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Singular solutions of doubly singular parabolic equations with absorption *

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Abstract

In this paper we study a doubly singular parabolic equation with absorption,

$$u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) - u^q$$

with $m > 0$, $p > 1$, $m(p-1) < 1$, and $q > 1$. We give a complete classification of solutions, which we call singular, that are non-negative, non-trivial, continuous in $\mathbb{R}^n \times [0,\infty)\backslash\{(0,0)\}$, and satisfy $u(x,0) = 0$ for all $x \neq 0$. Applications of similar but simpler equations show that these solutions are very important in the study of intermediate asymptotic behavior of general solutions.

1 Introduction

We are interested in the study of singular solutions to the doubly singular parabolic equation with absorption:

$$u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) - u^q \quad \text{in } \mathbb{R}^n \times (0,+\infty)\,, \tag{1.1}$$

where $m > 0$, $p > 1$, $m(p-1) < 1$, and $q > 1$.

Here by a **singular solution** we mean a non-negative and non-trivial solution which is continuous in $\mathbb{R}^n \times [0,+\infty)\backslash\{(0,0)\}$ and satisfies

$$\lim_{t\searrow 0}\sup_{|x|>\varepsilon} u(x,t) = 0 \quad \forall \varepsilon > 0\,. \tag{1.2}$$

A singular solution is called a **fundamental solution** (FS for short) if, for some $c > 0$,

$$\lim_{t\searrow 0}\int_{|x|\le\varepsilon} u(x,t)dx = c \quad \forall \varepsilon > 0\,. \tag{1.3}$$

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A singular solution is called a **very singular solution** (VSS for short) if

$$\lim_{t\searrow 0}\int_{|x|\le\varepsilon} u(x,t)dx=\infty \quad \forall ep>0\,. \tag{1.4}$$

By a **self-similar solution** we mean a solution u that has the form

$$u(x,t)=\Big(\frac{\alpha}{t}\Big)^{\alpha} f\left(|x|\Big(\frac{\alpha}{t}\Big)^{\alpha\beta}\right),\quad \alpha:=\frac{1}{q-1},\quad \beta:=\frac{q-m(p-1)}{p}, \tag{1.5}$$

where f as a function of $r=|x|(\alpha/t)^{\alpha\beta}$ is defined on $[0,+\infty)$ and solves

$$(|(f^m)'|^{p-2}(f^m)')'+\frac{n-1}{r}|(f^m)'|^{p-2}(f^m)'+\beta r f'+f-f^q=0 \quad \forall r>0. \tag{1.6}$$

Note that for u given by (1.5), the condition (1.2) is equivalent to

$$\lim_{r\to\infty} r^{1/\beta} f(r)=0. \tag{1.7}$$

Furthermore, if $q<m(p-1)+p/n$ (i.e. $n\beta<1$) and the solution f of (1.6) satisfies (1.7), then $u(x,t)$ given explicitly by (1.5) satisfies (1.4), i.e., it is a very singular self-similar solution of (1.1).

Recently, Leoni [14] proved that problem (1.6), (1.7) has a solution, that is, (1.1) has a self-similar VSS, if and only if $q<m(p-1)+p/n$. In the present paper we will give a complete classification for all singular solutions of (1.1), under the assumptions that $m>0$, $p>1$ satisfying $m(p-1)<1$ and $q>1$. More importantly, we obtain the existence and uniqueness of both FS and VSS, self-similar or otherwise.

Our main results read as follows:

Theorem 1.1 *Assume that $m>0$, $p>1$, $m(p-1)<1$,and $q>1$. Then the following statements hold:*

(i) *Every singular solution of (1.1) is either an FS or a VSS;*

(ii) *When $q\ge m(p-1)+p/n$, (1.1) does not have any singular solution;*

(iii) *When $q<m(p-1)+p/n$, (1.1) admits a unique VSS, u_∞ and for every $c>0$, a unique FS, u_c, with initial mass c. In addition, $u_{c_1}<u_{c_2}$ for any $c_1<c_2$ and $u_c\to u_\infty$ as $c\to\infty$;*

(iv) *When $p\le n(1+m)/(1+mn)$, (1.1) does not have any singular solution.*

Because $m(p-1)<1$, the equation (1.1) is called *doubly singular*, which resembles both the porous medium equations of fast diffusion

$$u_t=\Delta(u^m)-u^q, \tag{1.8}$$

and the p-Laplacian equations with $1<p<2$

$$u_t=\operatorname{div}(|\nabla u|^{p-2}\nabla u)-u^q. \tag{1.9}$$

There have been many works on the singular solutions of (1.8) and (1.9) and their applications in study intermediate limit of general solutions , see [1]-[5], [6]-[13], [15]-[19] and the references therein.

This paper is organized as follows. In §2, we study equation (1.6) and prove the existence and uniqueness of very singular self-similar solution. In §3 we show the existence and uniqueness of both FS and VSS, discuss various properties of such solutions and complete the proof of our theorem.

For the convenience of the reader, we list the following special constants that will be used in this paper:

$$\alpha = \frac{1}{q-1}, \quad \beta = \frac{q-m(p-1)}{p},$$
$$\mu = \frac{p}{1-m(p-1)}, \quad k = \frac{1}{p-n[1-m(p-1)]}.$$

Observe that $q < m(p-1) + p/n$ if and only if $n\beta < 1$. Also, $1 < q < m(p-1) + p/n$ and $m(p-1) < 1$ imply that $\mu > n$ and $k > 0$, which is equivalent to $p > n(1+m)/(1+mn)$.

We note in passing that the present case of (1.1) is very different from the case of $m(p-1) > 1$, which is similar to (1.8) with $m > 1$ or (1.9) with $p > 2$. In particular, when $m(p-1) > 1$, there exist compact supported solutions and such solutions have finite speed of propagation. Whereas for our case, the propagation speed is infinite and any nontrivial, nonnegative solution has $\mathbb{R}^n$ as its support for $t > 0$. As a matter of fact, the major effort is given to estimate the decay of singular solutions at $|x| = \infty$. Once we can do that, a lot of techniques in [8]-[11] which were developed for degenerate equations such as (1.8) with $m > 1$ and (1.9) with $p > 2$ can be adapted to study the present singular case.

2 Existence and Uniqueness of Very Singular Self-similar Solution

In this section we study (1.6) and prove the existence and uniqueness of very singular self-similar solution. Our proof of existence is different from the one given in [14]. In particular, through the classification of solutions in relation to their initial values, we prove the existence of self-similar VSS, rather than the shooting argument employed in [14].

We consider the solution of (1.6) with initial value

$$f(0) = a, \quad f'(0) = 0. \tag{2.1}$$

For each $a > 0$, (1.6), (2.1) has a unique solution $f(r; a)$ and the solution is continuously differentiable in a in a right neighbourhood of $r > 0$ (see Proposition 1 in Appendix). Since $a \geq 1$ implies that $f' \geq 0$ in its existence interval, we need only consider the case $a \in (0, 1)$. For $a \in (0, 1)$, if we denote by $(0, R(a))$

the maximal existence interval where $f > 0$, then $f' < 0$ in $(0, R(a))$ and either (i) $R(a) = \infty$ and $\lim_{r \searrow \infty} f(r; a) = 0$, or (ii) $R(a) < \infty$ and $f(R(a); a) = 0$. The main results of this section read as follows.

Theorem 2.1 *Assume that $m > 0$, $p > 1$, $m(p-1) < 1$ and $q > 1$. For each $a \in (0,1)$, let $f(r; a)$ be the solution of (1.6), (2.1). Then the following conclusions hold:*

(i) *If $n\beta \geq 1$, then $f > 0$ and $f' < 0$ in $(0, \infty)$ and $\liminf_{r\to\infty} r^{1/\beta} f(r; a) > 0$.*

(ii) *If $n\beta < 1$, then there exists $a^* \in (0,1)$ such that the following classification is valid:*
(a) *If $a \in (0, a^*)$, then there exists $R(a) < \infty$ such that $f' < 0$ in $(0, R(a)]$ and $f(R(a); a) = 0$.*
(b) *If $a \in (a^*, 1)$, then $f' < 0$, $f > 0$, $f_a := \frac{d}{da} f > 0$, and $(r^\mu f)' > 0$ in $(0, \infty)$. In addition, $\lim_{r\to\infty} r^{1/\beta} f(r; a)$ has a finite limit $k(a)$ which, as a function of a defined on $(a^*, 1)$, is positive, continuous and strictly increasing, and satisfies $\lim_{a \searrow a^*} k(a) = 0$ and $\lim_{a \nearrow 1} k(a) = \infty$.*
(c) *If $a = a^*$, then $\lim_{r\to\infty} r^{1/\beta} f(r; a) = 0$, $\lim_{r\to\infty} r^{m\mu} f^m(r; a) = F^*$, where*

$$F^* = \left\{ \frac{(m\mu)^{p-1}(\mu - n)[1 - m(p-1)]}{q-1} \right\}^{m/[1-m(p-1)]}$$

Nonexistence Results

Now we prove nonexistence results of very singular self-similar solutions. We note the same result was proved in [14]. For completeness we give a simple proof here.

Proof of Theorem 2.1(i). Multiplying (1.6) by $r^{1/\beta - 1}$ we have, for r in $(0, R(a))$,

$$(r^{1/\beta-1}|(f^m)'|^{p-2}(f^m)' + \beta r^{1/\beta} f)' = (n - 1/\beta) r^{1/\beta - 2}|(f^m)'|^{p-1} + r^{1/\beta-1} f^q > 0$$

since $n\beta \geq 1$. Thus the function $g(r) := r^{1/\beta-1}|(f^m)'|^{p-2}(f^m)' + \beta r^{1/\beta} f$ is strictly increasing in $(0, R(a))$. Note that $\lim_{r\searrow 0} g(r) = 0$, we get $g > 0$ in $(0, R(a))$. Since $f' < 0$ we conclude that $R(a) = \infty$ and $f \searrow 0$ as $r \nearrow \infty$. Since $g(r)$ is increasing,

$$\lim_{r\to\infty} (r^{1/\beta-1}|(f^m)'|^{p-2}(f^m)' + \beta r^{1/\beta} f) = \lim_{r\to\infty} g(r) = g_\infty$$

exists, where g_∞ is either a positive constant or ∞. Thus, $\liminf_{r\to\infty} r^{1/\beta} f > 0$. This completes the proof. Q.E.D.

A Monotonicity Lemma

Observe that $f = f(r;a)$ satisfies

$$-|(f^m)'|^{p-2}(f^m)' = \beta r f + r^{1-n}\int_0^r s^{n-1}[1-n\beta - f^{q-1}]f\,ds. \tag{2.2}$$

Since $f' < 0$, we have that

$$-(f^m)' = \left(\beta r f + r^{1-n}\int_0^r s^{n-1}[1-n\beta - f^{q-1}]f\,ds\right)^{1/(p-1)}.$$

Using $f(0) = a$ it follows that as $r \searrow 0$,

$$f^m(r;a) = a^m - \frac{p-1}{p}(a-a^q)^{1/(p-1)} n^{-1/(p-1)} r^{p/(p-1)}(1+\circ(r))\,. \tag{2.3}$$

To study the behavior of the solution $f(r;a)$, we introduce a function $F = F(r;a)$ defined by

$$F(r;a) := \{r^\mu f(r;a)\}^m, \quad \text{where} \quad \mu = \frac{p}{1-m(p-1)} > 0. \tag{2.4}$$

Then we have $(f^m)' = r^{-m\mu-1}(rF' - m\mu F)$ and

$$\begin{aligned}
&(|(f^m)'|^{p-2}(f^m)')' \\
&= (m\mu+1)(p-1)r^{-(m\mu+1)(p-1)-1}|m\mu F - rF'|^{p-2}(m\mu F - rF') \\
&\quad -(p-1)r^{-(m\mu+1)(p-1)}|m\mu F - rF'|^{p-2}(m\mu F' - F' - rF'').
\end{aligned}$$

Since $\mu = p/(1-m(p-1))$, substituting the above expressions into (1.6) gives

$$\begin{aligned}
&(p-1)r^2F'' + [n-1-2(p-1)m\mu]rF' + m\mu(\mu-n)F \\
&+(m\mu F - rF')^{2-p}\{\frac{\beta}{m}rF'F^{(1-m)/m} + (1-\beta\mu)F^{1/m} - r^{\mu(1-q)}F^{q/m}\} = 0\,.
\end{aligned} \tag{2.5}$$

In addition, a differentiation in a gives, for $F_a := \frac{\partial F}{\partial a}$,

$$\begin{aligned}
\mathcal{L}(F_a) := {}& (p-1)r^2F_a'' + [n-1-2(p-1)m\mu]rF_a' + m\mu(\mu-n)F_a \\
&+(2-p)(m\mu F - rF')^{1-p}(m\mu F_a - rF_a')\{\frac{\beta}{m}rF'F^{(1-m)/m} \\
&+(1-\beta\mu)F^{1/m} - r^{\mu(1-q)}F^{q/m}\} + (m\mu F - rF')^{2-p} \\
&\times\Big\{\frac{\beta}{m}rF_a'F^{(1-m)/m} + \frac{\beta(1-m)}{m^2}rF'F^{(1-2m)/m}F_a \\
&+\frac{1-\beta\mu}{m}F^{(1-m)/m}F_a - \frac{q}{m}r^{\mu(1-q)}F^{(q-m)/m}F_a\Big\} = 0\,.
\end{aligned} \tag{2.6}$$

Lemma 2.1 *If $F' > 0$ in a finite interval $(0, r_1)$, then $\mu a F_a > rF'$ in $(0, r_1)$ and $F_a > 0$ in $(0, r_1]$.*

Proof. Applying the differential operator $r\frac{d}{dr}$ to (2.5) and using the identity $r[r^2F'']' = r^2[rF']''$, we get

$$\mathcal{L}(rF') = \mu(1-q)(m\mu F - rF')^{2-p} r^{\mu(1-q)} F^{q/m} < 0 \quad \text{in} \quad (0, R(a)).$$

In the interval $(0, r_1)$, write $F_a = C(r)rF'$. Using the expansion (2.3) we have that, for all r sufficiently small,

$$\begin{aligned} C(r) &= (\mu a)^{-1}\Big\{1 + \frac{q-1}{mp} n^{-1/(p-1)} a^{q-m} (a - a^q)^{(2-p)/(p-1)} r^{p/(p-1)} \\ &\qquad + O(r^{2p/(p-1)})\Big\}. \end{aligned}$$

It then follows that $C(0) = (\mu a)^{-1}$, and $C'(r) > 0$ near the origin. Substituting $F_a = C(r)rF'$ into (2.6) we find

$$(p-1)r^2 C''[rF'] + C'[\cdots] + C\mathcal{L}(rF') = 0.$$

Because $\mathcal{L}(rF') < 0$ and $rF' > 0$ in $(0, r_1)$, we know that $C'(r)$ can not attain its first zero in $(0, r_1)$. Hence $C'(r) > 0$ in $(0, r_1)$. Consequently, $F_a = C(r)rF' > (\mu a)^{-1} rF' > 0$ in $(0, r_1)$.

It remains to show that $F_a > 0$ at r_1. For later application, here we provide an elaborated proof. Let $r_0 = \min\{1, r_1/2\}$ and ψ be the solution to $\mathcal{L}(\psi) = 0$ in $(0, R(a))$ with the initial values $\psi(r_0) = 0$ and $\psi'(r_0) = 1$. Then $\psi > 0$ in $(r_0, r_1]$ since between any two zeros of ψ there is a zero of F_a. Set $k_0 = C'rF'|_{r=r_0} > 0$ and $c_0 = C(r_0)$. We consider the function $\phi = F_a - k_0\psi$. It is obvious that $\mathcal{L}(\phi) = 0$ in $(0, R(a))$. In addition, at $r = r_0$, $\phi = F_a = c_0 rF'$ and $\phi' = \{C'rF' + C(rF')' - k_0\psi'\}|_{r=r_0} = c_0(rF')'|_{r=r_0}$. Writing $\phi = \overline{C}(r)rF'$, we have that $\overline{C}(r_0) = c_0$, $\overline{C}'(r_0) = 0$, and $\overline{C}$ satisfies the same equation as that for C. As $\overline{C}''(r_0) > 0$ (from the differential equation), we get that $\overline{C}' > 0$ in (r_0, r_1). Therefore $\phi = \overline{C}(r)rF' > 0$ in $[r_0, r_1)$. Consequently, $F_a \geq k_0\psi > 0$ in $(r_0, r_1]$. This completes the proof of the lemma. Q.E.D.

For convenience, we denote

$$\begin{aligned} \mathcal{A} &= \big\{a \in (0,1) : \text{there exists } R_1(a) \in (0, R(a)) \text{ such that } F'(R_1(a); a) = 0\big\} \\ \mathcal{B} &= \big\{a \in (0,1) : F'(\cdot; a) > 0 \quad \text{in } (0, \infty), \quad \lim_{r\to\infty} F(r; a) < \infty\big\} \\ \mathcal{C} &= \big\{a \in (0,1) : F'(\cdot; a) > 0 \quad \text{in } (0, \infty), \quad \lim_{r\to\infty} F(r; a) = \infty\big\}. \end{aligned}$$

Since $F'(r; a) > 0$ near the origin, then $F'(r; a) > 0$ in $(0, R(a))$ if $a \in (0,1)$ is not in $\mathcal{A}$, this implies that $R(a) = \infty$, so that $a \in \mathcal{B} \bigcup \mathcal{C}$. Thus, $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are disjoint with each other and $\mathcal{A} \bigcup \mathcal{B} \bigcup \mathcal{C} = (0, 1)$.

Characterization of the set $\mathcal{A}$

Lemma 2.2 *Let $a \in (0, 1)$. Then the following statements are equivalent:*

(i) $a \in \mathcal{A}$;

(ii) *there exists* $R_1 \in (0, R(a))$ *such that* $F'(r;a) > 0$ *in* $(0, R_1(a))$, $F''(R_1(a);a) < 0$, *and* $F'(r;a) < 0$ *in* $(R_1(a), R(a))$;

(iii) $\sup_{r\in(0,R(a))} F(r;a) < F^* = \left\{\frac{(m\mu)^{p-1}(\mu-n)[1-m(p-1)]}{q-1}\right\}^{m/[1-m(p-1)]}$;

(iv) *there exists* $r_1 \in (0, R(a))$ *such that* $\int_0^{r_1} s^{n-1}(1 - n\beta - f^{q-1}) f\,ds > 0$;

(v) $R(a) < \infty$ *and* $(f^m)'(R(a);a) < 0$;

(vi) $R(a) < \infty$.

Proof. **(i)** $\Rightarrow$ **(ii)**. Let $(0, R_1(a))$ be the maximal interval where $F' > 0$. Since $a \in \mathcal{A}$, $R_1(a) < R(a)$ and $F'(R_1(a);a) = 0$, we have that $F''(R_1(a);a) < 0$. In fact, if $F''(R_1(a);a) = 0$, then differentiating (2.5) with respect to r and evaluating the resulting equation at $r = R_1(a)$, it yields $F'''(R_1(a);a) < 0$. This contradicts the fact that $F' > 0$ in $(0, R_1(a))$. Therefore, $F''(R_1(a);a) < 0$.

Next we show that $F'(r;a) < 0$ in $(R_1(a), R(a))$. In fact, if this is not true, then there exists $R_2(a) \in (R_1(a), R(a))$ such that $F'(R_2(a);a) = 0$ and $F'(r;a) < 0$ in $(R_1(a), R_2(a))$. Evaluating (2.5) at $r = R_1(a)$ with $F'(R_1(a);a) = 0$ and $F''(R_1(a);a) < 0$, and at $r = R_2(a)$ with $F'(R_2(a);a) = 0$ and $F''(R_2(a);a) \geq 0$, and using the definition $F = r^{m\mu} f^m$, we obtain

$$\begin{aligned} &\Big\{\frac{q-1}{1-m(p-1)} + f^{q-1}\Big\} F^{(1-m(p-1))/m}\Big|_{r=R_1(a)} \\ &< \quad (m\mu)^{p-1}(\mu - n) \\ &\leq \quad \Big\{\frac{q-1}{1-m(p-1)} + f^{q-1}\Big\} F^{(1-m(p-1))/m}\Big|_{r=R_2(a)}. \end{aligned} \tag{2.7}$$

However, this is impossible since $F(R_1(a);a) > F(R_2(a);a)$ and $f(R_1(a);a) > f(R_2(a);a)$. Hence $F'(r;a) < 0$ in $(R_1(a), R(a))$.

(ii) $\Rightarrow$ **(iii)**. Note that the maximum of F is obtained at $r = R_1(a)$, so the assertion follows from the first inequality of (2.7).

(iii) $\Rightarrow$ **(iv)**. Assume for the contrary that $\int_0^r s^{n-1}(1 - n\beta - f^{q-1}) f\,ds \leq 0$ for all $r \in (0, R(a))$. Then from (2.2) we have that $-|(f^m)'|^{p-2}(f^m)' \leq \beta r f$, i.e. $-(f^m)' \leq (\beta r f)^{1/(p-1)}$ for all $r \in (0, R(a))$. Upon integrating this inequality over $(0, r)$ we have

$$f(r;a) \geq \left(a^{[m(p-1)-1]/(p-1)} + \frac{1-m(p-1)}{mp}\beta^{1/(p-1)} r^{p/(p-1)}\right)^{(p-1)/[m(p-1)-1]}$$

for all $r \in (0, R(a))$. Then it follows that $R(a) = \infty$ and, using (2.4), $\hat{F} := \liminf_{r\to\infty} F > 0$.

Note that either $F' > 0$ in $(0, \infty)$, or if F' changes sign, then $a \in \mathcal{A}$ and hence $F' < 0$ in $(R_1(a), \infty)$. In either case we have $\lim_{r\to\infty} F = \hat{F}$ and $\liminf_{r\to\infty} |rF'| = 0$.

Let $\{r_j\}_{j=1}^{\infty}$ be a sequence with $\lim_{j\to\infty} r_j = \infty$ and $\lim_{j\to\infty}(rF')|_{r=r_j} = 0$. We claim that $\{r_j\}$ can be chosen such that in addition $\lim_{j\to\infty}(r^2F'')|_{r=r_j} = 0$. In fact, if $|rF'|$, which is positive for all large r, oscillates infinitely many times, then one can choose $\{r_j\}$ to be the local minimum points of $|rF'|$ so that $0 = (rF')' = rF'' + F'$ on $\{r_j\}$. That is,

$$\lim_{j\to\infty}(r^2F'')|_{r=r_j} = -\lim_{j\to\infty}(rF')|_{r=r_j} = 0\,.$$

If $|rF'|$ does not oscillate infinitely many times, then $|rF'|$ eventually monotonically decreases to zero. So that, one can choose $\{r_j\}$ along which $r(|rF'|)'$ approaches zero, namely, $r^2F'' = r(rF')' - rF'$ approaches zero along the sequence $\{r_j\}$. Now evaluating (2.5) at r_j and sending $j\to\infty$ we obtain $\hat{F} = F^*$, which is a contradiction to the assumption $\sup_{r\in(0,R(a))} F < F^*$.

(iv) ⇒ (v). Since the function $z = 1 - n\beta - f^{q-1}$ is strictly increasing in $(0, R(a))$, by $\int_0^{r_1} s^{n-1} z f ds > 0$ we have that $z > 0$ for all $r \in [r_1, R(a))$. It then follows that for some $\delta > 0$, $\int_0^r s^{n-1}(1 - n\beta - f^{q-1}) f ds \ge \delta$ in $[r_1, R(a))$. From (2.2) we have

$$-|(f^m)'|^{p-2}(f^m)' \ge \beta r f + \delta r^{1-n} \quad \forall r \in [r_1, R(a)). \tag{2.8}$$

Since $1 < q < m(p-1) + p/n$ and $1 - m(p-1) > 0$, one can choose ε : $1 - m(p-1) < \varepsilon < \min\{1, p/n\}$. Therefore, $0 < \varepsilon < 1$ and satisfies

$$m - (1-\varepsilon)/(p-1) > 0, \quad 1 + (1 - n\varepsilon)/(p-1) > 0. \tag{2.9}$$

Using the inequality $\beta r f + \delta r^{1-n} \ge (\beta r f)^{1-\varepsilon}(\delta r^{1-n})^{\varepsilon} = \beta^{1-\varepsilon}\delta^{\varepsilon} f^{1-\varepsilon} r^{1-n\varepsilon}$, we obtain from (2.8) that

$$|(f^m)'|^{p-1} \ge \beta^{1-\varepsilon}\delta^{\varepsilon} f^{1-\varepsilon} r^{1-n\varepsilon} \quad \text{for all } r \in (r_1, R(a)).$$

Due to $(f^m)' < 0$, we have

$$-m f^{m-1-(1-\varepsilon)/(p-1)} f' \ge C r^{(1-n\varepsilon)/(p-1)}.$$

Integrating this inequality over $[r_1, r)$, $r < R(a)$, and using (2.9), we obtain immediately that $R(a) < \infty$. In addition, it follows from (2.8) that $(f^m)'(R(a); a) < 0$.

(v) ⇒ (vi) is trivially true. (vi) ⇒ (i) is also trivially true since $f(R(a); a) = 0$ implies that $F(r; a) = (r^{\mu} f(r; a))^m$ has an interior maximum in $(0, R(a))$. This completes the proof of the lemma. Q.E.D.

Lemma 2.3 *There exists $a_* \in ((1-n\beta)^{1/(q-1)}, 1]$ such that $\mathcal{A} = (0, a_*)$.*

Its proof is the same as that of Theorem 5.3 in [4].

Characterization of the set C

Lemma 2.4 *Let $a \in (0,1)$. Then $a \in \mathcal{C}$ if and only if*

$$\sup_{r\in(0,R(a))} F(r; a) > F^*\,.$$

Proof. The only if part follows from the definition of $\mathcal{C}$.
If $\sup_{r\in(0,R(a))} F(r;a) > F^*$, then by Lemma 2.2 (iii), $a \notin \mathcal{A}$, and so $a \in \mathcal{B}\bigcup\mathcal{C}$. However, if $\hat{F} := \lim_{r\to\infty} F(r;a)$ is finite, then a sequence $\{r_j\}$ can be found along which rF' and r^2F'' approach zero. It follows from (2.5) that $\hat{F} = F^*$. This contradicts the assumption that $\sup_{r\in(0,R(a))} F(r;a) > F^*$. Q.E.D.

Lemma 2.5 *There exists $a^* \in (0,1)$ such that $\mathcal{C} = (a^*,1)$. In addition, for every $a \in \mathcal{C}$ there exists $k(a) > 0$ such that*

$$\lim_{r\to\infty} r^{1/\beta} f(r;a) = k(a).$$

Furthermore, $k(a)$, as a function of $a \in (a^*,1)$, is positive, continuous, strictly increasing, and

$$\lim_{a\searrow a^*} k(a) = 0, \quad \lim_{a\nearrow 1} k(a) = \infty.$$

Proof. Step 1: We first prove that $\mathcal{C}$ is open and non-empty. Since, $a \in \mathcal{C}$ if and only if $\sup_{r\in(0,R(a))} F(r;a) > F^*$, by the continuous dependence of initial data, $\mathcal{C}$ is open. In view of $\lim_{a\nearrow 1} f(r;a) = f(r;1) \equiv 1$ uniformly in any compact subset of $[0,\infty)$ and
$\lim_{a\nearrow 1} F((2F^*)^{1/(m\mu)};a) = 2F^*$, we have $(1-\varepsilon,1) \subset \mathcal{C}$ for some sufficiently small positive ε.

Because $\mathcal{A} = (0,a_*)$, $[a_*,1) \subset \mathcal{B}\bigcup\mathcal{C}$, we know that $F'(r;a) > 0$ for all $r \in (0,\infty)$ and all $a \in [a_*,1)$. Consequently, by Lemma 2.1, $F_a(r;a) > 0$ for all $r \in (0,\infty)$ and all $a \in [a_*,1)$. This implies that $\mathcal{C} = (a^*,1)$ where $a^* = \inf\{a \geq a_* | \lim_{r\to\infty} F(r;a) > F^*\}$.

As a by-product, $\mathcal{B} = [a_*,a^*] = \{a | R(a) = \infty, \text{and} \quad F(r;a) \nearrow F^* \quad \text{as} \quad r \to \infty\}$.

Step 2. We are now in a position to study the behavior of the solution $f(\cdot;a)$ for $a \in \mathcal{C}$. For simplicity, we write $f(r;a)$ and $F(r;a)$ as $f(r)$ and $F(r)$ respectively.

It is convenient to use the variable $s = \ln r$. Because f is positive, we can write $f(e^s) = f(1)\exp(-\int_0^s G(\sigma)d\sigma)$. Since $f' < 0$ and $F' = r^{m\mu-1}[r(f^m)' + m\mu f^m] > 0$ for all $r > 0$, we get $0 < G(s) < \mu$ for all $s \in (-\infty,\infty)$. Substituting this transformation into (1.6) and using the relations $r\frac{d}{dr} = \frac{d}{ds}, r^2\frac{d^2}{dr^2} = \frac{d^2}{ds^2} - \frac{d}{ds}$ and $r^p f^{1-m(p-1)} = F^{(1-m(p-1))/m}$, we obtain, writing $\dot{G} = dG/ds$,

$$\begin{aligned}(p-1)\dot{G} &= H(G,s) \stackrel{\triangle}{=} m(p-1)G^2 + (p-n)G \\ &\quad + m^{1-p}G^{2-p}\{1 - \beta G - f^{q-1}\}F^{(1-m(p-1))/m}.\end{aligned}$$

Here we consider G as an unknown function, whereas $f = f(e^s)$ and $F = F(e^s)$ as known functions of s.

Since $a \in \mathcal{C}$, as $s \nearrow \infty$, $f \searrow 0$ and $F \nearrow \infty$. If we rewrite $H(G,s) = G^{2-p}[m(p-1)G^p + (p-n)G^{p-1} + m^{1-p}\{1 - \beta G - f^{q-1}\}F^{(1-m(p-1))/m}]$, it is easy to see that for any $\varepsilon > 0$ there exists $s_\varepsilon > 0$ such that $H(G,s) > 0$ for all

$G \in (0,(1-\varepsilon)/\beta]$ and $s > s_\varepsilon$; $H(G,s) < 0$ for all $G \in ((1+\varepsilon)/\beta, \mu]$ and $s > s_\varepsilon$. It then follows from an invariant region argument that

$$\lim_{s\to\infty} G(s) = 1/\beta.$$

Step 3. We show that, as $s \to \infty$, $G(s)$ approaches $1/\beta$ exponentially fast, with an exponent at least $\nu = \frac{1}{2}\min\{\frac{q-1}{\beta}, (1-m(p-1))(\mu - \frac{1}{\beta})\}$.

Considering the function $G^-(s) = \frac{1}{\beta}[1 - \frac{1}{2}e^{\nu(S-s)}]$ defined on $[S,\infty)$. We want to prove that $G^-(s)$ is a sub-solution of the equation $(p-1)\dot{G} = H(G,s)$ in $[S,\infty)$ provided that S is sufficiently large. For this purpose, first, we let S be large enough such that $f^{q-1}(e^S) < \frac{1}{4}$ and $G(s) > 1/(2\beta)$ for all $s > S$. Then

$$f^{q-1}(e^s) = f^{q-1}(e^S)\exp\{-(q-1)\int_S^s G(\sigma)d\sigma\} < \frac{1}{4}e^{\nu(S-s)} \quad \text{for all } s \geq S.$$

Next, by taking a larger S if necessary, we assume that $G(s) \leq 1/\beta + \frac{1}{2}(\mu - 1/\beta)$ for all $s \geq S$. Then

$$\begin{aligned} F^{(1-m(p-1))/m}(e^s) &= F^{(1-m(p-1))/m}(e^S)\exp\left\{[1-m(p-1)]\int_S^s(\mu - G)d\sigma\right\} \\ &\geq F^{(1-m(p-1))/m}(e^S)e^{\nu(s-S)} \quad \forall s \geq S. \end{aligned}$$

Hence,

$$\begin{aligned} &\{1 - \beta G^-(s) - f^{q-1}(e^s)\}F^{(1-m(p-1))/m}(e^s) \\ &\geq [\frac{1}{2}e^{\nu(S-s)} - f^{q-1}(e^s)]F^{(1-m(p-1))/m}(e^s) \\ &\geq \frac{1}{4}e^{\nu(S-s)}F^{(1-m(p-1))/m}(e^s) \\ &\geq \frac{1}{4}F^{(1-m(p-1))/m}(e^S) \quad \forall s \geq S\,. \end{aligned}$$

Using the fact that $1/(2\beta) < G^-(s) < 1/\beta$ we have, for all $s \geq S$,

$$(p-1)\frac{d}{ds}G^- - H(G^-,s) \leq \frac{\nu(p-1)}{2\beta} + \frac{n}{\beta} - \frac{1}{4}\lambda m^{1-p}F^{(1-m(p-1))/m)}(e^S) < 0$$

for all $s > S$ with S large enough, since $F(e^S) \to \infty$ as $S \to \infty$. Here

$$\lambda = \begin{cases} (2\beta)^{p-2}, & \text{if} p \leq 2, \\ \beta^{p-2}, & \text{if} p > 2. \end{cases}$$

Comparing $G(s)$ to $G^-(s)$ in $[S,\infty)$ we obtain that $G(s) \geq G^-(s) = \frac{1}{\beta}[1 - \frac{1}{2}e^{\nu(S-s)}]$ in $[S,\infty)$.

In a similar manner we can prove that $G(s) \leq G^+(s) = \frac{1}{\beta}[1 + \frac{1}{2}(\mu - 1/\beta)e^{\nu(S-s)}]$.

Therefore, $|G - 1/\beta| \leq \frac{1+\mu}{2\beta} e^{\nu(S-s)}$. Consequently, as $r \to \infty$,

$$\begin{aligned} r^{1/\beta} f(r) &= f(1)\exp\Big\{ -\int_0^{\ln r} (G(\sigma) - 1/\beta) d\sigma \Big\} \\ &\to f(1)\exp\Big\{ -\int_0^{\infty} (G(\sigma) - 1/\beta) d\sigma \Big\} =: k(a). \end{aligned}$$

Since $(f^m)_a = r^{-m\mu} F_a > 0$ in $(0, \infty)$, we know that $k(\cdot)$ is positive, continuous, and non-decreasing in $(a^*, 1)$.

As $G(s)$ approaches $1/\beta$ exponentially fast, we have that $r(r^{1/\beta} f)' = O(r^{-\nu})$ as $r \to \infty$.

Recall from Lemma 2.1 that $F_a \geq \frac{1}{a\mu} r F'$, which implies that

$$f_a \geq \frac{1}{a\mu}[\mu f + r f'],$$

i.e.

$$(r^{1/\beta} f)_a \geq \frac{1}{a\mu}[(\mu - 1/\beta) r^{1/\beta} f + r(r^{1/\beta} f)'].$$

Hence, for any $a^* < a_1 < a_2 < 1$,

$$\begin{aligned} k(a_2) - k(a_1) &= \lim_{r\to\infty} \int_{a_1}^{a_2} r^{1/\beta} f_a da \\ &\geq \lim_{r\to\infty} \int_{a_1}^{a_2} \frac{\mu - 1/\beta}{a\mu} r^{1/\beta} f da \\ &= \frac{\mu - 1/\beta}{a\mu} \int_{a_1}^{a_2} k(a) da. \end{aligned}$$

Because $\mu > 1/\beta$ and $k(a) > 0$, the above inequality show that $k(\cdot)$ is strictly increasing.

Now if $\lim_{a \searrow a^*} k(a) > 0$, it can be derived that $\sup_{r>0} r^{m\mu} f^m(r; a^*) > F^*$ because $\mu > 1/\beta$, which would imply that $a^* \in \mathcal{C}$. It contradicts to the definition of a^*. Therefore, $\lim_{a \searrow a^*} k(a) = 0$.

Finally, if $K = r^{1/\beta} f$ achieves a local maximum, say, at $r = r_1$, which is the first one, then at $r = r_1$, $K' = 0$, and $K'' \leq 0$, i.e. $\beta r f' + f = 0$ and $r^2 f'' \leq \frac{1+\beta}{\beta^2} f$. Substituting these two relations into (1.6) then yields

$$K(r_1; a) < K^* := \left\{ \left(\frac{m}{\beta}\right)^p \left[p - 1 + \frac{\beta}{m}(p - n) \right] \right\}^{1/[q - m(p-1)]}.$$

Hence, once the value of K exceeds K^*, then K monotonously increases thereafter. It then follows that $\lim_{a \nearrow 1} k(a) = \infty$ by $f(r; 1) \equiv 1$ and continuous dependence of solution on initial value in any finite interval. This completes the proof of the lemma. Q.E.D.

Characterization of the Set B.

Lemma 2.6 *$\mathcal{B} = \{a_*\} = \{a^*\}$ and $F(r; a^*) \nearrow F^*$ as $r \nearrow \infty$.*

Proof. From the previous discussion we know that $\mathcal{B} = [a_*, a^*]$, and that for all $a \in \mathcal{B}$, $F_a > 0$ for all $r > 0$, and $F(r;a) \nearrow F^*$ as $r \nearrow \infty$. It remains to show that $a_* = a^*$.

We claim that if $a \in \mathcal{B}$, then $\lim_{r\to\infty} F_a(r;a) = \infty$. To this aim we use the independent variable $s = \ln r$. Note that $rF' = \dot{F}$ vanishes as $s \to \infty$. The linear operator $\mathcal{L}$ in (2.6) takes the form, for s sufficiently large,

$$\mathcal{L}(\phi) = (p-1)\ddot{\phi} + [b + \circ(1)]\dot{\phi} - [c + \circ(1)]\phi$$

where $\circ(1) \to 0$ as $s \to \infty$, b is a certain constant, and $c = \mu(\mu - n)(1 + m - mp) > 0$ because $1 < q < m(p-1) + p/n$ and $m(p-1) < 1$. Since $c > 0$, it is easy to prove that the solution to $\mathcal{L}(\phi_1) = 0$ in (S, ∞) with initial value $\phi_1(S) = 0$, $\dot{\phi}_1(S) = 1$, with S large enough, will have the property that $\phi_1 \to \infty$ exponentially fast as $s \to \infty$.

Note that the function ψ, constructed in the proof of Lemma 2.1, is positive in (r_0, ∞). As F_a and ψ are linearly independent, one of them will be unbounded. Since $F_a \geq k_0\psi$, we have that $F_a \to \infty$ as $r \to \infty$.

Finally we prove that $a^* = a_*$. In fact, if $a^* > a_*$, then by Fatou's lemma,

$$\begin{aligned} 0 = \lim_{r\to\infty}(F(r;a^*) - F(r;a_*)) &= \lim_{r\to\infty} \int_{a_*}^{a^*} F_a(r;a)da \\ &\geq \int_{a_*}^{a^*} \liminf_{r\to\infty} F_a(r;a)da = \infty, \end{aligned}$$

which is impossible. This completes the proof of the lemma. Q.E.D.

The proof of Theorem 2.1: follows directly from Lemmas 2.3, 2.5 and 2.6.

3 Existence and Uniqueness of Singular Solutions

In this section we prove the existence and uniqueness of singular solutions of (1.1), and discuss their properties as well as those of the following equation

$$u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) \quad \text{in } \mathbb{R}^n \times (0, +\infty). \tag{3.1}$$

Properties of Singular Solutions and Non-existence Results

Lemma 3.1 *Assume that u is a singular solution of (1.1), or (3.1). Then either (1.3) or (1.4) holds. That is, every singular solution is either an FS or a VSS.*

Its proof is similar to that of Lemma 2.1 in [5]. We omit the details here.

Lemma 3.2 **(i)** *If u is singular solution of (1.1), then for $A := (\frac{1}{q-1})^{1/(q-1)}$,*

$$u(x,t) \leq At^{-1/(q-1)} \qquad \text{in } \mathbb{R}^n \times (0, \infty). \tag{3.2}$$

(ii) *If u is a singular solution to (1.1) or (3.1), then for* $B = \{nk(m\mu)^{p-1}\}^{\mu/p}$ *which is equal to* $\{[p-n(1-m(p-1))](mp/[1-m(p-1)])^{p-1}\}^{1/[1-m(p-1)]}$, *we have*

$$u(x,t) \le B(t^{1/p}|x|^{-1})^{\mu} = Bt^{1/[1-m(p-1)]}|x|^{-p/[1-m(p-1)]} \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{3.3}$$

Proof. (i) The proof is obvious since $At^{-1/(q-1)}$ is a solution of (1.1) with initial value ∞ in $\mathbb{R}^n$.

(ii) Direct calculation shows that for any $\varepsilon > 0$, the function $B(t+\varepsilon)^{\mu/p}(|x| - \varepsilon)^{-\mu}$ is a solution to $w_t = \operatorname{div}(|\nabla w^m|^{p-2}\nabla w^m)$ in $\{(x,t)| \ \ |x| > \varepsilon, \ \ t \ge 0\}$. Comparing this function with u in the domain $\{(x,t)| \ \ |x| > \varepsilon, \ \ t \ge 0\}$ then gives $u(x,t) \le B(t+\varepsilon)^{\mu/p}(|x|-\varepsilon)^{-\mu}$ for all $|x| > \varepsilon, \ \ t > 0$. Let $\varepsilon \searrow 0$ then yields $u(x,t) \le Bt^{\mu/p}|x|^{-\mu}$. The desired results are proved.

Lemma 3.3 *If (1.1) has a singular solution, then it must have a maximal singular solution u^* having the following properties:*

- **(i)** *Every singular solution of (1.1) is no bigger than u^*.*
- **(ii)** *u^* is self-similar; namely, there exists a smooth function $f(\cdot) : [0,\infty) \to [0,\infty)$ such that $u^* = (\alpha/t)^{\alpha} f(|x|(\alpha/t)^{\alpha\beta})$ and f solves (1.6).*

Proof. For any $\tau > 0$, let $u_\tau(x,t)$ be the solution of (1.1) in $\mathbb{R}^n \times (\tau,\infty)$ with initial value

$$u_\tau(x,\tau) = \min\{A\tau^{-1/(q-1)}, \ \ B(\tau^{1/p}|x|^{-1})^{\mu}\} \quad \text{on } \mathbb{R}^n \times \{t = \tau\}.$$

By comparison principle we have

$$u_\tau(x,t) \le \min\{At^{-1/(q-1)}, \ \ B(t^{1/p}|x|^{-1})^{\mu}\} \quad \text{on } \mathbb{R}^n \times [t,\infty). \tag{3.4}$$

Consequently, for any $\tau_1 > \tau_2 > 0$, $u_{\tau_1}(\cdot,\tau_1) \ge u_{\tau_2}(\cdot,\tau_1)$, so that by comparison, $u_{\tau_1} \ge u_{\tau_2}$ in $\mathbb{R}^n \times [\tau_1,\infty)$. Hence, $\lim_{\tau\searrow 0} u_\tau$ exists for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$. We denote this limit by u^*, which is necessarily a solution of (1.1). Since each u_τ satisfies (3.4), it follows that $u^*(x,t) \le \min\{At^{-1/(q-1)}, \ \ B(t^{1/p}|x|^{-1})^{\mu}\}$. It then yields that u^* satisfies (1.2).

To show that u^* is non-trivial, we only need to show that u^* is no lees than any singular solution of (1.1). In fact, if u is a singular solution of (1.1), then from Lemma 3.2 and comparison principle, $u \le u_\tau$ in $\mathbb{R}^n \times [s,\infty)$ for any $0 < \tau \le s$. Thus, $u \le u^*$ in $\mathbb{R}^n \times (0,\infty)$. Consequently, u^* is non-trivial, and is the maximal singular solution of (1.1) if (1.1) has one.

Due to the symmetry and the scaling invariance $u \to u_h(x,t)$ of the equation (1.1), here $u_h(x,t) = h^{1/(q-1)}u(h^{(q+m-pm)/[p(q-1)]}x, ht)$, we know that the maximal singular solution u^* must be self-similar and has the form (1.5). Q.E.D.

Theorem 3.1 **(i)** *If $q \ge m(p-1)+p/n$, then (1.1) does not have any singular solution.*

(ii) *If $p < n(1+m)/(1+mn)$, then neither (1.1) nor (3.1) has any singular solution.*

Proof. **(i)** By the results of §2 (see also [14]), we know that if $q \geq m(p-1)+p/n$ then (1.6), (1.7) has no positive solution. Using the Lemma 3.3 we know that assertion holds.
(ii) $p < n(1+m)/(1+mn)$ implies $\mu < n$. Suppose for the contrary that (1.1) or (3.1) has a singular solution u. Then for any $t > 0$, applying Lemma 3.1 we have

$$\int_{|x|\leq 1} u(x,t)dx \leq \int_{|x|\leq 1} Bt^{\mu/p}|x|^{-\mu}dx \leq \frac{1}{n-\mu}B\omega_n t^{\mu/p} \to 0 \quad \text{as} \quad t \searrow 0,$$

where ω_n is the area of unit sphere in $\mathbb{R}^n$. This contradicts Lemma 3.1. Q.E.D.

Singular Solutions of (3.1)

Theorem 3.2 *Assume that $p > n(1+m)/(1+mn)$. Then for any $c > 0$, (3.1) has a unique FS with initial mass c. It is given by*

$$E_c(x,t) := t^{-nk}\{a + b(|x|t^{-k})^{p/(p-1)}\}^{-\theta}, \tag{3.5}$$

where $b = k^{1/(p-1)}[1-m(p-1)]/(mp)$, $\theta = (p-1)/[1-m(p-1)]$ and $a = a(c) > 0$ is the unique constant such that $\int_{\mathbb{R}^n}(a + b|y|^{p/(p-1)})^{-\theta}dy = c$.

Proof. It is clear that $E_c(x,t)$ is an FS of (3.1) with initial mass c. We need only to prove the uniqueness. Assume that u is any FS of (3.1) with initial mass c, we shall show that $u = E_c$. The proof is divided into three steps.
Step 1. Consider the sequence $\{u^h\}_{h>0}$, where $u^h(x,t) = h^{nk}u(h^k x, ht)$. Direct calculation shows that u^h is a solution of (3.1), and

$$\int_{\mathbb{R}^n} u^h(x,t)dx = \int_{\mathbb{R}^n} u(y,ht)dx = c \quad \forall h > 0,\ t > 0\,.$$

In view of (3.3) we have $u^h(x,t) = h^{nk}u(h^k x, ht) \leq Bt^{\mu/p}|x|^{-\mu}$. By the regularity results (see [20]) we know that $\{u^h(\cdot,1)\}$ is equi-continuous in any bounded domain of $\mathbb{R}^n$, so there exists a subsequence of $\{u^h\}$, denote also by $\{u^h\}$, and a function u_0 such that $u^h(\cdot,1) \to u_0(\cdot)$ as $h \searrow 0$ uniformly in any compact subset of $\mathbb{R}^n$. Since $\mu > n$ and $u^h(x,1) \leq B|x|^{-\mu}$, the Lebesgue's dominated convergence theorem then gives $u^h(\cdot,1) \to u_0$ in $L^1(\mathbb{R}^n)$. Let $v(x,t)$ be the solution of (3.1) in $\mathbb{R}^n \times (1,\infty)$ with initial data $v(\cdot,1) = u_0$. Then the contraction principle yields, for all $t > 1$,

$$\int_{\mathbb{R}^n} |u^h(\cdot,t) - v(\cdot,t)| \leq \int_{\mathbb{R}^n} |u^h(\cdot,1) - v(\cdot,1)| \to 0 \quad \text{as} \quad h \to 0. \tag{3.6}$$

Step 2. Denote, for each $h > 0$,

$$e^h(t) = \int_{\mathbb{R}^n} |u^h(\cdot,t) - E_c(\cdot,t)|. \tag{3.7}$$

The contraction principle implies that $e^h(t)$ is non-increasing. Because $E_c = E_c^h$, we have

$$\begin{aligned} e^h(t) &= \int_{\mathbb{R}^n} |u^h(\cdot,t) - E_c^h(\cdot,t)| = h^{nk}\int_{\mathbb{R}^n} |u(h^k x, ht) - E(h^k x, ht)|dx \\ &= \int_{\mathbb{R}^n} |u(x,ht) - E(x,ht)|dx = e^1(ht). \end{aligned}$$

Thus $e^h(t)$ is non-increasing in both t and h. Since the initial mass of u and E_c is c, $e^h(t)$ is bounded by $2c$. It then follows that $\lim_{h\searrow 0} e^h(t)$ exists, and

$$\lim_{h\searrow 0} e^h(1) = \lim_{h\searrow 0} e^1(h) = \lim_{h\searrow 0} e^1(2h) = \lim_{h\searrow 0} e^h(2).$$

Denote this limit by e^0. Then, in view of (3.6) and (3.7) we obtain

$$\begin{aligned} e^0 &= \lim_{h\to 0} e^h(1) = \lim_{h\to 0}\int_{\mathbb{R}^n} |u^h(\cdot,1) - E_c(\cdot,1)| \\ &= \int_{\mathbb{R}^n} |v(\cdot,1) - E_c(\cdot,1)| \\ &= \lim_{h\to 0} e^h(2) = \int_{\mathbb{R}^n} |v(\cdot,2) - E_c(\cdot,2)|. \end{aligned} \tag{3.8}$$

Step 3. We first show that $e^0 = 0$. Suppose for the contrary that $e^0 > 0$. We define $\overline{u}$ and $\underline{u}$ as the solution of (3.1) in $\mathbb{R}^n \times (1,\infty)$ with initial data

$$\overline{u}(\cdot,1) := \max\{v(\cdot,1),\ E_c(\cdot,1)\}, \quad \underline{u}(\cdot,1) := \min\{v(\cdot,1),\ E_c(\cdot,)\}.$$

Then the comparison principle gives $\overline{u} \geq \max\{v, E_c\} \geq \min\{v, E_c\} \geq \underline{u}$ in $\mathbb{R}^n \times [1,\infty)$. Since $v(\cdot,2) \not\equiv E_c(\cdot,2)$ and $\int_{\mathbb{R}^n} E_c(\cdot,2) = \int_{\mathbb{R}^n} v(\cdot,2) = c$, it follows that,

$$\begin{aligned} \int_{\mathbb{R}^n} [\overline{u}(\cdot,2) - \underline{u}(\cdot,2)] &> \int_{\mathbb{R}^n} [\max\{v(\cdot,2),\ E_c(\cdot,2)\} - \min\{v(\cdot,2),\ E_c(\cdot,2)\}] \\ &= \int_{\mathbb{R}^n} |v(\cdot,2) - E_c(\cdot,2)| = e^0\,. \end{aligned}$$

On the other hand, by the contraction principle,

$$\int_{\mathbb{R}^n} |\overline{u}(\cdot,2) - \underline{u}(\cdot,2)| \leq \int_{\mathbb{R}^n} |\overline{u}(\cdot,1) - \underline{u}(\cdot,1)| = \int_{\mathbb{R}^n} |v(\cdot,1) - E_c(\cdot,1)| = e^0.$$

Here we obtain a contradiction. Therefore, $e^0 = 0$. Because $e^1(t)$ is non-increasing in t, then $0 = e^0 = \lim_{t\searrow 0} e^1(t)$ implies that $e^1(t) \equiv 0$. Consequently, $u \equiv E_c$. The proof is completed. Q.E.D.

Singular Solutions of (1.1)

Theorem 3.3 *Assume that $1 < q < m(p-1) + p/n$, which implies $p > n(1+m)/(1+mn)$. Then for any $c > 0$, (1.1) has a unique FS, denoted as u_c, with initial mass c. Moreover, u_c is monotone increasing in c and $u_c \to u_\infty$ as $c \to \infty$, and u_∞ is a VSS of (1.1).*

Proof. Step 1: Existence. Let $E_c(x,t)$ be given by (3.5), and $\phi_l(x) = E_c(x, 1/l)$. Then

$$\int_{\mathbb{R}^n} \phi_l(x)dx = c, \quad \text{and} \quad \lim_{l\to\infty} \phi_l(x) = 0 \quad \forall \quad x \neq 0.$$

That is, $\{\phi_l(x)\}$ is a $\delta-$sequence. Let $u_l(x,t)$ and $w_l(x,t)$ be the solution of (1.1) and (3.1) with initial data $\phi_l(x)$ respectively. Because $E_c(x, t+1/l)$ satisfies (3.1) and has initial data $\phi_l(x)$, by the uniqueness we have $w_l(x,t) = E_c(x, t+1/l)$. From comparison it yields $u_l(x,t) \leq E_c(x, t+1/l)$. This shows that for any $\varepsilon > 0$, $\{u_l(x,t)\}$ are uniformly bounded in $\mathbb{R}^n \times [\varepsilon, \infty)$. Consequently, by the regularity results (see [20]) it follows that $\{u_l\}$ is equi-continuous in any compact subset of $\mathbb{R}^n \times (0,\infty)\backslash\{(0,0)\}$. Hence, there exist a function u and a subsequence, denote still by $\{u_l\}$, such that $u_l \to u$ uniformly in any compact subset of $\mathbb{R}^n \times (0,\infty)\backslash\{(0,0)\}$. The limit function u is necessarily a (weak) solution of (1.1) in $\mathbb{R}^n \times (0,\infty)$.

Now we show that u is an FS of (1.1) with initial mass c. First, by (3.3) we have

$$u_l(x,t) \leq E_c(x, t+1/l) \leq B[(t+1/l)^{1/p}|x|^{-1}]^{\mu}. \tag{3.9}$$

It follows that $u(x,t) \leq E_c(x,t)$. Therefore, u satisfies (1.2). Next, by Fatou's lemma we have

$$\int_{\mathbb{R}^n} u(x,t)dx \leq \lim\inf_{l\to\infty} \int_{\mathbb{R}^n} u_l(x,t)dx \leq c \quad \forall \quad t > 0.$$

Now, we prove that for any $\delta > 0$,

$$\lim_{t\searrow 0} \int_{|x|<\delta} u(x,t)dx = c. \tag{3.10}$$

¿From the differential equation of (1.1) we obtain

$$\int_{\mathbb{R}^n} u_l(x,t)dx = \int_{\mathbb{R}^n} \phi_l(x)dx - \int_0^t \int_{\mathbb{R}^n} u_l^q(x,t)\,dx\,dt = c - \int_0^t \int_{\mathbb{R}^n} u_l^q(x,t)\,dx\,dt.$$

¿From (3.9) it follows that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} u_l^q(x,t)\,dx\,dt \\
&\leq \int_0^t \int_{\mathbb{R}^n} (t+\frac{1}{l})^{-qnk}\{a + b(|x|(t+\frac{1}{l})^{-k})^{p/(p-1)}\}^{-q\theta}\,dx\,dt \\
&= \int_0^t \int_{\mathbb{R}^n} (t+\frac{1}{l})^{(1-q)nk}\{a + b|x|^{p/(p-1)}\}^{-q\theta}\,dx\,dt \\
&= C[(t+\frac{1}{l})^{1-(q-1)nk} - (\frac{1}{l})^{1-(q-1)nk}],
\end{aligned}$$

and

$$\int_{|x|>\delta} u_l dx \leq \int_{|x|>\delta} B(t+1/l)^{\mu/p}|x|^{-\mu}dx = C_1\delta^{n-\mu}(t+1/l)^{\mu/p} \quad \text{since} \quad \mu > n.$$

Therefore,

$$\begin{aligned}
&\int_{|x|\le\delta} u_l(x,t)dx \\
&\quad= \int_{\mathbb{R}^n} u_l(x,t)dx - \int_{|x|>\delta} u_l(x,t)dx \\
&\quad\ge \Big\{c - C[(t+1/l)^{1-(q-1)nk} - (1/l)^{1-(q-1)nk}]\Big\} - C_1\delta^{n-\mu}(t+\frac{1}{l})^{\mu/p}.
\end{aligned}$$

Since $nk(q-1) < 1$, as $l \to \infty$ we obtain

$$\int_{|x|\le\delta} u_l(x,t)dx \ge c - Ct^{1-(q-1)nk} - C_1\delta^{n-\mu}t^{\mu/p}.$$

As $t \searrow 0$ we have (3.10). Hence, u is an FS with initial mass c.

Step 2: Uniqueness. To prove the uniqueness, we first show that for any FS u of (1.1) with initial mass c, $u(x,t) \le E_c(x,t)$.

Let $w(x,t) = B(t^{1/p}|x|^{-1})^{\mu}$, then (3.3) implies $u(x,t) \le w(x,t)$. For any $\tau > 0$, let u_τ be the solution to (3.1) for $t > \tau$ with initial value $u_\tau = u$ on $\{t = \tau\}$. Then by comparison, $u_\tau \ge u$ for all $t > \tau$. Therefore, when $\tau_1 \le \tau_2$, $u_{\tau_1} \ge u_{\tau_2}$ for all $t > \tau_2$, i.e., $\{u_\tau\}_{\tau>0}$ is monotone decreasing in τ. Consequently, the limiting function $v = \lim_{\tau\searrow 0} u_\tau$ exists.

Since $u_\tau(x,\tau) = u(x,\tau) \le w(x,\tau)$ and $w(x,t+\tau)$ satisfies (3.1). By comparison we have $u_\tau(x,t) \le w(x,t+\tau)$. In view of the regularity of solutions of (3.1) we conclude that for any $t > 0$, $u_\tau(\cdot,t) \to v(\cdot,t)$ as $\tau \searrow 0$, uniformly in any compact subset of $\mathbb{R}^n \times (0,\infty)\backslash\{(0,0)\}$ and in $L^1(\mathbb{R}^n)$. Since $\int_{\mathbb{R}^n} u_\tau(x,t)dx = \int_{\mathbb{R}^n} u(x,\tau)dx \to c$ as $\tau \to 0$, we assert that $\int_{\mathbb{R}^n} v(x,t)dx = c$ for all $t > 0$. Thus, v is an FS of (3.1) with initial mass c. By uniqueness of FS of (3.1), $v = E_c$. Consequently, $u \le \lim_{\tau\searrow 0} u_\tau = v = E_c$.

Let u_1 and u_2 be any two FSs of (1.1) with initial mass c. Then $u_i \le E_c$ for $i = 1, 2$. In view of contraction principle, for $t > s > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u_1(x,t) - u_2(x,t)|dx \\
&\quad\le \int_{\mathbb{R}^n} |u_1(x,s) - u_2(x,s)|dx \\
&\quad\le \int_{\mathbb{R}^n} \{|u_1(x,s) - E_c(x,s)| + |E_c(x,s) - u_2(x,s)|\}dx \\
&\quad= \int_{\mathbb{R}^n} \{[E_c(x,s) - u_1(x,s)] + [E_c(x,s) - u_2(x,s)]\}dx.
\end{aligned}$$

As $s \searrow 0$ we get that $u_1(x,t) = u_2(x,t)$.

Step 3. From the proof of Step 1 and the results of Step 2 we know that u_c is monotone increasing in c.

Step 4. By Lemma 3.2 we have

$$u_c(x,t) \le At^{-1/(q-1)} + B(t^{1/p}|x|^{-1})^{\mu} \quad \forall c > 0.$$

Similar to the arguments of Step 1 we have that the limit $\lim_{c\to\infty} u_c = u_\infty$ exists and u_∞ is a VSS of (1.1). This completes the proof of the theorem. Q.E.D.

Theorem 3.4 *Assume that* $1 < q < m(p-1) + p/n$, *then (1.1) has a unique VSS.*

Proof. Step 1. We first prove that each FS is no large than a VSS. Let u_c and U be an FS and a VSS of (1.1) respectively. By Lemma 3.3 we obtain the maximal singular solution $u^* = (\alpha/t)^\alpha f(|x|(\alpha/t)^{\alpha\beta})$, where f solves (1.6). Therefore,

$$\int_{\mathbb{R}^n} U(x,t)dx \le \int_{\mathbb{R}^n} u^*(x,t)dx = t^{\alpha(n\beta-1)}\alpha^{\alpha(1-n\beta)} \int_0^\infty f(y)dy.$$

For any $\sigma > 0$ we define the truncated VSS

$$U_\sigma(x,t) = \begin{cases} U(x,t) & \text{if } U(x,t) < \sigma, \\ \sigma & \text{if } U(x,t) \ge \sigma. \end{cases}$$

Because $n\beta < 1$ and U is a VSS, we conclude that there exist a sequence $\{\tau(l)\}$ with $\tau(l) \searrow 0$ and the corresponding $\{\sigma(l)\}$ such that

$$\int_{\mathbb{R}^n} U_{\sigma(l)}(x,\tau(l))dx = c.$$

Define $\psi_l(x) = U_{\sigma(l)}(x,\tau(l))$, and let v_l be the solution of (1.1) with initial value $\psi_l(x)$. Because $\psi_l(x) \le U(x,\tau(l))$, by the comparison principle we have $v_l(x,t) \le U(x,t+\tau(l)) \le A(t+\tau(l))^{-1/(q-1)} + B\{(t+\tau(l))^{1/p}|x|^{-1}\}^\mu$. Similar to the argument of Step 1 of the proof of Theorem 3.3 we have that limit $\lim_{l\to\infty} v_l = v$ exists and v is an FS of (1.1) with initial mass c, and $v(x,t) \le U(x,t)$. By the uniqueness of FS of (1.1) it follows $u_c = v \le U$.

Step 2. By Step 1 we know that the VSS u_∞ obtained by Theorem 3.3 is the minimal VSS. Similar to the proof of Lemma 3.3 we have that u_∞ is self-similar and has the form (1.5), and the corresponding function f solves (1.6). Theorem 2.1 shows that $u_\infty = u^*$. Therefore, VSS is unique. Q.E.D.

Appendix

In this appendix, we show that the initial value problem (1.6), (2.1) has a unique solution in a right neighbourhood of $r = 0$.

Proposition 3.1 *Suppose $q > 1$, $m > 0$, $p > 1$ and $m(p-1) < 1$. For each $a \in R$, the following problem*

$$\begin{aligned} (|(|f|^{m-1}f)'|^{p-2}(|f|^{m-1}f)')' + \frac{n-1}{r}|(|f|^{m-1}f)'|^{p-2}(|f|^{m-1}f)' & \\ +\beta rf' + f - |f|^{q-1}f &= 0 \\ (|f|^{m-1}f)'(0) = 0, \quad f(0) = a & \end{aligned} \tag{A.1}$$

has a unique solution with $|f|^{m-1}f$ in C^1 and $|(|f|^{m-1}f)'|^{p-2}(|f|^{m-1}f)'$ Hölder continuous in a right neighbourhood of $r = 0$. Furthermore, f is continuously differentiable in a for $a > 0$ and r sufficiently small and the following hold:

1. *If $a = 0$, then $f \equiv 0$;*
2. *If $a = 1$, then $f \equiv 1$;*
3. *If $a > 1$, then $f' > 0$.*

Proof: First we derive an integral equation which is equivalent to the initial value problem (A.1). Integrating (A.1) over $(0, r)$ multiplied by r^{n-1}, we find

$$\begin{aligned} -(|(|f|^{m-1}f)'|^{p-2}(|f|^{m-1}f)') &= \beta rf + \frac{1}{r^{n-1}}\int_0^r \rho^{n-1}[1 - n\beta - |f|^{q-1}]f d\rho \\ &=: \mathcal{G}[f](r) \end{aligned} \tag{A.2}$$

Integrating the $1/(p-1)-$th power of both sides then gives

$$|f|^{m-1}f = |a|^{m-1}a - \int_0^r |\mathcal{G}[f](\rho)|^{(2-p)/(p-1)}\mathcal{G}[f](\rho)d\rho =: \mathcal{H}[f](r) \tag{A.3}$$

or equivalently,

$$f = |\mathcal{H}[f](r)|^{(1-m)/m}\mathcal{H}[f](r).$$

Now we proceed to prove that (A.1) has a uniques solution with the desired smoothness.

The first case we consider is $1/(p-1) > 1$ and $m \le 1$. It is clear that $|\mathcal{G}[f](\rho)|^{(2-p)/(p-1)}\mathcal{G}[f](\rho)$ is continuously differentiable in f, hence the right hand side of (A.3) is continuously differentiable in $|f|^{m-1}f$. The existence and uniqueness of solution follows from classical Picard iteration and Gronwall's inequality.

Next, we consider the case of $1/(p-1) > 1$ and $m > 1$. For any two continuous functions f_1 and f_2,

$$\begin{aligned} &\Big||\mathcal{H}[f_1](r)|^{(1-m)/m}\mathcal{H}[f_1](r) - |\mathcal{H}[f_2](r)|^{(1-m)/m}\mathcal{H}[f_2](r)\Big| \\ &\le C(\|f_1\|_\infty, \|f_2\|_\infty)|\mathcal{H}[f_1](r)|^{p-2}\mathcal{H}[f_1](r) - |\mathcal{H}[f_2](r)|^{p-2}\mathcal{H}[f_2](r)| \\ &\le C(\|f_1\|_\infty, \|f_2\|_\infty)(\int_0^r |\mathcal{G}[f_1](\rho) - \mathcal{G}[f_2](\rho)|^{1/(p-1)}d\rho)^{p-1} \end{aligned}$$

by triangle inequality and our assumption that $m(p-1)<1$. Again, we have the existence and uniqueness.

At last, we deal with the case of $1/(p-1)<1$ and $m<1$. We show $|\mathcal{G}[f](r)|^{(2-p)/(p-1)}\mathcal{G}[f](r)$ is a Lipschitz continuous function of $|f|^{m-1}f$. It is clear that we can think of $\mathcal{G}[f](\rho)=\int_0^r F d\mu$ with certain measure μ, where $F(r)$ is continuously differentiable in f. In addition, for any two continuous functions f_1 and f_2, the corresponding F_1 and F_2 satisfy

$$\begin{aligned}
&\| \, |F_1|^{(2-p)/(p-1)}F_1-|F_2|^{(2-p)/(p-1)}F_2\|_\infty \\
&\quad\le\ C(\|f_1\|_\infty,\|f_2\|_\infty)\| \, |f_1|^{(2-p)/(p-1)}f_1-|f_2|^{(2-p)/(p-1)}f_2\|_\infty \\
&\quad\le\ C(\|f_1\|_\infty,\|f_2\|_\infty)\| \, |f_1|^{m-1}f_1-|f_2|^{m-1}f_2\|_\infty,
\end{aligned}$$

again by triangle inequality and our assumption that $m(p-1)<1$.

If $F_1,F_2\ge 0$, then

$$\begin{aligned}
&\Big| \, |\mathcal{G}[f_1](r)|^{(2-p)/(p-1)}\mathcal{G}[f_1](r)-|\mathcal{G}[f_2](r)|^{(2-p)/(p-1)}\mathcal{G}[f_2](r)\Big| \\
&\quad=\ |(\int_0^r F_1 d\mu)^{1/(p-1)}-(\int_0^r F_2 d\mu)^{1/(p-1)}| \\
&\quad\le\ (\int_0^r |F_1^{1/(p-1)}-F_2^{1/(p-1)}|^{p-1}d\mu)^{1/(p-1)} \\
&\quad\le\ C(\|f_1\|_\infty,\|f_2\|_\infty)\| \, |f_1|^{m-1}f_1-|f_2|^{m-1}f_2\|_\infty.
\end{aligned}$$

In general, if $a_1=:\int_0^r F_1^+d\mu\ge a_2=:\int_0^r F_1^-d\mu$ and $b_1=:\int_0^r F_2^+d\mu\ge a_2=:\int_0^r F_2^-d\mu$, $|(\int_0^r F_1 d\mu)^{1/p-1}-(\int_0^r F_2 d\mu)^{1/p-1}|=|(a_1-a_2)^{1/(p-1)}-(b_1-b_2)^{1/(p-1)}|$, by the elementary inequality,

$$\begin{aligned}
&|(a_1-a_2)^{1/(p-1)}-(b_1-b_2)^{1/(p-1)}| \\
&\quad\le\ C\left(|a_1^{1/(p-1)}-b_1^{1/(p-1)}|+|a_2^{1/(p-1)}-b_2^{1/(p-1)}|\right),
\end{aligned}$$

we reduce this case to the case of $F_1,F_2\ge 0$.

Finally, if $a_1\ge a_2$ but $b_2\ge b_1$,

$$|(\int_0^r F_1 d\mu)^{1/p-1}-(\int_0^r F_2 d\mu)^{1/p-1}|=|(a_1-a_2)^{1/(p-1)}+(b_2-b_1)^{1/(p-1)}|,$$

by the elementary inequality,

$$\begin{aligned}
&|(a_1-a_2)^{1/(p-1)}+(b_2-b_1)^{1/(p-1)}| \\
&\quad\le\ C\left(|a_1^{1/(p-1)}-b_1^{1/(p-1)}|+|a_2^{1/(p-1)}-b_2^{1/(p-1)}|\right),
\end{aligned}$$

we again reduce to the case of $F_1,F_2\ge 0$. Therefore, $|\mathcal{G}[f](r)|^{(2-p)/(p-1)}\mathcal{G}[f](r)$ is a Lipschitz continuous function of $|f|^{m-1}f$. This complete the part of existence and uniqueness of solution with desired smoothness.

The continuous differentiability of f to a when $a>0$ and r sufficiently small follows directly from the positivity of f and the continuous differentiability of $|f|^{m-1}f$ to a.

The assertions (1) and (2) follows from the uniqueness of solution. (3) can be seen directly from (A.2). This completes the proof of the proposition.

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