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STOCHASTIC NEWTONIAN EQUATIONS WITH MEAN BOUNDARY CONDITIONS

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Abstract. This article concerns stochastic Newtonian equations driven by the white noise with mean boundary conditions. We obtain sufficient conditions of the existence and uniqueness of solutions, and then solutions are adapted to the Brownian filtration. As applications, we show the existence and uniqueness of several stochastic differential equations with mean boundary conditions.

1. Introduction

We consider the stochastic Newtonian system

$$\frac{d^2X_t}{dt^2} = f(t, X_t) + g(t, X_t)\frac{dB_t}{dt}, \tag{1.1}$$

with mean boundary conditions

$$X(a) = 0, \quad \mathbf{E}X(b) = 0, \tag{1.2}$$

where the time parameter t runs on the interval $[a, b]$ $(b > a \geq 0)$. The drift term $f : [a, b]\times\mathbb{R}^l \to \mathbb{R}^l$ and the diffusion term $g : [a, b]\times\mathbb{R}^l \to \mathbb{R}^{l\times m}$ are continuous. $\{B_t\}$ is a standard m-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

In classical dynamics, the Newtonian equation

$$\frac{d^2X_t}{dt^2} = f(t, X_t), \tag{1.3}$$

is usually used to describe the effects of external forces. If the frictions are periodic with respect to time, then such systems can describe the behavior of a large class of oscillation processes such as the motion of spring pendulum or oscillating circuits.

Dirichlet boundary conditions are one of the basic boundary conditions appearing in mathematical physics. For deterministic boundary value problem with Dirichlet boundary condition (BVP)

$$\begin{gathered} y'' + f(t, y, y') = 0, \\ y(a) = A, \quad y(b) = B, \end{gathered} \tag{1.4}$$

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where $A, B \in \mathbb{R}^l$, $f : [a,b] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}^l$, one can utilize Green's function in reformulating (1.4) as an integral equation. By studying the properties of Green's function and the nonlinear terms, some conditions concerning the existence and uniqueness of the solutions can be obtained, see [4, 17, 18].

However, some mechanical processes in real world are always effected by unknown randomness, for instance breeze or inhomogeneous medium. In the study of random dynamical systems, these randomness is often modeled by white noise and the mechanical processes can be described as stochastic Newtonian system (1.1).

Stochastic boundary value problems (SBVPs) appear naturally in a variety of fields such as stochastic optimal control [27], valuation of boundary-linked assets [11], smoothing [25], the study of reciprocal processes [1] and maximum a posteriori estimation of trajectories of diffusions [26]. There has been a lot of literature on stochastic boundary value problems, see for example [2, 3, 5, 8, 9, 10, 20, 21, 22, 23, 24]. Nualart and Pardoux [21] studied a second-order stochastic differential equation

$$\frac{d^2X_t}{dt^2} + f\Big(X_t, \frac{dX_t}{dt}\Big) = \frac{dW_t}{dt}, \quad t \in [0,1], \tag{1.5}$$

subject to the Dirichlet type boundary condition $X_0 = a$ and $X_1 = b$, where a and b are fixed real numbers. Here $\{W_t\}$ is a one-dimensional Brownian motion starting at zero. They proved that pathwise existence and uniqueness of solutions assuming some smoothness and monotonicity conditions on f, and they also studied the Markov property of solutions using an extended version of the Girsanov theorem due to Kusuoka [16].

However, as far as we know, for the study of stochastic differential equations with boundary conditions, the solution process is usually not adapted to the Brownian filtration because of the deterministic boundary value conditions. Naturally, one asks that what type of boundary conditions would make the solution X_t of stochastic Newtonian system (1.1) adapt to the Brownian filtration.

We consider that boundary conditions should be uncertain and random due to noise. In this paper, we propose a kind of boundary conditions (1.2) with expectation at the end of the interval. Boundary conditions with expectation represent that the start point is fixed and the end point is random. In fact, such boundary value conditions are more natural than fixed ones. For instance, if a small ball in simple harmonic motion is affected by random perturbations, then one cannot forecast the location of the ball in stochastic systems. However, it is rational to give a "possible location" such as expectation at a time t in a statistical twist. Moreover, we verify that the process $\{X_t, a \leq t \leq b\}$ is an adapted process to Brownian filtration and the problem mentioned above is well-posed. These phenomena also lend support to that it is appropriate to consider mean boundary conditions in stochastic systems.

For Dirichlet boundary value problem of ordinary differential equations, suitable Lipschitz condition can guarantee the existence and uniqueness of solutions. However, for stochastic Newtonian system (1.1) with mean boundary conditions (1.2), what are sufficient conditions for the existence and uniqueness of solutions? This is exactly what we do in this present paper. In this paper, we give a new idea to study the above problems. We divide stochastic differential equation (1.1) into the deterministic part and the stochastic part. By constructing stochastic Picard sequences [12] for the stochastic part, we prove the convergence of numerical integrations in

$\mathcal{L}^2(\mathbf{P}, \mathbb{R}^l)$. We get some explicit conditions to obtain the existence and uniqueness of solutions for stochastic Newtonian systems with mean boundary conditions.

The dynamics of stochastic differential equations such as the well-posedness of periodic solutions has been the subject of much concerns recently. By finding new analysis tools or technology, the difficulties of the randomness on the orbit of stochastic system and the non-compactness of the space for stochastic processes are surmounted effectively. For example, Liu and Wang [19] obtained the existence of stochastic almost periodic solutions in distribution by Favard separation method. Chen et al [7] gave a weak Halanay's criterion and proved the existence of periodic solutions in distribution for SDEs. Ji et al [13] studied existence of periodic solutions in distribution for stochastic differential equations with irregular coefficients. Jiang and Li [15] studied the existence of periodic solutions in distribution of dissipative stochastic differential equations via the Wong-Zakai approximation method. They also verified a stochastic Levinson type conjecture in [14]. Under some assumptions on the coefficients, the existence of periodic solutions to semilinear SDEs has been established, see [6].

This article is organized as follows. In Section 2, we review some concepts, introduce some notation and state our main result, which shows the existence and uniqueness of solutions for system (1.1). In Section 3, we give the proof of the main result by constructing the stochastic Picard sequences. In particular, we obtain some explicit sufficient conditions of the existence and uniqueness of solutions. In Section 4, some examples are given to illustrate the theoretical results.

2. Preliminaries and main result

Throughout this paper, we assume that $(\Omega, \mathcal{F}, \mathbf{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $\mathcal{L}^2(\mathbf{P}, \mathbb{R}^l)$ stand for the space of all $\mathbb{R}^l$-valued random variables X such that

$$\mathbf{E}|X|^2 = \int_\Omega |X|^2 \, d\mathbf{P} < \infty.$$

For $X \in \mathcal{L}^2(\mathbf{P}, \mathbb{R}^l)$, we let

$$\|X\|_2 := \Big(\int_\Omega |X|^2 \, d\mathbf{P} \Big)^{1/2}.$$

Then $\mathcal{L}^2(\mathbf{P}, \mathbb{R}^l)$ is a Hilbert space equipped with the norm $\|\cdot\|_2$.

For the stochastic Newtonian equation (1.1) with mean boundary conditions (1.2), we have the following result. It shows that under some sufficient conditions, there exists a unique solution.

Theorem 2.1. *Assume that there exist positive constants L_1, L_2 and K such that*

(i) *(Lipschitz condition) for all $x, y \in \mathbb{R}^l$ and $t \in [a, b]$,*
$$|f(t,x) - f(t,y)| \leq L_1|x-y|, \quad |g(t,x) - g(t,y)| \leq L_2|x-y|;$$
(ii) *(Linear growth condition) for all $(t,x) \in [a,b] \times \mathbb{R}^l$,*
$$|f(t,x)| \vee |g(t,x)| \leq K(1+|x|);$$
(iii) *Lipschitz constants L_1 and L_2 satisfy*
$$(12 + \frac{1}{64})L_1^2(b-a)^4 + 3L_2^2(b-a)^3 < 1.$$

Then stochastic Newtonian system (1.1) *with mean boundary conditions* $X(a) = 0$ *and* $\mathbf{E}X(b) = 0$ *has a unique continuous solution* $X_t(\omega)$ *belonging to* $\mathcal{L}^2(\mathbf{P}, \mathbb{R}^l)$*, and* $X_t(\omega)$ *is adapted to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$.

Considering the influence of white noise for stochastic Newtonian system (1.1), the boundary conditions are uncertain and random. So we impose the boundary conditions with expectation at the end of the interval $t = b$. Here $X(a) = 0$ is deterministic.

The solution $\{X_t, a \leq t \leq b\}$ of stochastic Newtonian system (1.1) with mean boundary conditions (1.2) is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

If the mean boundary conditions (1.2) are generalized as

$$X(a) = A, \quad \mathbf{E}X(b) = B,$$

where A and B are arbitrary constants, then we can translate the variable X_t, and yields

$$X_t = Z_t + u(t), \quad u(t) = A + \frac{B - A}{b - a}(t - a).$$

Hence, $Z(t)$ satisfies mean boundary conditions $Z(a) = 0, \mathbf{E}Z(b) = 0$.

3. Proof of main results

Proof of Theorem 2.1. First of all, we rewrite stochastic Newtonian system (1.1) as

$$\mathbf{E}\ddot{X}(t) + \ddot{X}(t) - \mathbf{E}\ddot{X}(t) = f(t, \mathbf{E}X_t) + f(t, X_t) - f(t, \mathbf{E}X_t) + g(t, X_t)\dot{B}_t. \tag{3.1}$$

By the change of variables

$$\bar{x}(t) = \mathbf{E}X(t), \quad y(t) = X(t) - \mathbf{E}X(t),$$

system (3.1) can be written as

$$\ddot{\bar{x}}(t) + \ddot{y}(t) = f(t, \bar{x}(t)) + f(t, \bar{x}(t) + y(t)) - f(t, \bar{x}(t)) + g(t, \bar{x}(t) + y(t))\dot{B}_t. \tag{3.2}$$

We divide equation (3.2) into two parts, the deterministic system and the stochastic system. Here the deterministic system is

$$\ddot{\bar{x}}(t) = f(t, \bar{x}(t)), \tag{3.3}$$

and the stochastic system is

$$\ddot{y}(t) = f(t, \bar{x}(t) + y(t)) - f(t, \bar{x}(t)) + g(t, \bar{x}(t) + y(t))\dot{B}_t. \tag{3.4}$$

We divide the proof into 5 steps. First, using the Green's function, we study the existence of solutions of the deterministic system. Then we construct the stochastic Picard iterative sequences and studying their convergence in $\mathcal{L}^2(\mathbf{P}, \mathbb{R}^l)$, which shows the existence of solutions to the stochastic system. Third, we prove the existence of solutions of system (1.1). Fourth, the uniqueness of solutions of system (1.1) is obtained by studying the uniqueness of the solutions of the deterministic system and the stochastic system, respectively. Finally, we show that the solutions of system (1.1) is adapted to the filtration.

Step 1: Existence of solutions of the deterministic system. We consider the deterministic second-order boundary value problem

$$\begin{gathered} \ddot{\bar{x}} = f(t, \bar{x}(t)), \\ \bar{x}(a) = 0, \quad \bar{x}(b) = 0, \end{gathered} \tag{3.5}$$

where $f : [a, b] \times \mathbb{R}^l \to \mathbb{R}^l$ is a continuous function. It is easy to check that

$$\bar{x}(t) = \int_a^b G(t, s) f(s, \bar{x}(s)) ds, \quad t \in [a, b], \tag{3.6}$$

where $G : [a, b] \times [a, b] \to R$ is the Green's function

$$G(t, s) = \begin{cases} -\frac{(b-t)(s-a)}{b-a}, & a \le s \le t \le b, \\ -\frac{(b-s)(t-a)}{b-a}, & a \le t \le s \le b. \end{cases} \tag{3.7}$$

Obviously, $G(t, s)$ is non-positive and

$$\frac{a-b}{4} \le G(t, s) \le 0.$$

By the estimate for $G(t, s)$, we have

$$\begin{aligned} \int_a^b |G(t, s)| ds &= \int_a^t \frac{(b-t)(s-a)}{b-a} ds + \int_t^b \frac{(b-s)(t-a)}{b-a} ds \\ &= \frac{b-t}{b-a} \int_a^t (s-a) ds + \frac{t-a}{b-a} \int_t^b (b-s) ds \\ &= \frac{(b-t)(t-a)}{2} \\ &\le \frac{(b-a)^2}{8}. \end{aligned} \tag{3.8}$$

We define a Picard iterative sequences $\{\bar{x}_n(t)\}$ as follows

$$\begin{aligned} &\bar{x}_0(t) = 0, \\ &\bar{x}_n(t) = \int_a^b G(t, s) f(s, \bar{x}_{n-1}(s)) ds, \quad n = 1, 2, \dots, \end{aligned} \tag{3.9}$$

for $t \in [a, b]$. Here $\bar{x}_n(t) \in C([a, b]; \mathbb{R}^l)$, and there exists a constant $M > 0$ such that $|f(t, \bar{x}(t))| \le M$ for all $t \in [a, b]$. Note that

$$\begin{aligned} |\bar{x}_1(t) - \bar{x}_0(t)| &= \Big| \int_a^b G(t, s) f(s, \bar{x}_0(s)) ds \Big| \le \frac{M(b-a)^2}{8}, \\ |\bar{x}_2(t) - \bar{x}_1(t)| &= \Big| \int_a^b G(t, s) f(s, \bar{x}_1(s)) ds - \int_a^b G(t, s) f(s, \bar{x}_0(s)) ds \Big| \\ &\le M L_1 \big[\frac{(b-a)^2}{8} \big]^2. \end{aligned}$$

Suppose that

$$|\bar{x}_n(t) - \bar{x}_{n-1}(t)| \le M L_1^{n-1} \big[\frac{(b-a)^2}{8} \big]^n.$$

Then

$$\begin{aligned} |\bar{x}_{n+1}(t)-\bar{x}_n(t)| &= \Big|\int_a^b G(t,s)f(s,\bar{x}_n(s))ds-\int_a^b G(t,s)f(s,\bar{x}_{n-1}(s))ds\Big| \\ &\le L_1\Big|\int_a^b G(t,s)[\bar{x}_n(s)-\bar{x}_{n-1}(s)]ds\Big| \\ &\le ML_1^n\big[\frac{(b-a)^2}{8}\big]^n\Big|\int_a^b G(t,s)ds\Big| \\ &\le \frac{M(b-a)^2}{8}\big[\frac{L_1(b-a)^2}{8}\big]^n. \end{aligned} \tag{3.10}$$

Setting the condition $\frac{L_1(b-a)^2}{8}<1$, the partial sums

$$\bar{x}_n(t)=\bar{x}_0(t)+\sum_{i=0}^{n-1}[\bar{x}_{i+1}(t)-\bar{x}_i(t)]$$

converge uniformly in $t\in[a,b]$. Let $\bar{x}_n(t)\to\bar{x}(t)$ as $n\to\infty$. Then $\bar{x}(t)\in C([a,b];\mathbb{R}^l)$.

It remains to show that $\bar{x}(t)$ satisfies (3.6). Note that, for all $\varepsilon>0$,

$$\begin{aligned} &\Big|\int_a^b G(t,s)f(s,\bar{x}_n(s))ds-\int_a^b G(t,s)f(s,\bar{x}(s))ds\Big| \\ &\le L_1\Big|\int_a^b G(t,s)|\bar{x}_n(s)-\bar{x}(s)|ds\Big| \\ &\le L_1\max_{a\le t\le b}|\bar{x}_n(s)-\bar{x}(s)|\Big|\int_a^b G(t,s)ds\Big| \\ &< \frac{L_1(b-a)^2}{8}\varepsilon. \end{aligned} \tag{3.11}$$

Letting $n\to\infty$ in (3.9), we obtain that

$$\bar{x}(t)=\int_a^b G(t,s)f(s,\bar{x}(s))ds, \tag{3.12}$$

on $a\le t\le b$. Then $\bar{x}(t)$ is the solution of the deterministic system (3.5).

Step 2. Existence of solutions of the stochastic system. We consider the stochastic differential equation with mean boundary conditions

$$\begin{gathered} \ddot{y}(t)=f(t,\bar{x}(t)+y(t))-f(t,\bar{x}(t))+g(t,\bar{x}(t)+y(t))\dot{B}(t), \\ y(a)=0,\quad \mathbf{E}y(b)=0. \end{gathered} \tag{3.13}$$

By a change of variables, equation (3.13) can be written as

$$\dot{y}(t)=Y(t), \tag{3.14}$$

$$\dot{Y}(t)=f(t,\bar{x}(t)+y(t))-f(t,\bar{x}(t))+g(t,\bar{x}(t)+y(t))\dot{B}(t). \tag{3.15}$$

Integrating on both sides of (3.14) and (3.15) from a to t, $t\in[a,b]$, we have

$$y(t)=y(a)+\int_a^t Y(s)ds=\int_a^t Y(s)ds, \tag{3.16}$$

and

$$
\begin{aligned}
Y(t) = Y(a) &+ \int_a^t [f(s, \bar{x}(s) + y(s)) - f(s, \bar{x}(s))]ds \\
&+ \int_a^t g(s, \bar{x}(s) + y(s))dB(s).
\end{aligned} \tag{3.17}
$$

Taking expectations on both sides of $y(t)$, we have

$$
\begin{aligned}
\mathbf{E}y(t) &= \mathbf{E}\int_a^t Y(s)ds \\
&= \mathbf{E}\int_a^t \Big\{Y(a) + \int_a^s [f(u, \bar{x}(u) + y(u)) - f(u, \bar{x}(u))]du \\
&\quad + \int_a^s g(u, \bar{x}(u) + y(u))dB(u)\Big\}ds.
\end{aligned} \tag{3.18}
$$

Then

$$
\begin{aligned}
0 = \mathbf{E}y(b) &= \mathbf{E}\int_a^b Y(s)ds \\
&= \int_a^b Y(a)dt + \int_a^b ds \int_a^s \mathbf{E}[f(u, \bar{x}(u) + y(u)) - f(u, \bar{x}(u))]du \\
&\quad + \int_a^b ds\mathbf{E}\int_a^s g(u, \bar{x}(u) + y(u))dB(u) \\
&= Y(a)(b-a) + \int_a^b ds \int_a^s \mathbf{E}[f(u, \bar{x}(u) + y(u)) - f(u, \bar{x}(u))]du.
\end{aligned} \tag{3.19}
$$

We obtain

$$
Y(a) = -\frac{1}{b-a}\int_a^b ds \int_a^s \mathbf{E}[f(u, \bar{x}(u) + y(u)) - f(u, \bar{x}(u))]du. \tag{3.20}
$$

Hence, $y(t)$ can be represented as

$$
\begin{aligned}
y(t) &= \int_a^t Y(s)ds \\
&= -\frac{t-a}{b-a}\int_a^b ds \int_a^s \mathbf{E}[f(u, \bar{x}(u) + y(u)) - f(u, \bar{x}(u))]du \\
&\quad + \int_a^t ds \int_a^s [f(u, \bar{x}(u) + y(u)) - f(u, \bar{x}(u))]du \\
&\quad + \int_a^t ds \int_a^s g(u, \bar{x}(u) + y(u))dB(u).
\end{aligned} \tag{3.21}
$$

We define the stochastic Picard iterative sequences as follows: $y_0(t) = 0$ and

$$
\begin{aligned}
y_n(t) = &-\frac{t-a}{b-a}\int_a^b ds \int_a^s \mathbf{E}[f(u, \bar{x}_{n-1}(u) + y_{n-1}(u)) - f(u, \bar{x}_{n-1}(u))]du \\
&+ \int_a^t ds \int_a^s [f(u, \bar{x}_{n-1}(u) + y_{n-1}(u)) - f(u, \bar{x}_{n-1}(u))]du \\
&+ \int_a^t ds \int_a^s g(u, \bar{x}_{n-1}(u) + y_{n-1}(u))dB(u),
\end{aligned} \tag{3.22}
$$

for $t \in [a,b]$ and $n = 1, 2, \dots$. Note that

$$
\begin{aligned}
&\mathbf{E}|y_1(t) - y_0(t)|^2 \\
&= \mathbf{E}\Big| -\frac{t-a}{b-a}\int_a^b ds \int_a^s \mathbf{E}[f(u,\bar{x}_0(u) + y_0(u)) - f(u,\bar{x}_0(u))]du \\
&\quad + \int_a^t ds \int_a^s [f(u,\bar{x}_0(u) + y_0(u)) - f(u,\bar{x}_0(u))]du \\
&\quad + \int_a^t ds \int_a^s g(u,\bar{x}_0(u) + y_0(u))dB(u)\Big|^2 \\
&= \mathbf{E}\Big| \int_a^t ds \int_a^s g(u,0)dB(u)\Big|^2 \\
&\le (t-a)\int_a^t \mathbf{E}\Big|\int_a^s g(u,0)dB(u)\Big|^2 ds.
\end{aligned}
\tag{3.23}
$$

By Itô isometry formula and linear growth condition (ii), we obtain

$$
\begin{aligned}
\mathbf{E}|y_1(t) - y_0(t)|^2 &\le (t-a)\int_a^t \mathbf{E}\int_a^s |g(u,0)|^2\, du\, ds \\
&\le (t-a)\int_a^t K^2(s-a)ds \\
&\le \frac{K^2}{2}(b-a)^3.
\end{aligned}
$$

Next, for $n \ge 1$ and $t \in [a,b]$, we have

$$
\begin{aligned}
&\mathbf{E}|y_{n+1}(t) - y_n(t)|^2 \\
&= \mathbf{E}\Big| -\frac{t-a}{b-a}\int_a^b ds \int_a^s \mathbf{E}[f(u,\bar{x}_n(u) + y_n(u)) - f(u,\bar{x}_n(u))]du \\
&\quad + \int_a^t ds \int_a^s [f(u,\bar{x}_n(u) + y_n(u)) - f(u,\bar{x}_n(u))]du \\
&\quad + \int_a^t ds \int_a^s g(u,\bar{x}_n(u) + y_n(u))dB(u) \\
&\quad + \frac{t-a}{b-a}\int_a^b ds \int_a^s \mathbf{E}[f(u,\bar{x}_{n-1}(u) + y_{n-1}(u)) - f(u,\bar{x}_{n-1}(u))]du \\
&\quad - \int_a^t ds \int_a^s [f(u,\bar{x}_{n-1}(u) + y_{n-1}(u)) - f(u,\bar{x}_{n-1}(u))]du \\
&\quad - \int_a^t ds \int_a^s g(u,\bar{x}_{n-1}(u) + y_{n-1}(u))dB(u)\Big|^2 \\
&= \mathbf{E}\Big| -\frac{t-a}{b-a}\int_a^b ds \int_a^s \big\{\mathbf{E}[f(u,\bar{x}_n(u) + y_n(u)) - f(u,\bar{x}_{n-1}(u) + y_{n-1}(u))] \\
&\quad - \mathbf{E}[f(u,\bar{x}_n(u)) - f(u,\bar{x}_{n-1}(u))]\big\}du \\
&\quad + \int_a^t ds \int_a^s \big\{[f(u,\bar{x}_n(u) + y_n(u)) - f(u,\bar{x}_{n-1}(u) + y_{n-1}(u))] \\
&\quad - [f(u,\bar{x}_n(u)) - f(u,\bar{x}_{n-1}(u))]\big\}du
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_a^t ds \int_a^s [g(u, \bar{x}_n(u) + y_n(u)) - g(u, \bar{x}_{n-1}(u) + y_{n-1}(u))] dB(u)\Big|^2 \\
&\le 3\Big|\frac{t-a}{b-a}\int_a^b ds \int_a^s \Big\{\mathbf{E}[f(u, \bar{x}_n(u) + y_n(u)) - f(u, \bar{x}_{n-1}(u) + y_{n-1}(u))] \\
&- \mathbf{E}[f(u, \bar{x}_n(u) - f(u, \bar{x}_{n-1}(u))]\Big\} du\Big|^2 \\
&+ 3\mathbf{E}\Big|\int_a^t ds \int_a^s \{[f(u, \bar{x}_n(u) + y_n(u)) - f(u, \bar{x}_{n-1}(u) + y_{n-1}(u))] \\
&- [f(u, \bar{x}_n(u)) - f(u, \bar{x}_{n-1}(u))]\} du\Big|^2 \\
&+ 3\mathbf{E}\Big|\int_a^t ds \int_a^s [g(u, \bar{x}_n(u) + y_n(u)) - g(u, \bar{x}_{n-1}(u) + y_{n-1}(u))] dB(u)\Big|^2.
\end{aligned}
$$

Applying Hölder's inequality and Itô isometry formula, we have

$$
\begin{aligned}
&\mathbf{E}|y_{n+1}(t) - y_n(t)|^2 \\
&\le 3(\frac{t-a}{b-a})^2(b-a)\int_a^b \Big|\int_a^s \Big\{\mathbf{E}[f(u, \bar{x}_n(u) + y_n(u)) - f(u, \bar{x}_{n-1}(u) + y_{n-1}(u))] \\
&\quad - \mathbf{E}[f(u, \bar{x}_n(u) - f(u, \bar{x}_{n-1}(u))]\Big\} du\Big|^2 ds \\
&\quad + 3(t-a)\int_a^t \mathbf{E}\Big|\int_a^s \Big\{[f(u, \bar{x}_n(u) + y_n(u)) - f(u, \bar{x}_{n-1}(u) + y_{n-1}(u))] \\
&\quad - [f(u, \bar{x}_n(u)) - f(u, \bar{x}_{n-1}(u))]\Big\} du\Big|^2 ds \\
&\quad + 3(t-a)\int_a^t \mathbf{E}\Big|\int_a^s [g(u, \bar{x}_n(u) + y_n(u)) - g(u, \bar{x}_{n-1}(u) + y_{n-1}(u))] dB(u)\Big|^2 ds \\
&\le \frac{3(t-a)^2}{b-a}\int_a^b (s-a)\int_a^s |\mathbf{E}[f(u, \bar{x}_n(u) + y_n(u)) - f(u, \bar{x}_{n-1}(u) + y_{n-1}(u))] \\
&\quad - \mathbf{E}[f(u, \bar{x}_n(u) - f(u, \bar{x}_{n-1}(u))]|^2\, du\, ds \\
&\quad + 3(t-a)\int_a^t (s-a)\int_a^s \mathbf{E}|[f(u, \bar{x}_n(u) + y_n(u)) - f(u, \bar{x}_{n-1}(u) + y_{n-1}(u))] \\
&\quad - [f(u, \bar{x}_n(u) - f(u, \bar{x}_{n-1}(u))]\,|^2\, du\, ds \\
&\quad + 3(t-a)\int_a^t \int_a^s \mathbf{E}|g(u, \bar{x}_n(u) + y_n(u)) - g(u, \bar{x}_{n-1}(u) + y_{n-1}(u))|^2\, du\, ds \\
&\le \frac{6(t-a)^2}{b-a}\int_a^b (s-a)\int_a^s \Big[\mathbf{E}|f(u, \bar{x}_n(u) + y_n(u)) - f(u, \bar{x}_{n-1}(u) + y_{n-1}(u))|^2 \\
&\quad + \mathbf{E}|f(u, \bar{x}_n(u) - f(u, \bar{x}_{n-1}(u))|^2\Big]\, du\, ds \\
&\quad + 6(t-a)\int_a^t (s-a)\int_a^s \Big[\mathbf{E}|f(u, \bar{x}_n(u) + y_n(u)) - f(u, \bar{x}_{n-1}(u) + y_{n-1}(u))|^2 \\
&\quad + \mathbf{E}|f(u, \bar{x}_n(u) - f(u, \bar{x}_{n-1}(u))|^2\Big]\, du\, ds \\
&\quad + 3L_2^2(t-a)\int_a^t \int_a^s \mathbf{E}|\bar{x}_n(u) + y_n(u) - \bar{x}_{n-1}(u) - y_{n-1}(u)|^2\, du\, ds.
\end{aligned}
$$

By Lipchitz condition (i), we obtain

$$
\begin{aligned}
&\mathbf{E}|y_{n+1}(t)-y_n(t)|^2\\
&\le \frac{6L_1^2(t-a)^2}{b-a}\int_a^b(s-a)\int_a^s\Big[\mathbf{E}|\bar{x}_n(u)+y_n(u)-\bar{x}_{n-1}(u)-y_{n-1}(u)|^2\\
&\quad+|\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2\Big]\,du\,ds\\
&\quad+6L_1^2(t-a)\int_a^t(s-a)\int_a^s\Big[\mathbf{E}|\bar{x}_n(u)+y_n(u)-\bar{x}_{n-1}(u)-y_{n-1}(u)|^2\\
&\quad+|\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2\Big]\,du\,ds\\
&\quad+6L_2^2(t-a)\int_a^t\int_a^s\big[|\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2+\mathbf{E}|y_n(u)-y_{n-1}(u)|^2\big]\,du\,ds\\
&\le \frac{6L_1^2(t-a)^2}{b-a}\int_a^b(s-a)\int_a^s[3|\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2+2\mathbf{E}|y_n(u)-y_{n-1}(u)|^2\Big]\,du\,ds\\
&\quad+6L_1^2(t-a)\int_a^t(s-a)\int_a^s\Big[3|\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2+2\mathbf{E}|y_n(u)-y_{n-1}(u)|^2\Big]\,du\,ds\\
&\quad+6L_2^2(t-a)\int_a^t\int_a^s\Big[|\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2+\mathbf{E}|y_n(u)-y_{n-1}(u)|^2\Big]\,du\,ds\\
&\le 18L_1^2(t-a)\int_a^b(s-a)\int_a^s|\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2\,du\,ds\\
&\quad+18L_1^2(t-a)\int_a^t(s-a)\int_a^s|\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2\,du\,ds\\
&\quad+6L_2^2(t-a)\int_a^t\int_a^s|\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2\,du\,ds\\
&\quad+12L_1^2(t-a)\int_a^b(s-a)\int_a^s\mathbf{E}|y_n(u)-y_{n-1}(u)|^2\,du\,ds\\
&\quad+12L_1^2(t-a)\int_a^t(s-a)\int_a^s\mathbf{E}|y_n(u)-y_{n-1}(u)|^2\,du\,ds\\
&\quad+6L_2^2(t-a)\int_a^t\int_a^s\mathbf{E}|y_n(u)-y_{n-1}(u)|^2\,du\,ds.
\end{aligned}
$$

Note that

$$
|\bar{x}_n(t)-\bar{x}_{n-1}(t)|^2\le\frac{M^2(b-a)^4}{64}\cdot\big[\frac{L_1(b-a)^2}{8}\big]^{2n-2}:=\frac{M^2(b-a)^4}{64}q^{2n-2},\tag{3.24}
$$

and $q=\frac{L_1(b-a)^2}{8}<1$.

Then we have the following estimates:

$$
\begin{aligned}
&18L_1^2(t-a)\int_a^b(s-a)\int_a^s|\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2\,du\,ds\\
&\le\frac{9L_1^2M^2}{32}(b-a)^4(t-a)q^{2n-2}\int_a^b(s-a)^2ds\\
&=\frac{3L_1^2M^2}{32}(b-a)^8q^{2n-2},
\end{aligned}
$$

$$6L_2^2(t-a)\int_a^t\int_a^s |\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2\,du\,ds \le \frac{3L_2^2M^2}{64}(b-a)^7q^{2n-2}.$$

Then

$$\begin{aligned}
&18L_1^2(t-a)\int_a^b(s-a)\int_a^s |\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2\,du\,ds\\
&+18L_1^2(t-a)\int_a^t(s-a)\int_a^s |\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2\,du\,ds\\
&+6L_2^2(t-a)\int_a^t\int_a^s |\bar{x}_n(u)-\bar{x}_{n-1}(u)|^2\,du\,ds\\
&\le \big[\frac{3L_1^2(b-a)^8}{16}+\frac{3L_2^2(b-a)^7}{64}\big]M^2q^{2n-2} := M_1q^{2n-2},
\end{aligned}$$

where

$$M_1=\big[\frac{3L_1^2(b-a)^8}{16}+\frac{3L_2^2(b-a)^7}{64}\big]M^2.$$

Hence, for $n\ge 1$, we obtain that

$$\begin{aligned}
&\mathbf{E}|y_{n+1}(t)-y_n(t)|^2\\
&\le M_1q^{2n-2}+12L_1^2(t-a)\int_a^b(b-a)\int_a^b \mathbf{E}|y_n(u)-y_{n-1}(u)|^2\,du\,ds\\
&\quad+12L_1^2(t-a)\int_a^b(b-a)\int_a^b \mathbf{E}|y_n(u)-y_{n-1}(u)|^2\,du\,ds\\
&\quad+6L_2^2(t-a)\int_a^b\int_a^b \mathbf{E}|y_n(u)-y_{n-1}(u)|^2\,du\,ds\\
&= M_1q^{2n-2}+24L_1^2(b-a)^2(t-a)\int_a^b \mathbf{E}|y_n(u)-y_{n-1}(u)|^2du\\
&\quad+6L_2^2(t-a)(b-a)\int_a^b \mathbf{E}|y_n(u)-y_{n-1}(u)|^2du\\
&= M_1q^{2n-2}+\Big[24L_1^2(b-a)^2+6L_2^2(b-a)\Big](t-a)\int_a^b \mathbf{E}|y_n(u)-y_{n-1}(u)|^2du\\
&:= M_1q^{2n-2}+M_2(t-a)\int_a^b \mathbf{E}|y_n(u)-y_{n-1}(u)|^2du,
\end{aligned}$$

where $M_2=24L_1^2(b-a)^2+6L_2^2(b-a)$.

Moreover, we have

$$\begin{aligned}
\mathbf{E}|y_2(t)-y_1(t)|^2 &\le M_1q^0+M_2(t-a)\int_a^b\frac{K^2}{2}(b-a)^3du\\
&= M_1q^0+\frac{M_2K^2(b-a)^4}{2}(t-a),\\
\mathbf{E}|y_3(t)-y_2(t)|^2 &\le M_1q^2+M_2(t-a)\int_a^b\Big[M_1q^0+\frac{M_2K^2(b-a)^4}{2}(u-a)\Big]du\\
&= M_1q^2+M_2(t-a)\Big[M_1q^0(b-a)+\frac{M_2K^2(b-a)^6}{2^2}\Big],
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}|y_4(t)-y_3(t)|^2 &\le M_1q^4+M_2(t-a)\int_a^b\Big\{M_1q^2+M_2\Big[M_1q^0(b-a)\\
&\quad+\frac{M_2K^2(b-a)^6}{2^2}\Big](u-a)\Big\}du\\
&=M_1q^4+M_2(t-a)\Big[M_1q^2(b-a)\\
&\quad+\frac{M_2M_1q^0(b-a)^3}{2}+\frac{M_2^2K^2(b-a)^8}{2^3}\Big],\\
\mathbf{E}|y_5(t)-y_4(t)|^2 &\le M_1q^6+M_2(t-a)\int_a^b\Big\{M_1q^4+M_2\Big[M_1q^2(b-a)\\
&\quad+\frac{M_2M_1q^0(b-a)^3}{2}+\frac{M_2^2K^2(b-a)^8}{2^3}\Big](u-a)\Big\}du\\
&=M_1q^6+M_2(t-a)\Big[M_1q^4(b-a)+\frac{M_2M_1q^2(b-a)^3}{2}\\
&\quad+\frac{M_2^2M_1q^0(b-a)^5}{2^2}+\frac{M_2^3K^2(b-a)^{10}}{2^4}\Big],\\
\mathbf{E}|y_6(t)-y_5(t)|^2 &\le M_1q^8+M_2(t-a)\int_a^b\Big\{M_1q^6+M_2\Big[M_1q^4(b-a)\\
&\quad+\frac{M_2M_1q^2(b-a)^3}{2}+\frac{M_2^2M_1q^0(b-a)^5}{2^2}\\
&\quad+\frac{M_2^3K^2(b-a)^{10}}{2^4}\Big](u-a)\Big\}du\\
&=M_1q^8+M_2(t-a)\Big[M_1q^6(b-a)+\frac{M_2M_1q^4(b-a)^3}{2}\\
&\quad+\frac{M_2^2M_1q^2(b-a)^5}{2^2}+\frac{M_2^3M_1(b-a)^7}{2^3}+\frac{M_2^4K^2(b-a)^{12}}{2^5}\Big].
\end{aligned}$$

By induction we obtain that for $n>2$,

$$\begin{aligned}
\mathbf{E}|y_{n+1}(t)-y_n(t)|^2 &\le M_1q^{2n-2}+M_2\Big[M_1q^{2n-4}(b-a)+\frac{M_2^{n-1}K^2(b-a)^{2n+2}}{2^n}\\
&\quad+\sum_{k=1}^{n-2}\frac{M_2^kM_1q^{2n-2k-4}}{2^k}(b-a)^{2k+1}\Big](t-a).
\end{aligned} \tag{3.25}$$

Notice that

$$\begin{aligned}
\sum_{k=1}^{n-2}\frac{M_2^kM_1q^{2n-2k-4}}{2^k}(b-a)^{2k+1} &= M_1(b-a)\sum_{k=1}^{n-2}\frac{M_2^k(b-a)^{2k}}{2^k}\cdot q^{2(n-2-k)}\\
&=M_1(b-a)\sum_{k=1}^{n-2}\Big[\frac{M_2(b-a)^2}{2}\Big]^k\cdot q^{2(n-2-k)}\\
&\le M_1(b-a)\Big[\frac{M_2(b-a)^2}{2}+q^2\Big]^{n-2}.
\end{aligned}$$

Then

$$\mathbf{E}|y_{n+1}(t)-y_n(t)|^2 \le M_1q^{2n-2}+M_2\Big\{M_1q^{2n-4}(b-a)+\frac{M_2^{n-1}K^2(b-a)^{2n+2}}{2^n}$$

$$
+ M_1(b-a)\big[\frac{M_2(b-a)^2}{2}+q^2\big]^{n-2}\Big\}(t-a).
$$

If $p > n > 2$, we have

$$
\begin{aligned}
&\|y_p - y_n\|_{L^2}\\
&\le \sum_{k=n}^{p-1} \|y_{k+1}-y_k\|_{L^2}\\
&= \sum_{k=n}^{p-1} \Big[\mathbf{E}\int_a^b |y_{k+1}(t)-y_k(t)|^2 dt\Big]^{1/2}\\
&\le \sum_{k=n}^{p-1} \Big\{M_1(b-a)q^{2k-2} + \frac{M_2(b-a)^2}{2}[M_1 q^{2k-4}(b-a)\\
&\quad + \frac{M_2^{k-1}K^2(b-a)^{2k+2}}{2^k} + M_1(b-a)(\frac{M_2(b-a)^2}{2}+q^2)^{k-2}\Big\}^{1/2}\\
&\le \sum_{k=n}^{p-1} \Big[\sqrt{M_1(b-a)}q^{k-1} + \sqrt{\frac{M_1M_2(b-a)^3}{2}}q^{k-2}\\
&\quad + \frac{K}{\sqrt{2}}(b-a)^2\Big(\sqrt{\frac{M_2}{2}}(b-a)\Big)^k\\
&\quad + \frac{\sqrt{2M_1M_2(b-a)^3}}{M_2(b-a)^2+2q^2}\Big(\sqrt{\frac{M_2(b-a)^2}{2}+q^2}\Big)^k\Big].
\end{aligned}
$$

By the assumption $\frac{M_2(b-a)^2}{2}+q^2<1$, which means

$$
(12+\frac{1}{64})L_1^2(b-a)^4 + 3L_2^2(b-a)^3 < 1, \tag{3.26}
$$

we have

$$
\|y_p - y_n\|_{L^2} \to 0 \tag{3.27}
$$

as $p, n \to \infty$.

Therefore, $\{y_n(t)\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathcal{L}^2(\mathbf{P},\mathbb{R}^l)$ for all $t\in[a,b]$. Hence $\{y_n(t)\}_{n=0}^{\infty}$ is convergent in $\mathcal{L}^2(\mathbf{P},\mathbb{R}^l)$. We define

$$
y_t := \lim_{n\to\infty} y_n(t). \tag{3.28}
$$

Then y_t is $\mathcal{F}_t-$measurable for all $t\in[a,b]$.

Next, for $n\in N_+$ and $t\in[a,b]$, we have

$$
\begin{aligned}
y_{n+1}(t) = &-\frac{t-a}{b-a}\int_a^b ds\int_a^s \mathbf{E}[f(u,\bar{x}_n(u)+y_n(u)) - f(u,\bar{x}_n(u))]du\\
&+\int_a^t ds\int_a^s [f(u,\bar{x}_n(u)+y_n(u)) - f(u,\bar{x}_n(u))]du\\
&+\int_a^t ds\int_a^s g(u,\bar{x}_n(u)+y_n(u))dB(u).
\end{aligned}
$$

Then by the Cauchy-Schwartz inequality, we have

$$
\Big|\frac{t-a}{b-a}\int_a^b ds\int_a^s \mathbf{E}[f(u,\bar{x}_n(u)+y_n(u)) - f(u,\bar{x}_n(u))]du
$$

$$
\begin{aligned}
&- \frac{t-a}{b-a}\int_a^b ds\int_a^s \mathbf{E}[f(u,\bar{x}(u)+y(u))-f(u,\bar{x}(u))]du\Big|^2\\
&\leq \frac{(t-a)^2}{b-a}\int_a^b \Big|\int_a^s \{\mathbf{E}[f(u,\bar{x}_n(u)+y_n(u))-f(u,\bar{x}(u)+y(u))]\\
&\quad - \mathbf{E}[f(u,\bar{x}_n(u))-f(u,\bar{x}(u))]\}du\Big|^2 ds\\
&\leq \frac{(t-a)^2}{b-a}\int_a^b (s-a)\int_a^s |\mathbf{E}[f(u,\bar{x}_n(u)+y_n(u))-f(u,\bar{x}(u)+y(u))]\\
&\quad - \mathbf{E}[f(u,\bar{x}_n(u))-f(u,\bar{x}(u))]|^2\,du\,ds\\
&\leq \frac{2(t-a)^2}{b-a}\int_a^b (s-a)\int_a^s [\mathbf{E}|f(u,\bar{x}_n(u)+y_n(u))-f(u,\bar{x}(u)+y(u))|^2\\
&\quad + \mathbf{E}|f(u,\bar{x}_n(u))-f(u,\bar{x}(u))|^2]\,du\,ds\\
&\leq \frac{2L_1^2(t-a)^2}{b-a}\int_a^b (s-a)\int_a^s [\mathbf{E}|\bar{x}_n(u)+y_n(u)-\bar{x}(u)-y(u)|^2\\
&\quad + \mathbf{E}|\bar{x}_n(u)-\bar{x}(u)|^2]\,du\,ds\\
&\leq \frac{2L_1^2(t-a)^2}{b-a}\int_a^b (s-a)\int_a^s [3\mathbf{E}|\bar{x}_n(u)-\bar{x}(u)|^2+2\mathbf{E}|\bar{y}_n(u)-\bar{y}(u)|^2]\,du\,ds\\
&\leq 2L_1^2(b-a)^3\int_a^b [3\mathbf{E}|\bar{x}_n(u)-\bar{x}(u)|^2+2\mathbf{E}|\bar{y}_n(u)-\bar{y}(u)|^2]du\\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}
\tag{3.29}
$$

Using Hölder's inequality, we obtain

$$
\begin{aligned}
&\mathbf{E}\Big|\int_a^t ds\int_a^s [f(u,\bar{x}_n(u)+y_n(u))-f(u,\bar{x}_n(u))]du\\
&- \int_a^t ds\int_a^s [f(u,\bar{x}(u)+y(u))-f(u,\bar{x}(u))]du\Big|^2\\
&\leq (t-a)\int_a^t (s-a)\mathbf{E}\int_a^s |[f(u,\bar{x}_n(u)+y_n(u))-f(u,\bar{x}(u)+y(u))]\\
&\quad - [f(u,\bar{x}_n(u))-f(u,\bar{x}(u))]|^2\,du\,ds\\
&\leq 2(t-a)\int_a^t (s-a)\int_a^s [\mathbf{E}|f(u,\bar{x}_n(u)+y_n(u))-f(u,\bar{x}(u)+y(u))|^2\\
&\quad + \mathbf{E}|f(u,\bar{x}_n(u))-f(u,\bar{x}(u))|^2]\,du\,ds\\
&\leq 2L_1^2(t-a)\int_a^t (s-a)\int_a^s [3\mathbf{E}|\bar{x}_n(u)-\bar{x}(u)|^2+2\mathbf{E}|\bar{y}_n(u)-\bar{y}(u)|^2]\,du\,ds\\
&\leq 2L_1^2(b-a)^3\int_a^b [3\mathbf{E}|\bar{x}_n(u)-\bar{x}(u)|^2+2\mathbf{E}|\bar{y}_n(u)-\bar{y}(u)|^2]du\\
&\to 0 \quad \text{in } L^2(P).
\end{aligned}
\tag{3.30}
$$

By Itô isometry property, we have

$$
\mathbf{E}\Big|\int_a^t ds\int_a^s g(u,\bar{x}_n(u)+y_n(u))dB(u)-\int_a^t ds\int_a^s g(u,\bar{x}(u)+y(u))dB(u)\Big|^2
$$

$$\begin{aligned}
&\le (t-a)\int_a^t \mathbf{E}\Big|\int_a^s [g(u,\bar{x}_n(u)+y_n(u))-g(u,\bar{x}(u)+y(u))]dB(u)\Big|^2 ds \\
&= (t-a)\int_a^t \mathbf{E}\int_a^s |g(u,\bar{x}_n(u)+y_n(u))-g(u,\bar{x}(u)+y(u))|^2\,du\,ds \\
&\le L_2^2(t-a)\int_a^t \mathbf{E}\int_a^s |\bar{x}_n(u)+y_n(u)-\bar{x}(u)-y(u)|^2\,du\,ds \\
&\le 2L_2^2(t-a)\int_a^t\int_a^s [\mathbf{E}|\bar{x}_n(u)-\bar{x}(u)|^2+\mathbf{E}|y_n(u)-y(u)|^2]\,du\,ds \\
&\to 0 \quad \text{in } L^2(P).
\end{aligned} \tag{3.31}$$

Combining (3.29), (3.30) and (3.31), we conclude that for all $t\in[a,b]$,

$$\begin{aligned}
y(t) = &-\frac{t-a}{b-a}\int_a^b ds\int_a^s \mathbf{E}[f(u,\bar{x}(u)+y(u))-f(u,\bar{x}(u))]du \\
&+\int_a^t ds\int_a^s [f(u,\bar{x}(u)+y(u))-f(u,\bar{x}(u))]du \\
&+\int_a^t ds\int_a^s g(u,\bar{x}(u)+y(u))dB(u) \quad \text{a.s.}
\end{aligned} \tag{3.32}$$

Then $y(t)$ is the solution of the stochastic system (3.13).

Step 3: Existence of solutions of stochastic Newtonian equations with mean boundary conditions We define

$$\begin{gathered}
X_0(t)=0,\\
X_n(t)=\bar{x}_n(t)+y_n(t),
\end{gathered} \tag{3.33}$$

for $n\ge 1$ and $t\in[a,b]$. If $h>n>2$, we obtain

$$\begin{aligned}
\|X_h-X_n\|_{L^2} &= \|\bar{x}_h+y_h-\bar{x}_n-y_n\|_{L^2}\\
&\le \|\bar{x}_h-\bar{x}_n\|_{L^2}+\|y_h-y_n\|_{L^2}\to 0 \quad \text{as } h,n\to\infty.
\end{aligned} \tag{3.34}$$

Therefore, $\{X_n(t)\}_{n=0}^\infty$ is a Cauchy sequence in $\mathcal{L}^2(\mathbf{P},\mathbb{R}^l)$ and

$$\lim_{n\to\infty} X_n(t)=\lim_{n\to\infty}(\bar{x}_n+y_n)=\bar{x}(t)+y(t):=X_t. \tag{3.35}$$

We conclude that for all $t\in[a,b]$,

$$\begin{aligned}
X_t = &\int_a^b G(t,s)f(s,\bar{x}(s))ds-\frac{t-a}{b-a}\int_a^b ds\int_a^s \mathbf{E}[f(u,\bar{x}(u)+y(u))-f(u,\bar{x}(u))]du\\
&+\int_a^t ds\int_a^s [f(u,\bar{x}(u)+y(u))-f(u,\bar{x}(u))]du\\
&+\int_a^t ds\int_a^s g(u,\bar{x}(u)+y(u))dB(u)\\
=&\int_a^b G(t,s)f(s,\mathbf{E}X(s))ds-\frac{t-a}{b-a}\int_a^b ds\int_a^s \mathbf{E}[f(u,X(u))-f(u,\mathbf{E}X(u))]du\\
&+\int_a^t ds\int_a^s [f(u,X(u))-f(u,\mathbf{E}X(u))]du+\int_a^t ds\int_a^s g(u,X(u))dB(u),
\end{aligned}$$

and X_t is the solution of stochastic Newtonian system (1.1) with mean boundary conditions (1.2). The existence of solutions is complete.

Step 4 Uniqueness. Let $\bar{x}(t)$ and $\hat{x}(t)$ be two solutions of the deterministic second-order boundary value problem (3.5). Note that

$$
\begin{aligned}
|\bar{x}(t)-\hat{x}(t)| &= |\int_a^b G(t,s)f(s,\bar{x}(s))ds - \int_a^b G(t,s)f(s,\hat{x}(s))ds| \\
&\le \int_a^b |G(t,s)f(s,\bar{x}(s)) - f(s,\hat{x}(s))|ds \\
&\le \frac{(b-a)L_1}{4}\int_a^b |\bar{x}(s)-\hat{x}(s)|ds \\
&\le [\frac{(b-a)L_1}{4}]^2 \int_a^b\int_a^b |\bar{x}(s)-\hat{x}(s)|dsds_1 \\
&\le [\frac{(b-a)L_1}{4}]^k \underbrace{\int_a^b \cdots \int_a^b \int_a^b}_{k} |\bar{x}(s)-\hat{x}(s)|dsds_1\dots ds_{k-1}.
\end{aligned}
\tag{3.36}
$$

Set $A_1 = \int_a^b |\bar{x}(s)-\hat{x}(s)|ds$, then

$$
\begin{aligned}
|\bar{x}(t)-\hat{x}(t)| &\le [\frac{(b-a)L_1}{4}]^k \int_a^b \cdots \int_a^b A_1 ds_1 \dots ds_{k-1} \\
&= A_1(b-a)^{k-1}\cdot[\frac{(b-a)L_1}{4}]^k \\
&= \frac{A_1}{b-a}[\frac{(b-a)^2L_1}{4}]^k.
\end{aligned}
\tag{3.37}
$$

Under the condition $12L_1^2(b-a)^4<1$, we obtain

$$
\frac{L_1(b-a)^2}{4} < 1.
$$

Then

$$
|\bar{x}(t)-\hat{x}(t)| \le \frac{A_1}{b-a}[\frac{(b-a)^2L_1}{4}]^k \to 0 \quad \text{as } k\to\infty, \tag{3.38}
$$

Hence, $\bar{x}(t)\equiv\hat{x}(t)$ for all $a\le t\le b$.

Next, let $y(t)$ and $\hat{y}(t)$ be two solutions of the stochastic differential equation (3.13). They can be represented as follows

$$
\begin{aligned}
y(t) = &-\frac{t-a}{b-a}\int_a^b ds\int_a^s \mathbf{E}[f(u,\bar{x}(u)+y(u)) - f(u,\bar{x}(u))]du \\
&+\int_a^t ds\int_a^s [f(u,\bar{x}(u)+y(u)) - f(u,\bar{x}(u))]du \\
&+\int_a^t ds\int_a^s g(u,\bar{x}(u)+y(u))dB(u),
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{y}(t) = &-\frac{t-a}{b-a}\int_a^b ds\int_a^s \mathbf{E}[f(u,\bar{x}(u)+\hat{y}(u)) - f(u,\bar{x}(u))]du \\
&+\int_a^t ds\int_a^s [f(u,\bar{x}(u)+\hat{y}(u)) - f(u,\bar{x}(u))]du
\end{aligned}
$$

$$+\int_a^t ds\int_a^s g(u,\bar{x}(u)+\hat{y}(u))dB(u),$$

respectively. By Lipschitz condition, Hölder's inequality and Itô isometry property, we have

$$\begin{aligned}
&\mathbf{E}|y(t)-\hat{y}(t)|^2\\
&=\mathbf{E}\Big|-\frac{t-a}{b-a}\int_a^b ds\int_a^s \mathbf{E}[f(u,\bar{x}(u)+y(u))-f(u,\bar{x}(u)+\hat{y}(u))]du\\
&\quad+\int_a^t ds\int_a^s [f(u,\bar{x}(u)+y(u))-f(u,\bar{x}(u)+\hat{y}(u))]du\\
&\quad+\int_a^t ds\int_a^s [g(u,\bar{x}(u)+y(u))-g(u,\bar{x}(u)+\hat{y}(u))]dB(u)\Big|^2\\
&\le \frac{3(t-a)^2}{b-a}\int_a^b (s-a)\int_a^s \mathbf{E}|f(u,\bar{x}(u)+y(u))-f(u,\bar{x}(u)+\hat{y}(u))|^2\,du\,ds\\
&\quad+3(t-a)\int_a^t (s-a)\int_a^s \mathbf{E}|f(u,\bar{x}(u)+y(u))-f(u,\bar{x}(u)+\hat{y}(u))|^2\,du\,ds\\
&\quad+3(t-a)\int_a^t \mathbf{E}|\int_a^s [g(u,\bar{x}(u)+y(u))-g(u,\bar{x}(u)+\hat{y}(u))]dB(u)|^2 ds\\
&\le \frac{3L^2(t-a)^2}{b-a}\int_a^b (s-a)\int_a^s \mathbf{E}|y(u)-\hat{y}(u)|^2\,du\,ds\\
&\quad+3L^2(t-a)\int_a^t (s-a)\int_a^s \mathbf{E}|y(u)-\hat{y}(u)|^2\,du\,ds\\
&\quad+3L^2(t-a)\int_a^t \int_a^s \mathbf{E}|y(u)-\hat{y}(u)|^2\,du\,ds\\
&\le [6L_1^2(b-a)^2+3L_2^2(b-a)]\int_a^b\int_a^s \mathbf{E}|y(u)-\hat{y}(u)|^2\,du\,ds\\
&\le [6L_1^2(b-a)^2+3L_2^2(b-a)]^2\int_a^b\int_a^{s_1}\Big[\int_a^b\int_a^s \mathbf{E}|y(u)-\hat{y}(u)|^2\,du\,ds\Big]du_1ds_1\\
&\le [6L_1^2(b-a)^2+3L_2^2(b-a)]^k\\
&\quad\times\underbrace{\int_a^b\int_a^{s_{k-1}}\dots\int_a^b\int_a^{s_1}\Big[\int_a^b\int_a^s}_{2k}\mathbf{E}|y(u)-\hat{y}(u)|^2\,du\,ds\Big]du_1ds_1\dots du_{k-1}ds_{k-1}.
\end{aligned}$$

Setting $A_2=\int_a^b\int_a^s \mathbf{E}|y(u)-\hat{y}(u)|^2\,du\,ds$, we obtain

$$\begin{aligned}
&\mathbf{E}|y(t)-\hat{y}(t)|^2\\
&\le [6L_1^2(b-a)^2+3L_2^2(b-a)]^k\int_a^b\int_a^{s_{k-1}}\dots\int_a^b\int_a^{s_1} A_2du_1ds_1\dots du_{-1}ds_{k-1}\\
&\le A_2[6L_1^2(b-a)^2+3L_2^2(b-a)]^k\cdot[\frac{(b-a)^2}{2}]^{k-1}\\
&=\frac{2A_2}{(b-a)^2}[3L_1^2(b-a)^4+\frac{3}{2}L_2^2(b-a)^3)]^k\to 0\quad\text{as } k\to\infty,
\end{aligned}$$

then $y(t) \equiv \hat{y}(t)$ for all $a \le t \le b$, a.s.. Hence

$$X_t = \bar{x}(t) + y(t)$$

is a unique solution of the stochastic Newtonian system (1.1) with mean boundary conditions (1.2).

Step 5: Adaptability of the solution X_t to the filtration. The solution of the stochastic Newtonian system (1.1) with mean boundary conditions (1.2) can be represented as

$$\begin{aligned} X_t = &-\frac{t-a}{b-a}\int_a^b\int_a^v \mathbf{E}f(s,X_s)\,ds\,dv + \int_a^t\int_a^v f(s,X_s)\,ds\,dv \\ &+ \int_a^t\int_a^u g(s,X_s)dB_s du, \end{aligned} \tag{3.39}$$

for $t \in [a,b]$. Hence, the solution X_t is adapted to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$. The proof is complete. □

4. Examples

In this section, we give several examples to illustrate the theoretical results obtained in this paper. First, we consider the simplest care of SDE with mean boundary conditions.

Example 4.1. Consider one-dimensional stochastic differential equation on the interval $[0,1]$ with mean boundary conditions

$$\begin{gathered} \ddot{y}(t) = \lambda \dot{W}(t), \quad t \in [0,1], \\ y(0) = 0, \quad \mathbf{E}y(1) = 0. \end{gathered} \tag{4.1}$$

Here $\lambda \in (0,1/5)$ is a constant and $W(t)$ is a one-dimensional Wiener process. Let

$$\dot{y}(t) = z(t), \quad \dot{z}(t) = \lambda \dot{W}(t), \quad t \in [0,1].$$

Then $z(t) = z(0) + \lambda W(t)$ and

$$y(t) = y(0) + \int_0^t z(s)ds = tz(0) + \lambda t W(t) - \lambda \int_0^t s dW(s). \tag{4.2}$$

From the mean boundary condition $\mathbf{E}y(1) = 0$, we have $\mathbf{E}y(1) = z(0) = 0$. So

$$y(t) = \lambda t W(t) - \lambda \int_0^t s dW(s). \tag{4.3}$$

Therefore, by Theorem 2.1, there is a unique solution $y(t)$ of the stochastic system (4.1).

The stochastic system (4.1) can represent the oscillation of a wire blown by the wind. Next, let us consider linear stochastic differential equation with mean boundary conditions.

Example 4.2. We consider the linear SDE

$$\ddot{y}(t) + \lambda y(t) = \lambda \dot{W}(t), \quad t \in [0,1], \tag{4.4}$$

with mean boundary conditions

$$y(0) = 0, \quad \mathbf{E}y(1) = 0,$$

where $\lambda \in (0, 1/5)$ is a constant and $W(t)$ is a one-dimensional Wiener process. We change the variable $y(t)$, then

$$\dot{y}(t) = u(t), \quad \dot{u}(t) + \lambda y(t) = \lambda \dot{W}(t), \quad t \in [0, 1]. \tag{4.5}$$

Integrating from 0 to t on both sides of the equation (4.5) yields

$$u(t) = u(0) - \lambda \int_0^t y(s) ds + \lambda W(t).$$

Hence

$$\begin{aligned} y(t) &= y(0) + \int_0^t u(s) ds \\ &= tu(0) - \lambda \int_0^t \int_0^v y(s)\, ds\, dv + \lambda t W(t) - \lambda \int_0^t s dW(s). \end{aligned}$$

From the mean boundary condition $\mathbf{E}y(1) = 0$, we obtain

$$\mathbf{E}y(1) = u(0) - \lambda \int_0^1 \int_0^v \mathbf{E}y(s)\, ds\, dv = 0.$$

Then

$$u(0) = \lambda \int_0^1 \int_0^v \mathbf{E}y(s)\, ds\, dv.$$

So

$$y(t) = \lambda t \int_0^1 \int_0^v \mathbf{E}y(s)\, ds\, dv - \lambda \int_0^t \int_0^v y(s)\, ds\, dv + \lambda t W(t) - \lambda \int_0^t s dW(s).$$

When $\lambda \in (0, 1/5)$, the conditions in Theorem 2.1 are satisfied, and we have existence and uniqueness of system (4.4) with mean boundary conditions.

Let us consider the dynamics of stochastic flutter for a harmonic oscillator.

Example 4.3. We consider one-dimensional SDE:

$$\ddot{y}(t) + \lambda y(t) = \sin \pi t + \lambda \dot{W}(t), \quad t \in [0, 1], \tag{4.6}$$

with mean boundary conditions

$$y(0) = 0, \quad \mathbf{E}y(1) = 0.$$

Here $\lambda \in (0, \frac{1}{5})$ is a constant and $W(t)$ is a one-dimensional Wiener process. Let

$$\dot{y}(t) = v(t), \quad \dot{v}(t) + \lambda y(t) = \sin \pi t + \lambda \dot{W}(t), \quad t \in [0, 1].$$

Then

$$v(t) = v(0) - \lambda \int_0^t y(s) ds - \frac{1}{\pi} \cos \pi t + \frac{1}{\pi} + \lambda W(t),$$

$$\begin{aligned} y(t) &= y(0) + \int_0^t v(s) ds \\ &= tv(0) - \lambda \int_0^t \int_0^r y(s)\, ds\, dr - \frac{1}{\pi^2} \sin \pi t + \frac{t}{\pi} + \lambda \int_0^t W(s) ds. \end{aligned}$$

Since $\mathbf{E}y(1) = 0$, we have

$$\mathbf{E}y(1) = v(0) - \lambda \int_0^1 \int_0^r \mathbf{E}y(s)\, ds\, dr + \frac{1}{\pi} = 0.$$

Then

$$v(0) = \lambda \int_0^1 \int_0^r \mathbf{E}y(s)\,ds\,dr - \frac{1}{\pi}.$$

Hence

$$\begin{aligned} y(t) = \lambda t \int_0^1 \int_0^r \mathbf{E}y(s)\,ds\,dr - \lambda \int_0^t \int_0^r y(s)\,ds\,dr \\ - \frac{1}{\pi^2}\sin \pi t + \lambda t W(t) - \lambda \int_0^t s dW(s). \end{aligned}$$

When $\lambda \in (0, \frac{1}{5})$, the conditions in Theorem 2.1 are satisfied. Then there exists a unique solution $y(t)$ of system (4.6) with mean boundary conditions.

5. Conclusion

The main goal of this paper is to discuss the existence and uniqueness of solutions for a kind of second-order stochastic differential equations with mean boundary conditions. Based on the Picard iteration method, we introduce a new idea and obtain the sufficient conditions of the existence and uniqueness of solutions for stochastic Newtonian equations with stochastic boundary conditions. As applications, we show several types of SDEs with mean boundary conditions to illustrate our theoretical results. Comparing with the Dirichlet boundary conditions, stochastic boundary conditions with expectation are more random. And under this kind of stochastic boundary conditions, the solution of stochastic Newtonian systems is adapted to the Brownian filtration.

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