\,. \end{aligned}$$

Since the mapping R is compact, we have

$$\lim_{n\to\infty}\|R\sigma_N(\widetilde{u}^n - u)\|_{L^2(0,T:L^2(\Gamma_3))} = 0 \tag{4.15}$$

and

$$\liminf_{n\to\infty}\int_0^T j(u(t), \frac{d}{dt}u^n(t))dt \ge \int_0^T j(u(t), \dot{u}(t))dt\,,$$

see [4]. To prove (4.13) it suffices to use (4.15). From (4.6) we deduce for any $v \in L^2(0,T;V)$:

$$\lim_{n\to\infty}\int_0^T (\widetilde{f}^n(t), v(t) - \frac{d}{dt}u^n(t))_V dt = \int_0^T (f(t), v(t) - \dot{u}(t))_V dt.$$

whence (4.14) is proved. □

Passaging to the limit in inequality (4.8), we obtain the inequality

$$\begin{aligned} &\int_0^T \langle \mathcal{F}(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}(t))\rangle_Q dt \\ &+ \int_0^T j(u(t), v(t))dt - \int_0^T (j(u(t), \dot{u}(t)))dt \\ &\ge \int_0^T (f(t), v(t) - \dot{u}(t))_V dt + \int_0^T \langle \sigma_N(u(t)), v_N(t) - \dot{u}_N(t)\rangle dt \end{aligned} \tag{4.16}$$

In this inequality we set

$$v(s) = \begin{cases} z & \text{for } s \in (t, t+\lambda) \\ \dot{u}(s) & \text{elsewhere} \end{cases}$$

to obtain the inequality

$$\begin{aligned} &\frac{1}{\lambda}\int_t^{t+\lambda} (\langle \mathcal{F}(\varepsilon(u(s))), \varepsilon(z) - \varepsilon(\dot{u}(s))\rangle_Q + j(u(s), z) - j(u(s), \dot{u}(s)))ds \\ &\ge \frac{1}{\lambda}\int_t^{t+\lambda} (f(s), z - \dot{u}(s))_V ds + \frac{1}{\lambda}\int_t^{t+\lambda} \langle \sigma_N(u(s)), z_N - \dot{u}_N(s)\rangle ds\,. \end{aligned}$$

Passing to the limit, one obtains that u satisfies the inequality (2.14). To complete the proof, integrate on $(0,T)$ both sides of (3.1); that is,

$$\begin{aligned}&\int_0^T \langle \digamma(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v(t)) - \varepsilon(\widetilde{u}^n(t))\rangle_Q dt + \int_0^T j(\widetilde{u}^n(t), v(t) - \widetilde{u}^n(t))dt \\ &\geq \int_0^T (\widetilde{f}^n(t), v(t) - \widetilde{u}^n(t))_V dt \quad \forall v \in L^2(0,T;V)\end{aligned}$$

such that $v(t) \in K$, a.e. $t \in [0,T]$. Passaging to the limit in the above inequality, and using (4.6), (4.9), we obtain the inequality

$$\begin{aligned}&\int_0^T (\langle \digamma(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(u(t))\rangle_Q + j(u(t), v(t) - u(t)))dt \\ &\geq \int_0^T (f(t), v(t) - u(t))_V \quad \forall v \in L^2(0,T;V); v(t) \in K, \quad \text{a.e. } t \in [0,T]\end{aligned}$$

Following the same reasoning as previously done, we deduce that u satisfies the inequality

$$\begin{aligned}&\langle \digamma(\varepsilon(u(t))), \varepsilon(w) - \varepsilon(u(t))\rangle_Q + j(u(t), w - u(t)) \\ &\geq (f(t), w - u(t))_V \quad \forall w \in K, \text{a.e. } t \in [0,T].\end{aligned}$$

Using Green's formula in the above inequality, as in [4], we obtain that u satisfies the inequality (2.15) and consequently u is a solution of problem (P2).

Conclusion. In this article we have shown the existence of a solution to the quasistatic unilateral contact problem with nonlocal friction for nonlinear elastic materials for a small enough friction coefficient . As well known the problem of uniqueness of the solution still remains open.

References

[1] L.-E. Andersson; *A quasistatic frictional problem with normal compliance*, Nonlinear Anal. Th . Appl.16, 4, 347-369, 1991.

[2] L.-E. Andersson; *Existence Results for Quasistatic Contact Problems with Coulomb friction*, Appl.Math.Optim.42: 169-202, 2000.

[3] H. Brezis; *Equations et inéquations non linéaires dans les espaces vectoriels en dualité*, Annales Inst. Fourier, 18, 115-175, 1968.

[4] M. Cocou, E. Pratt, M. Raous; *Formulation and approximation of quasistatic frictional contact*, Int. J. Engng Sc.,34, 7, 783-798, 1996.

[5] S. Drabla, M. Sofonea; *Analysis of a Signorini problem with friction*, IMA Journal of Applied Mathematics, 63, 2, 113-130, 1999.

[6] G. Duvaut; *Equilibre d'un solide élastique avec contact unilatéral et frottement de Coulomb*, Cr Acad. Sci. Paris, Ser A, 290, 263-265, 1980.

[7] G. Duvaut, J.-L. Lions; *les inéquations en mécanique et en physique*, Dunod, Paris, 1972.

[8] W. Han, M. Sofonea; *Quasistatic Contact Problems in* Viscoelasticity and Viscoplasticity. Studies in advanced Mathematics 30, Americal Mathematical Society and International Press, 2002.

[9] A. Klarbring, A. Miklic, M. Shillor; *A global existence result for the quasistatic problem with normal compliance*. Internat. Ser. Numer. Math. 101. Birkhäuser Verlag. basel, 85-111, 1991.

[10] R. Rocca; *Existence of a solution for a quasistatic problem of unilateral contact with local friction*, CR. Acad. Sci Paris. Ser1, t. 328, 1253-1258, 1999.

[11] M. Rochi, M. Schillor, M. Sofonea; *Quasistatic viscoplastic contact with normal compliance and friction*, J. Elasticity, 51, 105-126, 1998.

Touzaline Arezki
Faculté de Mathématiques, USTHB BP 32 EL Alia, Bab-Ezzouar, 16111, Algérie
E-mail address: atouzaline@yahoo.fr

Mignot Alain
IRMAR Mathématiques, Campus de Beaulieu, Université de Rennes1, 35042, Rennes Cedex, France
E-mail address: alain.mignot@univ-rennes1.fr