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EXISTENCE OF GLOBAL SOLUTIONS FOR SYSTEMS OF SECOND-ORDER DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN

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Abstract. We obtain sufficient conditions for the existence of global solutions for the systems of differential equations

$$\big(A(t)\Phi_p(y')\big)' + B(t)g(y') + R(t)f(y) = e(t),$$

where $\Phi_p(y')$ is the multidimensional p-Laplacian.

1. Introduction

The p-Laplace differential equation

$$\operatorname{div}(\|\nabla v\|)^{p-2}\nabla v) = h(\|x\|, v) \tag{1.1}$$

plays an important role in the theory of partial differential equations (see e. g. [21]), where ∇ is the gradient, $p > 0$ and $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^n$, $n > 1$ and $h(y, v)$ is a nonlinear function on $\mathbb{R} \times \mathbb{R}$. Radially symmetric solutions of the equation (1.1) depend on the scalar variable $r = \|x\|$ and they are solutions of the ordinary differential equation

$$r^{1-n}(r^{n-1}|v'|)' = h(r, v), \tag{1.2}$$

where $v' = \frac{\mathrm{d}v}{\mathrm{d}r}$ and $p > 1$. If $p \neq n$ then the change of variables $r = t^{\frac{p-1}{p-n}}$ transforms the equation (1.2) into the equation

$$(\Psi_p(u'))' = f(t, u), \tag{1.3}$$

where $\Psi_p(u') = |u'|^{p-2}u'$ is so called one-dimensional, or scalar p-Laplacian [21], and

$$f(t, u) = \Big|\frac{p-1}{p-n}\Big|^p t^{\frac{p-n}{p(1-n)}} h(t^{\frac{p-1}{p-n}}, u)\,.$$

In [22] the existence of periodic solutions of the system

$$(\Phi_p(u'))' + \frac{\mathrm{d}}{\mathrm{d}t}\nabla F(u) + \nabla G(u) = e(t) \tag{1.4}$$

is studied, where

$$\Phi_p : \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi_p(u) = (|u_1|^{p-2}u_1, \dots, |u_n|^{p-2}u_n)^T.$$

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The operator $\Phi_p(u')$ is called multidimensional p-Laplacian. The study of radially symmetric solutions of the system of p-Laplace equations

$$\operatorname{div}(\|\nabla v_i\|^{p-2}\nabla v_i) = h_i(\|x\|, v_1, v_2, \dots, v_n), \quad i = 1, 2, \dots, n, \quad p > 1$$

leads to the system of ordinary differential equations

$$(|u_i'|^{p-2}u_i')' = f_i(t, u_1, u_2, \dots, u_n), \quad i = 1, 2, \dots, n, \quad p \neq n \tag{1.5}$$

where

$$f_i(t, u_1, u_2, \dots, u_n) = |\frac{p-1}{p-n}|^p t^{\frac{p-n}{p(1-n)}} h_i(t^{\frac{p-1}{p-n}}, u_1, u_2, \dots, u_n).$$

This system can be written in the form

$$(\Phi_p(u'))' = f(t, u), \tag{1.6}$$

where $f = (f_1, f_2, \dots, f_n)^T$ and $\Phi_p(u')$ is the n-dimensional p-Laplacian. Throughout this paper we consider the operator Φ_{p+1} with $p > 0$ and for the simplicity we denote it as Φ_p, i. e. $\Phi_p(u) = (|u_1|^{p-1}u_1, |u_2|^{p-1}u_2, \dots, |u_n|^{p-1}u_n)$.

We shall study the initial value problem

$$(A(t)\Phi_p(y'))' + B(t)g(y') + R(t)f(y) = e(t), \tag{1.7}$$

$$y(0) = y_0, \quad y'(0) = y_1, \tag{1.8}$$

where $p > 0$, $y_0, y_1 \in \mathbb{R}^n$, $A(t)$, $B(t)$, $R(t)$ are continuous, matrix-valued functions on $\mathbb{R}_+ := \langle 0, \infty)$, $A(t)$ is regular for all $t \in \mathbb{R}_+$, $e : \mathbb{R}_+ \to \mathbb{R}^n$ and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are continuous mappings. The equation (1.7) with $n = 1$ has been studied by many authors (see e. g. references in [21]). Many papers are devoted to the study of the existence of periodic solutions of scalar differential equation with $p-$Laplacian and in some of them it is assumed that $A(0) = 0$. We study the system without this singularity. ¿From the recently published papers and books see e.g. [12, 13, 21, 22]. The problems treated in this paper are close to those studied in [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 18, 20, 21, 22]. The aim of the paper is to study the problem of the existence of global solutions to (1.7) in the sense of the following definition.

Definition 1.1. A solution $y(t)$, $t \in \langle 0, T)$ of the initial value problem (1.7), (1.8) is called nonextendable to the right if either $T < \infty$ and $\lim_{t \to T^-}[\|y(t)\| + \|y'(t)\|] = \infty$, or $T = \infty$, i. e. $y(t)$ is defined on $\mathbb{R}_+ = \langle 0, \infty)$. In the second case the solution $y(t)$ is called global.

The main result of this paper is the following theorem.

Theorem 1.2. *Let $p > 0$, $A(t)$, $B(t)$, $R(t)$ be continuous matrix-valued functions on $\langle 0, \infty)$, $A(t)$ be regular for all $t \in \mathbb{R}_+$, $e : \mathbb{R}_+ \to \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous mappings and $y_0, y_1 \in \mathbb{R}^n$. Let*

$$R_0 = \int_0^\infty \|R(s)\| s^{m-1} ds < \infty \tag{1.9}$$

and there exist constants $K_1, K_2 > 0$ such that

$$\|g(u)\| \le K_1\|u\|^m, \quad \|f(v)\| \le K_2\|v\|^m, \quad u, v \in \mathbb{R}^n. \tag{1.10}$$

Then the following assertions hold:

1. *If $1 < m \le p$, then any nonextendable to the right solution $y(t)$ of the initial value problem* (1.7), (1.8) *is global.*

2. *Let $m > p, m > 1$,*

$$A_\infty := \sup_{0 \le t < \infty} \|A(t)^{-1}\|^{-1} < \infty,$$

$$E_\infty := \sup_{0 \le t < \infty} \| \int_0^t e(s)\,\mathrm{d}s \| < \infty$$

and

$$n^{p/2} \frac{m-p}{p} D^{\frac{m-p}{p}} \sup_{0 \le t < \infty} \int_0^t \Big(K_1 \|B(s)\| + 2^{m-1} K_2 \int_s^\infty \|R(\sigma)\| \sigma^{m-1}\,\mathrm{d}\sigma \Big)\,\mathrm{d}s < 1,$$

for all $t \in \langle 0, \infty)$, where

$$D = n^{p/2} A_\infty \Big(\|A(0)\Phi_p(y_1)\| + 2^{m-1} K_2 \|y_0\|^m R_0 + E_\infty \Big).$$

Then any nonextendable to the right solution $y(t)$ of the initial value problem (1.7)*,* (1.8) *is global.*

In [5] a solution $u : \langle 0, T) \to \mathbb{R}^n$ with $0 < T < \infty$ of the equation (1.7) with $n = 1$ is called singular of the second kind, if $\sup_{0<t<T} |y'(t)| = \infty$. By [5, Theorem 1] if $m = p > 0$ (we need to assume $m > 1$) and the condition (1.10) is fulfilled then there exists no singular solution of the second kind of (1.7) and all solutions of (1.7) are defined on $\mathbb{R}_+$, i. e. they are global. The proof of this result is based on the transformation $y_1(t) = y(t)$, $y_2(t) = A(t)|y'(t)|^{p-1} y'(t)$ transforming the scalar equation (1.7) into the form

$$y_1' = A(t)^{-\frac{1}{p}} |y_2|^{1/p} \operatorname{sgn} y_2, \quad y_2' = -B(t) g(A(t)^{-\frac{1}{p}} \operatorname{sgn} y_2) - R(t) f(y_1) + e(t). \tag{1.11}$$

An estimate of the function $v(t) = \max_{0, \le s \le t} |y_2(s)|$ proves the boundedness of $|y'(t)|$ on any bounded interval $\langle 0, T)$. By [5, Theorem 2], if $n = 1$, $R \in C^1(\mathbb{R}_+, \mathbb{R})$, $R(t) > 0$, $f(x)x > 0$ for all $t \in \mathbb{R}_+$ and either $|g(x)| \le |x|^p$ for $|x| \ge M$ for some $M \in (0, \infty)$ or $g(x)x \ge 0$ or $g(x) \ge 0$ for all $x \in \mathbb{R}_+$ then the equation (1.7) has no singular solution of the second kind and all its solutions are defined on $\mathbb{R}_+$, i. e. they are global. The method of proofs are based on the study of the boundedness from above of the scalar function $V(t) = \frac{A(t)}{R(t)} |y'(t)|^{p+1} + \frac{p+1}{p} \int_0^{y(t)} f(s)\,\mathrm{d}s$ on any bounded interval $\langle 0, T)$. We remark that in [5] the case $n = 1, m = p > 0$ is studied. The method of proofs applied in [5] is not applicable in the case $n > 1$. Our proof of Theorem 1.2 is completely different from that applied in [5]. The main tool of our proof is the discrete and also continuous version of the Jensen's inequality, Fubini theorem and a generalization of the Bihari theorem (see Lemma), proved in this paper. The application of the Jensen's inequality is possible only under the assumption $m > 1$. Therefore we do not study the case $0 < m < 1$. This means that the problem is open for $n > 1$ and $0 < m < 1$. The natural problem is to formulate sufficient conditions for the existence of solutions which are not global, or solutions which are not of the second kind. This problem is not solved even for the scalar case and it seems to be not simple. By [5, Remark 5] the existence of singular solutions of the second kind of (1.7) is an open problem even in the scalar case. M. Bartušek proved (see [1, Theorem 4]) that if $n = 1$, $0 < p < m$ then there exists a positive function $R(t)$, $t \ge 0$ such that the scalar equation (1.7) with $A(t) \equiv 1$, $B(t) \equiv 0$, $e(t) \equiv 0$ and $f(y) = |y|^p$ has a singular solution of the second kind. The case $0 < p < m, n = 1$, studied by Bartušek, corresponds to the assertion 2 of our Theorem 1.2, however for the example given by Barušek in [5] the assumptions of the assertion 2 are not satisfied. The function $R(t)$ is constructed using a continuous, piecewise polynomial function and the integral R_0 is not finite. Let us remark that for the case $p = 1$, i. e. for second order differential equations without p-Laplacian and also for higher order differential equations some sufficient conditions for the existence of singular solutions of the second kind are proved by Bartušek in the papers [2, 3, 4]. A result on the existence of singular solutions of the second kind for systems of nonlinear differential equations (without the p-Laplacian) are proved by Chanturia [7, Theorem 3] and also by Mirzov [18].

2. Proof of the main result

First we shall prove the following lemma.

Lemma 2.1. *Let $c > 0$, $m > 0$, $p > 0$, $t_0 \in \mathbb{R}$ be constants, $F(t)$ be a continuous, nonnegative function on $\mathbb{R}_+$ and $v(t)$ be a continuous, nonnegative function on $\mathbb{R}_+$ satisfying the inequality*

$$v(t)^p \le c + \int_{t_0}^{t} F(s)v(s)^m \,\mathrm{d}s, \quad t \ge t_0. \tag{2.1}$$

Then the following assertions hold:

1. *If $0 < m < p$ then*
$$v(t) \le \Big(c^{\frac{p-m}{p}} + \frac{p-m}{p}\int_{t_0}^{t} F(s)\,\mathrm{d}s\Big)^{\frac{1}{p-m}}, \quad t \ge t_0 \tag{2.2}$$
2. *If $m > p$, $m > 1$ and*
$$\frac{m-p}{p} c^{\frac{m-p}{p}} \sup_{t_0 \le t < \infty} \int_{t_0}^{t} F(s)\,\mathrm{d}s < 1$$
then
$$v(t) \le \frac{c}{\Big(1 - \frac{m-p}{p} c^{\frac{m-p}{p}} \int_{t_0}^{t} F(s)\,\mathrm{d}s\Big)^{\frac{1}{m-p}}}, \quad t \ge t_0. \tag{2.3}$$

Proof. Let $G(t)$ be the right-hand side of the inequality (2.1). Then $v(t)^m \le G(t)^{\frac{m}{p}}$ whihc yields

$$\frac{F(t)v(t)^m}{G(t)^{\frac{m}{p}}} \le F(t),$$

i. e.

$$\frac{G'(t)}{G(t)^{\frac{m}{p}}} \le F(t).$$

Integrating this inequality from t_0 to t we obtain

$$\begin{aligned}\int_{t_0}^{t} \frac{G'(s)}{G(s)^{\frac{m}{p}}}\,\mathrm{d}s &= \int_{G(t_0)}^{G(t)} \frac{\mathrm{d}\sigma}{\sigma^{\frac{m}{p}}} \\ &= \frac{p}{p-m}\Big(G(t)^{\frac{p-m}{p}} - G(t_0)^{\frac{p-m}{p}}\Big) \\ &\le \int_{t_0}^{t} F(s)\,\mathrm{d}s.\end{aligned}$$

Since $G(t_0) = c$ we obtain

$$v(t) \le G(t)^{1/p} \le \Big(c^{\frac{p-m}{p}} + \frac{p-m}{p}\int_{t_0}^{t} F(s)\,\mathrm{d}s\Big)^{\frac{1}{p-m}}.$$

The assertions (1.1) and (1.2) follow from this inequality. □

Remark 2.2. If $p = 1$, $m > 0$ then this lemma is a consequence of the well known Bihari inequality (see [6]). Some results on integral inequalities with power nonlinearity on their left-hand sides can be found in the B. G. Pachpatte monograph [19]. The idea of the proof of this lemma is based on that used in the proofs of results on integral inequalities with singular kernels and power nonlinearities on their left-hand sides, published in the papers [16, 17].

Let $y(t)$ be a solution of the initial value problem (1.7), (1.8) defined on an interval $\langle 0, T)$, $0 < T \le \infty$. If we denote $u(t) = y'(t)$ then

$$y(t) = y_0 + \int_0^t u(s)\,\mathrm{d}s, \tag{2.4}$$

and the equation (1.7) can be rewritten as the following integro-differential equation for $u(t)$:

$$\Big(A(t)\Phi_p(u(t))\Big)' + B(t)g(u(t)) + R(t)f\Big(y_0 + \int_0^t u(s)\,\mathrm{d}s\Big) = e(t) \tag{2.5}$$

with

$$u(0) = y_1. \tag{2.6}$$

Theorem 2.3. *Let $p > 0, A(t)$, $B(t)$, $R(t)$ be continuous matrix-valued functions on $\mathbb{R}_+$, $A(t)$ regular for all $t \in \mathbb{R}_+$, $e : \mathbb{R}_+ \to \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous mappings on $\mathbb{R}_+$, $y_0, y_1 \in \mathbb{R}^n$, $R_0 := \int_0^\infty \|R(s)\|s^{m-1}\,\mathrm{d}s < \infty$ and $0 < T < \infty$. Let the condition* (1.10) *be satisfied and let $u : \langle 0, T) \to \mathbb{R}^n$ be a solution of the equation* (2.5) *satisfying the condition* (2.6)*. Then the following assertions hold:*

1. *If $m = p > 1$, then*

$$\|u(t)\| \le d_T \mathrm{e}^{\int_0^t F_T(s)\,\mathrm{d}s}, \quad 0 \le t \le T$$

where

$$F_T(t) := n^{p/2}E_T\Big(K_1\|B(s)\| + 2^{m-1}K_2Q(s)\Big),$$
$$Q(s) = \int_s^\infty \|R(\sigma)\|\sigma^{m-1}\,\mathrm{d}\sigma,$$
$$E_T := \max_{0\le t\le T}\|E(t)\|,\ E(t) := \int_0^t e(s)\,\mathrm{d}s,$$
$$d_T = n^{p/2}A_T\Big(\|A(0)\Phi_p(y_1)\| + 2^{m-1}K_2\|y_0\|^m R_0 + E_T\Big),$$
$$A_T = \max_{0\le t\le T}\|A(t)^{-1}\|^{-1}.$$

2. *If $1 < m < p$, then*

$$\|u(t)\| \le \Big(d_T^{\frac{p-m}{p}} + \frac{p-m}{p}d_T\int_0^t F_T(s)\,\mathrm{d}s\Big)^{\frac{1}{p-m}}.$$

3. *Let* $m > p$, $m > 1$, $A_\infty := \sup_{T\geq 0} A_T < \infty$, $\sup_{0\leq t\leq\infty} E(t) < \infty$,

$$n^{p/2}\frac{m-p}{p}D^{\frac{m-p}{p}}\sup_{0\leq t<\infty}\int_0^t\Big(K_1\|B(s)\|+2^{m-1}K_2Q(s)\Big)\,\mathrm{d}s<1,$$

where

$$D = n^{p/2}A_\infty\Big(\|A(0)\Phi_p(y_1)\|+2^{m-1}K_2\|y_0\|^m R_0+E_\infty\Big).$$

then

$$\|u(t)\|\leq D\Big(1-n^{p/2}\frac{m-p}{p}D^{\frac{m-p}{p}}\int_0^t\Big(K_1\|B(s)\|+2^{m-1}K_2Q(s)\Big)\,\mathrm{d}s\Big)^{-\frac{1}{m-p}},$$

where $0\leq t\leq\infty$.

Proof. We shall give an explicit upper bound for the solution $u(t)$ of the equation (2.5), defined on the interval $\langle 0,T)$, satisfying (2.6). From the equation (2.5) and the condition (2.6) it follows that

$$\begin{aligned}\Phi_p(u(t)) = A(t)^{-1}\{A(0)\Phi_p(y_1)-\int_0^t B(s)g(u(s))\,\mathrm{d}s\\ +\int_0^t R(s)f\Big(y_0+\int_0^s u(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s+E(t)\},\end{aligned}\tag{2.7}$$

where $E(t)=\int_0^t e(s)\,\mathrm{d}s$. This inequality together with the conditions (1.10) yield

$$\begin{aligned}\|A(t)^{-1}\|\|\Phi_p(u(t))\| \leq \|A(0)\Phi_p(y_1)\|+K_1\int_0^t\|B(s)\|\|u(s)\|^m\,\mathrm{d}s\\ +K_2\int_0^t\|R(s)\|\Big(\|y_0\|+\int_0^t\|u(\tau)\|d\tau\Big)^m\,\mathrm{d}s+\|E(t)\|.\end{aligned}\tag{2.8}$$

We shall use the integral version of the Jensen' s inequality

$$\Big(\int_0^t H(s)\,\mathrm{d}s\Big)^\kappa\leq t^{\kappa-1}\int_0^t H(s)^\kappa\,\mathrm{d}s,\quad \kappa>1,\ t\geq 0\tag{2.9}$$

for $h\in C(\mathbb{R}_+,\mathbb{R}_+)$ (For a more general integral Jensen's inequality, see e.g. [15, Chapter VIII, Theorem 2]). Also we shall use its discrete version

$$(A_1+A_2+\dots+A_l)^\kappa\leq l^{\kappa-1}(A_1^\kappa+A_2^\kappa+\dots+A_l^\kappa),\tag{2.10}$$

for $A_1,A_2,\dots,A_l\geq 0$, $\kappa>1$ (see [15, Chapter VIII, Corollary 4]).

Let $m>1$. Then using the inequalities (2.9) and (2.10) we obtain the inequality

$$\begin{aligned}\Big(\|y_0\|+\int_0^s\|u(\tau)\|\,\mathrm{d}\tau\Big)^m &\leq 2^{m-1}\Big(\|y_0\|^m+\Big(\int_0^s\|u(\tau)\|\,\mathrm{d}\tau\Big)^m\Big)\\ &\leq 2^{m-1}\Big(\|y_0\|^m+s^{m-1}\int_0^s\|u(\tau)\|^m\,\mathrm{d}\tau\Big).\end{aligned}$$

Putting this inequality into (2.8) we obtain

$$
\begin{aligned}
&\|A(t)^{-1}\|\|\Phi_p(u(t))\| \\
&\le \|A(0)^{-1}\Phi_p(y_1)\| + K_1\int_0^t \|B(s)\|\|u(s)\|^m\,\mathrm{d}s + 2^{m-1}K_2\|y_0\|^m\int_0^t \|R(s)\|\,\mathrm{d}s \\
&\quad + 2^{m-1}K_2\int_0^t \|R(s)\|s^{m-1}\int_0^s \|u(\tau)\|^m\,\mathrm{d}\tau\,\mathrm{d}s\|E(t)\|.
\end{aligned}
\tag{2.11}
$$

Now we shall apply the following consequence of the Fubini theorem (see e. g. [23, Theorem 3.10 and Exercise 3.27]): If $h : \langle a, b\rangle \times \langle a, b\rangle \to \mathbb{R}$ is an integrable function then

$$
\int_a^b\int_a^y h(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b\int_x^b h(x,y)\,\mathrm{d}y\,\mathrm{d}x.
$$

If $h(\tau, s) = \|R(s)\|s^{m-1}\|u(\tau)\|^m$, $a = 0$, $b = t$, $y = s$, $x = \tau$ then

$$
\int_0^t\int_0^s h(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s = \int_0^t\int_\tau^t h(\tau,s)\,\mathrm{d}s\,\mathrm{d}\tau,
$$

i. e.

$$
\int_0^t\int_0^s \|R(s)\|s^{m-1}\|u(\tau)\|^m\,\mathrm{d}\tau\,\mathrm{d}s = \int_0^t \Big(\int_\tau^t \|R(s)\|s^{m-1}\,\mathrm{d}s\Big)\|u(\tau)\|^m\,\mathrm{d}\tau.
$$

This yields

$$
\int_0^t \|R(s)\|s^{m-1}\int_0^s \|u(\tau)\|^m\,\mathrm{d}\tau\,\mathrm{d}s \le \int_0^t Q(\tau)\|u(\tau)\|^m\,\mathrm{d}\tau, \tag{2.12}
$$

where

$$
Q(\tau) := \int_\tau^\infty \|R(s)\|s^{m-1}\,\mathrm{d}s
$$

for $\tau \ge 0$.

Let $0 < T < \infty$ and $t \in \langle 0, T)$. From the inequalities (2.11) and (2.12) it follows that

$$
\|A(t)^{-1}\|\|\Phi_p(u(t))\| \le c_T + \int_0^t F_0(s)\|u(s)\|^m\,\mathrm{d}s, \tag{2.13}
$$

where

$$
c_T = \|A(0)\Phi_p(y_1)\| + 2^{m-1}K_2\|y_0\|^m R_0 + E_T, \tag{2.14}
$$

$$
F_0(s) = K_1\|B(s)\| + 2^{m-1}K_2 Q(s), \tag{2.15}
$$

$$
E_T = \max_{0\le t\le T}\|E(t)\|. \tag{2.16}
$$

If $k \in \{1, 2, \dots, n\}$, then

$$
\begin{aligned}
|u_k(t)|^p \le \|\Phi_p(u(t))\| &= \big(u_1(t)^{2p} + u_2(t)^{2p} + \dots + u_n(t)^{2p}\big)^{1/2} \\
&\le A_T c_T + \int_0^t A_T F_0(s)\|u(s)\|^m\,\mathrm{d}s;
\end{aligned}
$$

i. e.,

$$
|u_k(t)|^p \le c_{0T} + \int_0^t F_{0T}(s)\|u(s)\|^m\,\mathrm{d}s, \tag{2.17}
$$

where

$$
A_T := \max_{0\le t\le T}\|A(t)^{-1}\|^{-1}, \quad \text{if } T < \infty, \tag{2.18}
$$

$$c_{0T} = A_T c_T, \quad F_{0T}(t) = A_T F_0(t). \tag{2.19}$$

This yields

$$\|u(t)\| \le n^{p/2}\Big(c_{0T} + \int_0^t F_{0T}(s)\|u(s)\|^m \,\mathrm{d}s\Big)^{1/p},$$

and therefore we have obtained the inequality

$$\|u(t)\|^p \le d_T + \int_0^t F_T(s)\|u(s)\|^m \,\mathrm{d}s, \tag{2.20}$$

where

$$d_T = n^{p/2}c_{0T}, F_T(t) = n^{p/2}F_{0T}(t). \tag{2.21}$$

Now applying Lemma 2.1 (the case $m = p$ follows from the Gronwall's lemma) to the inequality (2.20) we obtain the assertions 1. and 2. In the proof of the assertion 3. we use the assumptions $A_\infty := \sup_{0\le t<\infty} \|A(t)^{-1}\|^{-1} < \infty$, $\sup_{0\le t\le\infty} E(t) < \infty$. ¿From the inequality (2.20) we obtain the inequality,

$$\|u(t)\|^p \le D + \int_0^t G(s)\|u(s)\|^m \,\mathrm{d}s, \tag{2.22}$$

where D is defined in Theorem 1.2,

$$G(s) := K_1\|B(s)\| + 2^{m-1}K_2Q(s),$$

and $Q(s) = \int_s^t \|R(\sigma)\sigma^{m-1}\| \,\mathrm{d}\sigma$. Now if we put in Lemma $t_0 = 0$, $v(t) = \|u(t)\|$, $c = D$ and $F(t) = G(t)$ then we obtain the inequality from the assertion 3. □

Proof of Theorem 1.2. Let $y : \langle 0, T) \to \mathbb{R}^n$ be a nonextendable to the right solution of the initial value problem (2.5), (2.6) with $T < \infty$. Then $y(t) = y_0 + \int_0^t u(s)\,\mathrm{d}s$, where $u(t)$ is a solution of the equation (2.5) satisfying the condition (2.6). From Theorem 2.3 it follows that $M = \sup_{0\le t\le T} \|u(t)\| < \infty$ and since (2.4) yields $\|y(t)\| \le \|y_0\| + t\sup_{0\le s\le T} \|u(s)\|$ we obtain $\lim_{t\to T^-} \|y(s)\| < \infty$. This is a contradiction with nonextendability of $y(t)$. □

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