CHIP-FIRING ON SIGNED GRAPHS

by

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Contents

Abstract

Graphical chip-firing is a process where 'chips' are exchanged between vertices of a graph, the dynamics of which are governed by the graph Laplacian. Chip-firing is a well-developed field with applications to physics, computer science, and many areas of mathematics.

Guzmán and Klivans introduced a generalization of the graphical chip-firing model such that the dynamics can be governed by any invertible matrix. In this model, the set of allowable configurations is described by the lattice points of a rational convex cone given by a choice of M-matrix and the notions of criticality and superstability from classical chip-firing have analogues. A signed graph is a generalization of a simple graph with edges assigned to be either positive or negative. Signed graphs were first introduced by Harary in the context of social psychology, further studied by Zaslavsky, who investigated their matroidal properties, and have uses in a wide range of fields from data science to ecology.

Here we study the chip-firing model on signed graphs that results from applying the Guzmán-Klivans theory to the invertible signed graph Laplacian and the M-matrix graph Laplacian of the underlying graph. We investigate the behavior of this model and develop tools to compute examples.

1 Graphs and chip-firing

Chip-firing is widely understood as a game played on a graph and its study is interesting largely due to the way it relates to many other concepts in related fields of math. Physics and combinatorics have both introduced processes or systems that we now understand to share the same underlying DNA which we call chip-firing, with a portion of that DNA also modeling a well studied concept from computer science.

The chip-firing model as we will discuss it moving forward, will play out on a finite graph with a 'sink' vertex designated and relabeled q. Each vertex, with the exception of the sink, is assigned some number of 'chips' and if the vertex is ready to fire, may perform a 'firing move' to distribute its chips. The sink vertex does not fire and we do not track its chips.

1.1 Graphs and their Laplacians

A simple graph, G, is a pair of sets (V, E). Here V is the vertex set given by a set of points, $V = \{v_0, v_1, \ldots, v_n\}$, and E is the edge set of unordered pairs of distinct vertices, $E = \{(u, v)\}$. We will always specify v_0 as the sink vertex, and sometimes use q to refer to this vertex to match other conventions in the literature.

The graph Laplacian is a matrix representation of a simple graph that encodes the distribution dynamics of firing moves. The graph Laplacian represents the degree or number of neighbors of each vertex along the diagonals and the adjacency of vertices along the off diagonals.

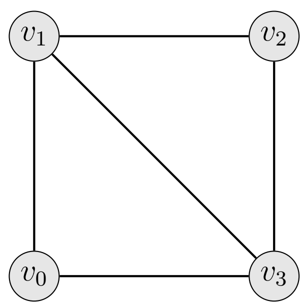
Definition 1. Let $G = (V, E)$ be a graph on $n + 1$ vertices $\{v_0, \ldots, v_n\}$. The *graph Laplacian* of G, $\Delta(G)$ is the $(n + 1) \times (n + 1)$ matrix given by

$$\Delta_{ij} \begin{cases} \deg(v_i) & i = j \\ -1 & v_i \neq v_j \text{ and } v_i, v_j \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

We can also take the reduced graph Laplacian by a vertex q of a graph which is necessary for the formulation of chip-firing we will be using.

Definition 2. Let $G = (V, E)$ be a graph. The *reduced graph Laplacian* of G with respect to the vertex q, is the $n \times n$ matrix $\Delta_q(G)$ that results from deleting the row and column associated with q from $\Delta(G)$.

We refer to Figure 1 for an example of a graph and the associated Laplacian matrices.



$$\Delta = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$$\Delta_{v_0} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Figure 1: A graph and its Laplacians.

1.2 Chip-firing

Next we define the chip-firing game that we will study. We can express the state of our chip-firing system, the number of chips at each vertex, as a vector. This state is referred to as the configuration.

Definition 3. Suppose G is a graph on vertex set $\{v_0, \ldots, v_n\}$ with sink vertex v_0. A *configuration* on G is a non-negative integer vector $\boldsymbol{c}$ representing the number of chips at each nonsink vertex of G.

$$\boldsymbol{c} = \{c_1, c_2, \ldots, c_n\} \in \mathbb{Z}^n$$

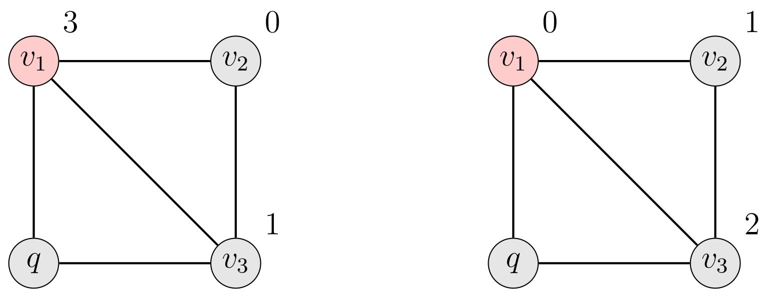
Definition 4. A vertex v in a graph G is *ready to fire* if the number chips at v is at least the number of its neighbors, so that

$$c_v \geq \deg(v).$$

When a vertex fires it sends one chip along each of its incident edges to each of its neighboring vertices. We can use the reduced graph Laplacian to model these redistribution dynamics. For a graph G, if a configuration $\boldsymbol{c}'$ is the result of firing vertex v_i from the configuration $\boldsymbol{c}$ then

$$\boldsymbol{c}' = \boldsymbol{c} - \Delta_q(G)e_i,$$

where e_i is the ith standard basis vector of $\mathbb{R}^n$.



$$\boldsymbol{c}' = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Figure 2: Firing v_1

In Figure 2 we see that when firing v_1, that vertex looses three chips, v_2 and v_3 gain one each, and the chip unaccounted for is lost to the sink. We will also be interested in the notion of firing a set (or even multiset) of vertices at once. See below for more details.

Chip-firing defines an equivalence relation $\sim$ on the set of all vectors $\boldsymbol{c}$ in $\mathbb{Z}^n$. Here two vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ are equivalent, written $\boldsymbol{c} \sim \boldsymbol{d}$, if one can obtain $\boldsymbol{c}$ from $\boldsymbol{d}$ via a sequence of firings and reverse firings (performed by $\boldsymbol{c}' = \boldsymbol{c} + \Delta_q(G)e_i$). Equivalently, we have $\boldsymbol{c} \sim \boldsymbol{d}$ if $\boldsymbol{c} - \boldsymbol{d} = \Delta(G)z$ for some vector $z \in \mathbb{Z}^n$.

One can check that the number of equivalence classes of chip configurations is counted by the determinant of the reduced Laplacian $\Delta_q(G)$. An important result in Graph Theory (Kirchoff's theorem or the Matrix-Tree theorem) says that this number is given by the number of spanning trees of the graph G.

Below we will see that these firing equivalence classes have interesting properties, described by certain representative configurations.

1.3 Special configurations

In this section we will describe certain special chip configurations, obtained by choosing elements from each equivalence class determined by $\Delta_q(G)$. Looking at the configuration $\boldsymbol{c}'$ from Figure 2 we notice that no site is ready to fire, configurations like this are called *stable*.

Definition 5. A configuration $\boldsymbol{c}$ on a graph G is *stable* if

$$c_i < \deg(v_i) \text{ for all } v_i \in V$$

or equivalently

$$\boldsymbol{c} - \Delta_q(G)e_i \not\geq 0 \text{ for } 1 \leq i \leq n.$$

There are two special cases of stable configurations, called critical and super-stable configurations.

Recall that stable configurations are those that do not allow any firings. It turns out that if we start our firings from a 'generic' starting point and stabilize, we obtain an important subclass of stable configurations as follows.

Definition 6. A configuration $\boldsymbol{c}$ on a graph G is *reachable* if there exists some configuration $\boldsymbol{d}$ such that

1. $\boldsymbol{d} - \Delta_q(G)e_i \geq 0$ for $\leq i \leq n$ and
2. $\boldsymbol{c} = \boldsymbol{d} - \sum_{j=1}^{k} \Delta_q(G)e_{i_j}$ such that $\boldsymbol{d} - \sum_{j=1}^{l} \Delta_q(G)e_{i_j} \geq 0$ for all $l \leq k$.

That is, a configuration $\boldsymbol{c}$ is reachable if there exists some other configuration $\boldsymbol{d}$ in which both every site can fire and $\boldsymbol{c}$ is the result of some sequence of firings on $\boldsymbol{d}$.

Definition 7. A configuration is *critical* if it is both stable and reachable.

An important property of critical configurations is that they form an abelian group under componentwise addition and stablization. This group is called the *critical group* of the graph G, denoted $K(G)$. One can also check that $K(G)$ is isomorphic to $\mathbb{Z}^n/\Delta_q(G)$. See [3] for more details.

Another important collection of configurations comes from the set of *superstable* configurations. While so far we have only been firing one vertex at a time, it is necessary to expand expand our rules to allow for simultaneous firings.

For $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq V$, a non-empty subset of the vertices of a graph, we let χ_S be the characteristic vector of S

$$\chi_S = \sum_{j=1}^{k} e_{i_j}.$$

For $S \subseteq V$ a non-empty subset of the vertices of a graph G, the configuration $\boldsymbol{c}'$ that is the result of the *set-firing* of S from the configuration $\boldsymbol{c}$ is given by the

$$\boldsymbol{c}' = \boldsymbol{c}\Delta_q(G)\chi_S,$$

where the set-firing of S is only *legal* if $\boldsymbol{c}' \geq 0$

With set-firings we can now define superstable configurations.

Definition 8. A configuration $\boldsymbol{c}$ is *superstable* if there are no legal set-firings, so that

$$\forall S \subseteq V, \boldsymbol{c} - \Delta_q(G)\chi_S \ngeq 0.$$

One can view the set of superstable configurations as solving a certain 'energy minimization' problem among all elements of the equivalence class defined by the Laplacian of a graph. This is the perspective taken in Baker-Shokrieh [2], and further developed in greater generality by Guzmán and Klivans in [5].This inspires the generalizations that we discuss in the next sections.

An important result in the theory of chip-firing is that each equivalence class determined by $\Delta(G)$ contains a unique critical and superstable configuration. In particular the cardinality of each of these sets is $\tau(G)$, where $\tau(G)$ is the number of spanning trees of G. Furthermore, the set of critical configurations and superstable configurations are in simple duality. If we let $\boldsymbol{k}$ denote the *canonical configuration* then one can check that $\boldsymbol{c}$ is critical if and only if $\boldsymbol{k} - \boldsymbol{c}$ is superstable.

1.4 Other important results in chip-firing

Although chip-firing has its origins in statistical physics and combinatorics, more recently it has seen connections to Algebraic Geometry, especially in light of a discretization of the Riemann-Roch theorem by Baker and Norine [1].

As we have seen the set of superstable configurations of a graph G is in bijection with $\tau(G)$, the set of spanning trees of G. Dhar [7] developed an efficient bijection between the superstable configurations of a graph and its spanning trees.

The set of superstable configurations is also related to the study of parking functions and their generalizations.

2 Signed graphs

We will briefly introduce signed graphs and their Laplacians. As far we know there is no previously studied interpretation for chip-firing on signed graphs, and this is partly what motivates our work. Signed graphs are widely studied and applicable to other fields of math and can be used to model social psychology, data science, and other complex systems. Zaslavsky [8] studied their relation to matroid theory where he also established a relevant formulation of the Matrix-Tree theorem for signed graphs.

Put simply, a signed graph is a graph in which each edge has been assigned either a positive or negative value.

Definition 9. A *Signed Graph* G is the pairing of a simple graph, denoted $|G|$, and a mapping $\sigma : E(|G|) \to \{\pm\}$.

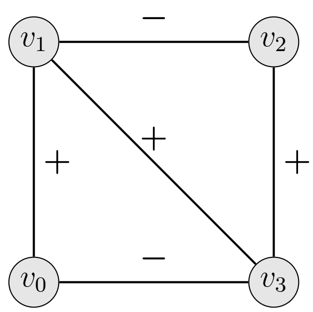
We define the signed graph Laplacian and reduced signed graph Laplacian as follows.

Definition 10. Let $G = (|G|, \sigma)$ be a signed graph on $n+1$ vertices. The *signed graph Laplacian* of G, $\Delta(G)$ is the $(n+1) \times (n+1)$ matrix given by

$$\Delta_{ij} \begin{cases} \deg(v_i) & i = j \\ -1 & v_i \neq v_j\ ,\ v_i, v_j \in E(G), \text{ and } \sigma((v_i, v_j)) = + \\ 1 & v_i \neq v_j\ ,\ v_i, v_j \in E(G), \text{ and } \sigma((v_i, v_j)) = - \\ 0 & \text{otherwise} \end{cases}$$

Definition 11. Let $G = (|G|, \sigma)$ be a signed graph. The *reduced signed graph Laplacian* of G with respect to the vertex q, $\Delta_q(G)$ is the $n \times n$ matrix that results from deleting the row and column associated with q from $\Delta(G)$.

In Zaslavsky's work [8] he studies the notions of balance, switching, restriction and contraction, and relates signed graphs to the notion of *double coverings* of ordinary graphs. He also describes the matroids that are underlying a signed graph and



$$\Delta = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 3 & 1 & -1 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -1 & 3 \end{bmatrix}$$

$$\Delta_{v_0} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Figure 3: A signed graph and its Laplacians

provides a description of their independent sets in graph theoretical terms. He also proves the following matrix-tree theorem for signed graphs.

Theorem 1. *[8] Let G be a signed graph on n vertices and let b_ℓ be the number of independent sets with n edges that contain ℓ circles. Then*

$$\det \Delta(G) = \sum_{\ell=0}^{n} 4^\ell b_\ell.$$

For our work we will be interested in a matrix-tree theorem for the determinant of the reduced Laplacian $\Delta_q(G)$. Here we recover $\det \Delta_q(G)$ as the sum over independent sets where there is only one 'balanced' component, which contains q. We refer to [8] for more details.

3 Generalized chip-firing

When looking at chip-firing through the perspective of configuration vectors and the graph Laplacian we do not need to consider the graph in the system, the redistribution dynamics of firing moves are all encoded within the matrix. This leads us to an interesting question. If we can use a graph Laplacian to chip-fire, can we play the game using other matrices? Matrices that do not even represent a graph?

As it turns out, we can. Guzmán and Klivans [6] established a theory for chip-firing that utilizes a pair of matrices, (L,M), where L is any invertible matrix and M is a so-called M-matrix (see below for a definition).

In their generalization of the graphical chip-firing model, any invertible integer matrix, L, can be used to govern the redistribution dynamics of the chips while an M-matrix of the same size, M, is used to define the set of valid configurations.

Definition 12. Suppose M is an invertible matrix with the property that $\mathrm{M}_{i,j} \leq 0$ for all $i \neq j$ and $\mathrm{M}_{i,i} > 0$ for all i. Then M is an *M-matrix* if any (and hence all) of the following are true:

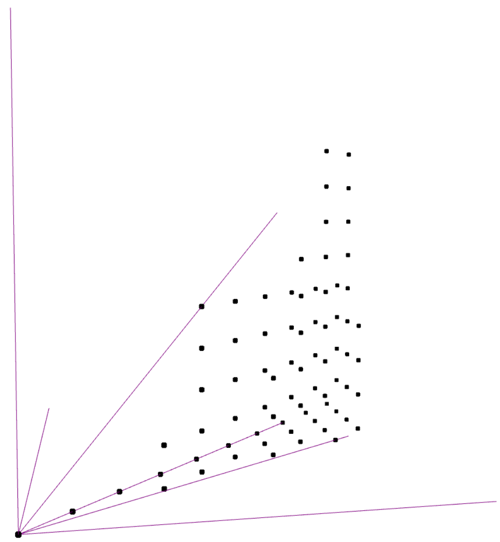
(a) M is avalanche finite.

(b) All entries of M^{-1} are non-negative.

(c) There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and M has all positive entries.

Here M is said to be *avalanche finite* if given any initial configuration the process of firing via the rule defined by M eventually stabilizes. These systems are closely related to chip-firing and exist as a generalization of graphical chip-firing with which non-graphical chip-firing models can be built. In particular reduced graph Laplacian matrices are M-matrices (and equivalently avalanche finite).

Recall that in graphical chip-firing each site has a non-negative number of chips which are collectively represented by the configuration vector. The set of valid con-



$$N = \mathrm{LM}^{-1} = \begin{pmatrix} 2 & 2 & 1 \\ \frac{5}{4} & 2 & \frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

Figure 4: An instance of S^+ for the given N

figurations then is $\mathbb{Z}^n_{\geq 0}$ which can be visualized as the lattice points contained within the positive orthant. In the Guzmán-Klivans theory we can visualize the set of valid configurations as the lattice points contained within the cone LM^{-1}, we call this set $S+$ and define it as follows for $\mathrm{N} := \mathrm{LM}^{-1}$.

$$S^+ = \{\mathrm{N}x | \mathrm{N}x \in \mathbb{Z}^n, x \in \mathbb{R}^n_{\geq 0}\}$$

We can see now that if, for a simple graph G, we apply the Guzmán-Klivans theory to the pair $(\Delta_q(G), \Delta_q(G))$, N will be the identity matrix, $S^+ = \mathbb{Z}^n_{\geq 0}$, and we get the graphical chip-firing model. It is also worth noting that S^+ may not be a subset of $\mathbb{Z}^n_{\geq 0}$, that is, we could have a valid configuration which includes a negative number of chips at some site.

With this new theory, and in particular this new understanding of what a valid configuration is, it is necessary to restate when a site is ready to fire. For a valid configuration $\boldsymbol{c} \in S^+$, the site i is ready to fire if $\boldsymbol{c} - \mathrm{L}e_i = \boldsymbol{c}' \in S^+$.

We will also restate the definitions for stable, reachable, critical and superstable in this setting.

Definition 13. A configuration $\boldsymbol{c} \in S^+$ is *stable* if $\boldsymbol{c} - Le_i \notin S^+$ for $1 \leq i \leq n$.

Definition 14. A configuration $\boldsymbol{c} \in S^+$ is *reachable* if there exists some configuration $\boldsymbol{d} \in S^+$ such that

1. $\boldsymbol{d} - Le_i \in S^+$ for $\leq i \leq n$ and
2. $\boldsymbol{c} = \boldsymbol{d} - \sum_{j=1}^{k} Le_{i_j}$ such that $\boldsymbol{d} - \sum_{j=1}^{l} Le_{i_j} \in S^+$ for all $l \leq k$.

Definition 15. A configuration in S^+ is *critical* if it is both stable and reachable.

Where the definitions for stable and critical configurations follow from graphical chip-firing we extend the definition of superstable configuration to include multiset-firing.

Definition 16. A configuration $\boldsymbol{c} \in S^+$ is *z-superstable* if $\boldsymbol{c} - Lz \notin S^+$ for every $z \in \mathbb{Z}^n_{\geq 0}$ and $z \neq 0$.

Here z represents a multiset-firing where the value z_i indicates how many times we will fire site i. In what follows we will refer to these configurations as *superstable* if the context is clear.

Guzmán and Klivans prove that several results from classical chip-firing extend to this more general setting. In particular, each equivalence class of $\mathbb{Z}^n/\mathrm{L}$ contains a unique critical configuration and a unique superstable configuration. From this it follows that each set has the same cardinality, given by the determinant of L.

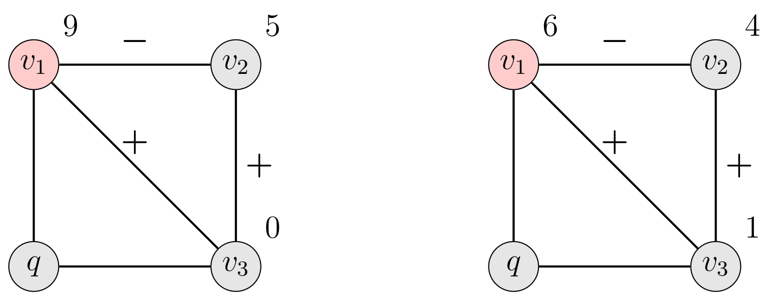
On the other hand, much of what we know from graphical chip-firing is now lost. Notably, we no longer have the duality between critical and superstable configurations and any connections with other areas of graph theory are no longer relevant in this setting such as the relationship between superstable configurations and spanning trees or parking functions.

For any fixed L we select to govern the firing rules of our game, we have free choice of an M-matrix to pair it with; each selection resulting in a different cone of valid configurations and thus a different chip-firing system. This brings us to the dilemma of what matrix to select for M, given an L. It is here that we turn to signed graphs where, where we can apply the Guzmán Klivans model nicely. We can fix L to be the reduced signed graph Laplacian, an invertible matrix, and since the reduced graph Laplacian is an M-matrix, we can pick M to be the reduced graph Laplacian of the underlying graph.

4 Chip-firing on signed graphs

Let G be a signed graph on n-vertices and $|G|$ be the underlying graph of G. In this section we will study the chip-firing model on $(\Delta_q(G), \Delta_q(|G|))$. Studying the Guzmán-Klivans theory on this pairing gives us a natural selection for for the M-matrix that also looks very similar to the $(\Delta_q(G), \Delta_q(G))$ pair from which we retrieve graphical chip firing. We also notice that in the case where all edges are positive we have exactly that same $(\Delta_q(G), \Delta_q(G))$ pair.

In Figure 5 we see that the effect of the negative edge (v_1, v_2) on the firing of v_1 is that both v_1 and v_2 loose a chip. Notice also from Figure 3 that the signs of edges incident to the sink are not preserved in the reduced signed graph Laplacian and as such have no effect on the chip-firing system.



$$\boldsymbol{c}' = \begin{bmatrix} 9 \\ 5 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

To check that $\boldsymbol{c}' \in S^+$ we must confirm that for $\boldsymbol{c}' = \mathrm{LM}^{-1}x$, $x \in \mathbb{R}^n_{\geq 0}$.

$$\mathrm{ML}^{-1}\boldsymbol{c}' = x \begin{bmatrix} \frac{4}{3} & -\frac{4}{3} & -\frac{1}{3} \\ -\frac{5}{6} & \frac{4}{3} & -\frac{1}{6} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{1}{6} \\ 1 \end{bmatrix}$$

Figure 5: Firing v_1

Recall in the general theory of Guzmán-Klivans the cone S^+ of allowable configurations can be any rational cone in $\mathbb{Z}^n$. In the case of signed graphs there are some restrictions.

Proposition 1. *For any signed graph G, the set of allowable configurations for G is contained in the nonnegative orthant.*

Proof. Let $\Delta_q(G)$ denote the (reduced) Laplacian of the signed graph G, and let $\Delta_q(|G|)$ denote the reduced Laplacian of the underlying graph. Since $\Delta_q(|G|)$ is an M-matrix we have that the entries of $\Delta_q(|G|)^{-1}$ are all nonnegative from Definition 12. Recall that the columns of $\Delta_q(G)\Delta_q(|G|)^{-1}$ define the cone of the allowable configurations. By definition the ith column of $\Delta_q(G)\Delta_q(|G|)^{-1}$ is given by $\Delta_q(G)\vec{m}_i$, where $\vec{m}_i$ is the ith column of $\Delta_q(|G|)$. Recall that $\Delta_q(G)$ is obtained from $\Delta_q(|G|)$ by changing some negative -1 entries to $+1$. Since $\Delta_q(|G|)\Delta_q(|G|)^{-1}$ has all nonnegative entries and $\Delta_q(|G|)^{-1}$ has nonnegative entires, this implies that $\Delta_q(G)\Delta_q(|G|)^{-1}$ has all nonnegative entries, and the claim follows. □

Recall that much of what is studied in chip-firing relates to the critical and superstable configurations and much of that theory is lost in the Guzmán-Klivans generalization. Applying their theory to signed graphs gives us an opportunity to try and recover what was lost. If we wish to study the critical and superstable configurations of a chip-firing system it is helpful to first calculate some examples.

4.1 Code

As we can see from the example in Figure 5, in order to perform firing moves, each resultant configuration must be confirmed to be an element of S^+(represented in Figure 4). In classical chip-firing this is as easy as checking that the configuration is non-negative and it may take only a few minutes to find the critical and superstable configurations of a small graph. Now a great deal more work must go into calculating these examples for a signed graph.

Klivans provided us with a function written in Matlab to calculate the critical configurations of a graph which, through experimentation, we found to provide inconsistent results. Using that code as a framework we rewrote the function to calculate the critical configurations of a signed graph and wrote an additional function to calculate the superstable configurations of a signed graph.

The `findcritical` function takes an initial configuration `v`, checks that `v` satisfies part 1 of Definition 14, and then performs firings, checking that each firing is legal, until the configuration has stabilized and is therefore critical. The function may need tuning on the `max_rand_value` to ensure it terminates.

```
function [lt]=findcritical(L,M)
  % input: L invertible matrix, M M-matrix
  % output: critical configurations

  max_rand_value=50;

  n = size(L,1);
  lambda = abs(det(L));
  Ninv = double(M*sym(inv(L)));
  lt=[];
  ltInitial=[];
  g=0;
  while g < lambda
    % Take a random sufficiently large integer vector.
    v = randi([0 max_rand_value],n,1);
    vInitial=v;

    % Check if v is in S+ equivalently, check if inv(N)*v is in R+
    if not(all(Ninv*v ≥ 0))
      continue
    end

    % Check that v is sufficiently large (every site can fire)
    vsufficient=true;
    for i=1:n
      k = zeros(n,1);
      k(i,1) = 1;

      if not(all(Ninv*(v - L*k) ≥ 0))
        vsufficient=false;
      end
    end
    if not(vsufficient)
      continue
    end

    % Chip-fire until stable - no single site can fire.
    t = 0;
    while t < n
      t = 0;
      for i=1:n
        k = zeros(n,1);
        k(i,1) = 1;
        testfiring = v - L*k;
        if all(Ninv*testfiring ≥ 0)
          v = testfiring;
```

```
        else
          t=t+1;
        end
      end
    end

    % Check that we are not adding a duplicate and add to the list
    valreadyfound=false;
    for i=1:g
      if v'==lt(i,:);
        valreadyfound=true;
      end
    end
    if not(valreadyfound)
      lt =[lt;v'];
      ltInitial=[ltInitial;vInitial'];
    end
      g=size(lt,1);
  end
  lt=sortrows(lt);
end
```

The `findzsuperstable` function takes each configuration `config` from the set of critical configurations for an (L, M) pair and performs consecutive multiset firings on `config` where each site may fire up to `mset_max` times, checking that each firing is legal, until no firings can occur and the configuration is superstable.

```
function [superedlts]=findzsuperstable(L,M,mset_max)
  %input: L invertible matrix, M M-matrix, mset_max the most ...
     times a vertex might be fired
  %output: critical configurations
  % When mset_max is 1, the function will return \chi-superstables

  Ninv = double(M*sym(inv(L)));

  % Calculate starting from the criticals
  lt = findcritical(L,M);

  % Get the multisets
  multisets=findmultisets(size(L,1),mset_max);
  multisets(1,:)=[];

  superedlts=zeros(size(lt));
  for i=1:size(lt,1)
    % config starts as the critical representation of the ...
       equivalence class under L dynamics for which we are ...
       looking for the superstable
    config=lt(i,:)';
    isconfigsuper=false;
    while not(isconfigsuper)
      isconfigsuper=true;
      % Cycle through all the multisets we are going to try ...
         firing on the config
      for mset=1:size(multisets,1)
        % testfiring is the config after firing this multiset
        testfiring=config-L*multisets(mset,:)';
        % Check that the firing was legal
        if all(Ninv*testfiring ≥ 0)
          % if the firing was legal we will make that the new ...
             config we are testing and start over with our multisets
          config=testfiring;
          % Make sure we do not falsely assume the config is ...
             superstable
          isconfigsuper=false;
          % Start over with the multiset firings
          break;
        end
      end
    end
    superedlts(i,:)=config';
  end
end
```

The `findmultisets` function is used by `findzsuperstable` and provides all integer vectors of dimension `n` that are componentwise less than or equal to `max`.

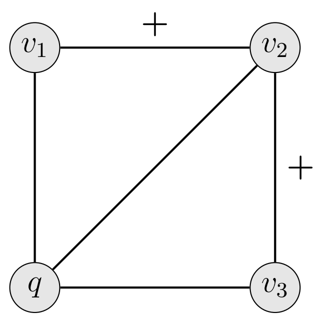
```
function [msets]=findmultisets(n,max)
  %input: n=numbner of elements in the multiset, max=maxmium ...
     value within the multiset
  %output: an array of vectors of length n including every every ...
     combination of 0-n for each value

  max=max+1;
  rangecol=zeros(max,1);
  for i=1:max
    rangecol(i,1)=i-1;
  end

  msets=rangecol;
  for i=1:n-1
    newmset=[];
    for j=1:size(msets,1)
      row=msets(j,:);
      rowexpansion=row;
      for k=1:max-1
        rowexpansion=[rowexpansion;row];
      end
      rowexpansion=[rowexpansion rangecol];
      newmset=[newmset;rowexpansion];
    end
    msets=newmset;
  end
end
```

4.2 Calculated examples

Using this code we calculated the critical and superstable configurations for a number of signed graphs. Below are those similar to the example we use in this text.

$K_4 \backslash \{e\}$ with $\deg(q) = 3$ and $\sigma : \emptyset \mapsto -$

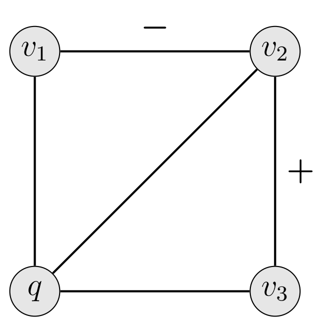
$$\mathrm{L} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \mathrm{LM}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$Criticals = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} z = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$K_4 \backslash \{e\}$ with $\deg(q) = 3$ and $\sigma : (1, 2) \mapsto -$

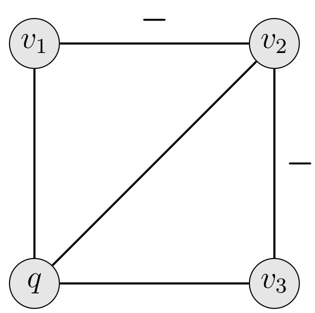
$$\mathrm{L} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \mathrm{LM}^{-1} = \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{5}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}$$



$$Criticals = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 3 & 1 \\ 3 & 4 & 0 \\ 3 & 4 & 1 \\ 4 & 4 & 1 \\ 4 & 5 & 0 \\ 4 & 5 & 1 \\ 5 & 6 & 0 \end{pmatrix} z = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 3 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

$K_4\backslash\{e\}$ with $\deg(q) = 3$ and $\sigma : (1,2),(2,3) \mapsto -$

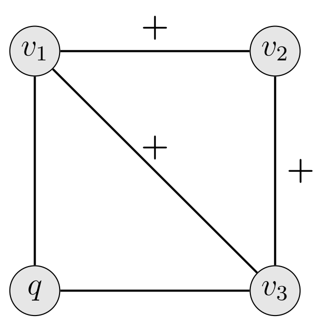
$$\mathrm{L} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \mathrm{LM}^{-1} = \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{3}{2} & 2 & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix}$$



$$Criticals = \begin{pmatrix} 2 & 3 & 2 \\ 2 & 4 & 2 \\ 3 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 6 & 3 \\ 4 & 7 & 4 \\ 4 & 8 & 5 \\ 5 & 8 & 4 \end{pmatrix} z = \begin{pmatrix} 2 & 3 & 2 \\ 2 & 4 & 2 \\ 0 & 0 & 0 \\ 3 & 6 & 4 \\ 4 & 6 & 3 \\ 1 & 2 & 1 \\ 3 & 4 & 2 \\ 2 & 4 & 3 \end{pmatrix}$$

$K_4\backslash\{e\}$ with $\deg(q) = 2$ and $\sigma : \emptyset \mapsto -$

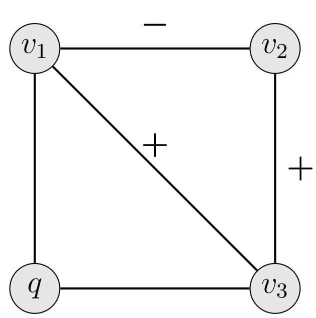
$$\mathrm{L} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \mathrm{LM}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$Criticals = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix} z = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$K_4\backslash\{e\}$ with $\deg(q) = 2$ and $\sigma : (1,2) \mapsto -$ (from Figure 5)

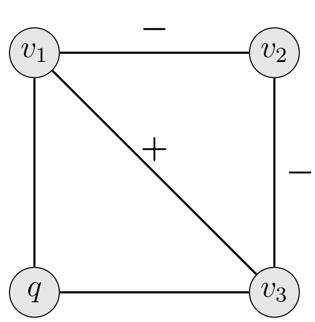
$$\mathrm{L} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \mathrm{LM}^{-1} = \begin{pmatrix} 2 & 2 & 1 \\ \frac{5}{4} & 2 & \frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}$$



$$Criticals = \begin{pmatrix} 6 & 4 & 1 \\ 6 & 5 & 2 \\ 7 & 5 & 1 \\ 7 & 5 & 2 \\ 7 & 6 & 1 \\ 7 & 6 & 2 \\ 8 & 6 & 0 \\ 8 & 6 & 1 \\ 8 & 6 & 2 \\ 9 & 7 & 0 \\ 9 & 7 & 1 \\ 9 & 7 & 2 \end{pmatrix} z = \begin{pmatrix} 3 & 2 & 0 \\ 6 & 4 & 0 \\ 4 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 7 & 5 & 0 \\ 4 & 3 & 2 \\ 5 & 4 & 0 \\ 2 & 2 & 0 \\ 5 & 4 & 2 \\ 6 & 5 & 0 \\ 3 & 3 & 0 \end{pmatrix}$$

$K_4\backslash\{e\}$ with $\deg(q) = 2$ and $\sigma : (1,2), (2,3) \mapsto -$

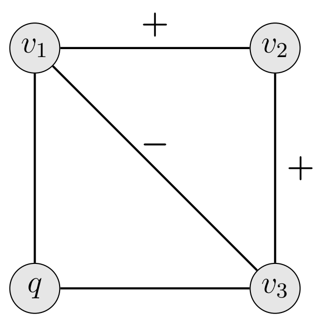
$$\mathrm{L} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \mathrm{LM}^{-1} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$



$$Criticals = \begin{pmatrix} 4 & 6 & 5 \\ 4 & 7 & 6 \\ 5 & 6 & 4 \\ 6 & 7 & 4 \\ 6 & 8 & 6 \\ 6 & 9 & 7 \\ 7 & 9 & 6 \\ 8 & 11 & 8 \end{pmatrix} z = \begin{pmatrix} 1 & 2 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \\ 2 & 4 & 4 \\ 0 & 0 & 0 \\ 3 & 5 & 4 \\ 4 & 5 & 3 \\ 2 & 3 & 2 \end{pmatrix}$$

$K_4\backslash\{e\}$ with $\deg(q) = 2$ and $\sigma : (1,3) \mapsto -$

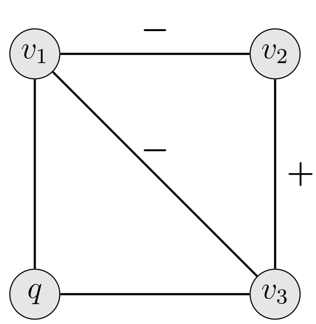
$$L = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 3 \end{pmatrix} M = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} LM^{-1} = \begin{pmatrix} \frac{7}{4} & 1 & \frac{5}{4} \\ 0 & 1 & 0 \\ \frac{5}{4} & 1 & \frac{7}{4} \end{pmatrix}$$



$$Criticals = \begin{pmatrix} 4 & 1 & 5 \\ 5 & 1 & 4 \\ 5 & 1 & 6 \\ 6 & 0 & 6 \\ 6 & 1 & 5 \\ 6 & 1 & 7 \\ 7 & 0 & 7 \\ 7 & 1 & 6 \\ 7 & 1 & 7 \\ 8 & 0 & 8 \\ 8 & 1 & 8 \\ 9 & 1 & 9 \end{pmatrix} z = \begin{pmatrix} 4 & 0 & 3 \\ 3 & 0 & 4 \\ 5 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 5 \\ 6 & 0 & 5 \\ 1 & 0 & 1 \\ 5 & 0 & 6 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{pmatrix}$$

$K_4\backslash\{e\}$ with $\deg(q) = 2$ and $\sigma : (1,2),(1,3) \mapsto -$

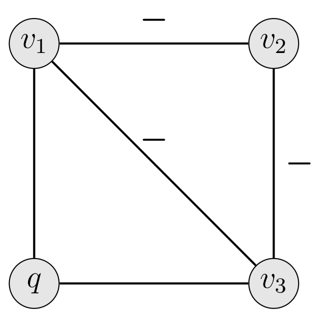
$$\mathrm{L} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 3 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \mathrm{LM}^{-1} = \begin{pmatrix} \frac{11}{4} & 3 & \frac{9}{4} \\ \frac{5}{4} & 2 & \frac{3}{4} \\ \frac{5}{4} & 1 & \frac{7}{4} \end{pmatrix}$$



$$Criticals = \begin{pmatrix} 10 & 4 & 6 \\ 12 & 5 & 7 \\ 13 & 6 & 7 \\ 14 & 6 & 8 \\ 15 & 7 & 8 \\ 16 & 7 & 9 \\ 17 & 8 & 9 \\ 19 & 9 & 10 \end{pmatrix} z = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \\ 4 & 2 & 2 \\ 5 & 3 & 2 \\ 6 & 3 & 3 \\ 7 & 4 & 3 \\ 9 & 5 & 4 \end{pmatrix}$$

$K_4\backslash\{e\}$ with $\deg(q) = 2$ and $\sigma : (1,2), (1,3), (2,3) \mapsto -$

$$\mathrm{L} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \mathrm{LM}^{-1} = \begin{pmatrix} \frac{11}{4} & 3 & \frac{9}{4} \\ 2 & 3 & 2 \\ \frac{9}{4} & 3 & \frac{11}{4} \end{pmatrix}$$

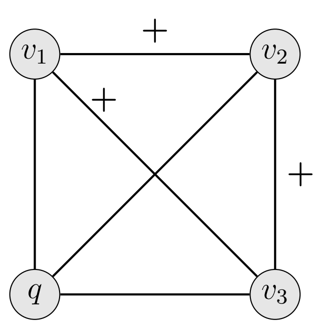


$$Criticals = \begin{pmatrix} 10 & 8 & 10 \\ 10 & 9 & 11 \\ 11 & 9 & 10 \\ 11 & 9 & 11 \\ 11 & 10 & 12 \\ 12 & 10 & 11 \\ 12 & 10 & 12 \\ 12 & 11 & 13 \\ 13 & 11 & 12 \\ 13 & 11 & 13 \\ 14 & 12 & 14 \\ 15 & 13 & 15 \end{pmatrix} z = \begin{pmatrix} 0 & 0 & 0 \\ 8 & 6 & 7 \\ 7 & 6 & 8 \\ 1 & 1 & 1 \\ 9 & 7 & 8 \\ 8 & 7 & 9 \\ 2 & 2 & 2 \\ 10 & 8 & 9 \\ 9 & 8 & 10 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \end{pmatrix}$$

K_4 with $\sigma : \emptyset \mapsto -$

$$\mathrm{L} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$\mathrm{LM}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

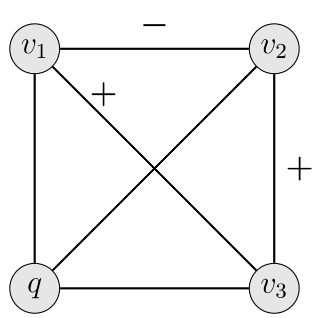


$$Criticals = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 0 \\ 2 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix} z = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

K_4 with $\sigma : (1,2) \mapsto -$

$$\mathrm{L} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$\mathrm{LM}^{-1} = \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

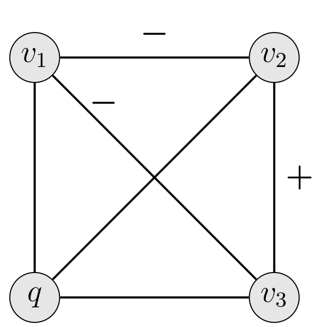


$$Criticals = \begin{pmatrix} 3 & 4 & 1 \\ 4 & 3 & 1 \\ 4 & 4 & 2 \\ 4 & 5 & 1 \\ 4 & 5 & 2 \\ 5 & 4 & 1 \\ 5 & 4 & 2 \\ 5 & 5 & 2 \\ 5 & 6 & 1 \\ 5 & 6 & 2 \\ 6 & 5 & 1 \\ 6 & 5 & 2 \\ 6 & 6 & 0 \\ 6 & 6 & 1 \\ 6 & 6 & 2 \\ 7 & 7 & 0 \\ 7 & 7 & 1 \\ 7 & 7 & 2 \\ 8 & 8 & 1 \\ 8 & 8 & 2 \end{pmatrix} z = \begin{pmatrix} 3 & 4 & 1 \\ 4 & 3 & 1 \\ 1 & 1 & 1 \\ 4 & 5 & 1 \\ 4 & 3 & 0 \\ 5 & 4 & 1 \\ 3 & 4 & 0 \\ 2 & 2 & 1 \\ 5 & 6 & 1 \\ 5 & 4 & 0 \\ 6 & 5 & 1 \\ 4 & 5 & 0 \\ 2 & 2 & 2 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \\ 3 & 3 & 2 \\ 4 & 4 & 0 \\ 1 & 1 & 0 \\ 5 & 5 & 0 \\ 2 & 2 & 0 \end{pmatrix}$$

K_4 with $\sigma : (12), (13) \mapsto -$

$$\mathrm{L} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$\mathrm{LM}^{-1} = \begin{pmatrix} 2 & \frac{3}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

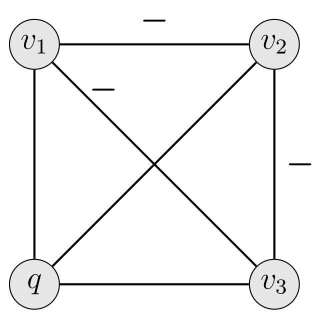


$$Criticals = \begin{pmatrix} 5 & 2 & 4 \\ 5 & 4 & 2 \\ 6 & 4 & 4 \\ 7 & 3 & 5 \\ 7 & 4 & 4 \\ 7 & 4 & 5 \\ 7 & 5 & 3 \\ 7 & 5 & 4 \\ 8 & 4 & 5 \\ 8 & 5 & 4 \\ 8 & 5 & 5 \\ 9 & 5 & 6 \\ 9 & 6 & 5 \\ 10 & 6 & 6 \\ 11 & 6 & 7 \\ 11 & 7 & 6 \end{pmatrix} z = \begin{pmatrix} 5 & 2 & 4 \\ 5 & 4 & 2 \\ 4 & 2 & 2 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \\ 7 & 4 & 5 \\ 3 & 1 & 3 \\ 7 & 5 & 4 \\ 8 & 4 & 5 \\ 8 & 5 & 4 \\ 3 & 2 & 2 \\ 4 & 2 & 3 \\ 4 & 3 & 2 \\ 0 & 0 & 0 \\ 6 & 3 & 4 \\ 6 & 4 & 3 \end{pmatrix}$$

K_4 with $\sigma : (12), (13), (23) \mapsto -$

$$\mathrm{L} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \mathrm{M} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$\mathrm{LM}^{-1} = \begin{pmatrix} 2 & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & 2 & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & 2 \end{pmatrix}$$



$$Criticals = \begin{pmatrix} 7 & 8 & 8 \\ 8 & 7 & 8 \\ 8 & 8 & 7 \\ 8 & 8 & 9 \\ 8 & 9 & 8 \\ 8 & 9 & 9 \\ 9 & 8 & 8 \\ 9 & 8 & 9 \\ 9 & 9 & 8 \\ 9 & 10 & 10 \\ 10 & 9 & 10 \\ 10 & 10 & 9 \\ 10 & 11 & 11 \\ 11 & 10 & 11 \\ 11 & 11 & 10 \\ 11 & 11 & 11 \\ 12 & 12 & 12 \\ 13 & 13 & 13 \\ 14 & 14 & 14 \\ 15 & 15 & 15 \end{pmatrix} z = \begin{pmatrix} 5 & 4 & 4 \\ 4 & 5 & 4 \\ 4 & 4 & 5 \\ 8 & 8 & 9 \\ 8 & 9 & 8 \\ 6 & 5 & 5 \\ 9 & 8 & 8 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \\ 7 & 6 & 6 \\ 6 & 7 & 6 \\ 6 & 6 & 7 \\ 8 & 7 & 7 \\ 7 & 8 & 7 \\ 7 & 7 & 8 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

5 Moving forward

Here we have laid out the Guzmán-Klivans theory of chip-firing on general invertible matrices as applied to signed graphs, described the basic behavior of the game, and developed tools to help in the exploration of these chip-firing systems. There is still a great deal left to explore in this new model. As we mentioned previously, much of what we know from the graphical chip-firing model is lost in generalization. With this application to signed graphs it may be possible to recover some of that theory.

We have already explored the possibility of recovering the duality between superstable and critical configurations and while there does seem to be some relation between the sets of superstable and critical configurations, it is apparent that there is no canonical configuration.

As we have seen, in the classical setting of chip firing on a simple graph, there is a nice duality between critical configurations and superstables. In particular there is a notion of a unique maximal stable configuration κ (sometimes called the 'canonical configuration') which is always critical. In the setting of signed graphs, we see from our examples that more than one critical configuration is maximal (in the sense of coordinate-wise partial order).

A related question is whether the superstable configurations are in fact 'closed under taking subsets'. In particular suppose $\boldsymbol{c}$ is a superstable configuration for a signed graph G and $\boldsymbol{d}$ is vector in the cone S^+ of valid configurations that is coordinate-wise less then $\boldsymbol{c}$. Is it true that $\boldsymbol{d}$ is also superstable?

Another very promising front is to form an analogue to the bijection between superstable configurations and spanning trees. In the context of chip-firing on regular graphs, there are many bijections from spanning trees to the set of superstable configurations [4], many that generalize the original burning algorithm of Dhar.

We know from Guzmán-Klivans and Zaslavsky that they are both counted by the determinant of the reduced graph Laplacian. In Theorem 1 we saw that for a signed

G the value of $\det \Delta_q(G)$ is given by a (weighted) count of combinatorial objects that generalize the set of spanning trees of G. We also know that the set of superstable configurations has cardinality given by this number. It would be interesting to find a map between these two sets.

We can also look at a number of special cases within signed graphs. From our examples and testing it seems that in the case of all negative edges there is some interesting behavior, especially regarding superstable configurations. An important area of interest within signed graphs are balanced graphs where we might find some interesting behavior. Sign-symmetric graphs may also have some relation to chip-firing on bipartite graphs.

The superstable configurations on signed graphs may also give us some idea of parking functions on signed graphs as they do for simple graphs.

Much of the questions and explorations we have would be aided by the development of more code to generate examples and help discover patterns. The code itself could be aided by the discovery of an upper-bound for the number of times a site may fire in a multiset-firing while stabilizing to a superstable configuration.

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