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# EXISTENCE OF POSITIVE PSEUDO-SYMMETRIC SOLUTIONS FOR ONE-DIMENSIONAL $p$-LAPLACIAN BOUNDARY-VALUE PROBLEMS

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ABSTRACT. We prove the existence of positive pseudo-symmetric solutions for four-point boundary-value problems with $p$-Laplacian. Also we present an monotone iterative scheme for approximating the solution. The interesting point here is that the nonlinear term $f$ involves the first-order derivative.

## 1. Introduction

In this paper, we consider the four-point boundary value problem

$$(\phi_p(u'))'(t) + q(t)f(t, u(t), u'(t)) = 0, \quad t \in (0, 1), \tag{1.1}$$

$$u(0) - \alpha u'(\xi) = 0, \quad u(\xi) - \gamma u'(\eta) = u(1) + \gamma u'(1 + \xi - \eta), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, \gamma \geq 0$, $\xi, \eta \in (0, 1)$ are prescribed and $\xi < \eta$.

The study of multipoint boundary-value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseer [7, 8]. Since then, the more general nonlinear multipoint boundary-value problems have been studied by many authors by using the Leray-Schauder continuation theorem, nonlinear alternative of Leray-Schauder and coincidence degree theory, we refer the reader to [1, 2, 3, 6] for some recent results. Recently, Avery and Henderson [4] consider the existence of three positive solutions for the problem

$$(\phi_p(u'))'(t) + q(t)f(t, u(t)) = 0, \quad t \in (0, 1), \tag{1.3}$$

$$u(0) = 0, \quad u(\eta) = u(1). \tag{1.4}$$

The definition of pseudo-symmetric was introduced in their paper. Based on this definition, Ma [9] studied the existence and iteration of positive pseudo-symmetric solutions for the problem (1.3)-(1.4). However, to the best of our knowledge, no work has been done for BVP (1.1)-(1.2) using the monotone iterative technique. The aim of this paper is to fill the gap in the relevant literatures. We obtain not only the existence of positive solutions for (1.1)-(1.2), but also give an iterative scheme for approximating the solutions. It is worth stating that the first term of our iterative scheme is a constant function or a simple function. Therefore, the

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iterative scheme is significant and feasible. At the same time, we give a way to find the solution which will be useful from an application viewpoint.

We consider the Banach space $E = C^1[0,1]$ equipped with norm

$$\|u\| := \max\{\|u\|_0, \|u'\|_0\},$$

where $\|u\|_0 = \max_{0\le t\le 1} |u(t)|$, $\|u'\|_0 = \max_{0\le t\le 1} |u'(t)|$. In this paper, a positive solution $u(t)$ of BVP (1.1), (1.2) means a solution $u(t)$ of (1.1), (1.2) satisfying $u(t) > 0$, for $0 < t < 1$.

We recall that a function $u$ is said to be concave on $[0,1]$, if

$$u(\lambda t_2 + (1-\lambda)t_1) \ge \lambda u(t_2) + (1-\lambda)u(t_1), \quad t_1, t_2, \lambda \in [0,1].$$

**Definition 1.1.** For $\xi \in (0,1)$ a function $u \in E$ is said to be pseudo-symmetric if $u$ is symmetric over the interval $[\xi, 1]$. That is, for $t \in [\xi, 1]$ we have $u(t) = u(1+\xi-t)$.

**Remark 1.2.** For $\xi \in (0,1)$, if $u \in E$ is pseudo-symmetric, we have $u'(t) = -u'(1+\xi-t)$, $t \in [\xi, 1]$.

Define the cone $K$ of $E$ as

$$K = \{u \in C^1[0,1] : u(t) \ge 0,\ u \text{ is concave on}[0,1] \text{ and } u \text{ is symmetric on } [\xi,1]\}.$$

For $x, y$ in $K$ a cone of $E$, recall that $x \le y$ if $y - x \in K$.

In the rest of the paper, we make the following assumptions:

- (H1) $q(t) \in L^1[0,1]$ is nonnegative and $q(t) = q(1+\xi-t)$, a.e. $t \in [\xi,1]$, and $q(t) \not\equiv 0$ on any subinterval of [0,1];
- (H2) $f \in C([0,1]\times[0,\infty)\times R, [0,\infty))$ and $f(t,x,y) = f(1+\xi-t, x, -y)$, $(t,x,y) \in [\xi,1]\times[0,\infty)\times R$. Moreover, $f(t,\cdot,y)$ is nondecreasing for $(t,y) \in [0, \frac{\xi+1}{2}]\times R$, $f(t,x,\cdot)$ is nondecreasing for $(t,x) \in [0,\frac{\xi+1}{2}] \times [0,\infty)$.

## 2. Existence Result

**Lemma 2.1** ([9]). *Each $u \in K$ satisfies the following properties:*

- (i) $u(t) \ge \frac{2}{1+\xi}\|u\|_0 \min\{t, 1+\xi-t\}$, $t \in [0,1]$;
- (ii) $u(t) \ge \frac{2\xi}{1+\xi}\|u\|_0$, $t \in [\xi, \frac{1+\xi}{2}]$;
- (iii) $\|u\|_0 = u(\frac{1+\xi}{2})$.

For $x \in K$, we define a mapping $T : K \to E$ given by

$$(Tx)(t) = \begin{cases} \alpha\phi_q(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau) \\ + \int_0^t \phi_q(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)ds, \quad 0 \le t \le \frac{1+\xi}{2}, \\ \alpha\phi_q(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau) \\ + \int_0^\xi \phi_q(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)ds \\ + \int_t^1 \phi_q(\int_{\frac{1+\xi}{2}}^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)ds, \quad \frac{1+\xi}{2} \le t \le 1. \end{cases} \tag{2.1}$$

Obviously, $Tx \in E$, and we can prove $Tx$ is a solution of the boundary-value problem

$$(\phi_p(u'))'(t) + q(t)f(t,x(t),x'(t)) = 0, \quad t \in (0,1), \tag{2.2}$$

$$u(0) - \alpha u'(\xi) = 0, \quad u(\xi) - \gamma u'(\eta) = u(1) + \gamma u'(1+\xi-\eta). \tag{2.3}$$

Therefore, each fixed point of $T$ is a solution of problem (1.1)-(1.2).

**Lemma 2.2.** *Suppose (H1), (H2) hold, then $T : K \to K$ is completely continuous and nondecreasing.*

*Proof.* For $t \in [\xi, \frac{1+\xi}{2}]$, we have $1 + \xi - t \in [\frac{1+\xi}{2}, 1]$. Therefore,

$$
\begin{aligned}
&(Tx)(1+\xi-t)\\
&= \alpha\phi_q\Big(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big) + \int_0^\xi \phi_q\Big(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\
&\quad + \int_{1+\xi-t}^1 \phi_q\Big(\int_{\frac{1+\xi}{2}}^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\
&= \alpha\phi_q\Big(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big) + \int_0^\xi \phi_q\Big(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\
&\quad + \int_\xi^t \phi_q\Big(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\
&= \alpha\phi_q\Big(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big) + \int_0^t \phi_q\Big(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\
&= (Tx)(t).
\end{aligned}
\tag{2.4}
$$

So that, $Tx$ is symmetric on $[\xi, 1]$ with respect to $\frac{1+\xi}{2}$. Obviously, $(Tx)(t) \geq 0, Tx$ is concave on $[0, 1]$. Therefore, $T : K \to K$. It is easy to see that $T : K \to K$ is completely continuous.

For $x_1(t), x_2(t) \in K$ and $x_1(t) \leq x_2(t)$, thus $x_2(t) - x_1(t) \in K$. So that $x_2'(t) - x_1'(t) \geq 0$, $t \in [0, \frac{1+\xi}{2}]$ and $x_2'(t) - x_1'(t) \leq 0$, $t \in [\frac{1+\xi}{2}, 1]$. Assumption (H2) implies $(Tx_1)(t) \leq (Tx_2)(t)$. □

**Lemma 2.3** ([5])**.** *Let $u_0, v_0 \in E$, $u_0 < v_0$ and $T : [u_0, v_0] \to E$ be an increasing operator such that*

$$u_0 \leq Tu_0, \quad Tv_0 \leq v_0.$$

*Suppose that one of the following conditions is satisfied:*

- (C1) *$K$ is normal and $T$ is condensing;*
- (C2) *$K$ is regular and $T$ is semicontinuous, i.e., $x_n \to x$ strongly implies $Tx_n \to Tx$ weakly.*

*Then $T$ has a maximal fixed point $x^*$ and a minimal fixed point $x_*$ in $[u_0, v_0]$; moreover*

$$x^* = \lim_{n\to\infty} v_n, \quad x_* = \lim_{n\to\infty} u_n,$$

*where $v_n = Tv_{n-1}$, $u_n = Tu_{n-1}$ $(n = 1, 2, 3, \dots)$, and*

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0.$$

Denote the positive quantities

$$A = 1/\max\Big\{\alpha\phi_q\Big(\int_\xi^{\frac{1+\xi}{2}} q(\tau)d\tau\Big) + \int_0^{\frac{1+\xi}{2}} \phi_q\Big(\int_s^{\frac{1+\xi}{2}} q(\tau)d\tau\Big)ds,\ \phi_q\Big(\int_0^{\frac{1+\xi}{2}} q(\tau)d\tau\Big)\Big\}, \tag{2.5}$$

$$B = 1/\int_\xi^{\frac{1+\xi}{2}} \phi_q\Big(\int_s^{\frac{1+\xi}{2}} q(\tau)d\tau\Big)ds. \tag{2.6}$$

**Theorem 2.4.** *Assume* (H1), (H2) *hold. If there exist two positive numbers $a, b$ with $\frac{2b}{1+\xi} < a$, such that*

$$\sup_{t\in[0,1]} f(t,a,a) \le \phi_p(aA), \qquad \inf_{t\in[\xi,\frac{1+\xi}{2}]} f(t,\frac{2\xi b}{1+\xi},0) \ge \phi_p(bB). \tag{2.7}$$

*Then, (1.1)-(1.2) has at least one positive pseudo-symmetric solution $v^* \in K$ with*

$$b \le \|v^*\|_0 \le a, \quad 0 \le \|(v^*)'\|_0 \le a, \quad \lim_{n\to\infty} T^n v_0 = v^*,$$

*where $v_0(t) = \frac{2b}{1+\xi}\min\{t, 1+\xi-t\}$, $t \in [0,1]$.*

*Proof.* We denote $K[b,a] = \{\omega \in K : b \le \|\omega\|_0 \le a,\ 0 \le \|\omega'\|_0 \le a\}$. Next, we first prove $TK[b,a] \subset K[b,a]$. Let $\omega \in K[b,a]$, then $0 \le \omega(t) \le \max_{t\in[0,1]}\omega(t) \le \|\omega\|_0 \le a$,
$t \in [0,1]$, $0 \le \omega'(t) \le \max_{t\in[0,1]}|\omega'(t)| = \|\omega'\|_0 \le a$, $t \in [0, \frac{1+\xi}{2}]$. By lemma 2.1 (ii), $\min_{t\in[\xi,\frac{1+\xi}{2}]}\omega(t) \ge \frac{2\xi}{1+\xi}\|\omega\|_0 \ge \frac{2\xi b}{1+\xi}$, $\min_{t\in[\xi,\frac{1+\xi}{2}]}\omega'(t) \ge \omega'(\frac{1+\xi}{2}) = 0$. So, by assumption (2.7), we have

$$0 \le f(t,\omega(t),\omega'(t)) \le f(t,a,a) \le \sup_{t\in[0,1]} f(t,a,a) \le \phi_p(aA), \quad t \in [0, \frac{1+\xi}{2}], \tag{2.8}$$

$$f(t,\omega(t),\omega'(t)) \ge f(t,\frac{2\xi b}{1+\xi},0) \ge \inf_{t\in[\xi,\frac{1+\xi}{2}]} f(t,\frac{2\xi b}{1+\xi},0) \ge \phi_p(bB), \quad t \in [\xi, \frac{1+\xi}{2}]. \tag{2.9}$$

By lemma 2.2, we know $T\omega \in K$. So, lemma 2.1 (iii) implies $\|T\omega\|_0 = (T\omega)(\frac{1+\xi}{2})$. As a result,

$$\begin{aligned}
&\|T\omega\|_0\\
&= (T\omega)(\frac{1+\xi}{2})\\
&= \alpha\phi_q(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau) + \int_0^{\frac{1+\xi}{2}} \phi_q(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)ds\\
&\le aA(\alpha\phi_q(\int_\xi^{\frac{1+\xi}{2}} q(\tau)d\tau) + \int_0^{\frac{1+\xi}{2}} \phi_q(\int_s^{\frac{1+\xi}{2}} q(\tau)d\tau)ds) \le a;
\end{aligned}$$

$$\begin{aligned}
\|(T\omega)'\|_0 &= (T\omega)'(0)\\
&= \phi_q(\int_0^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)\\
&\le aA\phi_q(\int_0^{\frac{1+\xi}{2}} q(\tau)d\tau) \le a;
\end{aligned}$$

$$\begin{aligned}
\|T\omega\|_0 &= (T\omega)(\frac{1+\xi}{2})\\
&\ge \int_\xi^{\frac{1+\xi}{2}} \phi_q\Big(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\
&\ge bB\int_\xi^{\frac{1+\xi}{2}} \phi_q\Big(\int_s^{\frac{1+\xi}{2}} q(\tau)d\tau\Big)ds = b.
\end{aligned}$$

Thus, $b \le \|T\omega\|_0 \le a$, $0 \le \|(T\omega)'\|_0 \le a$, which implies $TK[b,a] \subset K[b,a]$.

Let $\upsilon_0(t) = \frac{2b}{1+\xi}\min\{t, 1+\xi-t\}$, $t \in [0,1]$, then $\|\upsilon_0\|_0 = b$ and $\|\upsilon_0'\|_0 = \frac{2b}{1+\xi} < a$, so $\upsilon_0 \in K[b,a]$. Let $\upsilon_1 = T\upsilon_0$, then $\upsilon_1 \in K[b,a]$, we denote

$$\upsilon_{n+1} = T\upsilon_n = T^{n+1}\upsilon_0, \quad (n = 0,1,2,\dots). \tag{2.10}$$

Since $TK[b,a] \subset K[b,a]$, we have $\upsilon_n \in K[b,a]$, $(n = 0,1,2,\dots)$. From $\upsilon_1 \in K[b,a]$, thus

$$\upsilon_1(t) \ge \frac{2}{1+\xi}\|\upsilon_1\|_0 \min\{t, 1+\xi-t\} \ge \frac{2b}{1+\xi}\min\{t, 1+\xi-t\} = \upsilon_0(t), \quad t \in [0,1],$$

which implies $T\upsilon_0 \ge \upsilon_0$. $K$ is normal and $T$ is completely continuous. By Lemma 2.3, we have $T$ has a fixed point $\upsilon^* \in K[b,a]$. Moreover, $\upsilon^* = \lim_{n\to\infty}\upsilon_n$. Since $\|\upsilon^*\|_0 \ge b > 0$ and $\upsilon^*$ is a nonnegative concave function on $[0,1]$, we conclude that $\upsilon^*(t) > 0$, $t \in (0,1)$. Therefore, $\upsilon^*$ is a positive pseudo-symmetric solution of (1.1)-(1.2). ☐

**Corollary 2.5.** *Assume* (H1),(H2) *hold. If*

$$\limsup_{l\to 0}\inf_{t\in[\xi,\frac{1+\xi}{2}]}\frac{f(t,l,0)}{\phi_p(l)} \ge \phi_p\Big(\frac{1+\xi}{2\xi}B\Big), \tag{2.11}$$

*particularly,* $\limsup_{l\to 0}\inf_{t\in[\xi,\frac{1+\xi}{2}]}\frac{f(t,l,0)}{\phi_p(l)} = +\infty$*,*

$$\liminf_{l\to+\infty}\sup_{t\in[0,1]}\frac{f(t,l,l)}{\phi_p(l)} \le \phi_p(A), \tag{2.12}$$

*particularly,* $\liminf_{l\to+\infty}\sup_{t\in[0,1]}\frac{f(t,l,l)}{\phi_p(l)} = 0$*. Where $A, B$ are defined as (2.5), (2.6). Then there exist two positive numbers $a, b$ with $\frac{2b}{1+\xi} < a$, such that problem (1.1), (1.2) has at least one positive pseudo-symmetric solution $\upsilon^* \in K$ with*

$$b \le \|\upsilon^*\|_0 \le a,\ 0 \le \|(\upsilon^*)'\|_0 \le a,\ \lim_{n\to\infty}T^n\upsilon_0 = \upsilon^*,$$

*where* $\upsilon_0(t) = \frac{2b}{1+\xi}\min\{t, 1+\xi-t\}$*,* $t \in [0,1]$*.*

**Remark 2.6.** Problem (1.1)-(1.2) may have two positive pseudo-symmetric solutions $\omega^*, \upsilon^* \in K$, if we make another iteration by choosing $\omega_0(t) = a$ and $\omega_n = \lim_{n\to\infty}T^n\omega_0 = \omega^*$. However, $\omega^*$ and $\upsilon^*$ may be the same solution.

**Example 2.7.** We consider the problem

$$(|u'|^3u')'(t) + \frac{1}{t^{\frac12}(\frac43 - t)^{\frac12}}[(u'(t))^2 + \ln((u(t))^2+1)] = 0, \quad t \in (0,1), \tag{2.13}$$

$$u(0) - 2u'(\frac13) = 0, \quad u(\frac13) - 3u'(\frac12) = u(1) + 3u'(\frac56). \tag{2.14}$$

We notice that $p = 5$, $\alpha = 2$, $\xi = \frac13$, $\gamma = 3$, $\eta = \frac12$. Obviously, $f(t,u,u') = (u'(t))^2 + \ln((u(t))^2+1)$ is nondecreasing for $(t,u') \in [0,\frac23]\times R$, $f(t,u,u') = (u'(t))^2 + \ln((u(t))^2+1)$ is nondecreasing for $(t,u) \in [0,\frac23]\times[0,\infty)$, $q(t) = \frac{1}{t^{\frac12}(\frac43-t)^{\frac12}}$ is nonnegative and pseudo-symmetric about $\frac23$. So, conditions (H1),(H2) are satisfied.

On the other hand,

$$\limsup_{l\to 0}\inf_{t\in[\xi,\frac{1+\xi}{2}]}\frac{f(t,l,0)}{\phi_p(l)}=\limsup_{l\to 0}\inf_{t\in[\frac{1}{3},\frac{2}{3}]}\frac{\ln(l^2+1)}{l^4}=\infty,$$

$$\liminf_{l\to +\infty}\sup_{t\in[0,1]}\frac{f(t,l,l)}{\phi_p(l)}=\liminf_{l\to +\infty}\sup_{t\in[0,1]}\frac{l^2+\ln(l^2+1)}{l^4}=0.$$

Therefore, from Corollary 2.5, it follows that (2.13)-(2.14) has at least one positive pseudo-symmetric solution.

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