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LOCAL WELL-POSEDNESS FOR A HIGHER ORDER NONLINEAR SCHRÖDINGER EQUATION IN SOBOLEV SPACES OF NEGATIVE INDICES

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Abstract. We prove that the initial value problem associated with

$$\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + i\gamma|u|^2 u = 0, \quad x, t \in \mathbb{R},$$

is locally well-posed in H^s for $s > -1/4$.

1. Introduction

In this work, we study a particular case of the initial value problem (IVP)

$$\begin{gathered}\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + F(u) = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x)\,.\end{gathered} \tag{1.1}$$

Here u is a complex valued function, $F(u) = i\gamma|u|^2 u + \delta|u|^2\partial_x u + \epsilon u^2 \partial_x \overline{u}$, $\gamma, \delta, \epsilon \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$ are constants.

Hasegawa and Kodama [10, 14] proposed (1.1) as a model for propagation of pulse in optical fiber. We will study the IVP (1.1) in Sobolev space $H^s(\mathbb{R})$ under the condition $\delta = \epsilon = 0$, $\beta \neq 0$ (see case (iv) in Theorem 1.1 below). When $\gamma, \delta, \epsilon \in \mathbb{R}$, it was shown in [16] that the flow associated to the IVP (1.1) leaves the following quantity

$$I_1(v) = \int_{\mathbb{R}} |v|^2(x,t)\,dx, \tag{1.2}$$

conserved in time. Also, when $\delta - \gamma = \epsilon \neq 0$ we have the following quantity conserved:

$$I_2(v) = c_1 \int_{\mathbb{R}} |\partial_x v|^2(x,t)\,dx + c_2 \int_{\mathbb{R}} |v|^4(x,t)dx + c_3 \int_{\mathbb{R}} v(x,t)\partial_x \overline{v(x,t)}dx, \tag{1.3}$$

where $c_1 = 3\beta\epsilon$, $c_2 = -\epsilon(\epsilon+\delta)/2$ and $c_3 = i(\alpha(\epsilon+\delta) - 3\beta\gamma)$. These quantities were used in [16] to establish global well-posedness for (1.1) in $H^s(\mathbb{R})$, $s \geq 1$. Note that the quantity $i\int_{\mathbb{R}} v(x,t)\partial_x\overline{v(x,t)}dx$ in (1.3) is real since

$$\partial_t(i \int_{\mathbb{R}} v(x,t)\partial_x\overline{v(x,t)}dx) = 2\epsilon\ \mathrm{Im}(\int_{\mathbb{R}} [v(x,t)\partial_x\overline{v(x,t)}]^2 dx).$$

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We say that the IVP (1.1) is locally well-posed in X (Banach space) if the solution uniquely exists in certain time interval $[-T, T]$ (unique existence), the solution describes a continuous curve in X in the interval $[-T, T]$ whenever initial data belongs to X (persistence), and the solution varies continuously depending upon the initial data (continuous dependence) i.e. continuity of application $u_0 \mapsto u(t)$ from X to $\mathcal{C}([-T, T]; X)$. We say that the IVP (1.1) is globally well-posed in X if the same properties hold for all time $T > 0$. If some hypothesis in the definition of local well-posed fails, we say that the IVP is ill-posed.

Particular cases of (1.1) are the following:

• Cubic nonlinear Schrödinger equation (NLS), ($\alpha = \mp 1$, $\beta = 0$, $\gamma = -1$, $\delta = \epsilon = 0$).

$$iu_t \pm u_{xx} + |u|^2 u = 0, \quad x, t \in \mathbb{R}. \tag{1.4}$$

The best known local result for the IVP associated to (1.4) is in $H^s(\mathbb{R})$, $s \geq 0$, obtained by Tsutsumi [26].

• Nonlinear Schrödinger equation with derivative ($\alpha = -1$, $\beta = 0$, $\gamma = 0$, $\delta = 2\epsilon$).

$$iu_t + u_{xx} + i\lambda(|u|^2 u)_x = 0, \quad x, t \in \mathbb{R}. \tag{1.5}$$

The best known local result for the IVP associated to (1.5) is in $H^s(\mathbb{R})$, $s \geq 1/2$, obtained by Takaoka [24].

• Complex modified Korteweg-de Vries (mKdV) equation ($\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\delta = 1$, $\epsilon = 0$).

$$u_t + u_{xxx} + |u|^2 u_x = 0, \quad x, t \in \mathbb{R}. \tag{1.6}$$

If u is real, (1.6) is the usual mKdV equation and Kenig et al. [11] proved the IVP associated to it is locally well-posed in $H^s(\mathbb{R})$, $s \geq 1/4$.

• When $\alpha \neq 0$ is real and $\beta = 0$, we obtain a particular case of the well-known mixed nonlinear Schrödinger equation

$$u_t = i\alpha u_{xx} + \lambda(|u|^2)_x u + g(u), \quad x, t \in \mathbb{R}, \tag{1.7}$$

where g satisfies some appropriate conditions. Ozawa and Tsutsumi in [19] proved that for any $\rho > 0$, there is a positive constant $T(\rho)$ depending only on ρ and g, such that the IVP (1.7) is locally well-posed in $H^{1/2}(\mathbb{R})$, whenever the initial data satisfies

$$\|u_0\|_{\mathrm{H}^{1/2}} \leq \rho.$$

There are other dispersive models similar to (1.1). The interested readers can see the following works and the references therein [1, 7, 20, 21, 23].

Laurey [17, 16] proved that the IVP associated to (1.1) is locally well-posed in $H^s(\mathbb{R})$, $s > 3/4$. Staffilani [22] improved this result by proving the IVP associated to (1.1) is locally well-posed in $H^s(\mathbb{R})$, $s \geq 1/4$.

When α, β are functions of t, we proved in [2, 3] local well-posedness in $H^s(\mathbb{R})$, $s \geq 1/4$. Also we studied in [2, 5] the unique continuation property for the solution of (1.1). Regarding the ill-posedness of the IVP (1.1), we proved in [4] the following theorem.

Theorem 1.1. *The mapping data-solution $u_0 \mapsto u(t)$ for the IVP (1.1) is not $\mathcal{C}^3$ at origin in the following cases:*

(i) $\beta = 0$, $\alpha \neq 0$, $\delta = \epsilon = 0$, $\gamma \neq 0$ *for* $s < 0$.
(ii) $\beta = 0$, $\alpha \neq 0$, $\delta \neq 0$ *or* $\epsilon \neq 0$ *for* $s < 1/2$.
(iii) $\beta \neq 0$, $\delta \neq 0$ *or* $\epsilon \neq 0$ *for* $s < 1/4$.
(iv) $\beta \neq 0$, $\delta = \epsilon = 0$, $\gamma \neq 0$ *for* $s < -1/4$.

In this work, we consider the case (iv) and prove the following result.

Theorem 1.2. *Let $\beta \neq 0$ real and $\gamma \neq 0$ complex, then the following IVP*

$$\begin{gathered} \partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + i\gamma|u|^2 u = 0, \quad x,t \in \mathbb{R}, \\ u(x,0) = u_0, \end{gathered} \tag{1.8}$$

is locally well-posed in $H^s(\mathbb{R})$, $s > -1/4$.

The following trilinear estimate will be fundamental in the proof of Theorem 1.2.

Theorem 1.3. *Let $-1/4 < s \leq 0$, $b > 7/12$, $b' < s/3$, then we have*

$$\|uv\overline{w}\|_{\mathrm{X}^{s,b'}} \leq C\|u\|_{\mathrm{X}^{s,b}}\|v\|_{\mathrm{X}^{s,b}}\|w\|_{\mathrm{X}^{s,b}}, \tag{1.9}$$

where

$$\|u\|_{\mathrm{X}^{s,b}} = \|\langle\xi\rangle^s\langle\tau - \phi(\xi)\rangle^b\hat{u}\|_{\mathrm{L}^2_\xi\mathrm{L}^2_\tau}, \quad \langle\xi\rangle = 1 + |\xi|, \quad \phi(\xi) = \alpha\xi^2 + \beta\xi^3.$$

Theorem 1.4. *The trilinear estimate (1.9) fails if $s < -1/4$ and $b \in \mathbb{R}$.*

Remarks. • When $\gamma \in \mathbb{R}$, as (1.1) preserves L^2 norm, Theorem 1.2 permits to obtain global existence in L^2.
• From Lemma 2.3 we note that $b = 7/12+$ is the best possible for $s = -1/4+$, in the trilinear estimate (1.9).
• The trilinear estimate is valid for all $s > 0$, as it can be seen by combining $\langle\xi\rangle^s \leq \langle\xi - (\xi_2 - \xi_1)\rangle^s\langle\xi_2\rangle^s\langle\xi_1\rangle^s$ and the estimate (1.9) for $s = 0$.
• We will use the notation $\|u\|_{\{s,b\}} := \|u\|_{X^{s,b}}$.
• When $\alpha = 0, \beta = 1$, we have the usual bilinear estimate due to Kenig et al. [12],

$$\|(uv)_x\|_{\{-3/4+,-1/2+\}} \leq C\|u\|_{\{-3/4+,1/2+\}}\|v\|_{\{-3/4+,1/2+\}}.$$

Also we have the 1/4 trilinear estimate due to Tao [25],

$$\|(uvw)_x\|_{\{1/4,-1/2+\}} \leq C\|u\|_{\{1/4,1/2+\}}\|v\|_{\{1/4,1/2+\}}\|w\|_{\{1/4,1/2+\}}.$$

2. Proofs of Main Result

Proof of Theorem 1.4. As in [12] consider the set

$$B := \{(\xi,\tau); N \leq \xi \leq N + N^{-1/2}, |\tau - \phi(\xi)| \leq 1\},$$

where $\phi(\xi) = \alpha\xi^2 + \beta\xi^3$. We have $|B| \sim N^{-1/2}$. Let us consider $\hat{v} = \chi_B$, it is not difficult to see that $\|v\|_{\{s,b\}} \leq N^s|B|^{1/2}$. Moreover

$$\mathcal{F}(|v|^2\overline{v}) := \chi_B * \chi_B * \chi_{-B} \gtrsim \frac{1}{N}\chi_A,$$

where A is a rectangle contained in B such that $|A| \sim N^{-1/2}$.
Therefore

$$\|\,|v|^2\overline{v}\|_{\{s,b'\}} = \|\langle\xi\rangle^s\langle\tau - \phi(\xi)\rangle^{b'}\mathcal{F}(|v|^2\overline{v})\|_{\mathrm{L}^2_\xi\mathrm{L}^2_\tau} \gtrsim N^s\frac{1}{N}N^{-1/4} = N^{s-5/4}.$$

As a consequence, for large N the trilinear estimate fails if $3(s - 1/4) < s - 5/4$, i.e. if $s < -1/4$. □

Proof of Theorem 1.3. In Lemma 2.1 below, we gather some elementary estimates needed in the proof of Theorem 1.3, we need the following results from elementary calculus.

Lemma 2.1. (1) *If $b > 1/2$, $a_1, a_2 \in \mathbb{R}$ then*
$$\int_{\mathbb{R}} \frac{dx}{\langle x-a_1\rangle^{2b}\langle x-a_2\rangle^{2b}} \sim \frac{1}{\langle a_1-a_2\rangle^{2b}}. \tag{2.1}$$
(2) *If $0 < c_1, c_2 < 1$, $c_1 + c_2 > 1$, $a_1 \neq a_2$, then*
$$\int_{\mathbb{R}} \frac{dx}{|x-a_1|^{c_1}|x-a_2|^{c_2}} \lesssim \frac{1}{|a_1-a_2|^{(c_1+c_2-1)}}. \tag{2.2}$$
(3) *Let $a \in \mathbb{R}$, $c_1 \le c_2$, then*
$$\frac{|x|^{c_1}}{\langle ax\rangle^{c_2}} \le \frac{1}{|a|^{c_1}}. \tag{2.3}$$
(4) *Let $a, \eta \in \mathbb{R}$, $b > 1/2$, then*
$$\int_{\mathbb{R}} \frac{dx}{\langle a(x^2-\eta^2)\rangle^{2b}} \lesssim \frac{1}{|a\eta|}. \tag{2.4}$$

Now, let $f(\xi,\tau) = \langle \xi\rangle^s \langle \tau - \xi^3\rangle^b \hat{u}$, $g(\xi,\tau) = \langle \xi\rangle^s \langle \tau - \xi^3\rangle^b \hat{v}$, $h(\xi,\tau) = \langle \xi\rangle^s$, $\langle \tau - \xi^3\rangle^b \hat{w}$, $\eta = (\xi,\tau)$, $x = (\xi_1,\tau_1)$, $y = (\xi_2,\tau_2)$. We have
$$\begin{aligned}\|uv\overline{w}\|_{\{s,b'\}} =& \| \int_{\mathbb{R}^4} f(\eta+x-y)g(y)\overline{h}(x)K(\eta,x,y)dxdy\|_{\mathrm{L}^2_\eta}\\ \le& \|K(\eta,x,y)\|_{\mathrm{L}^\infty_\eta \mathrm{L}^2_{x,y}}\|f\|_{\mathrm{L}^2}\|g\|_{\mathrm{L}^2}\|h\|_{\mathrm{L}^2},\end{aligned}$$
where
$$K(\eta,x,y) = \frac{\langle \xi+\xi_1-\xi_2\rangle^\rho \langle \xi_2\rangle^\rho \langle \xi_1\rangle^\rho}{r(\xi,\tau)\langle \tau_1-\phi(\xi_1)\rangle^b\langle \tau_2-\phi(\xi_2)\rangle^b\langle \tau+\tau_1-\tau_2-\phi(\xi+\xi_1-\xi_2)\rangle^b}$$
and $r(\xi,\tau) = \langle \xi\rangle^\rho\langle \tau-\phi(\xi)\rangle^{-b'}$, $\rho = -s$. Using (2.1) we obtain
$$\begin{aligned}I(\xi,\tau) := \|K\|^2_{\mathrm{L}^2_{x,y}} \sim& \frac{1}{r(\xi,\tau)^2}\int_{\mathbb{R}^2}\frac{G_\rho(\xi,\xi_1,\xi_2)\ d\xi_1 d\xi_2}{\langle \tau-\phi(\xi+\xi_1-\xi_2)-\phi(\xi_2)+\phi(\xi_1)\rangle^{2b}}\\ =& \frac{1}{r(\xi,\tau)^2}\int_{\mathbb{R}^2}\frac{G_\rho(\xi,\xi_1,\xi_2)\ d\xi_1 d\xi_2}{\langle \tau-\phi(\xi)+g(\xi,\xi_1,\xi_2)\rangle^{2b}}\end{aligned}$$
where
$$\begin{gathered}G_\rho(\xi,\xi_1,\xi_2) := \langle \xi+\xi_1-\xi_2\rangle^{2\rho}\langle \xi_1\rangle^{2\rho}\langle \xi_2\rangle^{2\rho},\\ g(\xi,\xi_1,\xi_2) = (\xi_1-\xi_2)(\xi+\xi_2)(2\alpha+3\beta(\xi-\xi_1)).\end{gathered} \tag{2.5}$$
Assuming $y = \tau - \phi(\xi)$, to get Theorem 1.3 it is sufficient to prove the following lemma.

Lemma 2.2. *Let $0 \le \rho < 1/4$, $b > 7/12$, $b' < -\rho/3$. Then*
$$I(\xi,y) := \frac{1}{\langle \xi\rangle^{2\rho}\langle y\rangle^{-2b'}}\int_{\mathbb{R}^2}\frac{G_\rho(\xi,-\xi_1,-\xi_2)d\xi_1 d\xi_2}{\langle y+g(\xi,\xi_1,\xi_2)\rangle^{2b}} \le C(\rho,b,b') < \infty,$$
where $C(\rho,b,b')$ is a constant independent of ξ and y.

To prove Lemma 2.2 we need to prove the following lemmas.

Lemma 2.3. *Let $\rho < 1/4$. Then*

$$I(0,0) = \int_{\mathbb{R}^2} \frac{G_\rho(0,-\xi_1,-\xi_2)d\xi_1 d\xi_2}{\langle g(0,\xi_1,\xi_2)\rangle^{2b}} = \begin{cases} C(\rho,b) < \infty & \text{if } \rho + 1/3 < b \\ \infty & \text{if } \rho + 1/3 \geq b, \end{cases}$$

where $C(\rho,b)$ is a constant.

We have that if $b = 7/12$ and $\rho = -s \geq 1/4$ then $I(0,0) = \infty$, therefore Lemma 2.3 shows that $b = 7/12+$ is the best possible when $s = -1/4+$.

Lemma 2.4. *Let $0 \leq \rho < 1/4$, $b > 7/12$. Then*

$$I(\xi,0) = \frac{1}{\langle \xi\rangle^{2\rho}} \int_{\mathbb{R}^2} \frac{G_\rho(\xi,-\xi_1,-\xi_2)d\xi_1 d\xi_2}{\langle g(\xi,\xi_1,\xi_2)\rangle^{2b}} \leq C(\rho,b),$$

where $C(\rho,b)$ is a constant independent of ξ.

For clarity in exposition, we consider the case $\alpha = 0$, $\beta = 1$, i.e. $\phi(\xi) = \xi^3$ (see the observation at the end of the proof of Lemma 2.2).

In the definition of $I(\xi,y)$ if we make the change of variables $\xi - \xi_1 := \xi\xi_1$, $\xi + \xi_2 := \xi\xi_2$ and $y = \xi^3 z$, then $I(\xi,y)$ becomes

$$I(\xi,z) = p(\xi,z) \int_{\mathbb{R}^2} \frac{H_\rho(\xi,\xi_1,\xi_2)d\xi_1 d\xi_2}{\langle \xi^3(z + F(\xi_1,\xi_2))\rangle^{2b}}, \tag{2.6}$$

where $p(\xi,z) = \xi^2\langle \xi^3 z\rangle^{2b'}\langle \xi\rangle^{-2\rho}$, $F(\xi_1,\xi_2) = (2-(\xi_1+\xi_2))\xi_1\xi_2$ and

$$H_\rho(\xi,\xi_1,\xi_2) = \langle \xi(1-(\xi_1+\xi_2))\rangle^{2\rho}\langle \xi(1-\xi_1)\rangle^{2\rho}\langle \xi(1-\xi_2)\rangle^{2\rho}.$$

From here onwards we will suppose $z > 0$, because if $z < 0$ we can obtain the same result by symmetry (see Remark after Proposition 1).

Proof of Lemma 2.3. By symmetry it is sufficient to prove that the integrals

$$\begin{aligned} I_1(0,0) &:= \int_0^\infty \int_0^\infty \frac{G_\rho(0,-\xi_1,-\xi_2)d\xi_1 d\xi_2}{\langle g(0,\xi_1,\xi_2)\rangle^{2b}}, \\ I_2(0,0) &:= \int_0^\infty \int_0^\infty \frac{G_\rho(0,-\xi_1,\xi_2)d\xi_1 d\xi_2}{\langle g(0,\xi_1,-\xi_2)\rangle^{2b}} \end{aligned}$$

are finite. We will prove that $I_1(0,0)$ is finite only; the same proof works for $I_2(0,0)$. Also, by symmetry we can suppose that $0 \leq \xi_2 \leq \xi_1$. We have

$$\begin{aligned} \int_1^\infty d\xi_1 \int_0^{\xi_1} d\xi_2 \frac{G_\rho(0,-\xi_1,-\xi_2)}{\langle g(0,\xi_1,\xi_2)\rangle^{2b}} &= \int_1^\infty d\xi_1 \int_0^{\xi_1/2} d\xi_2 + \int_1^\infty d\xi_1 \int_{\xi_1/2}^{\xi_1} d\xi_2 \\ &= I_{1,1} + I_{1,2}. \end{aligned} \tag{2.7}$$

Since $0 \leq \xi_2 \leq \xi_1$, we have $G_\rho(0,-\xi_1,-\xi_2) \leq \langle \xi_1\rangle^{4\rho}\langle \xi_2\rangle^{2\rho}$. In $I_{1,1}$ we have $\xi_1/2 < \xi_1 - \xi_2 < \xi_1$, therefore if $b > \rho + 1/3$,

$$\begin{aligned} I_{1,1} &\lesssim \int_1^\infty \langle \xi_1\rangle^{4\rho} d\xi_1 \int_0^{\xi_1/2} \frac{\langle \xi_2\rangle^{2\rho} d\xi_2}{\langle 3\xi_1^2\xi_2\rangle^{2b}} \\ &\lesssim \int_1^\infty \langle \xi_1\rangle^{4\rho}\Big(\frac{1}{\xi_1^2} + \frac{1}{\xi_1^{2+4\rho}} + \frac{1}{\xi_1^{2+4\rho}} \int_1^{3\xi_1^3/2} \frac{x^{2\rho}dx}{(1+x)^{2b}}\Big)d\xi_1 \\ &= C(\rho,b) < \infty. \end{aligned}$$

Analogously we can prove that $I_{1,1} = \infty$ if $b \leq \rho + 1/3$. In $I_{1,2}$ we have $\xi_1/2 \leq \xi_2 \leq \xi_1$, so

$$\begin{aligned} I_{1,2} &\lesssim \int_1^\infty \langle \xi_1 \rangle^{4\rho} d\xi_1 \int_{\xi_1/2}^{\xi_1} \frac{\langle \xi_1 - \xi_2 \rangle^{2\rho} d\xi_2}{\langle (\xi_1 - \xi_2)\xi_1^2 \rangle^{2b}} \\ &= \int_1^\infty \langle \xi_1 \rangle^{4\rho} d\xi_1 \int_0^{\xi_1/2} \frac{\langle x \rangle^{2\rho} dx}{\langle \xi_1^2 x \rangle^{2b}} \\ &\lesssim C(\rho, b), \quad b > \rho + 1/3. \end{aligned}$$

□

The propositions will be useful for proving Lemmas 2.2 and 2.4.

Proposition 1. *Let $0 \leq \rho < 1/4$, $b > 1/3 + 2\rho/3$, then we have*

$$J_1 = \xi^{2+4\rho} \int_{\mathbb{R}^2} \frac{d\xi_1 d\xi_2}{\langle \xi^3 (z + F) \rangle^{2b}} \leq C,$$

where C is a constant independent of ξ and z.

Proof. If $\xi_1 \leq 0$, $\xi_2 \leq 0$, then $|z+F| \geq |\xi_1 + \xi_2||\xi_1\xi_2|$. Therefore by Lemma 2.3 and by symmetry, it is enough to consider $\xi_1 \geq 0$. We have $|z + F| = |\xi_1||(\xi_2 + (\xi_1 - 2)/2)^2 - (\xi_1 - 2)^2/4 - z/\xi_1|$. Let $l^2 = (\xi_1 - 2)^2/4 + z/\xi_1$, $c(\rho) = (2 + 4\rho)/3$, then making change of variable $\eta = \xi_2 + (\xi_1 - 2)/2$ and using (2.2) and (2.3) we have

$$\begin{aligned} J_1 &= \xi^{2+4\rho} \int_0^\infty d\xi_1 \int_{\mathbb{R}} \frac{d\eta}{\langle \xi^3 \xi_1 (\eta^2 - l^2) \rangle^{2b}} \\ &\lesssim \int_0^\infty d\xi_1 \int_{\mathbb{R}} \frac{l\, dx}{[|\xi_1| l^2 |x^2 - 1|]^{c(\rho)}} \\ &\lesssim \int_0^\infty \frac{d\xi_1}{|\xi_1|^{c(\rho)} |\xi_1 - 2|^{(1+8\rho)/3}} \int_{\mathbb{R}} \frac{dx}{|x^2 - 1|^{c(\rho)}} \\ &\lesssim 2^{-(20\rho+1)/3}, \quad 0 < \rho < 1/4. \end{aligned}$$

The case $\rho = 0$ follows from the case $0 < \rho < 1/4$, taking the limit. □

Remark. When $z < 0$, we make $\xi_1 := -\xi_1$, $\xi_2 := -\xi_2$ then $|z + F| = |\xi_1||(\xi_2 + (\xi_1 + 2)/2)^2 - (\xi_1 + 2)^2/4 + z/\xi_1|$ and the proof is similar.

Proposition 2. *Let $|\xi| > 1$, $b > 1/2$, $0 \leq \rho < 1/4$. Then*

$$J_2 = \xi^{2+4\rho} \int_0^\infty \xi_1^{4\rho} d\xi_1 \int_{\mathbb{R}} \frac{d\xi_2}{\langle \xi^3 (z + F) \rangle^{2b}} \leq C,$$

where C is a constant independent of ξ and z.

Proof. By Proposition 1 we can suppose $\xi_1 > 4$, so $(\xi_1 - 2) > \xi_1/2$. Using (2.4) and making change of variables as above, we have

$$J_2 \lesssim \frac{\xi^{2+4\rho}}{|\xi|^3} \int_4^\infty \frac{\xi_1^{4\rho}}{\xi_1\, l} d\xi_1 \leq C.$$

□

Proof of Lemma 2.4. **Case** $|\xi| \leq 1$. Let

$$\begin{aligned} A_1 &= \{(\xi_1, \xi_2)/|\xi_1| > 2, |\xi_2| > 2\}, \quad A_2 = \{(\xi_1, \xi_2)/|\xi_1| \leq 2, |\xi_2| \leq 2\}, \\ A_3 &= \{(\xi_1, \xi_2)/|\xi_1| \leq 2, |\xi_2| > 2\}, \quad A_4 = \{(\xi_1, \xi_2)/|\xi_1| > 2, |\xi_2| \leq 2\}. \end{aligned}$$

Consider $I(\xi,0)=\sum_{j=1}^4 I_j(\xi,0)$, where $I_j(\xi,0)$ is defined in the region A_j. Obviously $I_2(\xi,0)\le C$. In A_1 we have $|\xi-\xi_1|>|\xi_1|/2$ and $|\xi+\xi_2|>|\xi_2|/2$, therefore Lemma 2.3 gives $I_1(\xi,0)\le C$. In A_3 we have $|\xi+\xi_2|>|\xi_2|/2$, and consequently

$$\begin{aligned}
I_3(\xi,0) \lesssim & \frac{1}{\langle\xi\rangle^{2\rho}}\int_{A_3}\frac{\langle\xi_2\rangle^{4\rho}d\xi_1 d\xi_2}{\langle(\xi_1-\xi_2)\xi_2(\xi-\xi_1)\rangle^{2b}}\\
=&\frac{1}{\langle\xi\rangle^{2\rho}}\int_{A_3\cap\{|\xi_1-\xi_2|>|\xi_2|\}} + \frac{1}{\langle\xi\rangle^{2\rho}}\int_{A_3\cap\{|\xi_1-\xi_2|\le|\xi_2|\}}\\
=&I_{3,1}(\xi,0)+I_{3,2}(\xi,0).
\end{aligned}$$

In the first integral, for $\rho<1/4$, $b>1/2$ we have

$$\begin{aligned}
I_{3,1}(\xi,0) &\lesssim \frac{1}{\langle\xi\rangle^{2\rho}}\int_{|\xi_2|>2}\langle\xi_2\rangle^{4\rho}d\xi_2\int_{|\xi_1|\le 2}\frac{d\xi_1}{\langle\xi_2^2(\xi-\xi_1)\rangle^{2b}}\\
&\lesssim \frac{1}{\langle\xi\rangle^{2\rho}}\int_{|\xi_2|>2}\frac{\langle\xi_2\rangle^{4\rho}d\xi_2}{\xi_2^2}\le C.
\end{aligned}$$

To estimate $I_{3,2}(\xi,0)$ we make the change of variables $\eta_2=\xi_1-\xi_2$, $\eta_1=\xi_1$ and as $|\xi_1|\le 2$ we obtain the same estimate as that for $I_{3,1}(\xi,0)$.

By symmetry we can estimate $I_4(\xi,0)$ in the same manner as $I_3(\xi,0)$.

Case $|\xi|>1$. Let us consider $I(\xi,0)$ in the form (2.6) and let $B_1=\{|\xi_1+\xi_2|>4\}$ and $B_2=\{|\xi_1+\xi_2|\le 4\}$, then $I(\xi,0)=I_1(\xi)+I_2(\xi)$, where $I_j(\xi)$ is defined in B_j. In B_1 we have

$$|2-(\xi_1+\xi_2)|>|\xi_1+\xi_2|/2,\quad |1-(\xi_1+\xi_2)|\le 5|\xi_1+\xi_2|/4, \tag{2.8}$$

moreover $B_1\subset\{|\xi_1|\ge 2\}\cup\{|\xi_2|\ge 2\}=:B_{1,1}\cup B_{1,2}$ and therefore $I_1(\xi)\le I_{1,1}(\xi)+I_{1,2}(\xi)$, where $I_{1,j}(\xi)$ is defined in $B_{1,j}\cap B_1$. In $B_{1,1}$ we have $|\xi_1|/2\le|1-\xi_1|\le 3|\xi_1|/2$, therefore using (2.8), we obtain that $I_{1,1}(\xi)\lesssim I(0,0)\le C$ if $\rho<1/4$, $\rho+1/3<b$. In similar manner we have $I_{1,2}(\xi)\lesssim I(0,0)\le C$.

From definition of B_2 we have $H_\rho\lesssim\langle\xi\rangle^{2\rho}\langle\xi+\xi|\xi_1|\rangle^{4\rho}$, so using symmetry and Propositions 1 and 2, we have $I_2(\xi)\le C<\infty$ if $0\le\rho<1/4$, $b>\rho+1/3$. □

Proof of Lemma 2.2. Let $0\le\rho<1/4$, $b>7/12$, $b'<-\rho/3$. Using symmetry and Lemma 2.4 it is sufficient to prove

$$J=p(\xi,z)\int_0^\infty\int_{\mathbb{R}}\frac{H_\rho(\xi,\xi_1,\xi_2)d\xi_1 d\xi_2}{\langle\xi^3(z+F(\xi_1,\xi_2))\rangle^{2b}}\le C<\infty.$$

By Lemma 2.4 we can suppose $|\xi|^3 z\ge 1$; since if $|\xi|^3 z<1$,

$$\langle\xi^3(z+F)\rangle^{-2b}\le 2^{2b}\langle\xi^3F\rangle^{-2b}.$$

Also by symmetry we can suppose $|\xi_2|\le|\xi_1|$. Therefore

$$H_\rho(\xi,\xi_1,\xi_2)\lesssim 1+|\xi|^{6\rho}+|\xi|^{6\rho}|\xi_1|^{6\rho}.$$

Using Proposition 1 we can suppose $|\xi_1|>4$ ($l^{-1}\le|\xi_1|^{-1}$).

Case $|\xi||\xi_1|\le 1$. We have $H_\rho\lesssim\langle\xi\rangle^{6\rho}$ and therefore $J\le C<\infty$, by Proposition 1.

Case $|\xi||\xi_1|>1$.

i) If $|\xi_1|^3\le z$, $|\xi_1|\le z^{1/3}$, we have $H_\rho(\xi,\xi_1,\xi_2)\lesssim 1+|\xi|^{6\rho}+|z|^{2\rho/3}|\xi|^{6\rho}|\xi_1|^{4\rho}$.

Therefore using (2.4), in this region we have

$$\begin{aligned}\frac{\xi^{2+6\rho}|z|^{2\rho/3}}{\langle\xi^3 z\rangle^{-2b'}}\int_{1/|\xi|}^{|z|^{1/3}}|\xi_1|^{4\rho}d\xi_1\int_{\mathbb{R}}\frac{d\eta}{\langle\xi^3\xi_1(\eta^2-l^2)\rangle^{2b}} &\lesssim \frac{\xi^{2+6\rho}|z|^{2\rho/3}}{\langle\xi^3 z\rangle^{-2b'}|\xi|^3}\int_{1/|\xi|}^{\infty}\frac{|\xi_1|^{4\rho}d\xi_1}{|\xi_1|^2}\\ &\lesssim \frac{(|\xi|^3 z)^{2\rho/3}}{\langle\xi^3 z\rangle^{-2b'}}\\ &\le C.\end{aligned}$$

ii) If $|\xi_1|^3 \ge z$, $|\xi_1| \ge z^{1/3}$, we can proceed as follows. By Lemma 2.4 we can suppose $|z+F| \le |F|/2$, so $|F| \le 2z$, $|(2-(\xi_1+\xi_2))\xi_1\xi_2| \le 2z$. This implies that $|1-\xi_2||1-(\xi_1+\xi_2)| \lesssim 1+|\xi_1|+z^{2/3}$. Therefore

$$\begin{aligned}H_\rho \lesssim &(\langle\xi\rangle^{4\rho}+|\xi|^{6\rho})+|\xi|^{4\rho}|\xi_1|^{4\rho}+|\xi|^{6\rho}|\xi_1|^{2\rho}+|\xi|^{4\rho}|\xi_1|^{2\rho}+|\xi|^{6\rho}|\xi_1|^{4\rho}\\ &+|\xi|^{4\rho}z^{4\rho/3}+|\xi|^{6\rho}z^{4\rho/3}+|\xi|^{6\rho}z^{4\rho/3}|\xi_1|^{2\rho} =: \sum_{j=1}^{8} l_j.\end{aligned}$$

We have,

$$\frac{|\xi|^{6\rho}}{\langle\xi\rangle^{2\rho}} \le |\xi|^{4\rho}. \tag{2.9}$$

To estimate the term that contains $l_1 = \langle\xi\rangle^{4\rho}+|\xi|^{6\rho}$, we use (2.9) and Proposition 1.

For terms l_j, $j=2,\dots,5$, we use (2.9) and Propositions 1 and 2 if $|\xi|>1$. If $|\xi|<1$, we integrate in the region $\xi_1 > 1/|\xi|$ as above.

In $l_6 = |\xi|^{4\rho}z^{4\rho/3}$, we have

$$\frac{|\xi|^2|\xi|^{4\rho}z^{4\rho/3}}{\langle\xi^3 z\rangle^{-2b'}|\xi|^3\langle\xi\rangle^{2\rho}}\int_{z^{1/3}}^{\infty}\frac{d\xi_1}{\xi_1^2} \lesssim \frac{1}{(|\xi|^3 z)^{(1-4\rho)/3}} \le C.$$

We estimate $l_7 = |\xi|^{6\rho}z^{4\rho/3}$, as in l_6 using (2.9). Finally in $l_8 = |\xi|^{6\rho}z^{4\rho/3}|\xi_1|^{2\rho}$, we have

$$\frac{|\xi|^{2+6\rho}z^{4\rho/3}}{\langle\xi^3 z\rangle^{-2b'}\langle\xi\rangle^{2\rho}|\xi|^3}\int_{z^{1/3}}^{\infty}\frac{|\xi_1|^{2\rho}d\xi_1}{\xi_1^2} \lesssim \frac{(|\xi|^3 z)^{(6\rho-1)/3}}{\langle\xi^3 z\rangle^{-2b'}} \le C.$$

□

Remark. In the case $\alpha \ne 0$ under little modifications, the proofs of Propositions 1 and 2 and the proofs of Lemmas 2.2, 2.3 and 2.4 are similar to the case $\alpha = 0$. For example in order to prove Lemma 2.3 with $\alpha \ne 0$ we proceed as follows

In (2.5) we have $g(\xi,\xi_1,\xi_2) = (\xi_1-\xi_2)\xi_2(2\alpha-3\beta\xi_1))$. In order to obtain symmetry in ξ_1 and ξ_2, we consider the change of variable $2\alpha-3\beta\xi_1 := 3\beta\xi_1$. In this way we have

$$I(0,0) \lesssim C\left(\frac{\alpha}{\beta}\right)\int_{\mathbb{R}^2}\frac{\langle\xi_1+\xi_2\rangle^{2\rho}\langle\xi_1\rangle^{2\rho}\langle\xi_2\rangle^{2\rho}d\xi_1 d\xi_2}{\left\langle\beta\left(\frac{2\alpha}{\beta}-(\xi_1+\xi_2)\right)\xi_1\xi_2\right\rangle^{2\rho}}. \tag{2.10}$$

Now using symmetry, the rest of the proof is the same as that of Lemma 2.3, if we replace the lower limit 1 in the integrals in (2.7) by $4\alpha/3\beta$.

Proof of Theorem 1.2

Consider a cut-off function $\psi \in \mathcal{C}^\infty$, such that $0 \le \psi \le 1$,

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \le 1 \\ 0 & \text{if } |t| \ge 2, \end{cases}$$

and let $\psi_T(t) := \psi(t/T)$. To prove Theorem 1.2 we need the following result.

Proposition 3. *Let $-1/2 < b' \le 0 \le b \le b' + 1$, $T \in [0,1]$. Then*

$$\|\psi_1(t)U(t)u_0\|_{\{s,b\}} = C\|u_0\|_{\mathrm{H}^s}, \tag{2.11}$$

$$\|\psi_T(t)\int_0^t U(t-t')F(t',\cdot))dt'\|_{\{s,b\}} \le CT^{1-b+b'}\|F(u)\|_{\{s,b'\}}, \tag{2.12}$$

where $F(u) := i\gamma|u|^2u$.

The proof of (2.11) is obvious, and the proof of (2.12) is practically done in [8].

Let us consider (1.8) in its equivalent integral form:

$$u(t) = U(t)u_0 - \int_0^t U(t-t')F(u)(t',\cdot)dt'. \tag{2.13}$$

Note that, if for all $t \in \mathbb{R}$, $u(t)$ satisfies:

$$u(t) = \psi_1(t)U(t)u_0 - \psi_T(t)\int_0^t U(t-t')F(u)(t',\cdot)dt',$$

then $u(t)$ satisfies (2.13) in $[-T,T]$. Let $a > 0$ and

$$X_a = \{v \in X^{s,b}; \|v\|_{\{s,b\}} \le a\}.$$

For $v \in X_a$ fixed, let us define

$$\Phi(v) = \psi_1(t)U(t)u_0 - \psi_T(t)\int_0^t U(t-t')F(v)(t',\cdot)dt'.$$

Let $\epsilon = 1 - b + b' > 0$, $b - 1 < b' < s/3$ (this implies $7/12 < b < 11/12$) using Proposition 3 and Theorem 1.3 we obtain

$$\|\Phi(v)\|_{s,b} \le C\|u_0\|_{\mathrm{H}^s} + CT^\epsilon\|F(v)\|_{s,b'} \le C\|u_0\|_{\mathrm{H}^s} + CT^\epsilon a^3 \le a,$$

where $a = 2C\|u_0\|_{\mathbf{H}^s}$ and $T^\epsilon \le 1/(2Ca^2)$.

We can prove that Φ is a contraction in an analogous manner. The proof of Theorem 1.2 follows by using a standard argument, see for example [11, 12].

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