Electronic Journal of Differential Equations, Vol. 2005(2005), No. 130, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

METHOD OF STRAIGHT LINES FOR A BINGHAM PROBLEM AS A MODEL FOR THE FLOW OF WAXY CRUDE OILS

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ABSTRACT. In this work, we develop a method of straight lines for solving a Bingham problem that models the flow of waxy crude oils. The model describes the flow of mineral oils with a high content of paraffin at temperatures below the cloud point (i.e. the crystallization temperature of paraffin) and more specifically below the pour point at which the crystals aggregate themselves and the oil takes a jell-like structure. From the rheological point of view such a system can be modelled as a Bingham fluid whose parameters evolve according to the volume fractions of crystallized paraffin and the aggregation degree of crystals. We prove that the method is well defined for all times, a monotone property, qualitative behaviour of the solution, and a convergence theorem. The results are compared with numerical experiments at the end of this article.

1. INTRODUCTION

As a justification for using this model, we quote statements made by Farina and Fasano in [9]:

> Crude oils in many reservoirs throughout the world contain significant quantities of wax which can crystallize during production, transportation, and storage [24]. This can cause severe difficulties in pipelining and storage. At sufficiently high temperatures, the waxy crude oils (i.e., oils which contain a great deal of wax), although chemically very complex, are simple Newtonian fluids. As the temperature is reduced, the flow properties of these crudes can radically change from the simple Newtonian flow to a very complex behavior due to the crystallization of waxes [6]. The waxes basically consist of n-alkanes, usually ranging from $C_{18}H_{38}$ to $C_{40}H_{82}$, which crystallize (as soon as the equilibrium temperature and pressure is reached), forming an interlocking structure of plate, needle, or malformed crystals [11]. When the oil is cooled to a temperature lower than the crystallization point (generally called pour point), the crystals, growing and agglomerating, entrap the oil into a jell-like structure. Consequently, the flow properties of the oil become distinctly non-Newtonian. A yield-stress (the minimum

2000 *Mathematics Subject Classification.* 35A40, 35B40, 35R35, 65M20, 65N40.
Key words and phrases. Bingham fluid; straight lines; non-newtonian fluids.

Submitted April 15, 2005. Published November 24, 2005.

> stress required to start the flow) can be detected. Moreover, the flow properties are complicated by their critical dependence upon their mechanical and thermal 'history'. The viscosity of the waxy crudes can be greatly reduced by a continued shear. This fact seems to indicate a kind of thixotropy. The disintegration of large wax agglomerates appears to be the primary cause of the lower viscosity [26].

This paper deals with the model proposed by Farina and Fasano in [9]. In that work they study the low temperature behavior of a waxy crude oil in a laboratory experimental loop. In the second section we describe the physical model for this class of fluids, and the study of the related mathematical problem. In the third section we describe the mathematical problem, while in the fourth we remember some results present in [10]. In the fifth section a method of straight lines for a Bingham problem as a model for the flow of a waxy crude oils is developed. We prove that the method is well defined for all times, a monotone property, qualitative behaviour of the solution, and asymptotic convergence for large times. In the sixth section the results are compared with numerical experiments.

2. The physical problem

First, we state the physical assumptions.

(F1) Low, uniform, and constant temperature. A first simplifying assumption is that the temperature is uniform, constant, and below the so-called "pour point", so that the density of the crystallized wax is constant in space and time. Therefore, the non-Newtonian behaviour of the fluid has to be only atributed to the aglomeration of wax crystals. If the temperature field is denoted by $T(\vec{x}, t)$, in a domain V in $\mathbb{R}^2$:

$$\text{constant} = T(\vec{x}, t) \le T_{pp}, \quad \vec{x} \in V, \quad t \ge 0, \tag{2.1}$$

where T_{pp} is the "pour point".

(F2) Incompressible Fluid. A very reasonable assumption, consistent with the previous one, is to take,

$$\rho = \text{constant}, \tag{2.2}$$

where ρ is the oil density. If $\vec{v}(\vec{x}, t)$ is the velocity field of the fluid, using (2.2) and the continuity equation, we conclude that $\nabla \cdot \vec{v} = 0$ in V.

(F3) Laminar Flow. This assumption is justified by the fact that, for low temperatures, the Reynolds number (evaluated for typical pipelines values) is less than the threshold of turbulent flow.

Now let us pass to define a rheological model which takes into account the experimental data. Waxy crude oils show, at low temperature, the presence of a yield-stress [7]. According to this experimental evidence, we describe them as Bingham fluids. Roughly speaking, a Bingham fluid is a non-newtonian fluid which behaves like a rigid body when the shear stress τ is less than a threshold value τ_0, while it behaves like a viscous fluid when the stress exceeds τ_0, and for which the relationship between the stress τ and the shear strain γ is linear, that is

$$\tau = \tau_0 + \eta\gamma \tag{2.3}$$

where η is the viscosity.

To consider the evolution of the sheared system exhibiting a kind of "thixotropy" we introduce a time-dependent parameter α, defined as the ratio between the aggregated solid paraffin mass by volume unit, and the total mass of paraffin present in the fluid. Therefore, α is a quantity ranging in the interval $[0,1]$. We assume that the yield-stress τ is influenced only by the agglomeration factor, that is

$$\tau_0 = \tau_0(\alpha), \tag{2.4}$$

with

- (Y1) $\tau_0 : [0,1] \to [\tau_m, \tau_M]$, where $0 \le \tau_m < \tau_M < +\infty$.
- (Y2) $\tau_0 \in C^1([0,1])$.
- (Y3) τ_0 is a non-decreasing function.

A natural way of writing down an evolution equation for α accounting both for the spontaneous aggregation of paraffin crystals and agglomerates fragmentation (explaining "thixotropy") is the following

$$\begin{gathered} \alpha'(t) = K_1(1-\alpha(t)) - K_2\alpha(t)|\overline{W}(t)|, \\ \alpha(0) = \alpha_0. \end{gathered} \tag{2.5}$$

where

$$\overline{W}(t) = \frac{1}{V}\int_V W(\vec{x},t)d\vec{x}$$

is the power dissipated in the flow by the viscous force, and K_1, K_2 are constants (they can be obtained experimentally).

3. The mathematical problem

In this section we will consider the Bingham problem in plane geometry. We consider a fluid between two parallel plates. Using the Navier-Stokes equation for the viscous region and the Newton's law for the rigid zone, we model the behavior of the system. The boundary that separates the two regions is an unknown that evolves in time. It is one of the most important unknown quantities of the problem.

We assume that the fluid is incompressible, laminar, and with constant density ρ. Fixing the x coordinate along the direction of motion, y the perpendicular coordinate to the plates, and z the remaining coordinate, we make the following assumptions:

1. The pressure gradient, ∇p, is applied in only one direction, that is, $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$.
2. The fluid is laminar, that is, the velocities v and w satisfy $v = w = 0$.
3. The non-zero component of the velocity u depends only on time, t, and on the perpendicular position, y, that is, $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} = 0$.
4. There is no transport of fluid through the free boundary, $y = s(t)$. This is a condition of no deformation, that is, $u_y(s(t),t) = 0$, for all $t > 0$.
5. The velocity of the fluid u at the walls of the plates is zero. This is an adherence condition.

Using the above hypotheses, we obtain a system of partial differential equations, which we call problem (P),

$$\begin{gathered}
\rho u_t - \eta u_{yy} = f(t), \quad s(t) < y < L,\ t > 0, \\
u(L,t) = 0, \quad t > 0, \\
u(y,0) = u_0(y), \quad s(0) = s_0,\ 0 < s_0 < y < L, \\
u_y(s(t),t) = 0, \quad t > 0, \\
u_t(s(t),t) = \frac{1}{\rho}\big(f(t) - \frac{\tau_0}{s(t)}\big), \quad t > 0.
\end{gathered} \tag{3.1}$$

where $f(t)$ represents $-\partial p/\partial x$. We add to the problem above the equation (2.5) for the evolution of α. In this case the dissipated power is

$$W(t) = \frac{\eta}{L}\int_{s(t)}^{L} u_y^2(y,t)dy + \frac{\tau_0(\alpha(t))}{L} u(s(t),t). \tag{3.2}$$

Then the evolution equation for α is

$$\alpha'(t) = K_1(1-\alpha(t)) - \alpha(t)\frac{K_2}{L}\Big|\eta\int_{s(t)}^{L} u_y^2(y,t)dy + \tau_0(\alpha(t))u(s(t),t)\Big|, \tag{3.3}$$

with the initial condition

$$\alpha(0) = \alpha_0. \tag{3.4}$$

Now we have the following model for the problem (3.1), called (P^α),

$$\rho u_t - \eta u_{yy} = f(t), \quad s(t) < y < L, t > 0, \tag{3.5}$$
$$u(L,t) = 0, \quad t > 0, \tag{3.6}$$
$$u_y(s(t),t) = 0, \quad t > 0, \tag{3.7}$$
$$u_t(s(t),t) = \frac{1}{\rho}\big(f(t) - \frac{\tau_0(\alpha)}{s(t)}\big), \quad t > 0, \tag{3.8}$$
$$s(0) = s_0, \tag{3.9}$$
$$u(y,0) = u_0(y), \quad s_0 \le y \le L, \tag{3.10}$$
$$\alpha'(t) = K_1(1-\alpha(t)) - \frac{\alpha(t)K_2}{L}\big|\eta\int_{s(t)}^{L} u_y^2 dy - \tau_0(\alpha(t))\int_{s(t)}^{L} u_y dy\big|, \quad t > 0, \tag{3.11}$$
$$\alpha(0) = \alpha_0. \tag{3.12}$$

To get the equation (3.11) we use the equation (3.3) and the fact that $u(y,t) = -\int_y^L u_y(\xi,t)d\xi$.

The following problems will be useful for obtaining the discrete solution. We transform the problem (P^α) using the function $w = u_y$ to obtain a new problem, denoted by (P_y^α),

$$\rho w_t - \eta w_{yy} = 0, \quad s(t) < y < L, \quad t > 0, \tag{3.13}$$

$$w_y(L,t) = -f(t)/\eta, \quad t > 0, \tag{3.14}$$

$$w(s(t),t) = 0, \quad t > 0, \tag{3.15}$$

$$w_y(s(t),t) = -\tau_0(\alpha)/\eta s(t), \quad t > 0, \tag{3.16}$$

$$s(0) = s_0, \tag{3.17}$$

$$w(y,0) = u_0'(y), \quad 0 < s_0 \le y \le L, \tag{3.18}$$

$$\alpha'(t) = K_1(1-\alpha(t)) - \frac{\alpha(t)K_2}{L}\Big|\eta\int_{s(t)}^L w^2 dy - \tau_0(\alpha(t))\int_{s(t)}^L w dy\Big|, \quad t > 0, \tag{3.19}$$

$$\alpha(0) = \alpha_0. \tag{3.20}$$

If $z = u_t$ then the function z satisfies the problem, (P_t^α),

$$\rho z_t - \eta z_{yy} = f'(t), \quad s(t) < y < L, \quad t > 0, \tag{3.21}$$

$$z(L,t) = 0, \quad t > 0, \tag{3.22}$$

$$z(s(t),t) = \frac{1}{\rho}\big(f(t) - \frac{\tau_0(\alpha)}{s(t)}\big), \quad t > 0, \tag{3.23}$$

$$z_y(s(t),t) = \frac{\tau_0(\alpha)s'(t)}{\eta s(t)}, \quad t > 0, \tag{3.24}$$

$$s(0) = s_0, \tag{3.25}$$

$$z(y,0) = \frac{\eta}{\rho}u_0''(y) + \frac{f(0)}{\rho}, \quad 0 < s_0 \le y \le L, \tag{3.26}$$

$$\alpha'(t) = K_1(1-\alpha(t)) - \frac{\alpha(t)K_2}{L}\Big|\eta\int_{s(t)}^L w^2 dy - \tau_0(\alpha(t))\int_{s(t)}^L w dy\Big|, \quad t > 0, \tag{3.27}$$

$$\alpha(0) = \alpha_0. \tag{3.28}$$

In [10] a theorem of local existence and uniqueness of the problem (P_t^α) is proved.

4. Theoretical results

Now, we recall some hypotheses and we will add a few more.

Y-Hypotheses on τ_0:

- (Y1) $\tau_0 : [0,1] \to [\tau_m, \tau_M]$, with $0 < \tau_m < \tau_M < \infty$.
- (Y2) $\tau_0 \in C^1([0,1])$.
- (Y3) τ_0 is monotone non-decreasing.
- (Y4) $\tau_m \le \tau_0(\alpha_0) \le \tau_M$.
- (Y5) τ_0 is Lipschitz with constant N; that is,

$$|\tau_0(\alpha_1) - \tau_0(\alpha_2)| \le N|\alpha_1 - \alpha_2|. \tag{4.1}$$

P-Hypotheses on f:

- (P1) $0 < f_m < f(t) < f_M$ for all $t > 0$.
- (P2) Operability condition: $f_m > \tau_M/L$.

This implies that the pressure gradient is bounded, but is big enough to allow that the fluid can circulate inside the pipe.

A-Hypotheses on α_0 and s_0:

(A1) $s_m < s_0 < s_M$ with $s_m = \tau_m/f_M$ y $s_M = \tau_M/f_m$.
(A2) $0 \le \alpha_0 \le 1$.

The aggregation factor is in the interval $[0,1]$ by definition.
U-Hypotheses on $u_0(y)$:

(U1) $u_0(y) \in C^3([s_0, L])$.
(U2) $u_0(y) \ge 0$ for $s_0 \le y \le L$; $u_0(L) = 0$.
(U3) $u_0'(y) \le 0$ for $s_0 \le y \le L$; $u_0'(s_0) = 0$. $u_m'(y) \le u_0'(y) \le u_M'(y)$ in $s_0 \le y \le L$ where

$$u_m'(y) = \begin{cases} 0, & 0 \le y \le s_m, \\ -\frac{f_M}{\eta}(y - s_m), & s_m \le y \le L, \end{cases} \tag{4.2}$$

$$u_M'(y) = \begin{cases} 0, & 0 \le y \le s_M, \\ -\frac{f_m}{\eta}(y - s_M), & s_M \le y \le L. \end{cases} \tag{4.3}$$

(U4) $u_0''(y) < 0$ in $s_0 < y < L$; $u_0''(s_0) = -\tau_0(\alpha_0)/s_0$. These are conditions of smoothness on u_0.

As a consequence of the Maximum Principle, and the Hopf's Lemma, we have

$$u(y,t) \ge 0, \quad s(t) < y < L, \quad 0 < t < T_0, \tag{4.4}$$

$$u_y(y,t) \le 0, \quad s(t) < y < L, \quad 0 < t < T_0, \tag{4.5}$$

$$u_{yy}(y,t) \le 0, \quad s(t) < y < L, \quad 0 < t < T_0. \tag{4.6}$$

where T_0 is the maximum time of existence. Besides that, properties of the free boundary problem and the relationship of the velocity field with respect to the initial conditions can be proved.

$$s_m < s(t) < s_M < L, \tag{4.7}$$

$$u_m'(y) \le u_y(y,t) \le u_M'(y), \tag{4.8}$$

$$u_M(y) \le u(y,t) \le u_m(y), \tag{4.9}$$

where

$$u_M(y) = -\frac{f_m}{2\eta}(y^2 - L^2) + \frac{f_m s_M}{\eta}(y - L), \quad s_M < y < L, \tag{4.10}$$

$$u_m(y) = -\frac{f_M}{2\eta}(y^2 - L^2) + \frac{f_M s_m}{\eta}(y - L), \quad s_m < y < L. \tag{4.11}$$

Let's suppose now that the problem (P^α) has a solution for all time, and that for t big enough, the solution tends asimptotically to a stationary solution (time independent). Therefore, we get the following stationary problem (P_E^α),

$$u_\infty''(y) = -f_\infty/\eta, \quad y \in [s_\infty, L], \tag{4.12}$$

$$u_\infty(L) = 0, \tag{4.13}$$

$$u_\infty'(s_\infty) = 0, \tag{4.14}$$

$$f_\infty - \tau_0(\alpha_\infty)/s_\infty = 0, \tag{4.15}$$

$$K_1(1 - \alpha_\infty) - \frac{K_2}{L}\alpha_\infty \Big| \eta \int_{s_\infty}^L u_\infty'^2 dy + \tau_0(\alpha_\infty) u_\infty(s_\infty) \Big| = 0. \tag{4.16}$$

The unknowns are u_∞, s_∞ and α_∞. From the equations (4.12)-(4.14) we can deduce

$$u_\infty(y) = \frac{f_\infty}{2\eta}(L^2 - y^2) + \frac{f_\infty}{\eta} s_\infty (y - L), \quad s_\infty \le y \le L. \tag{4.17}$$

We use the other two equations to obtain the values of α_∞ and s_∞. Manipulating algebraically we have

$$1 - \alpha_\infty \Big[1 + \frac{K_2 f_\infty^2}{6K_1 \eta L}(L - s_\infty)^2 (2L + s_\infty)\Big] = 0. \tag{4.18}$$

From now on we need to know the function τ_0. As a first approximation, we can suppose a linear relation

$$\tau_0(\alpha) = \tau_m + \alpha(\tau_M - \tau_m). \tag{4.19}$$

Using this information, and combining the equations (4.15) and (4.18) we have the identity

$$(s_\infty f_\infty - \tau_m)\Big[1 + \frac{K_2 f_\infty^2}{6K_1 \eta L}(L - s_\infty)^2 (2L + s_\infty)\Big] = \tau_M - \tau_m. \tag{4.20}$$

Let's define the function

$$g(s) = (s f_\infty - \tau_m)\Big[1 + \frac{K_2 f_\infty^2}{6K_1 \eta L}(L - s)^2 (2L + s)\Big]. \tag{4.21}$$

If the following condition holds

$$\frac{\tau_M}{L} < f_\infty < \frac{\sqrt{6}}{L}\sqrt{\frac{\eta K_1}{K_2}}, \tag{4.22}$$

there exists a unique stationary solution s_∞ and α_∞, where s_∞ can be calculated as the only root of g in the interval $[0, L]$, and α_∞ is obtained computing

$$\alpha_\infty = \frac{s_\infty f_\infty - \tau_m}{\tau_M - \tau_m}. \tag{4.23}$$

This is possible because the function g satisfies $g(0) < 0$, $g(L) > 0$ and $g'(s) > 0$ for $s \in [0, L]$.

5. Method of the Straight Lines.

Some examples of this method can be found in [1]- [23]. In this case, the idea is to discretize the time, and get an ordinary differential equation system. The difficulty is that we do not know if the numerical solution exists, because the domain is also an unknown.

Choosing a fixed time step $\Delta t > 0$, we define:

$$\begin{gathered}
t_n = (n-1)\Delta t, \quad n \in \mathbb{N}, \\
s_n = s(t_n), \quad n \in \mathbb{N}, \\
f_n = f(t_n), \quad n \in \mathbb{N}, \\
w_n(r) = w(r, t_n), \quad n \in \mathbb{N}, \\
q = \sqrt{\frac{1}{\Delta t}}.
\end{gathered}$$

We approximate time derivatives with the incremental quotient, and the evolution equation for α is discretized with the Euler's method. In this way, the (P_y^α) system is transformed into the $(P_{y,d}^\alpha)$ system. For all n in $\mathbb{N}$ we have:

$$w''_{n+1} - \frac{\rho}{\eta}q^2 w_{n+1} = -\frac{\rho}{\eta}q^2 w_n, \quad s_{n+1} < y < L, \quad n \in \mathbb{N}, \tag{5.1}$$

$$w_{n+1}(s_{n+1}) = 0, \quad n \in \mathbb{N}, \tag{5.2}$$

$$w'_{n+1}(s_{n+1}) = -\tau_0(\alpha_{n+1})/\eta s_{n+1}, \quad n \in \mathbb{N}, \tag{5.3}$$

$$w'_{n+1}(L) = -f_{n+1}/\eta, \quad n \in \mathbb{N}, \tag{5.4}$$

$$s_1 = s_0, \tag{5.5}$$

$$w_1(y) = u'_0(y), \quad s_0 \le y \le L, \tag{5.6}$$

$$\alpha_{n+1} = \alpha_n + \frac{1}{q^2}\Big(K_1(1-\alpha_n) - \alpha_n\frac{K_2}{L}\Big|\eta\int_{s_n}^L w_n^2 dy - \tau_0(\alpha_n)\int_{s_n}^L w_n dy\Big|\Big), \quad n \in \mathbb{N}. \tag{5.7}$$

$$\alpha_1 = \alpha_0. \tag{5.8}$$

Now we will try to prove existence and uniqueness of the solution of the problem $(P_{y,d}^\alpha)$, but before we will prove a technical lemma.

Lemma 5.1. *Consider the system of equations*

$$\begin{gathered} w'' - \frac{\rho}{\eta}q^2 w = g, \quad s < y < L, \\ w(s) = 0, \\ w'(s) = -T/\eta s, \end{gathered} \tag{5.9}$$

where ρ, η, q, s, L y T are positive numbers and g is a continuous function. Then

$$w(y) = -\frac{T}{\sqrt{\rho\eta}sq}\sinh\big(\sqrt{\frac{\rho}{\eta}}q(y-s)\big) + \int_s^y \frac{g(\xi)}{\sqrt{\rho/\eta}q}\sinh\big(\sqrt{\frac{\rho}{\eta}}q(y-\xi)\big)d\xi, \tag{5.10}$$

$$w'(y) = -\frac{T}{\eta s}\cosh\big(\sqrt{\frac{\rho}{\eta}}q(y-s)\big) + \int_s^y g(\xi)\cosh\big(\sqrt{\frac{\rho}{\eta}}q(y-\xi)\big)d\xi. \tag{5.11}$$

Proof. The proof of this lemma can be found in [22]. The only thing to do is to convert the second order differential equation into a differential equation system of first order. The system can be uncoupled and the problem is reduced to the resolution of two first order ordinary differential equations. □

We have just deduced that the solution of the equations (5.1)-(5.3) is

$$\begin{aligned} w_{n+1}(y) = &-\frac{\tau_0(\alpha_{n+1})}{\sqrt{\rho\eta}s_{n+1}q}\sinh\big(\sqrt{\frac{\rho}{\eta}}q(y-s_{n+1})\big) \\ &- \int_{s_{n+1}}^y \sqrt{\frac{\rho}{\eta}}q w_n(\xi)\sinh\big(\sqrt{\frac{\rho}{\eta}}q(y-\xi)\big)d\xi, \end{aligned} \tag{5.12}$$

$$\begin{aligned} w'_{n+1}(y) = &-\frac{\tau_0(\alpha_{n+1})}{\eta s_{n+1}}\cosh\big(\sqrt{\frac{\rho}{\eta}}q(y-s_{n+1})\big) \\ &- \int_{s_{n+1}}^y \frac{\rho}{\eta}q^2 w_n(\xi)\cosh\big(\sqrt{\frac{\rho}{\eta}}q(y-\xi)\big)d\xi. \end{aligned} \tag{5.13}$$

We need to know the value of s_{n+1} (we do not know yet if it exists). Replacing our solution (5.13) into the equation (5.4), we deduce that

$$\begin{aligned} -\frac{f_{n+1}}{\eta} &= w'_{n+1}(L) \\ &= -\frac{\tau_0(\alpha_{n+1})}{\eta s_{n+1}} \cosh\big(\sqrt{\frac{\rho}{\eta}} q(L-s_{n+1})\big) \\ &\quad - \int_{s_{n+1}}^{L} \frac{\rho}{\eta} q^2 w_n(\xi) \cosh\big(\sqrt{\frac{\rho}{\eta}} q(L-\xi)\big) d\xi. \end{aligned}$$

We define

$$\begin{aligned} F_{n+1}(s) = \frac{f_{n+1}}{\eta} &- \frac{\tau_0(\alpha_{n+1})}{\eta s} \cosh\big(\sqrt{\frac{\rho}{\eta}} q(L-s)\big) \\ &- \int_{s}^{L} \frac{\rho}{\eta} q^2 w_n(\xi) \cosh\big(\sqrt{\frac{\rho}{\eta}} q(L-\xi)\big) d\xi. \end{aligned} \tag{5.14}$$

Then s_{n+1} must be a root of F_{n+1} in the interval $(0, L)$.

Theorem 5.2. *If the hypotheses* (Y1)-(Y5), (P1)-(P2), (A1)-(A2),(U1)-(U4)*, hold and we have the following condition, for the discretization parameter,*

$$\max\Big(K_1, \frac{K_2}{L}\eta \int_{s_m}^{L} u_m^{\prime 2} dy - \frac{K_2}{L}\tau_M \int_{s_m}^{L} u'_m dy\Big) \le q^2, \tag{5.15}$$

then there exists a unique solution of the problem $(P^{\alpha}_{y,d})$*.*

Proof. We will prove the theorem by induction. For $n = 1$, the quantities α_1, s_1 and w_1 are defined by the initial conditions. Moreover, it is true that

$$0 \le \alpha_1 \le 1, \tag{5.16}$$

$$w_1 \equiv 0, \quad \text{in } [0, s_1], \tag{5.17}$$

$$w_1 \le 0, \tag{5.18}$$

$$w'_1 \le 0, \tag{5.19}$$

$$s_1 \in (0, L). \tag{5.20}$$

$$u'_m \le w_1 \le u'_M \tag{5.21}$$

$$s_m \le s_1 \le s_M \tag{5.22}$$

We observe that (5.17) is satisfied since we can extend the function u'_0 continuously by zero in the interval $[0, s_0]$. Then, suppose that exists a unique solution until the level n with the following properties: $0 \le \alpha_n \le 1$, $w_n \equiv 0$ in $[0, s_n]$, $w_n \le 0$, $w'_n \le 0$ and $s_n \in (0, L)$, $u'_m \le w_n \le u'_M$ and $s_m \le s_n \le s_M$. Now we consider the level $n + 1$.

► $0 \le \alpha_{n+1} \le 1$. Using the inductive hypothesis we have

$$\alpha_{n+1} \ge \frac{\alpha_n}{q^2}\Big[q^2 - \frac{K_2}{L}\eta \int_{s_n}^{L} w_n^2 dy + \frac{K_2}{L}\tau_0(\alpha_n) \int_{s_n}^{L} w_n dy\Big]. \tag{5.23}$$

Using the bounds and sign of w_n and the bounds of s_n, we obtain

$$\alpha_{n+1} \ge \frac{\alpha_n}{q^2}\Big[q^2 - \frac{K_2}{L}\eta \int_{s_m}^{L} u_m^{\prime 2} dy + \frac{K_2}{L}\tau_0(\alpha_n) \int_{s_m}^{L} u'_m dy\Big]. \tag{5.24}$$

Using the bounds of τ_0 and the condition on q,

$$\alpha_{n+1} \geq \frac{\alpha_n}{q^2}\Big[q^2 - \frac{K_2}{L}\eta\int_{s_m}^{L} u_m'^2 dy + \frac{K_2}{L}\tau_M\int_{s_m}^{L} u_m' dy\Big] \geq 0. \tag{5.25}$$

Moreover using the signs of w_n,

$$\alpha_{n+1} \leq \frac{1}{q^2}\left(\alpha_n q^2 + K_1(1-\alpha_n)\right). \tag{5.26}$$

Due to the condition on q, we obtain

$$\alpha_{n+1} \leq \frac{1}{q^2}\left(\alpha_n q^2 + q^2(1-\alpha_n)\right) = \alpha_n \leq 1. \tag{5.27}$$

Then $\tau_0(\alpha_{n+1})$ is a well-defined number.

▶ There exists at least a root of F_{n+1} in $(0, L)$. The function F_{n+1} is continuous in the interval $(0, L]$. Moreover:

$$F_{n+1}(L) = \frac{1}{\eta}\big(f_{n+1} - \frac{\tau_M}{L}\big) > 0,$$

due to the operability condition;

$$\lim_{s\to 0^+} F_{n+1}(s) = -\infty,$$

because the integral is bounded. Then, by continuity, it exists a root of F_{n+1} in the interval $(0, L)$.

▶ F_{n+1} may have more than one root in $(0, L)$. From (5.14) we obtain

$$\begin{aligned} F'_{n+1}(s) =& \frac{\tau_0(\alpha_{n+1})}{\eta s^2}\cosh\Big(\sqrt{\frac{\rho}{\eta}}q(L-s)\Big) + \frac{\tau_0(\alpha_{n+1})}{\eta s}\sqrt{\frac{\rho}{\eta}}q\sinh\Big(\sqrt{\frac{\rho}{\eta}}q(L-s)\Big) \\ &+ \frac{\rho}{\eta}q^2 w_n(s)\cosh\Big(\sqrt{\frac{\rho}{\eta}}q(L-s)\Big). \end{aligned} \tag{5.28}$$

If F_{n+1} has no critical points the proof is concluded. Otherwise, suppose that there exists at least a s_* such that $F'_{n+1}(s_*) = 0$; that is,

$$\begin{aligned} 0 =& \frac{\tau_0(\alpha_{n+1})}{\eta s_*^2}\cosh\Big(\sqrt{\frac{\rho}{\eta}}q(L-s_*)\Big) + \frac{\tau_0(\alpha_{n+1})}{\eta s_*}\sqrt{\frac{\rho}{\eta}}q\sinh\Big(\sqrt{\frac{\rho}{\eta}}q(L-s_*)\Big) \\ &+ \frac{\rho}{\eta}q^2 w_n(s_*)\cosh\Big(\sqrt{\frac{\rho}{\eta}}q(L-s_*)\Big). \end{aligned} \tag{5.29}$$

Clearly $s_* \neq 0$ since $w_n \equiv 0$ in $[0, s_n]$. Multiplying (5.29) by the inverse of the first member of the sum, we have that s_* is a zero of the function

$$B(s) = h(s) + \frac{\rho q^2 s^2 w_n(s)}{\tau_0(\alpha_{n+1})}, \tag{5.30}$$

where

$$h(s) = 1 + s\sqrt{\frac{\rho}{\eta}}q\tanh\Big(\sqrt{\frac{\rho}{\eta}}q(L-s)\Big). \tag{5.31}$$

It can be shown that the function h is concave and positive in the interval $(0, L)$. See [22]. The function B is a sum of a positive concave function h (where $h(0) = h(L) = 1$), and a negative decreasing function. Now, the second term of the right hand side of the equation (5.30) could eventually equal the function h in several points in the interval where h is increasing. Therefore the statement is concluded.

► s_{n+1} can be chosen as the minimum root of F_{n+1}. We already know that there exists at least one root of F_{n+1} in the interval $(0, L)$, and that $\lim_{s\to 0^+} F_{n+1}(s) = -\infty$ (that is, $s = 0$ can not be a root of F_{n+1}). Now, let's suppose that we have a set $R = \{r_i\}$ (finite or infinite) of roots of F_{n+1}. If R is finite, we define s_{n+1} as the minimum of the roots of F_{n+1}. If R is infinite, then we take s_{n+1} as the infimum of R. Then there exists a subsequence of roots that converges to s_{n+1}. As F_{n+1} is a continuous function, s_{n+1} is also a root and therefore the minimum of R.

Now we can solve w_{n+1} using (5.12) and (5.13). Until now we have solved the level $n + 1$, but we need the properties of w_{n+1} and s_{n+1} in order to continue the inductive step.

► $w_{n+1} \equiv 0$ in $[0, s_{n+1}]$. We extend w_{n+1} by zero in $[0, s_{n+1}]$ and due to (5.2) this extension is continuous, but the derivative does not exits at s_{n+1} because of (5.3).

► $w_{n+1} \le 0$ in $[0, L]$. We know that $w_{n+1} = 0$ in $[0, s_{n+1}]$. If we denote w_{n+1} by A, s_{n+1} by s, and $\rho q^2/\eta$ by γ^2, it is clear that w_{n+1} satisfies the system

$$\begin{gathered} A'' - \gamma^2 A \ge 0, \quad s < y < L, \\ A(s) \le 0, \\ A'(L) \le 0. \end{gathered} \tag{5.32}$$

Here we have used (5.1), (5.2) and (5.4), and the inductive hypothesis. In [22] it is proved that $A \le 0$ in the interval $[0, L]$, that is equivalent to $w_{n+1} \le 0$ in $[s_{n+1}, L]$. That means the statement is finished.

► $w'_{n+1} \le 0$ in $[0, L]$. In $[0, s_{n+1})$ is clear that $w'_{n+1} = 0$. In $[s_{n+1}, L]$, it holds

$$\begin{gathered} w'''_{n+1} - \frac{\rho}{\eta} q^2 w'_{n+1} = -\frac{\rho}{\eta} q^2 w'_n, \quad s_{n+1} < y < L, \\ w'_{n+1}(s_{n+1}) = -\tau_0(\alpha_{n+1})/\eta s_{n+1}, \\ w'_{n+1}(L) = -f_{n+1}/\eta. \end{gathered} \tag{5.33}$$

We denote w'_{n+1} by A, s_{n+1} by s and $\rho q^2/\eta$ by γ^2, then w'_{n+1} satisfies the system

$$\begin{gathered} A'' - \gamma^2 A \ge 0, \quad s < y < L, \\ A(s) < 0, \\ A(L) < 0. \end{gathered} \tag{5.34}$$

In [22] it is proved that $A \le 0$ in the interval $[s, L]$, that means that $w'_{n+1} \le 0$ in $[s_{n+1}, L]$. The statement is proved.

► $s_m \le s_{n+1} \le s_M$. Let s be such that $0 < s < s_m$. Note that the function

$$\frac{1}{s} \cosh\Big(\sqrt{\frac{\rho}{\eta}} q(L - s)\Big) \tag{5.35}$$

is decreasing in the interval $(0, L]$. Taking into account that $w_n = 0$ in $[s, s_m]$ (because $s_m \le s_n$ using inductive hypothesis), we can deduce that

$$F_{n+1}(s) \le F_{n+1}(s_m) \tag{5.36}$$

Now, using that $\tau_0 \ge \tau_m$ and $w_n \ge u'_m$ we can obtain

$$\begin{aligned} F_{n+1}(s_m) \le & \frac{f_{n+1}}{\eta} - \frac{\tau_m}{\eta s_m} \cosh\Big(\sqrt{\frac{\rho}{\eta}} q(L - s_m)\Big) \\ & + \int_{s_m}^{L} \frac{\rho}{\eta} q^2 \frac{f_M}{\eta} (\xi - s_m) \cosh\Big(\sqrt{\frac{\rho}{\eta}} q(L - \xi)\Big) d\xi. \end{aligned} \tag{5.37}$$

Using integration by parts in the second member of the inequality in the integral (5.37) we obtain

$$F_{n+1}(s_m) \le \frac{1}{\eta}\,(f_{n+1} - f_M) < 0. \tag{5.38}$$

Then, $F_{n+1}(s) \le F_{n+1}(s_m) < 0$ for all s such that $0 < s \le s_m$. Then $s_m \le s_{n+1}$.

In a similar way, we take now $s \in [s_M, L]$. Using that $w_n \le u'_M$ in $[s_M, L]$ we get

$$\begin{aligned} F_{n+1}(s) \ge& \frac{f_{n+1}}{\eta} - \frac{\tau_0(\alpha_{n+1})}{\eta s}\cosh\Big(\sqrt{\frac{\rho}{\eta}}q(L-s)\Big) \\ &+ \int_s^L \frac{\rho}{\eta}q^2\frac{f_m}{\eta}(\xi - s_M)\cosh\Big(\sqrt{\frac{\rho}{\eta}}q(L-\xi)\Big)d\xi. \end{aligned} \tag{5.39}$$

Integrating by parts in the second member of the inequality in the integral (5.39), and using that $\tau_0 \le \tau_M$ we obtain

$$F_{n+1}(s) \ge \frac{1}{\eta}(f_{n+1} - f_m) > 0. \tag{5.40}$$

Then $F_{n+1}(s) > 0$ for all s in $[s_M, L]$; therefore, $s_{n+1} \le s_M$, so we conclude the statement.

► $w_{n+1} \le u'_M$. Let $B_{n+1} = w_{n+1} - u'_M$ be. If $y \in [0, s_{n+1}]$ then $B_{n+1}(y) = 0$. If $y \in [s_{n+1}, s_M]$ then $B_{n+1}(y) = w_{n+1}(y) \le 0$. If $y \in [s_M, L]$ it satisfies

$$\begin{gathered} B''_{n+1} - \frac{\rho}{\eta}q^2 B_{n+1} \ge 0, \quad s_M < y < L, \\ B_{n+1}(s_M) \le 0, \\ B'_{n+1}(L) \le 0. \end{gathered} \tag{5.41}$$

As before we can prove that $B_{n+1} \le 0$, so the statement is proved.

► $u'_m \le w_{n+1}$. It follows in the same way.

So the proof is complete. □

Corollary 5.3. *Assume the conditions of theorem 5.2 are satisfied. Then:*

- $w_n \le 0$, *in* $[0, L]$, *for all* $n \in \mathbb{N}$.
- $w'_n \le 0$, *in* $[0, L]$, *for all* $n \in \mathbb{N}$.
- $w_n < 0$, *in* $(s_n, L]$ *for all* $n \in \mathbb{N}$.
- $s_m < s_n < s_M$ *for all* $n \in \mathbb{N}$.
- $u'_m \le w_n \le u'_M$ *for all* $n \in \mathbb{N}$.

We observe that the properties of the discrete solutions agree with the classical solution.

Proposition 5.4. *The stationary solution of problem* (P^α_y) *with the equations* (3.13)-(3.20) *is*

$$s_\infty = \frac{\tau_0(\alpha_\infty)}{f_\infty}, \quad w_\infty(y) = \begin{cases} 0, & \text{if } y \in [0, s_\infty), \\ \frac{-f_\infty}{\eta}(y - s_\infty), & \text{if } y \in [s_\infty, L], \end{cases} \tag{5.42}$$

where α_∞ *and* s_∞ *satisfy*

$$K_1(1-\alpha_\infty) - \frac{K_2}{L}\alpha_\infty\Big|\eta\int_{s_\infty}^L w_\infty^2 dy - \tau_0(\alpha_\infty)\int_{s_\infty}^L w_\infty dy\Big| = 0. \tag{5.43}$$

Proof. We observe that the system (3.13)-(3.20) has a stationary solution that satisfies

$$
\begin{gathered}
w''_\infty = 0, \quad s_\infty < y < L, \\
w'_\infty(L) = -f_\infty/\eta, \\
w_\infty(s_\infty) = 0, \\
w'_\infty(s_\infty) = -\tau_0(\alpha_\infty)/\eta s_\infty,
\end{gathered}
\tag{5.44}
$$

added to the equality (5.43). Clearly, the solution of the system (5.44) is the stationary solution. □

Theorem 5.5. *Assume the hypotheses of Theorem 5.2 are satisfied. If there exists a unique stationary solution, and if*

$$
\lim_{n\to\infty} \alpha_n = \alpha_*, \quad \lim_{n\to\infty} s_n = s_*, \quad \lim_{n\to\infty} w_n = w_*, \tag{5.45}
$$

then $\alpha_* = \alpha_\infty$, $s_* = s_\infty$ and $w_* = w_\infty$.

Proof. We take $\lim_{n\to\infty}$ in (5.12) to obtain

$$
w_*(y) = -\frac{\tau_0(\alpha_*)}{\sqrt{\rho\eta} s_* q} \sinh\Big(\sqrt{\frac{\rho}{\eta}} q(y - s_*)\Big) - \int_{s_*}^{y} \sqrt{\frac{\rho}{\eta}} q w_*(\xi) \sinh\Big(\sqrt{\frac{\rho}{\eta}} q(y-\xi)\Big) d\xi. \tag{5.46}
$$

When we compute the derivatives of w_*, we obtain

$$
w''_* = 0, \quad y \in [s_*, L], \quad w_*(s_*) = 0, \quad w'_*(s_*) = -\frac{\tau_0(\alpha_*)}{\eta s_*}. \tag{5.47}
$$

Moreover if we take $\lim_{n\to\infty}$ in (5.13), we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} w'_{n+1}(y) \\
&= -\frac{\tau_0(\alpha_*)}{\eta s_*} \cosh\Big(\sqrt{\frac{\rho}{\eta}} q(y - s_*)\Big) - \int_{s_*}^{y} \frac{\rho}{\eta} q^2 w_*(\xi) \cosh\Big(\sqrt{\frac{\rho}{\eta}} q(y-\xi)\Big) d\xi.
\end{aligned}
\tag{5.48}
$$

From (5.46) and (5.48) we deduce that $\lim_{n\to\infty} w'_n(y) = w'_*(y)$. Then

$$
w'_*(L) = \lim_{n\to\infty} w'_{n+1}(L) = -f_{n+1}/\eta = -f_\infty/\eta.
$$

We take limit as $n \to \infty$ in (5.7) and we obtain (5.43). We have proved that w_*, s_* and α_* satisfy the system (5.44) with (5.43). That means that $w_* = w_\infty$, $s_* = s_\infty$ and $\alpha_* = \alpha_\infty$, because there exists only one stationary solution. □

6. Numerical results

The numerical experiments shown below prove that the algorithm reproduces the physical behaviour of the solution. In the following figures there are examples of several cases.

In Figure 1, from left to right and from top to bottom we have the following data

- Example 1: $s_0 = 0.7$, $\alpha_0 = 0.25$, $f(t) = 4$, $u'_0(y) = -4(y - 0.7), \tau_0(\alpha) = 1 + \alpha$.
- Example 2: $s_0 = 0.1$, $\alpha_0 = 0.25$, $f(t) = 4$, $u'_0(y) = -4(y - 0.1), \tau_0(\alpha) = 1 + \alpha$.
- Example 3: $s_0 = 0.7$, $\alpha_0 = 0.25$, $f(t) = 4 + 0.25t$, $u'_0(y) = -4(y - 0.7), \tau_0(\alpha) = 1 + \alpha$.

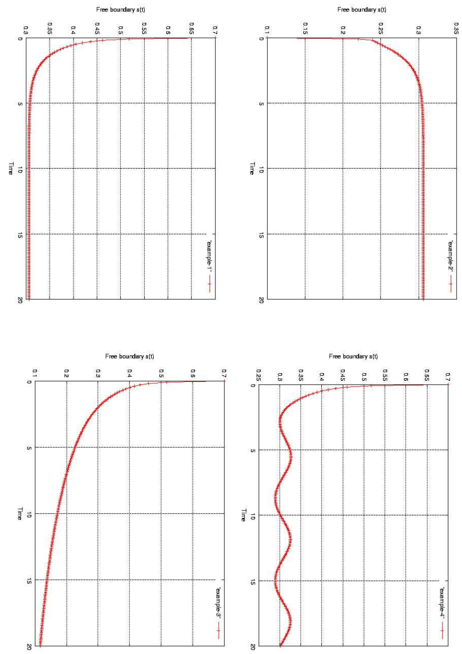

FIGURE 1. Method of straight lines - Plane geometry - τ_0 variable.

- Example 4: $s_0 = 0.7$, $\alpha_0 = 0.25$, $f(t) = 4 + \sin(t)/4$, $u_0'(y) = -4(y - 0.7)$, $\tau_0(\alpha) = 1 + \alpha$.

Concluding Remarks. Waxy crude oils are highly non-Newtonian fluids known to cause pipeling difficulties because their rheological properties are strongly affected by paraffin crystallization. On the basis of experimental data, a physical model has been used to describe the behaviour of these crudes. The mathematical problem has been studied in planar geometry, and a method of lines with the time as a discrete variable has been developed. We prove that the method is well defined for all times, a monotone property, qualitative behaviours of the solution, and asymptotic convergence for large times. In the experiments we tested four examples with different initial conditions for the free boundary, and different pressure gradients. In the first and second examples we can see that, no matter the initial position of the free boundary is, the pressure gradient pushes the solution towards the asymptotic solution. In the third example we observe that the free boundary tends to zero as the pressure gradient tends to infinity, and in the last example the pressure gradient and the free boundary behave periodically. Clearly, these examples show that, asymptotically, the free boundary $s(t)$ behaves like $\tau_0(\alpha(t))/f(t)$, as predicted theoretically.

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