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ON THE SECOND EIGENVALUE OF A HARDY-SOBOLEV OPERATOR

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Abstract. In this note, we study the variational characterization and some properties of the second smallest eigenvalue of the Hardy-Sobolev operator $L_\mu := -\Delta_p - \frac{\mu}{|x|^p}$ with respect to an indefinite weight $V(x)$.

1. Introduction

Let Ω be a domain in $\mathbb{R}^N$ containing 0. We recall the classical Hardy-Sobolev inequality which states that, for $1 < p < N$,

$$\int_\Omega |\nabla u|^p dx \geq \Big(\frac{N-p}{p}\Big)^p \int_\Omega \frac{|u|^p}{|x|^p} dx, \quad \forall u \in C_c^\infty(\Omega). \tag{1.1}$$

Let $D_0^{1,p}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|u\|_{1,p} = \|\nabla u\|_{L^p(\Omega)}$. The Hardy-Sobolev operator L_μ on $D_0^{1,p}(\Omega)$ is defined as

$$L_\mu u := -\Delta_p u - \frac{\mu}{|x|^p}|u|^{p-2}u, \quad 0 < \mu < \Big(\frac{N-p}{p}\Big)^p,$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, is the p-Laplacian.

We are interested in the variational characterization and some properties of the second smallest eigenvalue of the problem

$$\begin{gathered} L_\mu u = \lambda V(x)|u|^{p-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

On the weight on $V(x)$, we assume the following:

- (H1) $V \in L^1_{\rm loc}(\Omega)$, $V^+ = V_1 + V_2 \not\equiv 0$ with $V_1 \in L^{N/p}(\Omega)$ and V_2 is such that $\lim_{x\to y,\, x\in\Omega} |x-y|^p V_2(x) = 0$ for all $y \in \overline{\Omega}$, $\lim_{|x|\to\infty, x\in\Omega} |x|^p V_2(x) = 0$, where $V^+(x) = \max\{V(x), 0\}$.
- (H2) There exists $r > N/p$ and a closed subset S of measure zero in $\mathbb{R}^N$ such that $\Omega\backslash S$ is connected and $V \in L^r_{\rm loc}(\Omega\backslash S)$.

Here we note that there is no global integrability condition assumed on V^-.

This work is motivated by the work in [8]. The eigenvalue problem with indefinite weights has been studied for the case $\mu = 0$ by Szulkin-Willem [8]. However, some important properties, of the smallest eigenvalue λ_1, such as simplicity and

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being isolated were shown only for $p = 2$. Recently the author in [6] proved the simplicity of λ_1 and sign changing nature of eigenfunctions corresponding to other eigenvalues when Ω is bounded. Infact in [6] the author studied these properties for L_μ. Following the same arguments, one can prove these results in the present case. However, showing that λ_1 is isolated and characterization of the second smallest eigenvalue, were open questions. To prove these properties, we follow the ideas in [5] and in [3]. Here we should mention that our results are new even for the case $\mu = 0$. We use the following results in later sections.

Propositioin 1.1 (Boccardo-Murat [1]). *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $u_n \in W^{1,p}(\Omega)$ satisfy*

$$-\Delta_p u_n = f_n + g_n \quad \text{in } \mathcal{D}'(\Omega)$$

and

(i) $u_n \to u$ *weakly in* $W^{1,p}(\Omega)$
(ii) $u_n \to u$ *in* $L^p(\Omega)$
(iii) $f_n \to f$ *in* $W^{-1,p'}$
(iv) g_n *is a bounded sequence of Radon measures.*

Then there exists a subsequence $\{u_n\}$ of $\{u_n\}$ such that $\nabla u_n \to \nabla u$ a.e. in Ω.

Propositioin 1.2 (Brezis-Lieb [2]). *Let $f_n \to f$ a.e in Ω as $n \to \infty$ and f_n be bounded in $L^p(\Omega)$, for some $p > 1$. Then*

$$\lim_{n\to\infty} \{\|f_n\|_p - \|f_n - f\|_p\} = \|f\|_p.$$

Let X be a Banach space and let $M = \{u \in X \mid g(u) = 0\}$ with $g \in C^1$. Also let $f : X \to \mathbb{R}$ be a C^1 functional and let $\tilde{f}$ be the restriction of f to M. Then we have the following form of the Mountain pass Theorem [7].

Propositioin 1.3. *Let $u, v \in M$ with $u \not\equiv v$ and suppose that*

$$c := \inf_{h\in\Gamma} \max_{w\in h(t)} f(w) > \max\{f(u), f(v)\}$$

where

$$\Gamma := \{h \in C([-1,+1], M)\, |h(-1) = u \quad \text{and} \quad h(1) = v\} \neq \emptyset$$

Also suppose that $\tilde{f}$ satisfies Pailse-Smale (PS) condition on M. Then c is a critical value of $\tilde{f}$.

We define the norm

$$\|\tilde{f}'\|_* = \inf\{\|f'(u) - tg'(u)\|_{X^*} : t \in \mathbb{R}\}.$$

The variational characterization of the smallest eigenvalue is given by

$$\lambda_1 = \inf_{0\not\equiv u\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p dx - \int_\Omega \frac{|u|^p}{|x|^p} dx}{\int_\Omega |u|^p V(x) dx}$$

and the corresponding eigenfunction is denoted by ϕ_1, which is unique under the condition $\int_\Omega |\phi|^p V(x) dx = 1$ (see [6]). We will prove the following property.

Theorem 1.4. *The eigenvalue λ_1 is isolated in the spectrum of L_μ.*

We will establish the following variational characterisation of the second smallest eigenvalue:

$$\lambda_2 = \inf_{\gamma\in\Gamma}\sup_{u\in\gamma}\frac{\int_\Omega |\nabla u|^p dx - \int_\Omega \frac{|u|^p}{|x|^p}dx}{\int_\Omega |u|^p V(x)dx},$$

where $\Gamma = \{\gamma \in C([-1,1]:M) \,|\, \gamma(-1) = -\phi_1, \gamma(1) = \phi_1\}$ and M is defined as in the next section. We show also the following property of λ_2.

Theorem 1.5. *If $V_a \le V_b$, then $\lambda_2(V_a) \ge \lambda_2(V_b)$.*

2. Proofs of results

In this section we show that λ_1 is siolated and give a variational characterization for second smallest eigenvalue of L_μ.

Lemma 2.1. *The mapping $u \longmapsto \int_\Omega V^+|u|^p dx$ is weakly continuous.*

The proof of this lemma follows from (1.1) and (H1). We refer the reader to [8] for more details.
Now, we consider the set

$$M = \Big\{u \in D_0^{1,p}(\Omega) \,\Big|\, \int_\Omega |u|^p V(x) = 1\Big\}.$$

Since M is not a manifold in $D_0^{1,p}(\Omega)$, we define $X = \{u \in D_0^{1,p}(\Omega) \,|\, \|u\|_X < \infty\}$, where

$$\|u\|_X^p := \int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p V^- dx.$$

Then M is a C^1-manifold as a subset of the space X. On this space, we define the functional

$$J_\mu(u) = \frac{\int_\Omega |\nabla u|^p\, dx - \int_\Omega \frac{\mu}{|x|^p}|u|^p\, dx}{\int_\Omega |u|^p V\, dx}.$$

Let $\tilde J_\mu$ denote the restriction of J_μ to M. and let $\|u\|^p_{L^p(V)} = \int_\Omega |u|^p V(x)dx$.

Lemma 2.2. *The functional $\tilde J_\mu$ satisfies the Palais-Smale condition at any positive level.*

Proof. Let $\{u_n\}$ be a sequence in M such that $J_\mu(u_n) \to \lambda > 0$ and

$$\langle J_\mu(u_n), \phi\rangle - J_\mu(u_n)\int_\Omega |u_n|^{p-2}u_n\phi V dx = o(1). \tag{2.1}$$

Using Hardy-Sobolev inequality and $u_n \in M$, it follows that u_n is bounded in X which gives the existence of a subsequence $\{u_n\}$ of $\{u_n\}$ and u such that $u_n \to u$ weakly in $D_0^{1,p}(\Omega)$. Since $\lambda > 0$ we may assume that $J_\mu(u_n) \ge 0$. Using Lemma 2.1 and (2.1), we get

$$\langle J'_\mu(u_n) - J'_\mu(u), u_n - u\rangle + J_\mu(u_n)\int_\Omega \big[|u_n|^{p-2}u_n - |u|^{p-2}u\big](u_n - u)V^- dx = o(1).$$

By Fatou's Lemma,

$$\begin{aligned}0 &= \int_\Omega \lim_{n\to\infty}\big[|u_n|^{p-2}u_n - |u|^{p-2}u\big][u_n - u]V^- \\ &\le \liminf_{n\to\infty}\int_\Omega \big[|u_n|^{p-2}u_n - |u|^{p-2}u\big][u_n - u]V^- dx.\end{aligned}$$

Also, u_n satisfies

$$-\Delta_p u_n - \frac{\mu}{|x|^p}|u_n|^{p-2}u_n - J_\mu(u_n)|u_n|^{p-2}u_n V(x) = o(1) \quad \text{in} \quad \mathcal{D}'(\Omega_m),$$

where Ω_m is a bounded domain such that $\Omega = \cup_{m=1}^{\infty}\Omega_m$. By Proposition 1.1, noting that $\frac{\mu}{|x|^p}|u_n|^{p-2}u_n + J_\mu(u_n)|u_n|^{p-2}u_n V^-$ is a bounded sequence of Radon measures, there exists a subsequence $\{u_n^m\}$ of $\{u_n\}$ an u such that $\nabla u_n^m \to \nabla u$ a.e., in Ω_m. By the process of diagonalization we can choose a subsequence $\{u_n\}$ such that $\nabla u_n \to \nabla u$ a.e. in Ω. By Proposition 1.2, we have

$$\|u_n - u\|_{1,p}^p = \|u_n\|_{1,p}^p - \|u\|_{1,p}^p + o(1) \tag{2.2}$$

$$\|\frac{u_n - u}{|x|}\|_{L^p(1)}^p = \|\frac{u_n}{|x|}\|_{L^p(1)}^p - \|\frac{u}{|x|}\|_{L^p(1)}^p + o(1). \tag{2.3}$$

We also have, by Fatau's lemma,

$$\begin{aligned}
&\int_\Omega V^-(|u_n|^p + |u|^p - |u_n|^{p-2}u_n u - |u|^{p-2}u u_n)dx \\
&\geq \int_\Omega V^-\,(|u_n|^p + |u|^p) - \Big(\int_\Omega V^-|u_n|^p\Big)^{(p-1)/p}\Big(\int_\Omega V^-|u|^p\Big)^{1/p} \\
&\quad - \Big(\int_\Omega V^-|u|^p\Big)^{(p-1)/p}\Big(\int_\Omega V^-|u_n|^p\Big)^{1/p} \\
&= \Big[\Big(\int_\Omega V^-|u_n|^p\Big)^{(p-1)/p} - \Big(\int_\Omega V^-|u|^p\Big)^{(p-1)/p}\Big] \\
&\quad \times \Big[\Big(\int_\Omega V^-|u_n|^p\Big)^{1/p} - \Big(\int_\Omega V^-|u|^p\Big)^{\frac{1}{p}}\Big] \geq 0\,.
\end{aligned}$$

Now using (2.2) and (2.3),

$$\begin{aligned}
o(1) &= \langle J_\mu(u_n) - J_\mu(u), (u_n - u)\rangle + J_\mu(u_n)\int_\Omega [|u_n|^{p-2}u_n - |u|^{p-2}u](u_n - u)V^- dx \\
&\geq \int_\Omega |\nabla u_n - \nabla u|^p - \int_\Omega \frac{\mu}{|x|^p}|u_n - u|^p + o(1) \\
&\geq \big(1 - \frac{\mu}{\lambda_N}\big)\|u_n - u\|_{1,p} + o(1).
\end{aligned}$$

i.e., $u_n \to u$ in $D_0^{1,p}(\Omega)$. Notice that

$$\begin{aligned}
o(1) &= \langle J_\mu(u_n) - J_\mu(u), u_n - u\rangle \\
&= \int_\Omega V^-\,\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\,(u_n - u)dx + o(1) \geq 0\,.
\end{aligned}$$

Therefore, $\int_\Omega V^-|u_n|^p dx \to \int_\Omega V^-|u|^p dx$ and hence $\|u_n\|_X \to \|u\|_X$. ☐

Observe that $\tilde{J}_\mu(u) \geq \lambda_1$ and $\tilde{J}_\mu(\pm\phi_1) = \lambda_1$. So $+\phi_1$ and $-\phi_1$ are two global minima of $\tilde{J}_\mu$. Now consider

$$\Gamma = \{\gamma \in C([-1,1]; M) \,|\, \gamma(-1) = -\phi_1, \gamma(1) = \phi_1\}.$$

By Proposition 1.3, there exists $u \in X$ such that $\tilde{J}'_\mu(u) = 0$ and $J_\mu(u) = \mathcal{C}$, where

$$\mathcal{C} = \inf_{\gamma\in\Gamma}\sup_{u\in\gamma}\tilde{J}_\mu(u). \tag{2.4}$$

Lemma 2.3. (i) *M is locally arc wise connected*

(ii) *Any connected open subset B of M is arcwise connected*
(iii) *if B' is a component of an open set A, then $\partial B' \cap B$ is empty.*

The proof of this lemma follows from the fact that M is a Banach Manifold. For a proof we refer the reader to [3]. Define $\mathcal{O} = \{u \in M \,\big|\, \tilde{J}_\mu(u) < r\}$

Lemma 2.4. *Each component of $\mathcal{O}$ contains a critical point of $\tilde{J}_\mu$.*

Proof. Let $\mathcal{O}_1$ be a component of $\mathcal{O}$ and let $d = \inf\{\tilde{J}_\mu(u), u \in \mathcal{O}_1\}$, where $\overline{\mathcal{O}_1}$ is X-closure of $\mathcal{O}$. Suppose this infimum is achieved by $v \in \overline{\mathcal{O}_1}$. Then by Lemma 2.3 this cannot be in $\partial\mathcal{O}_1$ and hence v is in $\mathcal{O}_1$ and is a critical point of $\tilde{J}_\mu$.

Now we show that d is achieved. Let $u_n \in \mathcal{O}_1$ be a minimizing sequence with $\tilde{J}_\mu(u_n) \le d + \frac{1}{n^2}$. By Ekeland Variational Principle, we get $v_n \in \mathcal{O}_1$ such that

$$\tilde{J}_\mu(v_n) \le \tilde{J}_\mu(u_n), \tag{2.5}$$

$$\|v_n - u_n\|_X \le \frac{1}{n}, \tag{2.6}$$

$$\tilde{J}_\mu(v_n) \le \tilde{J}_\mu(v) + \frac{1}{n}\|v_n - v\|_X, \quad \forall v \in \mathcal{O}_1\,. \tag{2.7}$$

From (2.5) it follows that $\tilde{J}_\mu(v_n)$ is bounded. Now we claim that $\|\tilde{J}_\mu{}'(v_n)\|_* \to 0$. We fix n and choose $w \in X$ tangent to M at v_n, i.e., $\int_\Omega |v_n|^{p-2}v_n wV = 0$. Now we consider the path

$$u_t = \frac{v_n + tw}{\|v_n + tw\|_{L^p(V)}}.$$

Since $\tilde{J}_\mu(v_n) \le d + \frac{1}{n} < r$ for n large, we have $v_n \in \overline{\mathcal{O}}_1$ and by Lemma 2.3 (iii), $v_n \notin \partial\mathcal{O}_1$. So $u_t \in \mathcal{O}_1$ for $|t|$ small. Taking $v = u_t$ in (2.7) we obtain

$$\begin{aligned} &\frac{\tilde{J}_\mu(v_n) - \tilde{J}_\mu(v_n + tw)}{t} \\ &\le \frac{1}{nt}\|v_n(\frac{1}{r(t)} - 1)\|_X + \frac{1}{n}\|w\| + \frac{1}{t}\big(\frac{1}{r(t)^p} - 1\big)\tilde{J}_\mu(v_n + tw), \end{aligned} \tag{2.8}$$

where $r(t) = \|v_n + tw\|_{L^p(V)}$. The last term in (2.8) involves $\frac{r(t)^p - 1}{t}$ which can be calculated as

$$\frac{d}{dt}r(s)^p\big|_{s=0} = \lim_{t\to 0}\frac{r(t)^p - 1}{t}.$$

On the other hand since w is tangent to M at v_n,

$$\frac{d}{dt}r(s)^p\big|_{s=0} = p\int_\Omega |v_n|^{p-2}v_n wV(x)dx = 0.$$

Therefore, we have $\frac{r(t)^p-1}{t} \to 0$ as $t \to 0$ and that the second term goes to 0. Similarly, the first term also goes to zero as $t \to 0$. Taking limit $t \to 0$ in (2.8) we get

$$\langle J'_\mu(v_n), w\rangle \le \frac{1}{n}\|w\|_X, \quad \text{for all } w \in X \text{ tangent to } M \text{ at } v_n.$$

Now if w is arbitrary in X. We choose α_n so that $(w - \alpha_n v_n)$ is tangent to M at v_n. i.e., $\alpha_n = \int_\Omega |v_n|^{p-2}v_n wV(x)dx$. So (2.8) gives,

$$|\langle J'_\mu(v_n), w\rangle - \langle J'_\mu(v_n), v_n\rangle \int_\Omega |v_n|^{p-2}v_n w| \le \frac{1}{n}\|w - \alpha_n v_n\|_X$$

Since $\|\alpha_n v_n\|_X \leq C.\|w\|_X$, we have

$$|\langle J'_\mu(v_n), w\rangle - t_n \int_\Omega |v_n|^{p-2} v_n w V(x) dx| \leq \epsilon_n \|w\|_X$$

where $t_n = \langle J'_\mu(v_n), v_n\rangle$ and $\epsilon_n \to 0$. Therefore, $\|\tilde{J_\mu}'(v_n)\|_* \to 0$ and v_n is a Palais-Smale sequence. Hence by Lemma 2.2, $\{v_n\}$ has a convergent subsequence with limit, say, v. Then d is achieved at v. □

Lemma 2.5. *The number $\mathcal{C}$ defined by* (2.4) *is the second smallest eigenvalue of* L_μ

Proof. We follow the proof in [3]. Assume by contradiction that there exists an eigenvalue δ such that $\lambda_1 < \delta < \mathcal{C}$. In other words, $\tilde{J}_\mu$ has a critical value δ with $\lambda_1 < \delta < \mathcal{C}$. We will construct a path in Γ on which $\tilde{J}_\mu$ remains $\leq \delta$, which yields a contradiction with the definition of $\mathcal{C}$. Let $u \in$ M satisfies the equation

$$-\Delta_p u - \frac{\mu}{|x|^p}|u|^{p-2}u = \delta V(x)|u|^{p-2}u \quad \text{in } \mathcal{D}'(\Omega),$$

and u changes sign in Ω. Taking u^+ and u^- as test function we get

$$\int_\Omega |\nabla u^+|^p\, dx - \int_\Omega \frac{\mu}{|x|^p}|u^+|^p\, dx = \delta \int_\Omega (u^+)^p V(x)\, dx$$
$$\int_\Omega |\nabla u^-|^p\, dx - \int_\Omega \frac{\mu}{|x|^p}(u^-)^p\, dx = \delta \int_\Omega (u^-)^p V(x)\, dx\,.$$

Consequently

$$\tilde{J}_\mu(u) = \tilde{J}_\mu(\frac{u^+}{\|u^+\|_{L^p(V)}}) = \tilde{J}_\mu(\frac{-u^-}{\|u^-\|_{L^p(V)}}) = \tilde{J}_\mu(\frac{u^-}{\|u_-\|_{L^p(V)}}) = \delta.$$

We will consider the following three paths in M, which go respectively from u to $\frac{u^+}{\|u^+\|_{L^p(V)}}$, from $\frac{u^+}{\|u^+\|_{L^p(V)}}$ to $\frac{u^-}{\|u^-\|_{L^p(V)}}$ and $\frac{-u^-}{\|u^-\|_{L^p(V)}}$ to u:

$$u_1(t) = \frac{tu + (1-t)u^+}{\|tu + (1-t)u^+\|_{L^p(V)}},$$
$$u_2(t) = \frac{tu^+ + (1-t)u^-}{\|tu + (1-t)u^-\|_{L^p(V)}},$$
$$u_3(t) = \frac{-tu^- + (1-t)u}{\| - tu^- + (1-t)u\|_{L^p(V)}}.$$

Also we have

$$\tilde{J}_\mu(u_1(t)) = \tilde{J}_\mu(u_2(t)) = \tilde{J}_\mu(u_3(t)) = \delta.$$

By joining the paths $u_1(t)$ and $u_2(t)$ we get a new path which connects u and $\frac{u^-}{\|u^-\|_{L^p(V)}}$ and stays at levels $\leq \delta$. Call this path as $u_4(t)$. Now we define $O = \{v \in M \bigm| \tilde{J}_\mu(v) < \delta\}$. Clearly $\phi_1, -\phi_1 \in O$. Since $\frac{u^-}{\|u^-\|_{L^p(V)}}$ does not change sign and vanishes on a set of positive measure it is not a critical point of $\tilde{J}_\mu$. So $\frac{u^-}{\|u^-\|_{L^p(V)}}$ is a regular value of $\tilde{J}_\mu$, and consequently there exists a C^1 path $\eta : [-\epsilon, \epsilon] \to M$ with $\eta(0) = \frac{u^-}{\|u^-\|_{L^p(V)}}$ and $\frac{d}{dt}(\tilde{J}_\mu(\eta(t))\big|_{t=0} \neq 0$. choose a point $v \in O$ on this path (this is possible because $\tilde{J_\mu}'(\eta(t))\big|_{t=0} \neq 0$) we can thus move from $\frac{u^-}{\|u^-\|_{L^p(V)}}$ to v through this path which lies at levels $< \delta$. Taking the component of O which contains v and applying Lemma 2.3 together with Lemma 2.4, we can connect v to $+\phi_1$ (or to $-\phi_1$) with a path in M at levels $< \delta$. Let us assume that this is $+\phi_1$ which is reached in this way. Now call this path connecting $\frac{u^-}{\|u^-\|_{L^p(V)}}$ and ϕ_1 as $u_5(t)$, and consider the symmetric path $-u_5(t)$, which goes from $-\frac{u^-}{\|u^-\|_{L^p(V)}}$ to $-\phi_1$. We evaluate the functional $\tilde{J}_\mu$ along $-u_5(t)$. Since $\tilde{J}_\mu$ is even,

$$\tilde{J}_\mu(-u_5(t)) = \tilde{J}_\mu(u_5(t)) \leq \delta.$$

Finally with $u_3(t)$ we can connect $-\frac{u^-}{\|u^-\|_{L^p(V)}}$ with u by a path which stays at level δ . Putting every thing together we get a path connecting $-\phi_1$ and ϕ_1 staying at levels $\leq \delta$. This concludes the proof. □

Note that Theorem 1.4 is an immediate consequence of Lemma 2.5. So we have the following characterization of λ_2, the second smallest eigenvalue of L_μ,

$$\lambda_2 = \inf_{\gamma\in\Gamma} \sup_{u\in\gamma} \int_\Omega |\nabla u|^p dx - \int_\Omega \frac{|u|^p}{|x|^p} dx.$$

Now let μ_k be the sequence of eigenvalues obtained in [6] which are characterized as

$$\mu_k = \inf_{\mathcal{A}\in\mathcal{F}} \sup_{u\in\mathcal{A}} \int_\Omega |\nabla u|^p - \int_\Omega \frac{\mu}{|x|^p}|u|^p \,,$$

where $\mathcal{F} = \{\mathcal{A} \subset M \mid \text{ the genus of } \mathcal{A} \geq k\}$.

Corollary 2.6. *With the notation above, $\mu_2 = \lambda_2$.*

Proof. Let γ be a curve in Γ. By joining this with its symmetric path $-\gamma(t)$ we can get a set of genus ≥ 2 where J_μ does not increase its values. Therefore, $\lambda_2 \geq \mu_2$. But by Theorem 1.4, there is no eigenvalue between λ_1 and λ_2. Hence $\lambda_2 = \mu_2$. □

Lemma 2.7. *Let $u \in X$ be a solution of (1.2) and let $\mathcal{O}$ be a component of $\{x \in \Omega \,\big|\, u(x) > 0\}$. Then $u\big|_{\mathcal{O}} \in D_0^{1,p}(\mathcal{O})$*

Proof. Let $u_n \in C_c(\Omega) \cap D_0^{1,p}(\Omega)$ such that $u_n \to u$ in $D_0^{1,p}(\Omega)$. Then $u_n^+ \to u^+$ in $D_0^{1,p}(\Omega)$. Let $v_n = \min(u_n, u)$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that

$$\phi(t) = \begin{cases} 0 & \text{for } t \leq 1/2 \\ 1 & \text{for } t \geq 1 \end{cases}$$

and $|\phi'| \leq 1$. Let $\psi_r(x) = \phi\big(d(x,S)/r\big)$ where $d(x,S) = \operatorname{dist}(x,S)$. Then

$$\psi_r(x) \begin{cases} 0 & \text{for } d(x,S) \leq r/2 \\ 1 & \text{for } d(x,S) \geq r \end{cases}$$

and $|\nabla\psi_r(x)| \leq C/r$ for some constant C. Now we define $w_{n,r}(x) = \psi_r v_n(x)\big|_{\mathcal{O}}$. Since $\psi_r v_n \in C(\overline{\Omega})$, we have $w_{n,r} \in C(\overline{\mathcal{O}})$ and vanishes on the boundary $\partial\mathcal{O}$. Indeed for $x \in \partial\mathcal{O} \cap S$ then $\psi_r(x) = 0$ and so $w_{n,r}(x) = 0$. If $x \in \partial\mathcal{O} \cap \Omega$ and $x \notin S$ then $u(x) = 0$(since u is continuous except at 0) and so $v_n(x) = 0$. If $x \in \partial\Omega$ then $u_n(x) = 0$ and hence $v_n(x) = 0$. So in all the cases $w_{n,r}(x) = 0$ for $x \in \partial\mathcal{O}$.

Therefore, $w_{n,r} \in D_0^{1,p}(\mathcal{O})$.

$$\begin{aligned}\int_\Omega |\nabla(w_{n,r}) - \nabla(\psi_r u)|^p &= \int_{\mathcal{O}} |(\nabla\psi_r)v_n + \psi_r\nabla v_n - (\nabla\psi_r)u - \psi_r\nabla u|^p\,dx\\ &\le \|\nabla\psi_r v_n - \nabla\psi_r u\|^p_{L^p(\mathcal{O})} + \|\psi_r\nabla v_n - \psi_r\nabla u\|^p_{L^p(\mathcal{O})}\end{aligned}$$

which goes to 0 as $n \to \infty$. i.e., $w_{n,r} \to \psi_r u\big|_{\mathcal{O}}$ in $D_0^{1,p}(\mathcal{O})$. Now

$$\int_{\mathcal{O}} |\nabla\psi_r u + \psi_r\nabla u - u|^p \le \int_{\mathcal{O}} |\psi_r\nabla u - \nabla u|^p + \int_{\mathcal{O}\cap\{r/2<|x|<r\}} |\nabla\psi_r|^p u$$

$\to 0$ as $r \to 0$ by (1.1). Therefore, $u\big|_{\mathcal{O}} \in D_0^{1,p}(\mathcal{O})$. □

Proof of Theorem 1.5. We denote $\tilde{J}_\mu$ corresponding to V_b with $\tilde{J}_{\mu,b}$. Let u_a be a solution to

$$-\Delta_p u - \frac{\mu}{|x|^p}|u|^{p-2}u = \lambda_2 V_a(x)|u|^{p-2}u \quad \text{in } \mathcal{D}'(\Omega).$$

Assuming that the claim below is true, we have

$$\tilde{J}_{\mu,b}\left(\frac{v^+}{\|v^+\|_{L^p(V_b)}}\right) < \lambda_2(V_a), \tilde{J}_{\mu,b}\left(\frac{v^-}{\|v^-\|_{L^p(V_b)}}\right) < \lambda_2(V_a), \tilde{J}_{\mu,b}\,(v) < \lambda_2(V_a).$$

Define $\mathcal{O}_b = \{u \in X, \int_\Omega |u|^p V_b = 1, \tilde{J}_{\mu,b}\,(v) < \lambda_2(V_a).\}$ Now we proceed as in Lemma 2.5, to define the paths $v_i(t), i = 1, ..., 5$ on which $\tilde{J}_{\mu,b} < \lambda_2(V_a)$. We join these paths in a way described in Lemma 2.5 to obtain a path $\gamma(t)$ in Γ_b (the family of paths corresponds to V_b) such that $\tilde{J}_{\mu,b}(\gamma(t)) < \lambda_2(V_a)$. This completes the proof.

Claim: There exists $v \in X$, which changes sign and

$$\begin{gathered}\frac{\int_\Omega |\nabla v^+|^p\,dx - \int_\Omega \frac{\mu}{|x|^p}|v^+|^p\,dx}{\int_\Omega (v^+)^p V_b\,dx} < \lambda_2(V_a),\\ \frac{\int_\Omega |\nabla v^-|^p\,dx - \frac{\mu}{|x|^p}|v^-|^p\,dx}{\int_\Omega (v^-)^p V_b\,dx} < \lambda_2(V_a).\end{gathered} \tag{2.9}$$

Proof of Claim: Since u_a is an eigenfunction corresponding to $\lambda_2 > \lambda_1$, it has to change sign in Ω (see [6]). Let O_1 and O_2 be positive and negative nodal domains of u_a respectively such that

$$\int_{O_1} V_a\,(u_a^+)^p\,dx < \int_{O_1} V_b\,(u_a^+)^p\,dx \quad \text{and} \quad \int_{O_2} V_a\,(u_a^-)^p\,dx \le \int_{O_2} V_b\,(u_a^-)^p\,dx.$$

By Lemma 2.7, $u_a\big|_{O_1} \in D_0^{1,p}(O_1)$ and also in $L^p(O_1, V^-)$. We have

$$\lambda_1(O_1, V_b) \le \frac{\int_{O_1} |\nabla u_a|^p - \frac{\mu}{|x|^p}|u_a|^p}{\int_{O_1} |u_a|^p V_b} < \lambda_2(V_a).$$

Therefore, $\lambda_1(O_1, V_b) < \lambda_2(V_a)$. Simillarly $\lambda_1(O_2, V_b) \le \lambda_2(V_a)$. Now we modify O_1 and O_2 to get $\tilde{O_1}$ and $\tilde{O_2}$ with empty intersection and $\lambda_1(\tilde{O_1}, V_b) < \lambda_2(V_a)$ and $\lambda_1(\tilde{O_2}, V_b) < \lambda_2$. For $\eta > 0$, let $O_1(\eta) = \{x \in O_1 \mid dist(x, O_1^c) > \eta\}$. Then $\lambda_1(O_1(\eta), V_b) \ge \lambda_1(O_1, V_b)$ and $\lambda_1(O_1(\eta), V_b) \to \lambda_1(O_1, V_b)$ as $\eta \to 0$. Therefore, there exists $\eta_0 > 0$ such that $\lambda_1(O_1(\eta), V_b) < \lambda_2(V_a)$ for $0 < \eta < \eta_0$. Let $x \in \partial O_2 \cap \Omega$ and $0 < \eta < \min\{\eta_0, \operatorname{dist}(x_0, \Omega^c)\}$. Now define $\tilde{O_2} = O_2 \cup B(x_0, \eta/2)$. Then $\tilde{O_2} \cap O_1(\eta) = \emptyset$, $\lambda_1(\tilde{O_2}, V_b) < \lambda_1(O_2, V_b) < \lambda_2(V_a)$. Now we consider the function

$v = v_1 - v_2$, where v_i are the extensions by zero outside $\tilde{O}_i$ of the eigenfunctions associated to $\lambda_1(\tilde{O}_i, V_b)$. Then v satisfies (2.9). $\square$

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