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NONEXISTENCE OF NONNEGATIVE SOLUTIONS FOR PARABOLIC INEQUALITIES IN THE HALF-SPACE

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Abstract. Based on the method of nonlinear capacity, we study the nonexistence of nonnegative monotonic solutions for the quasilinear parabolic inequality $u_t - \Delta_p u \geq u^q$. Also we study generalizations in the half-space in terms of parameters p and q.

1. Introduction

The question about nonexistence of nontrivial nonnegative global solutions to nonlinear equation $u_t - Au = g(x)u^q$ and the inequality $u_t - Au \geq g(x)u^q$, where A is an elliptic operator, in different domains is of substantial interest. Such inequalities can be understood as nonlinear heat equations with a supplementary external source term $f(x,t) = u_t - Au - g(x)u^q \geq 0$. The aim of the study is to find the range of values of q such that the equation or inequality in question has no-nontrivial nonnegative global solutions, i.e. the extra heat source leads to blow-up of a local solution.

The results in the whole space $\mathbb{R}^n$ go back to Fujita [11] who established that solutions to the equation $u_t - \Delta u = u^q$ do not exist for $1 < q < 1 + \frac{2}{n}$. Similar nonexistence ranges for much more general operators were obtained later in [16]. As for the half-space, up to our knowledge, so far only stationary solutions have been considered. The first results in this direction were obtained by Berestycki, Capuzzo Dolcetta and Nirenberg [2] who proved nonexistence of solutions to the inequality $-\Delta u \geq u^q$ for $1 < q < \frac{n+1}{n-1}$. The optimality of these results was shown by Birindelli and Mitidieri [3]. Inequalities of the form $Au \geq u^q$ with $A = -\Delta_p$, where $p > 1$ and Δ_p is the p-Laplace operator defined by $\Delta_p u := \operatorname{div}(|Du|^{p-2}Du)$, in the half-space with a punched point or a removed neighborhood of a point on the boundary were studied by Bidaut-Véron and Pohozaev [4], and later by Véron and A. Porretta [18]. They obtained results on nonexistence of solutions in the domains under study and consequently in the whole half-space for $p - 1 < q < q_{\rm cr}(p,n)$, where $q_{\rm cr}(p,n) = p - 1 + \frac{p}{\beta_{p,n}}$, and $\beta_{p,n}$ is the growth rate of singular solutions near zero, obtained explicitly only for $n = 2$ ($\beta_{p,2} = \frac{3-p+\sqrt{(p-1)^2+2-p}}{3(p-1)}$). One should also note the papers of Filippucci [10] on critical exponents for semilinear inequalities of

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the form $-\operatorname{div}(u^\alpha |x|^\beta Du) \geq |x|^\gamma u^q$ in the half-space, of Dancer, Du and Efendiev [5] and of Zou [20] on nonexistence of solutions to the Dirichlet problem

$$\begin{gathered} -\Delta_p u = u^q, \quad x \in \mathbb{R}^n_+, \\ u(x) = 0, \quad x \in \partial\mathbb{R}^n_+, \end{gathered} \tag{1.1}$$

for a nonlinear equation with a p-Laplace operator in a half-space, as well as those of Farina, Montoro and Sciunzi [6]–[9] on monotonicity of essentially bounded solutions of the same problem, which implies their nonexistence for a certain range of q. Elliptic problems with singular coefficients near unbounded sets were considered, in particular, in [12, 13].

In this article we consider the nonexistence of nonnegative solutions for the parabolic inequality $u_t - \Delta_p u \geq a x_n^\gamma u^q$ in the half-space. Based on the method of nonlinear capacity [16, 17], we obtain sufficient conditions for nonexistence of solutions. Similar results for elliptic inequalities and systems can be found in [14].

The rest of this article consists of three sections. §2 has our main results, §3 contains a proof in the semilinear case, and §4 the quasilinear case.

2. Formulation of main results

Denote $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$. Let $p > 1$, $q > p-1$, $a > 0$, $\gamma \in \mathbb{R}$, and let $u_0 \in C(\mathbb{R}^n_+)$ be a nonnegative function. Consider the problem

$$\begin{gathered} u_t - \Delta_p u \geq a x_n^\gamma u^q, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n_+, \\ u(x,t) \geq 0, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_+. \end{gathered} \tag{2.1}$$

We understand its weak solutions in the following sense.

Definition 2.1. A weak solution of problem (2.1) is a nonnegative function $u \in C^{2,1}(\mathbb{R}^n_+ \times \mathbb{R}_+)$, which satisfies the integral inequality

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} (|Du|^{p-2}(Du, D\varphi) - u\varphi_t)\,dx\,dt \geq \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} a x_n^\gamma u^q \varphi\,dx\,dt + \int_{\mathbb{R}^n_+} u_0 \varphi\,dx$$

for any nonnegative $\varphi \in C^\infty(\mathbb{R}^n_+ \times \mathbb{R}_+)$ such that $\varphi(x,t) \equiv 0$ for $(x,t) \in \partial\mathbb{R}^n_+ \times \mathbb{R}_+$ (that is, for $x_n = 0$).

Weak solutions of the problems considered below are defined in a similar way. In the case $p = 2$, we obtain the following result.

Theorem 2.2. *Let $a > 0$,$\gamma > -2$, and $1 < q \leq 1 + \frac{\gamma+2}{n+1}$. Then (2.1) with $p = 2$:*

$$\begin{gathered} u_t - \Delta u \geq a x_n^\gamma u^q, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n_+, \\ u(x,t) \geq 0, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \end{gathered} \tag{2.2}$$

has no nonnegative nontrivial weak solutions u.

For other values of $p \neq 2$, we obtain a nonexistence result in a smaller functional class of solutions (with an additional property of monotonicity).

Theorem 2.3. *Let $a > 0$, $\gamma > -p$, $q \geq \max(1, p-1)$, $\gamma(p-2) > p(1-q)$, and*

$$[(n+1)(q-1) - \gamma](q-p+1) - p(q-1) - \gamma(p-2) < 0.$$

Then (2.1) *has no nonnegative nontrivial weak solutions u such that $u(x', \cdot, t)$ is monotonic in x_n for each $x' \in \mathbb{R}^{n-1}$ and $t > 0$.*

Corollary 2.4. *Let $a > 0$ and $\max(1, p-1) \le q \le p - 1 + \frac{p}{n+1}$. Then the problem*

$$
\begin{gathered}
u_t - \Delta_p u \ge a u^q, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_+, \\
u(x,0) = u_0(x), \quad x \in \mathbb{R}^n_+, \\
u(x,t) \ge 0, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_+
\end{gathered}
\tag{2.3}
$$

(that is, (2.1) *with $\gamma = 0$) has no nonnegative nontrivial weak solutions u such that $u(x', \cdot, t)$ is monotonic in x_n for each $x' \in \mathbb{R}^{n-1}$ and $t > 0$.*

Evidently, the above corollary follows from Theorem 2.3 in the case $\gamma = 0$.

Remark 2.5. Nonexistence results can be obtained in the same class of monotonic solutions for the problem

$$
\begin{gathered}
u_t + \Delta_p u \ge a x_n^{\gamma} u^q, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_+, \\
u(x,0) = u_0(x), \quad x \in \mathbb{R}^n_+, \\
u(x,t) \ge 0, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_+,
\end{gathered}
\tag{2.4}
$$

where the operator Δ_p has the opposite sign (see [14]). Although the result in [14] is formulated for monotonically nondecreasing solutions, its proof is valid for non-increasing ones as well.

3. Proof of Theorem 2.2

We use the method of nonlinear capacity [16, 17]. We choose a family of nonnegative test functions $\xi^{\lambda}_{R,T} \in C^1_0(\mathbb{R}^n)$ such that $\lambda > 0$ (to be specified below), R and T are some positive parameters, and $\xi_{R,T}(x) = \prod_{k=1}^{N-1} \chi_R(x_k) \cdot \chi_R(x_n - 3R) \cdot \chi_T(t)$ with

$$
\chi_R(s) = \begin{cases} 1 & \text{if } s \le R, \\ 0 & \text{if } s \ge 2R, \end{cases}
\tag{3.1}
$$

where

$$
|D\chi_R(s)| \le c R^{-1}, \quad s \in \mathbb{R}_+.
\tag{3.2}
$$

Multiply both sides of (2.2) by $\xi^{\lambda}_{R,T} x_n$ and integrate by parts. After elementary transformations we obtain

$$
\begin{aligned}
&a \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^q \xi^{\lambda}_{R,T} x_n^{\gamma+1} \, dx \, dt \\
&\le \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u \cdot |\Delta(\xi^{\lambda}_{R,T} x_n)| \, dx \, dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u \cdot \Big| \frac{\partial \xi^{\lambda}_{R,T}}{\partial t} \Big| x_n \, dx \, dt.
\end{aligned}
\tag{3.3}
$$

Application of the parametric Young inequality to both integrals on the right-hand side of (3.3) yields

$$
\begin{aligned}
\frac{a}{2} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^q \xi^{\lambda}_{R,T} x_n^{\gamma+1} \, dx \, dt &\le c \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} |D\xi_{R,T}|^{\frac{2q}{q+1}} \xi_{R,T}^{\lambda - \frac{2q}{q+1}} x_n^{\frac{q-\gamma-1}{q-1}} \, dx \, dt \\
&\quad + c \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} |\chi'_T(t)|^{\frac{q}{q-1}} \chi_T^{\lambda - \frac{q}{q-1}} x_n^{-\frac{\gamma+1}{q-1}} \, dx \, dt \\
&:= I_1(R,T) + I_2(R,T).
\end{aligned}
\tag{3.4}
$$

For $\lambda > \frac{2q}{q-1}$, the integral $I_1(R,T)$ can be estimated as

$$I_1(R,T) \le R^{n-\frac{q+\gamma+1}{q-1}} T \tag{3.5}$$

and $I_2(R,T)$ as

$$I_2(R,T) \le R^{n-\frac{\gamma+1}{q-1}} T^{1-\frac{q}{q-1}}. \tag{3.6}$$

From (3.4)–(3.6) we obtain

$$\frac{a}{2}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^q \xi^\lambda_{R,T} x_n^{\gamma+1}\,dx\,dt \le c\big(R^{n-\frac{q+\gamma+1}{q-1}} T + R^{n-\frac{\gamma+1}{q-1}} T^{1-\frac{q}{q-1}}\big). \tag{3.7}$$

Choosing $T = R^\theta$ with $\theta > 0$ such that both terms are of the same order and taking $R \to \infty$, we obtain

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^q x_n^{\gamma+1}\,dx\,dt = 0,$$

which contradicts the assumption of non-triviality of the solution. This completes the proof of Theorem 2.2.

4. Proof of Theorem 2.3

Now, using the same family of test functions $\xi_{R,T}$ as in the previous proof, we multiply both parts of (2.3) by $u^\alpha \xi^\lambda_{R,T} x_n$, where $\alpha < 0$ will be specified below, and integrate by parts. After elementary transformations we obtain

$$\begin{aligned} &a\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{q+\alpha}\xi^\lambda_{R,T} x_n^{\gamma+1}\,dx\,dt + |\alpha|\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{\alpha-1}|Du|^p \xi^\lambda_{R,T} x_n\,dx\,dt \\ &\le \int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^\alpha |Du|^{p-1}|D\xi^\lambda_{R,T}| x_n\,dx\,dt + \int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u_t u^\alpha \xi^\lambda_{R,T} x_n\,dx\,dt \\ &\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^\alpha |Du|^{p-2}\frac{\partial u}{\partial x_n}\xi^\lambda_{R,T}\,dx\,dt. \end{aligned} \tag{4.1}$$

Application of the parametric Young inequality to the first integral on the right-hand side of (4.1) yields

$$\begin{aligned} &a\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{q+\alpha}\xi^\lambda_{R,T} x_n^{\gamma+1}\,dx\,dt + \frac{|\alpha|}{2}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{\alpha-1}|Du|^p \xi^\lambda_{R,T} x_n\,dx\,dt \\ &\le c\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{\alpha+p-1}|D\xi^\lambda_{R,T}|^p \xi^{\lambda(1-p)}_{R,T} x_n\,dx\,dt \\ &\quad + \frac{1}{\alpha+1}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{\alpha+1}(\xi^\lambda_{R,T})_t x_n\,dx\,dt \\ &\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^\alpha |Du|^{p-2}\frac{\partial u}{\partial x_n}\xi^\lambda_{R,T}\,dx\,dt. \end{aligned} \tag{4.2}$$

Applying the parametric Young inequality to the first two integrals on the right-hand side of (4.2) once more, we obtain

$$\begin{aligned}
&\frac{a}{2}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{q+\alpha}\xi^\lambda_{R,T}x_n^{\gamma+1}\,dx\,dt+\frac{|\alpha|}{2}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{\alpha-1}|Du|^p\xi^\lambda_{R,T}x_n\,dx\,dt\\
&\le c\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+}|D\xi_{R,T}|^{\frac{p(q+\alpha)}{q-p+1}}\xi_{R,T}^{\lambda-\frac{p(q+\alpha)}{q-p+1}}x_n^{\frac{q+\alpha-(\alpha+p-1)(\gamma+1)}{q-p+1}}\,dx\,dt\\
&\quad+c\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+}|\chi_T'(t)|^{\frac{q+\alpha}{q-1}}\chi_T^{\lambda-\frac{q+\alpha}{q-1}}x_n^{\frac{q+\alpha-(\alpha+1)(\gamma+1)}{q-1}}\,dx\,dt\\
&\quad+\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+}u^\alpha|Du|^{p-2}\frac{\partial u}{\partial x_n}\xi^\lambda_{R,T}\,dx\,dt\\
&:=I_1(R,T)+I_2(R,T)+I_3(R,T).
\end{aligned}\tag{4.3}$$

For $\lambda>\frac{pq}{q-p+1}$ and

$$\alpha>\frac{n(q-p+1)-(q+\gamma-1)(p-1)}{p+\gamma}\tag{4.4}$$

the integral $I_1(R,T)$ and $I_2(R,T)$ can be estimated as

$$I_1(R,T)\le R^{n-\frac{(p-1)(q+\alpha)+(\alpha+p-1)(\gamma+1)}{q-p+1}}T,\tag{4.5}$$

$$I_2(R,T)\le R^{n+\frac{q+\alpha-(\alpha+1)(\gamma+1)}{q-1}}T^{1-\frac{q+\alpha}{q-1}}.\tag{4.6}$$

If $\frac{\partial u}{\partial x_n}\ge 0$, then $I_3(R,T)<0$. Estimate the integral $I_3(R,T)$ in the case $\frac{\partial u}{\partial x_n}\le 0$. In case $p<2$, using the Hölder inequality and integrating by parts, we have

$$\begin{aligned}
I_3(R,T)&=-\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+}u^\alpha|Du|^{p-2}\frac{\partial u}{\partial x_n}\xi^\lambda_{R,T}\,dx\,dt\\
&\le\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+}u^\alpha\Big(-\frac{\partial u}{\partial x_n}\Big)^{p-1}\xi^\lambda_{R,T}\,dx\,dt\\
&\le c\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+}\Big(-\frac{\partial u^{1+\frac{\alpha}{p-1}}}{\partial x_n}\Big)^{p-1}\xi^\lambda_{R,T}\,dx\,dt\\
&\le c\Big(-\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+}\frac{\partial u^{1+\frac{\alpha}{p-1}}}{\partial x_n}\xi_R^{\frac{\lambda}{p-1}}\,dx\,dt\Big)^{p-1}R^{n(2-p)}\\
&=c\Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+}u^{1+\frac{\alpha}{p-1}}\frac{\partial\xi_R^{\frac{\lambda}{p-1}}}{\partial x_n}\,dx\,dt\Big)^{p-1}R^{n(2-p)}\\
&\le c\Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+}u^{1+\frac{\alpha}{p-1}}\Big|\frac{\partial\xi_R^{\frac{\lambda}{p-1}}}{\partial x_n}\Big|\,dx\,dt\Big)^{p-1}R^{n(2-p)}\\
&\le c\Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+}u^{q+\alpha}\xi^\lambda_{R,T}x_n^{\gamma+1}\,dx\,dt\Big)^{\frac{\alpha+p-1}{q+\alpha}}\cdot R^{n(2-p)}\\
&\quad\times\Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+}\Big|\frac{\partial\xi^\lambda_{R,T}}{\partial x_n}\Big|^{\frac{(q+\alpha)(p-1)}{(q+\alpha-1)(p-1)-\alpha}}\\
&\quad\times\big(\xi_{R,T}^{\lambda(q-p+1)-(q+\alpha)}x_n^{-(\gamma+1)(\alpha+p-1)}\big)^{\frac{1}{(q+\alpha-1)(p-1)-\alpha}}\,dx\,dt\Big)^{\frac{(q+\alpha-1)(p-1)-\alpha}{q+\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{a}{4}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{q+\alpha}\xi^\lambda_{R,T}x_n^{\gamma+1}\,dx\,dt \\
&\quad + cR^{\frac{n[(2-p)(q+\alpha)+(q+\alpha-1)(p-1)-\alpha]-(q+\alpha+\gamma+1)(p-1)-(\gamma+1)\alpha}{q-p+1}} \\
&= \frac{a}{4}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{q+\alpha}\xi^\lambda_{R,T}x_n^{\gamma+1}\,dx\,dt + cR^{n-\frac{(q+\alpha)(p-1)+(\gamma+1)(\alpha+p-1)}{q-p+1}},
\end{aligned}
$$

i.e.

$$
I_3(R,T) \le \frac{a}{4}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{q+\alpha}\xi^\lambda_{R,T}x_n^{\gamma+1}\,dx\,dt + cR^{n-\frac{(q+\alpha)(p-1)+(\gamma+1)(\alpha+p-1)}{q-p+1}}T\,. \tag{4.7}
$$

From (4.3)–(4.7) we obtain

$$
\begin{aligned}
&\frac{a}{2}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{q+\alpha}\xi^\lambda_{R,T}x_n^{\gamma+1}\,dx\,dt + \frac{|\alpha|}{2}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{\alpha-1}|Du|^p\xi^\lambda_{R,T}x_n\,dx\,dt \\
&\le c\big(R^{n-\frac{(q+\alpha)(p-1)+(\gamma+1)(\alpha+p-1)}{q-p+1}}T + R^{n-\frac{(\gamma+1)(\alpha+1)}{q-1}}T^{1-\frac{q+\alpha}{q-1}}\big).
\end{aligned} \tag{4.8}
$$

Choosing $T = R^\theta$ with $\theta > 0$ such that both terms are of the same order and taking $R \to \infty$, for α satisfying (4.4) we obtain

$$
\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{q+\alpha}x_n^{\gamma+1}\,dx\,dt = 0,
$$

which contradicts the assumption of non-triviality of the solution. This proves the theorem in the case $p < 2$.

In the case $p > 2$, estimates (4.3) and (4.5) are still valid, and for the integral $I_3(R,T)$ in the case $\frac{\partial u}{\partial x_n} \le 0$ we have

$$
\begin{aligned}
I_3(R,T) &= \int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^\alpha|Du|^{p-2}\frac{\partial u}{\partial x_n}\xi^\lambda_{R,T}\,dx\,dt \\
&= -\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^\alpha|Du|^{p-2}\Big(-\frac{\partial u}{\partial x_n}\Big)^{\frac{p-2}{p-1}}\Big(+\frac{\partial u}{\partial x_n}\Big)^{\frac{1}{p-1}}\xi^\lambda_{R,T}\,dx\,dt \\
&\le \int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^\alpha|Du|^{p-2+\frac{p-2}{p-1}}\Big(-\frac{\partial u}{\partial x_n}\Big)^{\frac{1}{p-1}}\xi^\lambda_{R,T}\,dx\,dt
\end{aligned}
$$

and by the Young inequality, similarly to the previous argument,

$$
\begin{aligned}
I_3(R,T) &\le \frac{|\alpha|}{4}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{\alpha-1}|Du|^p\xi^\lambda_{R,T}x_n\,dx\,dt \\
&\quad + c\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{\alpha+p-2}\frac{\partial u}{\partial x_n}x_n^{2-p}\xi^\lambda_{R,T}\,dx\,dt \\
&\le \frac{|\alpha|}{4}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{\alpha-1}|Du|^p\xi^\lambda_{R,T}x_n\,dx\,dt \\
&\quad + cR^{2-p}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{\alpha+p-1}\Big|\frac{\partial \xi_{R,T}}{\partial x_n}\Big|\xi_R^{\lambda-1}\,dx\,dt \\
&\le \frac{|\alpha|}{4}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{\alpha-1}|Du|^p\xi^\lambda_{R,T}x_n\,dx\,dt + \frac{a}{4}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n_+} u^{q+\alpha}\xi^\lambda_{R,T}x_n^{\gamma+1}\,dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+ cR^{\frac{(2-p)(q+\alpha)}{q-p+1}} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} x_n^{-\frac{(\gamma+1)(\alpha+p-1)}{q-p+1}} \Big| \frac{\partial \xi_{R,T}}{\partial x_n} \Big|^{\frac{q+\alpha}{q-p+1}} \xi_{R,T}^{\lambda-\frac{q+\alpha}{q-p+1}} \, dx\, dt \\
&\le \frac{a}{4} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^{q+\alpha} \xi_{R,T}^{\lambda} x_n^{\gamma+1} \, dx\, dt + cR^{n-\frac{(p-1)(q+\alpha)+(\alpha+p-1)(\gamma+1)}{q-p+1}} T,
\end{aligned}
$$

i. e.

$$
\begin{aligned}
I_3(R,T) \le \frac{|\alpha|}{4} &\int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^{\alpha-1} |Du|^p \xi_{R,T}^{\lambda} x_n \, dx\, dt \\
&+ \frac{a}{4} \int_{\mathbb{R}_+} \int_{\mathbb{R}^n_+} u^{q+\alpha} \xi_{R,T}^{\lambda} x_n^{\gamma+1} \, dx\, dt + cR^{n-\frac{(p-1)(q+\alpha)+(\alpha+p-1)(\gamma+1)}{q-p+1}} T,
\end{aligned} \tag{4.9}
$$

which together with (4.3) and (4.5) yields (4.8) again. The proof can be completed similarly to the previous case.

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