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A SPATIALLY PERIODIC KURAMOTO-SIVASHINSKY EQUATION AS A MODEL PROBLEM FOR INCLINED FILM FLOW OVER WAVY BOTTOM

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Abstract. The spatially periodic Kuramoto-Sivashinsky equation (pKS)

$$\partial_t u = -\partial_x^4 u - c_3\partial_x^3 u - c_2\partial_x^2 u + 2\delta\partial_x(\cos(x)u) - \partial_x(u^2),$$

with $u(t,x) \in \mathbb{R}$, $t \geq 0$, $x \in \mathbb{R}$, is a model problem for inclined film flow over wavy bottoms and other spatially periodic systems with a long wave instability. For given $c_2, c_3 \in \mathbb{R}$ and small $\delta \geq 0$ it has a one dimensional family of spatially periodic stationary solutions $u_s(\cdot; c_2, c_3, \delta, u_m)$, parameterized by the mass $u_m = \frac{1}{2\pi}\int_0^{2\pi} u_s(x)\,\mathrm{d}x$. Depending on the parameters these stationary solutions can be linearly stable or unstable. We show that in the stable case localized perturbations decay with a polynomial rate and in a universal nonlinear self-similar way: the limiting profile is determined by a Burgers equation in Bloch wave space. We also discuss linearly unstable u_s, in which case we approximate the pKS by a constant coefficient KS-equation. The analysis is based on Bloch wave transform and renormalization group methods.

1. Introduction

The inclined film problem concerns the flow of a viscous liquid film down an inclined plane, driven by gravity. This has various engineering applications, where often the bottom plate is not flat but has a wavy profile, for instance $y = \delta\cos(x)$. Over an infinitely long flat ($\delta = 0$) bottom the problem can be reduced in a hierarchy of (formal) reductions to a variety of simpler equations, such as Boundary Layer equations, Integral Boundary Layer equations (IBL), also called Shkadov models, and KdV and Kuramoto–Sivashinsky (KS) type of equations, see [3] and the references therein. Moreover, there exist approximation results [11] concerning the validity of some of these simplified equations, and results on special nontrivial solutions such as pulse trains and their stability; see [3], and [6]. Finally, for the problem over a flat incline it is shown in [12] that in the linearly stable case localized perturbation of the trivial (Nusselt) solution decay in a universal way to zero, with limiting profile determined by the Burgers equation.

Over wavy bottoms the problem becomes much more complicated. For experimental results we refer to [15, 1]. Analytically, first of all, the basic spatially

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periodic stationary solution U_s (Nusselt solution) is not known in closed form. In [15] an expansion of U_s in the film thickness is given, and the associated critical Reynolds number R_c is calculated, such that U_s is linearly stable for $\mathrm{R}\leq\mathrm{R}_c$ and unstable for $\mathrm{R}>\mathrm{R}_c$. However, for thicker films over wavy bottoms no analytical results are known. Therefore, simplifications of the full Navier–Stokes problem are much desired. In [13] the problem is reduced to a two dimensional quasilinear parabolic system with spatially periodic coefficients, the periodic Integral Boundary Layer equation (pIBL), and stationary solutions U_S of the pIBL are calculated which show good agreement with experiments. Moreover, preliminary numerical simulations of the dynamic pIBL yield a variety of interesting regimes, two of which are:

(i) self–similar decay of localized perturbations of U_s to zero in the case of linearly stable U_s;
(ii) modulated pulses in the linearly unstable case.

Here we prove, for a model problem, a rigorous nonlinear stability result which explains the behaviour in (i). Moreover, we remark on (formal) explanations for (ii). Our model problem is an extension of the KS equation as a model problem for inclined film flow over flat bottom. In particular, it has dynamics similar to (i), (ii) above, see Fig. 1. In detail, our model problem is

$$\partial_t u = -\partial_x^4 u - c_3\partial_x^3 u - c_2\partial_x^2 u + 2\delta\partial_x(\cos(x)u) - \partial_x(u^2), \tag{1.1}$$

with $t \geq 0$, $x \in \mathbb{R}$, $u(t,x) \in \mathbb{R}$; i.e., a spatially periodic Kuramoto–Sivashinsky equation (pKS) over the infinite line. In context with the inclined film problem, u should be interpretated as the film height, while the parameters c_2, c_3, δ, have the following meaning: $c_2 \in \mathbb{R}$ corresponds to $\mathrm{R}-\mathrm{R}_c$; i.e., to the distance from criticality, and δ models the amplitude of the bottom, thus, w.l.o.g $\delta \geq 0$. The linear terms $-\partial_x^4 u - c_2\partial_x^2 u$ model a long wave instability (for $c_2 > 0$) with short wave saturation, while $-c_3\partial_x^3$ models 3rd order dispersion. The nonlinearity $-\partial_x(u^2)$ is the standard convective one. An important feature of the inclined film problem is the conservation of mass; i.e., $\partial_t \int_{\mathbb{R}} h(t,x) - h_0 \,\mathrm{d}x = 0$, where h is the film height and h_0 the reference film height. This also holds for (1.1): the right hand side is a total derivative. Consequently the mass (or average film height)

$$u_m := \lim_{M\to\infty} \frac{1}{2M}\int_{-M}^{M} u(t,x)\,\mathrm{d}x$$

can be seen as a 4^{th} parameter. Given $c_2, c_3, u_m \in \mathbb{R}$ and small $\delta > 0$, (1.1) has a unique spatially 2π periodic stationary solution

$$u_s(x) = u_s(x; c_2, \delta, u_m) = u_m + \delta u_1(x) + \mathcal{O}(\delta^2)$$

which can be calculated by expansion in δ (see sec.2).

Remark 1.1. From the modelling point of view it appears reasonable to also include a parameter γ for the bottom wave number; i.e., to consider

$$\partial_t u = -\partial_x^4 u - c_3\partial_x^3 u - c_2\partial_x^2 u + 2\delta\partial_x(\cos(\gamma x)u) - \partial_x(u^2). \tag{1.2}$$

However, rescaling $v(\tau,y) = \alpha u(\beta\tau,\gamma y)$ with $\beta = \gamma^4$ and $\alpha = \gamma^3$ yields that v fulfills (1.1) with c_3, c_2, δ replaced by $\tilde{c}_3 = \gamma^2 c_3$, $\tilde{c}_2 = \gamma^2 c_2$ and $\tilde{\delta} = \gamma^3\delta$. Hence γ is not independent.

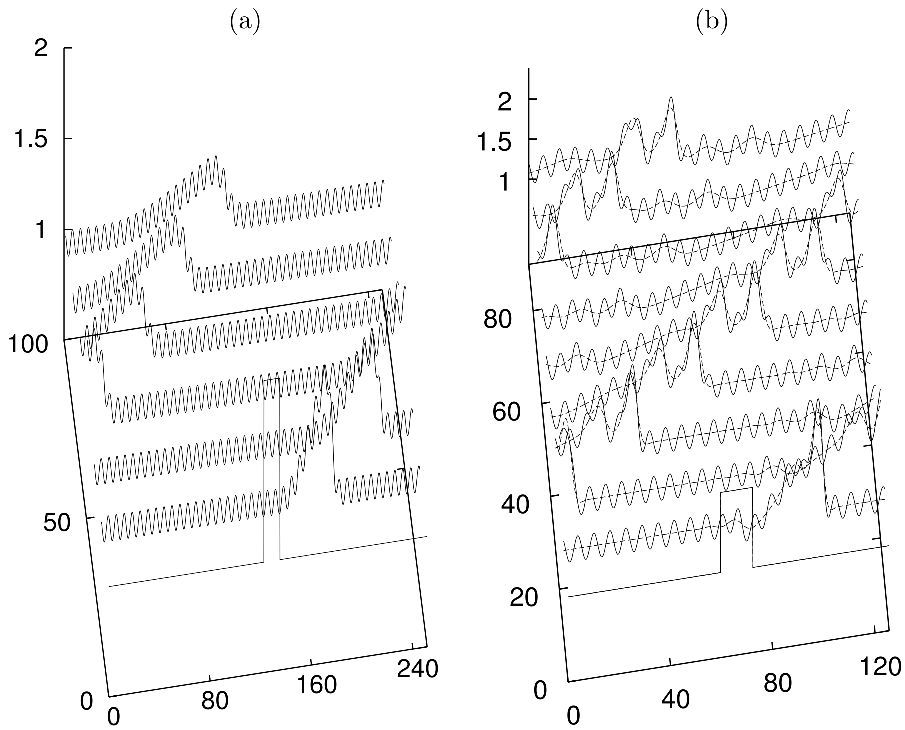

FIGURE 1. Numerical simulations of (1.1) with periodic boundary conditions on large domains. (a) $(\delta, c_2, c_3) = (0.1, -1, 0)$ (stable case), $x \in [0, 80\pi]$, initial data $u_0(x) = 2$ for $x \in [39\pi, 41\pi]$, $u_0(x) = 1$ else. The solution decays to u_s in a self–similar way determined by the Burgers equation (1.8) below. (b) $(\delta, c_2, c_3) = (0.1, 0.2, -1)$ (unstable case), $x \in [0, 40\pi]$, initial data $u(0,x)=2$ for $19\pi \le x \le 21\pi$ and $u(0,x)=1$ else (full line). Modulated pulses emerge and travel forever. The dotted line shows the solution of the associated amplitude equation (5.3), see Appendix 5.

Moreover, the KdV term $-c_3\partial_x^3$ plays no essential role in case (i) and hence for the main results of our paper; however, in the unstable case (ii), the KdV term becomes important, in particular in the limit of large c_3, see [6] for the constant coefficient case. Therefore we also include it in (1.1).

1.1. **Linear and Nonlinear diffusive stability.** Setting $u(t,x) = u_s(x) + v(t,x)$ we obtain

$$\partial_t v = \mathcal{L}(x)v - \partial_x(v^2) \tag{1.3}$$

with 2π periodic linear operator

$$\mathcal{L}(x)v = -\partial_x^4 v - c_3\partial_x^3 v - c_2\partial_x^2 v + 2\delta\partial_x(\cos(x)v) - 2(u_s'(x)v + u_s(x)\partial_x v), \tag{1.4}$$

where $u_s' = \partial_x u_s$. To calculate the eigenfunctions of the linearization $\partial_t v = \mathcal{L}(x)v$ we make a Bloch wave ansatz [7]

$$v(t,x) = \mathrm{e}^{\lambda(\ell)t + \mathrm{i}\ell x}\tilde{v}(\ell, x).$$

Here $\ell \in [-1/2, 1/2)$, which is called the first Brillouin zone, and $\tilde{v}(\ell, x+2\pi) = \tilde{v}(\ell, x)$ and $\tilde{v}(\ell+1, x) = e^{ix}\tilde{v}(\ell, x)$. Then $\lambda(\ell)$ and $\tilde{v}(\ell, \cdot)$ are determined from the linear eigenvalue problem

$$\begin{aligned}\lambda(\ell)\tilde{v}(\ell,x) &= \tilde{\mathcal{L}}(\ell,x)\tilde{v}(\ell,x)\\ &:= \big[-(\partial_x+i\ell)^4 - c_3(\partial_x+i\ell)^3 - c_2(\partial_x+i\ell)^2 + 2\delta(\cos(x)(\partial_x+i\ell) - \sin(x))\big]\tilde{v}\\ &\quad -2\big[u_s'(x)+u_s(x)(\partial_x+i\ell)\big]\tilde{v}\end{aligned}$$

over the bounded domain $x \in (0, 2\pi)$. Thus we obtain curves of eigenvalues $\ell \mapsto \lambda_n(\ell)$, $n \in \mathbb{N}$, which we sort by $\operatorname{Re}\lambda_n(\ell) \geq \operatorname{Re}\lambda_{n+1}(\ell)$, with associated eigenfunctions $\tilde{v}_n(\ell, x)$. The $\lambda_n(\ell)$ can again be calculated by perturbation analysis in δ, again see sec.2 for details. Clearly, $\operatorname{Re}\lambda_n(\ell) \to -\infty$ as $n \to \infty$, and u_s is linearly stable if $\operatorname{Re}\lambda_1(\ell) \leq 0$ for all $\ell \in [-1/2, 1/2)$. Since, for given c_2, c_3, δ, we have a 1–parameter family $u_s(x; c_2, c_3, \delta, u_m)$ of stationary solutions, parametrized by u_m, we always have $\lambda_1(0) = 0$ with

$$\tilde{v}_1(0, x) = \partial_{u_m} u_s(x; c_2, c_3, \delta, u_m).$$

This corresponds to conservation of mass in the inclined film problem. Next writing

$$\lambda_1(\ell) = -id_1\ell - d_2\ell^2 + \mathcal{O}(\ell^3) \tag{1.5}$$

and assuming that $\operatorname{Re}\lambda_1(\ell) < 0$ outside some neighborhood of $\ell = 0$ we find that u_s is linearly stable for small δ if $d_2 > 0$. In this case, the continuous spectrum up to the imaginary axis yields diffusive decay of localized perturbations to zero; i.e., for solutions v of $\partial_t v(t, x) = \mathcal{L}(x)v(t, x)$ with $v_0 \in L^1(\mathbb{R})$ we have

$$v(t,x) = \frac{z}{\sqrt{4\pi d_2 t}}\exp(-(x-d_1t)^2/4d_2t)\tilde{v}_1(0,x) + \mathcal{O}(t^{-1}), \tag{1.6}$$

where $z = \int_{\mathbb{R}} v_0(x)\,dx$ is the mass of the perturbation and where $d_1 = 2u_m + \mathcal{O}(\delta)$ is the speed of the comoving frame.

In contrast to exponential decay rates, the algebraic decay (1.6) is too weak to control arbitrary nonlinear terms. For instance, solutions to $\partial_t v = \partial_x^2 v + v^2$ on the real line may blow up in finite time [14], even for arbitrary small initial data. On the other hand, for $\partial_t v = \partial_x^2 v + v^{p_1}(\partial_x v)^{p_2}$ with $p_1 + 2p_2 > 3$ it is well known that solution to small localized initial data decay asymptotically as for the linear problem $\partial_t v = \partial_x^2 v$ (cf. (1.6) with $d_1 = 0, d_2 = 1$). This is called nonlinear diffusive stability, and the nonlinearity is called asymptotically irrelevant. A very robust method to prove such results is the renormalization group [2], which uses an iterative rescaling argument and has been applied to a variety of diffusive stability problems [8, 9, 12].

The case $p_1 + 2p_2 = 3$ is called marginal. In fact, we show that the asymptotics of solutions of (1.3) to small localized initial conditions are not given by Gaussian decay as in (1.6) but are determined by a non Gaussian profile related to the Burgers equation

$$\partial_t v = d_2\partial_x^2 v + b\partial_x(v^2) \quad \text{with} \quad b = -1 + \mathcal{O}(\delta^2). \tag{1.7}$$

This profile is obtained by Cole Hopf transformation. Setting

$$\psi(t,x) = \exp\Big(\frac{b}{d_2}\int_{-\infty}^{\sqrt{d_2}x} v(t,\xi)\,d\xi\Big), \quad v(t,x) = \frac{\sqrt{d_2}}{b}\frac{\psi_y(t,y)}{\psi(t,y)}, \quad y = x/\sqrt{d_2}$$

the Burgers equation is transformed into the linear diffusion equation $\partial_t\psi = \partial_x^2\psi$, $\psi|_{t=0} = \psi_0$. For $\lim_{x\to-\infty}\psi_0(x) = 1$ and setting $\lim_{x\to\infty}\psi_0(x)=z+1$; i.e.,

$$\ln(z+1)=\frac{b}{d_2}\int_{\mathbb{R}} v_0(t,\xi)\,\mathrm{d}\xi,$$

it is well known that $1 + z\,\mathrm{erf}(x/\sqrt{t})$ with $\mathrm{erf}(x) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\xi^2/4}\,\mathrm{d}\xi$ is an exact solution of $\partial_t\psi = \partial_x^2\psi$. It follows that

$$v^{(z)}(t,x) = t^{-1/2} f_z(x/\sqrt{t}) \quad \text{with} \quad f_z(y) = \frac{\sqrt{d_2}}{b}\frac{z\,\mathrm{erf}'(y)}{1 + z\,\mathrm{erf}(y)} \tag{1.8}$$

is an exact solution of the Burgers equation. Moreover,

$$\psi(t,x) = \frac{1}{\sqrt{4\pi t}}\int \mathrm{e}^{-(x-y)^2/(4t)}\psi_0(y)\,\mathrm{d}y = 1 + z\,\mathrm{erf}(x/\sqrt{t}) + \mathcal{O}(t^{-1/2})$$

as $t \to \infty$, for initial conditions $\psi_0 \in L^\infty(\mathbb{R})$ with $\lim_{\xi\to-\infty}\psi(\xi) = 1$ and with $\lim_{\xi\to\infty}\psi(\xi) = 1+z$. Therefore the so called renormalized solution of (1.7) satisfies

$$t^{1/2}v(t, t^{1/2}x) = f_z(x) + \mathcal{O}(t^{-1/2}); \tag{1.9}$$

i.e., it converges towards a non-Gaussian limit. It has been shown in [2] that the dynamics (1.9) in the Burgers equation is stable under addition of higher order terms. Similarly, our basic idea is that after a suitable transform (see (3.6) below), (1.3) in the linearly stable case ($d_2 > 0$ in (1.5)) can be interpretated as a higher order perturbation of the Burgers equation (1.7).

1.2. **The nonlinear stability result.** Throughout this paper we denote many different constants that are independent of δ and the rescaling parameter $L{>}0$ (see below) by the same symbol C. For $m, n \in \mathbb{N}$ we define the weighted spaces $H^m(n)=\{u \in L^2(\mathbb{R}) : \|u\|_{H^m(n)}{<}\infty\}$ with $\|u\|_{H^m(n)} = \|u\rho^n\|_{H^m(\mathbb{R})}$, where $\rho(x) = (1+|x|^2)^{1/2}$ and $H^m(\mathbb{R})$ is the Sobolev space of functions with derivatives up to order m in $L^2(\mathbb{R})$. With an abuse of notation we sometimes write, e.g., $\|u(t,x)\|_{H^m(n)}$ for the $H^m(n)$ norm of the function $x \mapsto u(t,x)$. For the bounded domain $(0, 2\pi)$ with periodic boundary conditions we also write $\mathcal{T}_{2\pi}$; i.e., $\int_{\mathcal{T}_{2\pi}} u(x)\,\mathrm{d}x := \int_0^{2\pi} u(x)\,\mathrm{d}x$. Fourier transform is denoted by $\mathcal{F}$, e.g., if $u \in L^2(\mathbb{R})$, then $\hat{u}(k) := \mathcal{F}(u)(k) = \frac{1}{2\pi}\int \mathrm{e}^{-\mathrm{i}kx}u(x)\,\mathrm{d}x$. From $\mathcal{F}(\partial_x u)(k) = \mathrm{i}k\hat{u}(k)$ and Parseval's identity we have that $\mathcal{F}$ is an isomorphism between $H^m(n)$ and $H^n(m)$; i.e., the weight in x–space yields smoothness in Fourier space and vice versa. This smoothness in k is essential for the proof of the following theorem, where for convenience we take initial conditions at $t = 1$.

Theorem 1.2. *Assume that the parameters $c_2, c_3, u_m \in \mathbb{R}$ and $\delta > 0$ small are chosen in such a way that $d_2 > 0$ in the expansion (1.5), and $\mathrm{Re}\,\lambda_n(\ell) < 0$ for all $n \in \mathbb{N}$ and all $\ell \in [-1/2, 1/2)$, except for $\lambda_1(0) = 0$. Let $p \in (0, 1/2)$. There exist $C_1, C_2 > 0$ such that the following holds. If $\|v_0\|_{H^2(2)} \le C_1$, then there exists a unique global solution v of (1.3) with $v|_{t=1} = v_0$, and*

$$\sup_{x\in\mathbb{R}}\Big|v(t,x) - t^{-1/2}f_z(t^{-1/2}(x - d_1 t))\tilde{v}_1(0,x)\Big| \le C_2 t^{-1+p}, \quad t \in [1,\infty), \tag{1.10}$$

with $d_1 = 2u_m + \mathcal{O}(\delta)$ from (1.5), $f_z(y) = \frac{\sqrt{d_2}}{b}\frac{z\,\mathrm{erf}'(y)}{1+z\,\mathrm{erf}(y)}$ from (1.8), and $\ln(1+z) = \frac{b}{d_2}\int_{\mathbb{R}} v_0(x)\,\mathrm{d}x$.

Thus we have the asymptotic $(H^2(2), L^\infty)$–stability of $v = 0$; i.e., for all $\varepsilon > 0$ there exists a $\nu>0$ such that $\|v_0\|_{H^r(2)} \le \nu$ implies $\|v(t)\|_{L^\infty} \le \varepsilon$, for all $t\ge 1$, and $\|v(t)\|_{L^\infty} \to 0$ with rate $t^{-1/2}$. The perturbations decay in an universal manner determined by the decay of localized initial data in the Burgers equation. Theorem 1.2 is in fact a corollary to the more detailed Theorem 3.1 stated in Bloch space in §3.3. In §2 we give examples such that the assumption $d_2 > 0$ holds. In particular, we shall see that $d_2 > 0$ may hold for $c_2 > 0$; i.e., the critical "Reynolds number" may be larger than 0 which is the critical Reynolds number in the spatially homogeneous case. A similar effect is also known in the full inclined film problem [15].

The remainder of this paper is organized as follows. In §2 we briefly review the properties of stationary solutions to (1.1), explain the set–up of Bloch waves, and give examples for $\lambda_1(\ell)$ from (1.5) for some chosen parameter values. In §3 we review the concept of irrelevant nonlinearities and the idea of renormalization, give a formal derivation of the Burgers equation as the amplitude equation for the critical mode $\tilde{v}_1(0, x)$ for (1.3) in the linearly stable case, and introduce Bloch spaces with weights to formulate our precise result Theorem 3.1. In §4 we set up a renormalization process to prove Theorem 3.1. In Appendix 5 we give some remarks on the unstable case (ii).

2. Spectral analysis

2.1. **Expansion of the stationary solutions.** To calculate u_s we expand in δ. We set

$$u_s(x) = u_m + \delta u_1(x) + \delta^2 u_2(x) + \mathcal{O}(\delta^3),$$

where u_j for $j \ge 1$ is 2π–periodic and has zero mean; i.e., u_m is considered as an additional parameter. Thus we write $u_s(x) = u_s(x; c_2, c_3, \delta, u_m)$. We obtain a hierarchy of linear inhomogeneous equations of the form

$$\mathcal{L}_0 u_j(x) = g(x), \quad \mathcal{L}_0 u = -\partial_x^4 u - c_3\partial_x^3 u - c_2\partial_x^2 u - 2u_m\partial_x u,$$

where $g(x)$ comes from the previous step. At $\mathcal{O}(\delta)$ we have $g(x) = 2u_m \sin(x)$, hence we use the ansatz $u_1(x) = \alpha_1 \cos(x) + \beta_1 \sin(x)$ to obtain the linear system

$$\begin{pmatrix} \mu_1 & \nu_1 \\ -\nu_1 & \mu_1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2u_m \end{pmatrix}, \quad \mu_j = -j^4 + c_2 j^2, \quad \nu_j = c_3 j^3 - 2j u_m, \tag{2.1}$$

with solution

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \frac{1}{d_1}\begin{pmatrix} -(c_3 - 2u_m)2u_m \\ (-1 + c_2)2u_m \end{pmatrix}, \quad d_j = (-j^4 + c_2 j^2)^2 + (c_3 j^3 - 2u_m j)^2.$$

At $\mathcal{O}(\delta^2)$ we have $g(x) = 2u_1' u_1 - 2\partial_x(\cos(x)u_1)$, hence $u_1 = \alpha_2 \cos(2x) + \beta_2 \sin(2x)$, which again yields a 2×2 linear system for (α_2, β_2), while at $\mathcal{O}(\delta^3)$, the right hand side contains harmonics $\mathrm{e}^{\mathrm{i}jx}$ with $j = 1, 2, 3$. Thus we need the ansatz

$$u_3(x) = \alpha_{31} \cos(x) + \alpha_{32} \cos(2x) + \alpha_{33} \cos(3x) + \beta_{31} \sin(x) + \beta_{32} \sin(2x) + \beta_{33} \sin(3x),$$

and have to solve a 6×6 linear system. This can be continued to any order in δ and the resulting systems can conveniently be solved using some symbolic algebra package. Moreover, from the diagonals of the linear systems we obtain the convergence of the Fourier series for u_s.

The maximum amplitude of u_s and the phase–shift with the "bottom profile" $\cos(x)$ depend on the parameters c_2, c_3, δ, u_m in a rather complicated way. Here, instead of giving explicit formulas we plot some solutions u_s in fig.2 on page 8, together with eigenvalue curves for the associated linearizations.

2.2. **Bloch wave analysis.** To calculate the spectrum of the linearization $\mathcal{L}$ of (1.1) around u_s we use the Bloch wave transform. The basic idea is to write

$$\begin{aligned} v(x) &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx}\hat{v}(k)\,\mathrm{d}k \\ &= \sum_{j\in\mathbb{Z}} \int_{-1/2+j}^{1/2+j} \mathrm{e}^{\mathrm{i}kx}\hat{v}(k)\,\mathrm{d}k \\ &= \int_{-1/2}^{1/2} \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}(\ell+j)x}\hat{v}(\ell+j)\,\mathrm{d}\ell \\ &= \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}\ell x}\tilde{v}(\ell,x)\,\mathrm{d}\ell =: (\mathcal{J}^{-1}\tilde{v})(x) \end{aligned} \tag{2.2}$$

where $\tilde{v}(\ell,x) = (\mathcal{J}v)(\ell,x) = \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}jx}\hat{v}(\ell+j)$. By construction we have

$$\tilde{v}(\ell,x) = \tilde{v}(\ell,x+2\pi) \quad \text{and} \quad \tilde{v}(\ell,x) = \tilde{v}(\ell+1,x)\mathrm{e}^{\mathrm{i}x}. \tag{2.3}$$

Bloch transform is an isomorphism between $H^s(\mathbb{R},\mathbb{C})$ and $L^2((-1/2,1/2],H^s(\mathcal{T}_{2\pi}))$ [7], where

$$\|\tilde{v}\|_{L^2((-1/2,1/2],H^s(\mathcal{T}_{2\pi}))} = \Big(\int_{-1/2}^{1/2} \|\tilde{v}(\ell,\cdot)\|^2_{H^s(\mathcal{T}_{2\pi})} d\ell\Big)^{1/2}.$$

Multiplication $u(x)v(x)$ in x-space corresponds in Bloch space to the "convolution"

$$(\tilde{u}*\tilde{v})(\ell,x) = \int_{-1/2}^{1/2} \tilde{u}(\ell-m,x)\tilde{v}(m,x)\,\mathrm{d}m, \tag{2.4}$$

where (2.3) has to be used for $|\ell-m|>1/2$. However, if $\chi:\mathbb{R}\to\mathbb{R}$ is 2π periodic, then $\mathcal{J}(\chi u)(\ell,x) = \chi(x)(\mathcal{J}u)(\ell,x)$.

In Bloch space the linear eigenvalue problem for $\mathcal{L}(x)$ thus becomes

$$\begin{aligned} \lambda(\ell)\tilde{v}(\ell,x) &\overset{!}{=} \tilde{\mathcal{L}}(\ell,x)\tilde{v}(\ell,x) := \mathrm{e}^{-\mathrm{i}\ell x}\big[\mathcal{L}(x)\mathrm{e}^{\mathrm{i}\ell x}\tilde{v}(\ell,x)\big] \\ &= \big[-(\partial_x+\mathrm{i}\ell)^4 - c_3(\partial_x+\mathrm{i}\ell)^3 - c_2(\partial_x+\mathrm{i}\ell)^2 \\ &\quad + 2\delta(\cos(x)(\partial_x+\mathrm{i}\ell) - \sin(x))\big]\tilde{v} - 2\big[u_s'(x) + u_s(x)(\partial_x+\mathrm{i}\ell)\big]\tilde{v} \end{aligned} \tag{2.5}$$

over the bounded domain $\mathcal{T}_{2\pi}$. This yields curves of eigenvalues $\lambda_n(\ell)$, with ℓ in $(-1/2,1/2)$ and $n\in\mathbb{N}$. To calculate $\lambda_n(\ell)$ we let

$$\tilde{v}(\ell,x) = \sum_{j\in\mathbb{Z}} b_j(\ell)\mathrm{e}^{\mathrm{i}jx},$$

which yields the infinite coupled system

$$\mathrm{i}\delta(j+\ell)(1-2a_1)b_{j-1} + m_j(\ell)b_j + \mathrm{i}\delta(j+\ell)(1-2\overline{a}_1)b_{j+1} + \mathcal{O}(\delta^2)b_k = \lambda(\ell)b_j, \tag{2.6}$$

$j,k\in\mathbb{Z}$, where $m_j(\ell) = -(j+\ell)^4 + c_3\mathrm{i}(j+\ell)^3 + c_2(j+\ell)^2 - 2\mathrm{i}u_m(j+\ell)$ and $a_1 = \frac{1}{2}(\alpha_1 - \mathrm{i}\beta_1)$ from $u_s = u_m + \delta(\alpha_1\cos(x) + \beta_1\sin(x)) + \mathcal{O}(\delta^2)$. For $\delta=0$ we have eigenvalues $m_j(\ell)$ with $\phi_j(\ell,x) = \mathrm{e}^{\mathrm{i}jx}$. In particular, $m_0(0)=0$ with

$\phi_0(x) \equiv 1$. For $\delta > 0$, and returning to counting $\lambda_n(\ell)$ with $n \in \mathbb{N}$, we still always have $\lambda_1(0) = 0$, with

$$\tilde{v}_1(0,x) = \partial_{u_m} u_s(x; c_2, c_3, \delta, u_m) = 1 + \delta[b_1(0)\mathrm{e}^{\mathrm{i}x} + b_{-1}(0)\mathrm{e}^{-\mathrm{i}x}] + \mathcal{O}(\delta^2). \tag{2.7}$$

To order δ^2 the eigenvalue problem (2.6) yields

$$\det(A(\ell) - \lambda(\ell)) = 0 \tag{2.8}$$

with

$$A = \begin{pmatrix} m_{-1}(\ell) & \mathrm{i}\delta(\ell-1)(1-2\overline{a}_1) & 0 \\ \mathrm{i}\delta\ell(1-2a_1) & m_0(\ell) & \mathrm{i}\delta\ell(1-2\overline{a}_1) \\ 0 & \mathrm{i}\delta(1+\ell)(1-2a_1) & m_1(\ell) \end{pmatrix}.$$

The truncated eigenvalue problem (2.8) can again be solved explicitly using some algebra package, but as mentioned above, instead of giving the explicit formulas, in fig. 2 we plot u_s and $\lambda_1(\ell)$ for some parameters values.

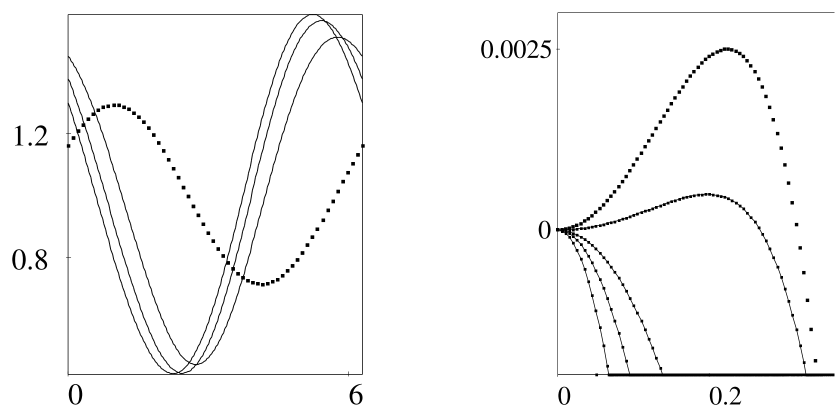

FIGURE 2. Left: parametric dependence of stationary solutions $u_s(x; c_2, c_3, \delta, u_m)$ on c_2 for $(c_3, \delta, u_m) = (0, 0.5, 1)$. Top down at $x = 0$: $c_2 = 0.5, 0, -0.5$, and $\partial_{u_m} u_s(x; 0, 0, 0.5, 1)$ (dotted curve). Right: $\operatorname{Re}\lambda_1(\ell)$ as obtained from (2.8) for $(\delta, u_m) = (0.5, 1)$; left to right at $\operatorname{Re}\lambda = -0.00125$: $c_2 = -0.5, 0, 0.1, 0.2$, and $\operatorname{Re} m_0(\ell)$ for $(u_m, c_2, c_3) = (1, 0.1, 0)$ (dotted curve); $u_s(\cdot, 0.1, 0.5, 1)$ is still spectrally stable, while clearly in the homogeneous case ($\delta = 0$) we have $u_s \equiv u_m$ unstable for $c_2 > 0$. The imaginary part of $\lambda_1(\ell)$ only depends very weakly on c_2 and δ and is given by $\operatorname{Im}\lambda_1(\ell) = -2u_m\ell - \mathrm{i}d_3\ell^3 + \mathcal{O}(\ell^5)$ with $d_3 = -c_3 + \mathcal{O}(\delta)$.

3. Nonlinear analysis in Bloch wave space

3.1. **The idea of renormalization.** To explain the idea of irrelevant nonlinearities and the renormalization group we consider

$$\partial_t v = \partial_x^2 v - \partial_x(v^2) + \alpha h(v), \quad v|_{t=1} = v_0, \quad h(v) = v^{p_1}(\partial_x v)^{p_2}, \tag{3.1}$$

with $p_1 + 2p_2 \geq 4$. For $\alpha = 0$ we know that $\|v(t,x) - t^{-1/2} f_z(x/t^{1/2})\|_{L^\infty} = \mathcal{O}(t^{-1})$ as $t \to \infty$ with $\ln(1+z) = -\int v_0 \,\mathrm{d}x$. To show a similar behaviour for $\alpha \neq 0$ we may use the renormalization group. These calculations are well documented in the literature, but we briefly repeat them here as the template for treating (1.3).

For $L > 0$ we define the rescaling operators $\mathcal{R}_L$ with $\mathcal{R}_L v(x) = v(Lx)$, and for $L > 1$ chosen sufficiently large we let

$$v_n(\tau, \xi) := L^n v(L^{2n}\tau, L^n\xi) = L^n \mathcal{R}_{L^n} v(L^{2n}\tau, \xi). \tag{3.2}$$

Then v_n satisfies

$$\partial_\tau v_n = \partial_\xi^2 v_n + \partial_\xi(v_n^2) + \alpha h_n(v_n) \quad \text{with} \quad h_n(v_n) = L^{(3-p_1-2p_2)n} v_n^{p_1} \partial_\xi(v_n^{p_2}), \tag{3.3}$$

and solving (3.1) for $t \in [1, \infty)$ is equivalent to iterating

$$\text{solve (3.3) on } \tau \in [L^{-2}, 1] \text{ with initial data } v_n(L^{-2}, \xi) = L\mathcal{R}_L v_{n-1}(1, \xi) \in X, \tag{3.4}$$

where X is a suitable Banach space. For $p_1 + 2p_2 \geq 4$ the term h_n in (3.3) formally goes to zero. Thus, in the limit $n \to \infty$ we recover the Burgers equation for v_n, with family of exact solutions $\{v_z(\tau, \xi) = \tau^{-1/2} f_z(\xi/\sqrt{t}) : z > -1\}$. In particular, these solutions are fixed points of the renormalization map $v(1/L^2, \cdot) \mapsto Lv(1, L\cdot)$ where v solves the Burgers equation.

It turns out that this line of fixed points is attractive in suitable spaces X, for instance $X = H^2(2)$, cf. the definition on p. 5. Moreover, this also holds for the flow of the perturbed Burgers equation. For more details concerning problems of type (3.1) we refer to [2, 12]. However, two observations are most important: (a) In (3.3) we see that derivatives in x, corresponding to factors $\mathrm{i}k$ in Fourier space according to $\mathcal{F}(\partial_x u)(k) = \mathrm{i}k\hat{u}(k)$, give additional factors L^{-1} in the rescaling; (b) The diffusive spreading in x space corresponds to concentration at $k = 0$ in Fourier space according to $\mathcal{F}(L\mathcal{R}_L u)(k) = \hat{u}(k/L)$. Therefore, only the parabolic shape of the spectrum $\lambda(k) = -k^2$ of the operator ∂_x^2 locally near $k = 0$ is relevant, as well as only the local behaviour of the nonlinearity near $k = 0$. For (1.1) Fourier analysis has to be replaced by Bloch wave analysis, where similar ideas apply: a factor $\mathrm{i}\ell$ corresponds to a derivative in x, and spreading in x corresponds to localization at $\ell = 0$. This is made rigorous in Lemma 4.2 below.

3.2. **Formal derivation of the Burgers equation.** In order to (formally) derive the Burgers equation as the amplitude equation for the critical mode $\tilde{v}_1(0, x)$ for (1.3) in the linearly stable case – and in order to later justify this and rigorously prove Theorem 1.2 – we consider (1.3) in Bloch space, i.e.

$$\partial_t \tilde{v}(t, \ell, x) = \tilde{\mathcal{L}}(l, x)\tilde{v}(t, \ell, x) + N(\tilde{v}(t))(\ell, x), \tag{3.5}$$

with

$$N(\tilde{v}(t))(\ell, x) = \mathcal{J}(-\partial_x v^2(t))(\ell, x) = -(\partial_x + \mathrm{i}l)(\tilde{v}(t) * \tilde{v}(t))(\ell, x).$$

To motivate the next transform we recall that the curve $\lambda_1(\ell) = -\mathrm{i}d_1\ell - d_2\ell^2 + \mathcal{O}(\ell^3)$ with critical mode $\tilde{v}_1(\ell, \cdot)$ corresponds to $\partial_t v = (-d_1\partial_x + d_2\partial_x^2)v$; i.e., the linear diffusion equation in the comoving frame $y = x - d_1 t$. Thus, it is tempting to simply go into this comoving frame in (1.3). However, this would give a space and time periodic operator $\mathcal{L}(y + d_1 t)$ in (1.3) which would make the subsequent analysis more complicated. Instead we introduce

$$\tilde{u}(t, \ell, x) = \mathrm{e}^{\mathrm{i}\ell d_1 t}\tilde{v}(t, \ell, x) \tag{3.6}$$

which fulfills

$$\partial_t \tilde{u}(t, \ell, x) = \tilde{\mathcal{M}}(l, x)\tilde{v}(t, \ell, x) + N(\tilde{v}(t))(\ell, x), \quad \tilde{\mathcal{M}}(l, x) = \tilde{\mathcal{L}}(l, x) + \mathrm{i}d_1\ell. \tag{3.7}$$

Clearly, $\tilde{\mathcal{M}}$ has the same eigenfunctions $\tilde{v}_j$ as $\tilde{\mathcal{L}}$ with eigenvalues $\mu_j(\ell) = \lambda_j(\ell)+\mathrm{i}d_1\ell$. In particular

$$\mu_1(\ell) = -d_2\ell^2 + \mathcal{O}(\ell^3).$$

In general, (3.6) does not correspond to a simple transform in x–space. However, if $\tilde{u}$ has the special form $\tilde{u}(t,\ell,x) = \tilde{\alpha}(t,l)g(x)$ then

$$v(t,x) = \int_{\mathcal{T}_{2\pi}} \mathrm{e}^{\mathrm{i}\ell(x-d_1t)}\tilde{\alpha}(t,\ell)g(x)\,\mathrm{d}\ell = \alpha(t,x-d_1t)g(x),$$

which we will exploit to prove Theorem 1.2.

Next we introduce mode filters to extract the critical mode $\tilde{v}_1(\cdot,\cdot)$ from $\tilde{u}$. Let $\rho > 0$ be sufficiently small such that $\mu_1(\ell)$ is isolated from the rest of the spectrum of $\tilde{\mathcal{L}}(\ell,\partial_x)$ for $|\ell| \le \rho$, and let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function with $\chi(\ell) = 1$ for $|\ell| \le \rho/2$ and $\chi(\ell) = 0$ for $|\ell| > \rho$. Then define

$$\tilde{E}_c(\ell)\tilde{v}(\ell,x) = \chi(\ell)\langle\tilde{v}(\ell,\cdot),\tilde{u}_1(\ell,\cdot)\rangle\tilde{v}_1(\ell,x).$$

Here $\langle v,w\rangle = \int_{\mathcal{T}_{2\pi}} v(x)\overline{w}(x)\,\mathrm{d}x$, and $\tilde{u}_1$ is the critical eigenfunction of the $L^2(\mathcal{T}_{2\pi})$-adjoint operator

$$\tilde{\mathcal{M}}^*(\ell,x) = -(\partial_x+\mathrm{i}\ell)^4+c_3(\partial_x+\mathrm{i}\ell)^3 - c_2(\partial_x+\mathrm{i}\ell)^2 - 2(\partial_x+\mathrm{i}\ell)(\delta\cos(x)\tilde{u}-u_s\tilde{u}) - \mathrm{i}d_1\ell, \tag{3.8}$$

normalized such that $\langle\tilde{v}_1(\ell),\tilde{u}_1(\ell)\rangle = 1$. Let $\tilde{E}_s = \mathrm{Id} - \tilde{E}_c$. Moreover, define auxiliary mode filters $\tilde{E}_c^h(\ell)\tilde{u}(\ell,x) = \chi(2\ell)\langle\tilde{u}(\ell,\cdot),\tilde{u}_1(\ell,\cdot)\rangle\tilde{v}_1(\ell,x)$ and $\tilde{E}_s^h(\ell)\tilde{u}(\ell,x) = \tilde{v}(\ell,x) - \chi(\ell/2)\langle\tilde{u}(\ell,\cdot),\tilde{u}_1(\ell,\cdot)\rangle\tilde{v}_1(\ell,x)$. Thus $\tilde{E}_c^h\tilde{E}_c = \tilde{E}_c$ and $\tilde{E}_s^h\tilde{E}_s = \tilde{E}_s$, which will be used to substitute for missing projection properties of $\tilde{E}_c$ and $\tilde{E}_s$. Finally, define the scalar mode filter $\tilde{E}_c^*$ by $\tilde{E}_c(\ell)\tilde{u}(\ell,x) = (\tilde{E}_c^*(\ell)\tilde{u}(\ell,\cdot))\tilde{v}_1(\ell,x)$.

Thus, if $(\tilde{\alpha},\tilde{u}_s)$ satisfies

$$\begin{aligned}\partial_t\tilde{\alpha}(t,\ell) &= \mu_1(\ell)\tilde{\alpha}(t,\ell) + \tilde{E}_c^*N(\tilde{v}(t))(\ell),\\ \partial_t\tilde{u}_s(t,\ell,x) &= \tilde{\mathcal{M}}_s\tilde{v}(t,\ell,x) + \tilde{E}_sN(\tilde{v}(t))(\ell,x),\end{aligned} \tag{3.9}$$

then $\tilde{u}(t,\ell,x) = \tilde{v}_c(t,\ell,x) + \tilde{u}_s(t,\ell,x)$ satisfies equation (3.7), where $\tilde{u}_c(t,\ell,x) = \tilde{\alpha}(t,\ell)\tilde{v}_1(\ell,x)$. The idea of this splitting is that $\tilde{u}_s$ is linearly exponentially damped. Thus we may expect that the dynamics of (3.7) and hence of (1.3) are dominated by the dynamics of $\tilde{\alpha}$. This will be made rigorous in §4. Here we first formally derive the Burgers equation for $\tilde{\alpha}$, ignoring $\tilde{u}_s$. Then the nonlinearity in (3.9) is given by (suppressing t for now)

$$\begin{aligned}&\tilde{E}_c^*N(\tilde{u}_c)(\ell)\\ &= -\chi(\ell)\int_{\mathcal{T}_{2\pi}}(\partial_x+\mathrm{i}\ell)\int_{-1/2}^{1/2}\tilde{\alpha}(\ell-m)\tilde{v}_1(\ell-m,x)\tilde{\alpha}(m)\tilde{v}_1(m,x)\,\mathrm{d}m\,\overline{\tilde{u}}_1(\ell,x)\,\mathrm{d}x\\ &= \int_{-1/2}^{1/2}K(\ell,\ell-m,m)\tilde{\alpha}(\ell-m)\tilde{\alpha}(m)\,\mathrm{d}m\end{aligned}$$

with

$$K(\ell,\ell-m,m) = \chi(\ell)\int_0^{2\pi}\tilde{v}_1(\ell-m,x)\tilde{v}_1(m,x)(\mathrm{i}\ell\overline{\tilde{u}}_1(\ell,x) + \partial_x\overline{\tilde{u}}_1(\ell,x))\,\mathrm{d}x. \tag{3.10}$$

Automatically we have $K(0,0,0)$. This is due to the following abstract argument [8, 9, 10]. If we consider (1.3) over $\mathcal{T}_{2\pi}$, then there exists a one dimensional center manifold $\mathcal{W}_c = \{v(x) = \gamma\tilde{v}_1(0,x) + h(\gamma)(x) : \gamma \in (-\gamma_0, \gamma_0)\}$, and the flow on $\mathcal{W}_c$ is given by the reduced equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma = P_c\big[\mathcal{M}(\gamma\tilde{v}_1 + h(\gamma)) + N(\gamma\tilde{v}_1 + h(\gamma))\big] = \langle -\partial_x(\tilde{v}_1^2), w_1\rangle\gamma^2 + \text{h.o.t.},$$

where h.o.t. denotes higher order terms and the projection P_c is $\tilde{E}_c^*(0)$. However, $\mathcal{W}_c$ coincides with the one-dimensional family of stationary solutions $\{u_s(\cdot, c_2, \delta, m) : m \approx u_m\}$, hence $\frac{\mathrm{d}}{\mathrm{d}t}\gamma{=}0$ and the projection vanishes. Alternatively, we can inspect (3.8) to see that $\tilde{u}_1(0,x) \equiv const$, which implies $K(0,0,0)$ as well.

We expand $K(\ell, \ell-m, m) = \partial_1 K(\mathbf{0})\ell + \partial_2 K(\mathbf{0})(\ell-m) + \partial_3 K(\mathbf{0})m + \mathcal{O}((\ell+m)^2)$. Ignoring for now the $\mathcal{O}((\ell+m)^2)$ terms we obtain

$$\tilde{E}_c^* N(\tilde{u}_c)(\ell) = \chi(\ell)\mathrm{i}b\ell(\tilde{\alpha}^{*2})(\ell)$$

with

$$\begin{aligned} b &= -\mathrm{i}\Big(\partial_1 K(0) + \frac{1}{2}\partial_2 K(0) + \frac{1}{2}\partial_3 K(0)\Big) \\ &= -\mathrm{i}\Big(\int_{\mathcal{T}_{2\pi}} \tilde{v}_1^2(0,x)(-\mathrm{i}\overline{\tilde{u}}_1(0,x) + \partial_\ell\partial_x\overline{\tilde{u}}(0,x)) + (\partial_\ell\tilde{v}_1(0,x))\tilde{v}_1(0,x)\partial_x\overline{\tilde{u}}_1(0,x)\,\mathrm{d}x\Big) \\ &= -1 + \mathcal{O}(\delta^2) \in \mathrm{i}\mathbb{R}, \end{aligned} \tag{3.11}$$

where we used the facts that $\tilde{v}_1(0,x) = 1 + \mathcal{O}(\delta) \in \mathbb{R}$, that $\tilde{u}_1(0,x) = 1/2\pi$ due to the normalization $\langle\tilde{v}_1(0,\cdot), \tilde{u}_1(0,\cdot)\rangle = 1$, and that $\mathrm{i}\partial_\ell\tilde{u}_1(0,x) \in \mathbb{R}$, see (3.8).

The result of these calculations (ignoring $\tilde{u}_s$ and the $\mathcal{O}((l+m)^2)$ terms in $K(\ell, \ell-m, m)$) is that $\tilde{\alpha}$ fulfills

$$\partial_t\tilde{\alpha}(t,\ell) = \mu_1(\ell)\tilde{\alpha}(t,\ell) + \mathrm{i}b\ell\tilde{\alpha}^{*2}(t,\ell), \tag{3.12}$$

Motivated by §3.1 we may for now also discard the $\mathcal{O}(\ell^3)$ terms in $\mu_1(\ell)$ to see that $\alpha(t,x) = (\mathcal{J}^{-1}\tilde{\alpha}(t))(x)$ fulfills the Burgers equation

$$\partial_t\alpha = d_2\partial_x^2\alpha + b\partial_x(\alpha^2).$$

In a nutshell, this, combined with §3.1, explains why the "comoving frame Burgers profile" $t^{-1/2}f_z((x-d_1t)/\sqrt{t})\tilde{v}_1(0,x)$ gives the lowest order asymptotics for (1.3).

3.3. **The result in Bloch wave space.** To make the formal calculations from §3.2 rigorous and thus prove Theorem 1.2 we need scaled Bloch spaces with regularity and weights. We first collect a number of definitions and basic properties. Let $\rho(\ell) = (1+|\ell|^2)^{1/2}$. For $L > 1$ and $m, n, b \geq 0$ define

$$\begin{aligned} \mathcal{B}_L(n,m,b) &:= \{\tilde{v} \in H^n((-L/2, L/2), H^m(\mathcal{T}_{2\pi}) : \|\tilde{v}\|_{\mathcal{B}_L(n,m,b)} < \infty\}, \\ \|\tilde{v}\|^2_{\mathcal{B}_L(n,m,b)} &= \sum_{\alpha\le n}\sum_{\beta\le m} \|(\partial_\ell^\alpha\partial_x^\beta\tilde{v})\rho^b\|^2_{L^2((-L/2,L/2),L^2(\mathcal{T}_{2\pi}))}. \end{aligned}$$

Let $\mathcal{B}(n,m,b) := \mathcal{B}_1(n,m,b)$. Based on Parseval's identity we have that $\mathcal{J}$ is an isomorphism between $H^m(n)$ and $\mathcal{B}(n,m,b)$, with arbitrary $b \geq 0$, see, e.g., [9, Lemma 5.4]. Indeed, for fixed $L > 0$ the weight ρ is irrelevant since due to the bounded wave number domain all norms $\|\cdot\|_{\mathcal{B}_L(n,m,b_1)}$ and $\|\cdot\|_{\mathcal{B}_L(n,m,b_2)}$ are equivalent, but the constants depend on b_1, b_2 and L, see (4.7), which will be crucial in our analysis. Next we define the scaling operators

$$\mathcal{R}_{1/L} : \mathcal{B}(n,m,b) \to \mathcal{B}_L(n,m,b), \quad \mathcal{R}_{1/L}\tilde{v}(\ell,x) = \tilde{v}(\ell/L, x).$$

Only ℓ is rescaled, and x is not, and as in (3.6) this in general does not correspond to a simple rescaling in x–space. In Bloch space our main result now reads as follows.

Theorem 3.1. *Assume that the parameters $c_2, c_3, u_m \in \mathbb{R}$ and $\delta > 0$ sufficiently small are chosen in such a way that $d_2 > 0$ in the expansion* (1.5) *and that $\operatorname{Re}\lambda_n(\ell) < 0$ for all $n \in \mathbb{N}$ and all $\ell \in [-1/2, 1/2)$, except for $\lambda_1(0) = 0$. Let $p \in (0, 1/2)$. There exist $C_1, C_2 > 0$ such that the following holds. If $\|v_0\|_{H^2(2)} \le C_1$, then*

$$\left\|(\ell, x) \mapsto \left[\tilde{v}(t, \ell/\sqrt{t}, x) - \mathrm{e}^{\mathrm{i}\ell d_1 t}\tilde{f}_z(\cdot)\tilde{v}_1(0, x)\right]\right\|_{\mathcal{B}_{\sqrt{t}}(2,2,2)} \le C_2 t^{-1/2+p}, \tag{3.13}$$

with $d_1 = 2u_m + \mathcal{O}(\delta)$ from (1.5), *$\tilde{f}_z(\ell) = \mathcal{F}(f_z)(\ell)$, where $f_z(y) = \frac{\sqrt{d_2}}{b}\frac{z\,\mathrm{erf}'(y)}{1+z\,\mathrm{erf}(y)}$ from* (1.8) *with $z = \frac{b}{d_2}\int_{\mathbb{R}} v_0(x)\,\mathrm{d}x$.*

Before proving this theorem we translate (3.13) back into x-space. In $L^\infty(\mathbb{R})$ we have

$$\begin{aligned}
v(t, x) &= \int_{-1/2}^{1/2} \exp(\mathrm{i}\ell x)\tilde{v}(t, \ell, x)\,\mathrm{d}\ell = \int_{-1/2}^{1/2} \exp(\mathrm{i}\ell(x - d_1 t))\tilde{u}(t, \ell, x)\,\mathrm{d}\ell \\
&= t^{-1/2}\int_{-\sqrt{t}/2}^{\sqrt{t}/2} \exp(\mathrm{i}\ell t^{-1/2}(x - d_1 t))\tilde{u}(t, t^{-1/2}\ell, x)\,\mathrm{d}\ell \\
&= t^{-1/2}\int_{-\sqrt{t}/2}^{\sqrt{t}/2} \exp(\mathrm{i}\ell t^{-1/2}(x - d_1 t))\tilde{f}_z(\ell)\,\mathrm{d}\ell\,\tilde{v}_1(0, x) + \mathcal{O}(t^{-1+p/2}) \\
&= t^{-1/2} f_z(t^{-1/2}(x - d_1 t))\tilde{v}_1(0, x) + \mathcal{O}(t^{-1+p/2}).
\end{aligned}$$

This proves Theorem 1.2.

4. Proof of Theorem 3.1

4.1. **The rescaled systems.** To prove Theorem 3.1 we now start with the system (3.9). Similar to (3.2) we introduce scaled variables

$$\alpha_n(\tau, \kappa) = \mathcal{R}_{L^{-n}}\tilde{\alpha}(L^{2n}\tau, \kappa) \quad \text{and} \quad w_n(\tau, \kappa, x) = L^{n(1-p)}\mathcal{R}_{L^{-n}}\tilde{u}_s(L^{2n}\tau, \kappa, x). \tag{4.1}$$

Here we "blow up" w_n since by this we can more directly control the terms involving w_n in the equation for α_n, see Lemma 4.3 below. We obtain

$$\begin{aligned}
\partial_\tau \alpha_n(t, \kappa) &= L^{2n}\mu_1(\kappa/L^n)\alpha_n(\tau, \kappa) + L^{2n}N_n^c(\alpha_n, w_n), \\
\partial_\tau w_n(\tau, \kappa, x) &= L^{2n}\tilde{\mathcal{M}}_n^s w_n + L^{(3-p)n}N_n^s(\alpha_n, w_n),
\end{aligned} \tag{4.2}$$

where $\tilde{\mathcal{M}}_n^s = L^{2n}\mathcal{R}_L^{-n}\tilde{\mathcal{M}}_s\mathcal{R}_{L^n}$ and

$$\begin{aligned}
N_n^c(\alpha_n, w_n)(\kappa, x) &= \mathcal{R}_{L^{-n}}\tilde{E}_c N\left((\mathcal{R}_{L^n}\alpha_n)\tilde{v}_1(\kappa, x) + L^{-n(1-p)}\mathcal{R}_{L^n}w_n\right), \\
N_n^s(\alpha_n, w_n)(\kappa, x) &= \mathcal{R}_{L^{-n}}\tilde{E}_s N\left((\mathcal{R}_{L^n}\alpha_n)\tilde{v}_1(\kappa, x) + L^{-n(1-p)}\mathcal{R}_{L^n}w_n\right).
\end{aligned} \tag{4.3}$$

Similar to (3.4), we consider the following iteration:

$$\begin{gathered}
\text{solve (4.2) on } \tau \in [L^{-2}, 1] \text{ with initial data} \\
\begin{pmatrix}\alpha_n \\ w_n\end{pmatrix}(L^{-2}, \kappa, \xi) = \mathcal{R}_{1/L}\begin{pmatrix}\alpha_{n-1} \\ L^{1-p}w_{n-1}\end{pmatrix}(1, \kappa, \xi).
\end{gathered} \tag{4.4}$$

As phase space for (4.2) we choose $\mathcal{X}_n \times \mathcal{X}_n$ with $\mathcal{X}_n = \mathcal{B}_{L^n}(2,2,2)$, where for α_n we can identify $\mathcal{X}_n$ with the Fourier space $H^2(2)$ since α_n is independent of x. Moreover, $\operatorname{supp}\alpha_n \subset \{|\ell| \le L^n\rho\}$. To treat (4.4) we note a number of estimates.

Lemma 4.1. *For $b_2 \ge b_1 \ge 0$ there exists a $C > 0$ such that in the critical part we have*

$$\|\mathrm{e}^{L^{2n}\mu_1(\cdot/L^n)(\tau-\tau')}\alpha_n\|_{B_{L^n}(2,2,b_1)} \le C(\tau-\tau')^{(b_1-b_2)/2}\|\alpha_n\|_{B_{L^n}(2,2,b_1)}. \tag{4.5}$$

The stable part is linearly exponentially damped; i.e., there exists a $\gamma_0 > 0$ such that

$$\|\mathrm{e}^{L^{2n}\tilde{\mathcal{M}}_n^s(\tau-\tau')}w_n\|_{\mathcal{B}_{L^n}(k,b,b)} \le C\mathrm{e}^{-\gamma_0 L^{2n}(\tau-\tau')}(\tau-\tau')^{-1/2}\|w_n\|_{\mathcal{B}_{L^n}(k,b-1,b-1)}. \tag{4.6}$$

Proof. Inequality (4.5) follows from the locally parabolic shape of $L^{2n}\mu_1(\kappa/L^n) = -d_2\kappa^2 + \mathcal{O}(\kappa/L^n)$ near $\kappa = 0$. Inequality (4.6) follows from $\operatorname{Re}\sigma(\tilde{\mathcal{M}}_s) \le -\gamma_0$. In fact, $\tilde{\mathcal{M}}_s$ is a 4^{th} order operator and therefore has better smoothing properties than stated in (4.6), but this estimate is sufficient in the following. $\square$

Next we note

$$\|\mathcal{R}_{1/L}\tilde{v}\|_{\mathcal{B}_L(2,2,b)} \le CL^{b+1/2}\|\tilde{v}\|_{\mathcal{B}(2,2,b)}, \tag{4.7}$$

and, for $\tilde{u}, \tilde{v} \in \mathcal{B}_L(n,m,0)$ with $n,m \ge 1/2$ and $\ell \in (-L/2, L/2)$,

$$\begin{aligned}
&\mathcal{R}_{1/L}(\mathcal{R}_L\tilde{u} * \mathcal{R}_L\tilde{v})(\ell,x)\\
&= \int_{-1/2}^{1/2} \tilde{u}(\ell - Lm, x)v(Lm,x)\,\mathrm{d}m\\
&= L^{-1}\int_{-L/2}^{L/2} \tilde{u}(\ell-m,x)\tilde{v}(m,x)\,\mathrm{d}m =: L^{-1}(\tilde{u} *_L \tilde{v})(\ell,x),
\end{aligned}$$

which will be used to express the rescaled nonlinear terms. Henceforth we will drop the subscript $_L$ in $*_L$. To estimate the nonlinearity $\partial_x(v^2)$ in Bloch space we need to exploit the derivative using the following Lemma [8, Lemma 14].

Lemma 4.2. *Let $\tilde{K} \in C_b^2([-1/2,1/2)^2, H^2(\mathcal{T}_{2\pi}))$ with*

$$\|\tilde{K}(\kappa-\ell,\ell)\|_{H^2(\mathcal{T}_{2\pi})} \le C(|\kappa-\ell|+|\ell|)^\gamma.$$

Then

$$(\tilde{v},\tilde{u}) \mapsto (\mathcal{M}_{1/L}K)(\tilde{v},\tilde{u})(\kappa) := \int \big(\mathcal{R}_{1/L}\tilde{K}(\kappa-\ell,\ell,x)\big)\tilde{v}(\kappa,x)\tilde{u}(\kappa-\ell,x)\,\mathrm{d}\ell$$

defines a bilinear mapping $(\mathcal{M}_{1/L}K) : \mathcal{B}_L(2,2,2) \times \mathcal{B}_L(2,2,2) \to \mathcal{B}_L(2,2,2)$. There exists a $C > 0$ such that for all $L > 1$ we have

$$\|(\mathcal{M}_{1/L}K)(\tilde{v},\tilde{u})\|_{\mathcal{B}_L(2,2,2-\gamma)} \le CL^{-\min\{\gamma,1\}}\|\tilde{v}\|_{\mathcal{B}_L(2,2,2)}\|\tilde{u}\|_{\mathcal{B}_L(2,2,2)}.$$

Lemma 4.3. *For $p \in (0,1/2)$ there exists a $C > 0$ such that for all $(\alpha_n, w_n) \in X_n$ we have $L^{2n}N_n^c(\alpha_n,w_n) = s_1 + s_2 + s_3 + s_4$ with $s_1(\kappa) = \mathrm{i}b\kappa\alpha_n^{*2}(\kappa)$ and*

$$\|s_2\|_{B_{L^n}(2,2,2p)} \le CL^{-n(1-2p)}\|\alpha_n\|_{\mathcal{X}_n}^2, \tag{4.8}$$

$$\|s_3\|_{B_{L^n}(2,2,1)} \le CL^{-n(1-p)}\|\alpha_n\|_{\mathcal{X}_n}\|w_n\|_{\mathcal{X}_n}, \tag{4.9}$$

$$\|s_4\|_{B_{L^n}(2,2,1)} \le CL^{-2n(1-p)}\|w_n\|_{\mathcal{X}_n}^2. \tag{4.10}$$

Moreover,

$$\begin{aligned} &L^{(3-p)n}\|N_n^s(\alpha_n,w_n)\|_{\mathcal{B}_{L^n}(2,1,1)}\\ &\le C\big(L^{(1-p)n}\|\alpha_n\|^2_{\mathcal{X}_n}+\|\alpha_n\|_{\mathcal{X}_n}\|w_n\|_{\mathcal{X}_n}+L^{-(1-p)n}\|w_n\|^2_{\mathcal{X}_n}\big). \end{aligned} \tag{4.11}$$

Proof. Explicitly we have

$$L^{2n}N_n^c(\kappa)=L^{2n}\chi(\frac{\kappa}{L^{2n}})\int_{x\in\mathcal{T}_{2\pi}}\Big(\int_{m=-1/2}^{1/2}(\partial_x+\frac{\mathrm{i}\kappa}{L^n})\Pi(\kappa,m,x)\,\mathrm{d}m\Big)\overline{\tilde{u}}_1(\frac{\kappa}{L^n},x)\,\mathrm{d}x, \tag{4.12}$$

with

$$\begin{aligned} \Pi(\kappa,m,x)=&\Big(\alpha_n(\kappa-L^nm)\tilde{v}_1(\frac{\kappa}{L^n}-m,x)+L^{-n(1-p)n}w_n(\kappa-L^nm)\Big)\\ &\times\Big(\alpha_n(L^nm)\tilde{v}_1(m,x)+L^{-n(1-p)n}w_n(L^nm)\Big)\,. \end{aligned}$$

Thus, substituting $L^nm\to m$ in (4.12) yields $L^{2n}N_n^c(\kappa)=s_1+s_2+s_3+s_4$ with

$$\begin{aligned} s_1+s_2&=L^n\int_{-L^n/2}^{L^n/2}K(\frac{\kappa}{L^n},\frac{\kappa-m}{L^n},\frac{m}{L^n})\alpha_n(\kappa-m)\alpha_n(m)\,\mathrm{d}m\\ &=\mathrm{i}b\kappa\alpha_n^{*2}+L^n\int_{-L^n/2}^{L^n/2}(\mathcal{R}_{L^{-n}}M(\kappa,m))\alpha_n(\kappa-m)\alpha_n(m)\,\mathrm{d}m, \end{aligned}$$

where $M\in C^2$ with $M(\ell,m)\le C((\ell+m)^\gamma)$ with $1<\gamma\le 2$. Now using Lemma 4.2 with $\gamma=2-2p$ we obtain (4.8).

Similarly,

$$\begin{aligned} &\|s_3(\kappa)\|_{\mathcal{B}_{L^n}(2,2,1)}\\ &=L^nL^{-n(1-p)}\big\|\chi(\frac{\kappa}{L^n})\langle(\partial_x+\frac{\mathrm{i}\kappa}{L^n})(w_n*_{L^n}(\mathrm{e}^{\mathrm{i}d_1L^n\cdot\tau}\alpha_n\tilde{v}(\cdot/L^n))),\tilde{u}_1(\frac{\kappa}{L^n})\rangle\big\|_{\mathcal{B}_{L^n}(2,2,1)} \end{aligned}$$

which shows (4.9) by again using that $\partial_x\tilde{u}_1(\frac{\kappa}{L^n})=\mathcal{O}(\frac{\kappa}{L^n})$, and (4.10) follows in the same way, as well as the estimate (4.11) in the stable part. $\square$

The terms involving α_n in (4.11) do not decay. However, combining (4.11) with the exponential decay of the stable semigroup we still get a local existence result for (4.2) with bounds independent of n.

Lemma 4.4. *There exist $C_1,C_2>0$ and $L_0>1$ such that for $L>L_0$ the following holds. Let*

$$\rho_{n-1}:=\|(\alpha_{n-1},w_{n-1})(1)\|_{\mathcal{X}_{n-1}}\le C_1L^{-5/2}.$$

Then there exists a local solution $(\alpha_n,w_n)\in C([1/L^2,1],\mathcal{X}_n)$ of (4.2), with

$$\sup_{\tau\in[1/L^2,1]}\|(\alpha_n,w_n)\|_{\mathcal{X}_n}\le C_2L^{5/2}\rho_{n-1}. \tag{4.13}$$

Proof. The variation of constant formula for (4.2) yields

$$\begin{aligned} \alpha_n(\tau)=&\mathrm{e}^{(\tau-L^{-2})L^{2n}\mu_1(\kappa/L^n)}\mathcal{R}_{1/L}\alpha_{n-1}(1)\\ &+L^{2n}\int_{1/L^2}^{\tau}\mathrm{e}^{(\tau-s)L^{2n}\mu_1(\kappa/L^n)}N_n^c(\alpha_n(s),w_n(s))\,\mathrm{d}s, \end{aligned} \tag{4.14}$$

$$w_n(\tau)=\mathrm{e}^{(\tau-L^{-2})\mathcal{M}_n^s}\mathcal{R}_{1/L}w_{n-1}(1)+L^{(3-p)n}\int_{1/L^2}^{\tau}\mathrm{e}^{(\tau-s)\mathcal{M}_n^s}N_n^c(\alpha_n(s),w_n(s))\,\mathrm{d}s. \tag{4.15}$$

Combining (4.7) with Lemmas 4.1 and 4.3 and applying the contraction mapping theorem yields the result. $\square$

4.2. **Splitting and iteration.** Due to the loss of $L^{5/2}$ in Lemma 4.4 we need to refine our estimate of the solutions of (4.2). Therefore let

$$\alpha_n(\tau,\kappa) = \alpha_n^{(z)}(\tau,\kappa) + \gamma_n(\tau,\kappa) \tag{4.16}$$

where $\alpha_n^{(z)}(\tau,\kappa) := \chi(\kappa/L^n)\hat{v}_z(\tau,\kappa)$, $\hat{v}_z(\tau,\kappa) = \hat{f}_z(\tau^{1/2}\kappa)$, with z defined by $\ln(z+1) = \frac{b}{d_2}\alpha_n(1/L^2,\kappa)|_{\kappa=0}$. Since $N(\cdot)(\ell,x)$ in (3.7) vanishes at $\ell=0$, so do N_n^c and N_n^s at $\kappa=0$, which corresponds to the conservation of mass by the nonlinearity $-\partial_x(v^2)$ in (1.1). Therefore $\gamma_n(\tau,0)=0$ for all $n\in\mathbb{N}$ and all $\tau\in[1/L^2,1]$. We obtain

$$\partial_\tau\gamma_n = L^{2n}\mu_1(\cdot/L^n)\gamma_n + L^{2n}(N_n^c(\alpha_n,w_n) - N_n^c(\alpha_n^{(z)},0)) + \operatorname*{Res}_n, \tag{4.17}$$

where

$$\operatorname*{Res}_n = -\partial_\tau\alpha_n^{(z)} + L^{2n}(\mu_1(\cdot/L^n)\alpha_n^{(z)} + N_n^c(\alpha_n^{(z)},0)).$$

Lemma 4.5. *Let $|z|<1$. There exists a $C>0$ such that*

$$\sup_{\tau\in[L^{-2},1]}\|\operatorname*{Res}_n\|_{\mathcal{X}_n}\le CL^{-n}|z|.$$

Proof. By construction, $L^{2n}\mu_1(\kappa/L^n) = -d_2\kappa^2 + \mathcal{O}(\kappa^3/L^n)$ and $L^{2n}N_n^c(\alpha_n^{(z)},0)) = (\mathrm{i}b\kappa + \mathcal{O}((\kappa/L^n)^2))(\alpha_n^{(z)}*\alpha_n^{(z)})$. Combining this with $\partial_\tau\hat{v}_z = -d_2\ell^2\hat{v}_z + \mathrm{i}b\kappa(\hat{v}_z*\hat{v}_z)$ yields

$$\operatorname*{Res}_n = CL^{-n}(\mathcal{O}(\kappa^3)\alpha_n^{(z)} + \mathcal{O}(\kappa^2(\alpha_n^{(z)}*\alpha_n^{(z)}))$$

which can be estimated in $\mathcal{X}_n = \mathcal{B}_{L^n}(2,2,2)$ by $CL^{-n}|z|$ since $\hat{v}_z$ is an analytic and exponentially decaying function. $\square$

Next write

$$\alpha_n(1,\kappa) = \alpha_n^{(z)}(1,\kappa) + g_{n,c}(\kappa), \quad w_n(1,\kappa,x) = g_{n,s}(\kappa,x).$$

By construction $g_{n,c}(0)=0$, and finally we use the contraction properties of the linear semigroup $\mathrm{e}^{L^{2n}\mu_1(\cdot/L^n)(1-L^2)}\mathcal{R}_{1/L}$ when acting on functions $g(\cdot)$ with $g(0)=0$, i.e.

$$\|\mathrm{e}^{L^{2n}\mu_1(\cdot/L^n)(1-L^2)}\mathcal{R}_{1/L}g\|_{\mathcal{B}_{L^n}(2,2,2)} \le CL^{-1}\|g\|_{\mathcal{B}_{L^{n-1}}(2,2,2)}. \tag{4.18}$$

Here we need the smoothness in ℓ. Using $g(\ell/L,x) = g(0,x) + (\ell/L)\partial_\ell g(\tilde{l},x) = (\ell/L)\partial_\ell g(\tilde{l},x)$ we have

$$\|g(\ell/L,\cdot)\|_{H^2(\mathcal{T}_{2\pi})} \le \frac{\ell}{L}\|g\|_{C^1(\mathcal{T}_{L^n},H^2(\mathcal{T}_{2\pi}))} \le C\frac{\ell}{L}\|g\|_{B_{L^n}(2,2,2)},$$

cf., e.g., [10, Lemma 28]. Thus, combining (4.5), (4.7), (4.8)–(4.10) and (4.18) we obtain in the critical part

$$\rho_{n,c} := \|g_{n,c}\|_{\mathcal{X}_n} \le CL^{-1}\|g_{n-1,c}\|_{\mathcal{X}_{n-1}} + C(|z|L^{5/2}\rho_{n-1} + (L^{5/2}\rho_{n-1})^2 + L^{-n}|z|), \tag{4.19}$$

while in the stable part we have, for L sufficiently large,

$$\begin{aligned}\rho_{n,s} &:= \|g_{n,s}\|_{\mathcal{X}_n}\\ &\le C\mathrm{e}^{-\gamma_0 L^{2n}(1-L^{-2})}\big[L^{5/2}\|g_{n-1,s}\|_{\mathcal{X}_{n-1}} + L^{(1-p)n}(L^{5/2}\rho_{n-1})^2\big]\\ &\le L^{-1}\rho_{n-1}\end{aligned} \tag{4.20}$$

Proof of Theorem 3.1. This now follows from a simple iterative argument. Let $\rho_0 \le L^{-m_0} =: \varepsilon$, hence also $|z| \le CL^{-m_0}$, and let $L \ge L_0$ with L_0 sufficiently large such that $CL^{-1} \le L^{-(1-p)}$. Then (4.19) implies $\rho_{n,c} \le L^{-(m_n-np)} + L^{-n(1-p)}|z|$ with

$$m_n = \min\{m_{n-1}+1, m_0+m_{n-1}-5/2, 2m_{n-1}-5\},$$

while (4.20) yields $\rho_{n,s} \le L^{-n(1-p)}$. Letting, e.g., $m_0 = 6$ yields $m_1 = 7$, $m_2 = 8, \dots$, hence $\rho_{n,c} \le L^{-n(1-p)}(1+|z|)$ and $\rho_n \le C|z| + L^{-m_n}$. For $\tilde v_n(\kappa, x) := \tilde v(L^{2n}, \kappa/L^n, x)$ this yields

$$\begin{aligned}\|v_n - \alpha_n^{(z)}\mathcal{R}_{1/L^n}\tilde v_1\|_{\mathcal{X}_n} &= \Big\|v_n(1) - \chi(\frac{\cdot}{L^n})\hat f_z(\cdot)\tilde v_1(\frac{\cdot}{L^n},\cdot)\Big\|_{\mathcal{X}_n}\\ &\le \|g_{n,c} + L^{-n(1-p)}g_{n,s}\|_{\mathcal{X}_n} \le 2L^{-n(1-p)}.\end{aligned}$$

This is (3.13) for $t = L^{2n}$, and the local existence Lemma 4.4 yields the result for all $t \in [L^{2n}, L^{2(n+1)}]$. □

5. Remarks on the unstable case

In the unstable case $d_2 < 0$ in (1.5) we expand $\lambda_1(\ell)$ further to obtain

$$\lambda_1(\ell) = -\mathrm{i}d_1\ell - d_2\ell^2 - \mathrm{i}d_3\ell^3 - d_4\ell^4 + \mathcal{O}(\ell^5) \tag{5.1}$$

with $d_3 = -c_3 + \mathcal{O}(\delta) \in \mathbb{R}$ and $d_4 = -1 + \mathcal{O}(\delta) < 0$. Then, with the same ansatz as in §3.2; i.e.,

$$\tilde v(t,\ell,x) = \mathrm{e}^{-\mathrm{i}\ell d_1 t}\tilde\alpha(t,\ell)\tilde v_1(\ell,x), \tag{5.2}$$

we may formally derive the constant coefficient Kuramoto-Sivashinsky equation

$$\partial_t\alpha = (d_2\partial_x^2 + d_3\partial_x^3 - d_4\partial_x^4)\alpha - b\partial_x(\alpha^2) \tag{5.3}$$

as the amplitude equation for the critical mode $\tilde v_1(\ell,x)$, which gives

$$\partial_t\alpha = (-d_1\partial_x + d_2\partial_x^2 + d_3\partial_x^3 - d_4\partial_x^4)\alpha - b\partial_x(\alpha^2) \tag{5.4}$$

as the amplitude equation in the laboratory frame. Equation (5.3) (or (5.4)) is only slightly simpler than (1.1), but, importantly, (5.3) is a well known (at least with $d_3 = 0$), much studied, generic amplitude equation for long wave instabilities; see, e.g., [5, 3] and the references therein, and [4] for recent progress.

However, it is not clear a priori if (5.3) is a useful approximation in our problem, in contrast to the stable case, where the Burgers equation (1.7) is used to "guess" the lowest order asymptotics of small localized solutions of (1.1) which is then proved rigorously a posteriori. In the unstable case no such behaviour can be expected: solutions of (5.3) are $\mathcal{O}(1)$ in general and do not decay but show complicated dynamical behaviour, see the references above, and [6].

Thus, first of all, already for the formal derivation of (5.3) an amplitude parameter ε should be introduced. This can be done by defining

$$\varepsilon := \operatorname{Re}\lambda_1(\ell_c) \quad \text{with} \quad \partial_\ell \operatorname{Re}\lambda_1|_{\ell=\ell_c} = 0;$$

i.e., ε is defined as the maximum growth rate in (1.5). Then the so called justification of (5.3) as the amplitude equation for (1.1) should be studied, namely: over what time–scales (relative to ε) and in what spaces do solutions of (5.3) via (5.2) approximate solutions of (1.1)? Here we refrain from this analysis, which would first require a number of assumptions on the coefficients c_2, c_3, δ in (1.1); we refer to [11] and the references therein for related work in this direction.

Instead, here we report some numerical simulations concerning the approximation of (1.1) by (5.2) and (5.4). The full line in Fig.1b) shows a numerical solution to (1.1) with

$$(\delta, c_2, c_3) = (0.1, -0.2, -1) \quad \text{and mass} \quad u_m = 1 \tag{5.5}$$

domain and initial condition u_m as noted. For the perturbation analysis with parameters (5.5) we numerically find that a good 4^{th} order approximation for λ_1 is $\lambda_1(\ell) = -2\mathrm{i}\ell + 0.186\ell^2 - \mathrm{i}\ell^3 - \ell^4$. Next we approximate $b = -1 + \mathcal{O}(\delta^2)$ by -1 and thus consider (5.3) with

$$(d_1, d_2, d_3, d_4, b) = (2, -0.186, -1, -1, -1). \tag{5.6}$$

The KS-equation (5.3) has boost (or Galilean) invariance: if $\alpha(t,x)$ solves (5.3) then $\beta(t,x) = \alpha(t,x) + c$ solves $\partial_t\beta = (bc\partial_x + d_2\partial_x^2 + d_3\partial_x^3 - d_4\partial_x^4)\beta - b\partial_x(\beta^2)$. Therefore the amplitude α for the approximation u_α can be calculated in two different ways: First we may set $\alpha(0,x) = u(0,x)$ and integrate (5.3). This gives the dotted line in Fig. 1b), while Fig.3a) compares the solutions $u(100,x)$ and $\alpha(100,x)$ thus obtained.

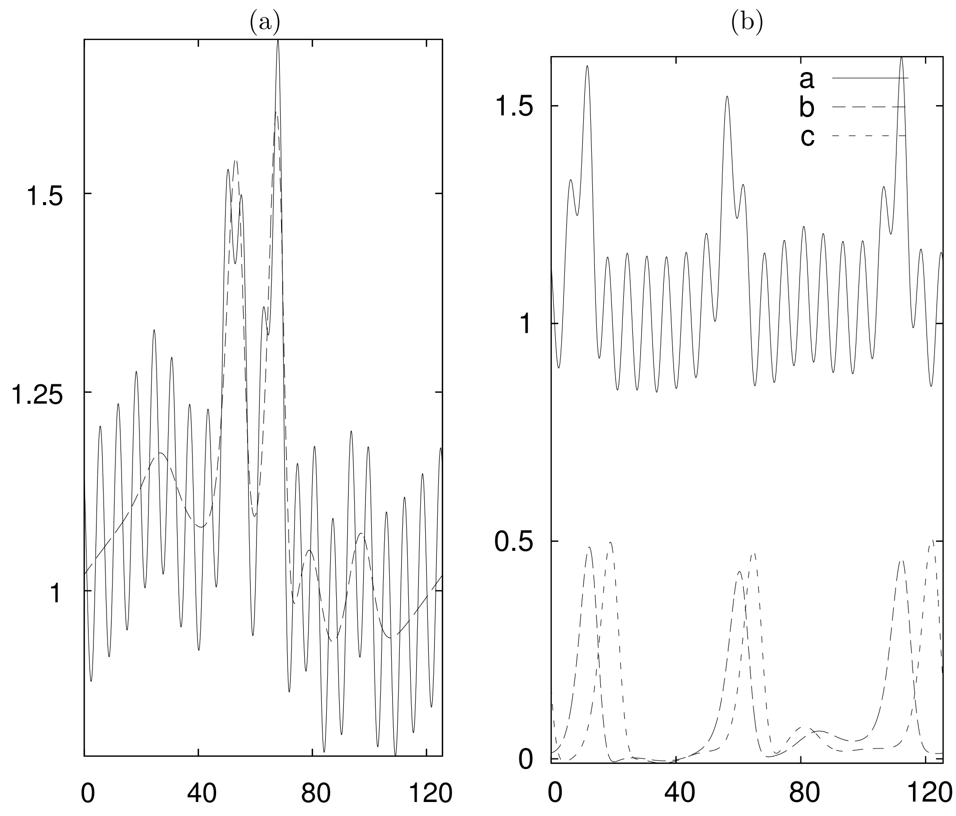

FIGURE 3. (a) the numerical solutions of (1.1) (full line) and (5.3) (dotted line) from Fig.1b) at $t = 100$. (b) the solution of (1.1) (a), of (5.4) with $\alpha(0,x) = u(0,x) - 1$ (b), and of (5.4) with $d_2 = -0.2$ (c), all at $t = 500$.

Equivalently, but more in the spirit of amplitude equations, we may set $\alpha(0,x) = u(0,x)-1$ such that α represents the amplitude of the perturbation, and integrate (5.4). The result at t=500 is shown in Fig. 3b) (curve b), together with $u|_{t=500}$ (curve a) and finally compared with the solution of (5.3) with $d_2 = -c_2 = -0.2$ (curve c). This last solution corresponds to simply setting $\delta = 0$ in (1.1). Clearly, $d_2 = -0.186$ gives a much better approximation. The reason is that $d_2{=}-0.2$ gives a stronger instability than the "effective instability" with $d_2{=}-0.186$; therefore the humps in the amplitude curve c are larger and hence travel faster than in curve b, which over large times in particular leads to the incorrect shift in curve c. Similar results were obtained in all our simulations which covered a variety of parameter-regimes and initial conditions, in particular also for larger δ.

In summary, we see that the formal derivation of (5.3) gives a useful amplitude equation, in contrast to just setting $\delta = 0$ in (1.1) which can be seen as a (very) naive averaging. Whether (5.3) allows to show interesting rigorous results for (1.1) in the unstable case remains to be seen.

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