Electronic Journal of Differential Equations, Vol. 2005(2005), No. 53, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

THROUGHOUT POSITIVE SOLUTIONS OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

ZHENGUO ZHANG, CHUNJIAO WANG, QIAOLUAN LI, FANG LI

Abstract. In this paper, we consider the second-order nonlinear and the nonlinear neutral functional differential equations

$$(a(t)x'(t))' + f(t, x(g(t))) = 0, \quad t \geq t_0$$
$$(a(t)(x(t) - p(t)x(t-\tau))')' + f(t, x(g(t))) = 0, \quad t \geq t_0\,.$$

Using the Banach contraction mapping principle, we obtain the existence of throughout positive solutions for the above equations.

1. Introduction

Recently, there has been an increasing interest in the study of the oscillation and nonoscillation of solutions of second-order ordinary and delay neutral differential and difference equations. Also eventually positive solutions and asymptotic behavior of nonoscillatory solutions have been investigated widely. Delay differential equations play a very important role in many practical problems. The papers [3, 4, 7, 8, 11, 12, 15] discuss the oscillation of second order differential and difference equations. The papers [1, 5] discuss the oscillation and non-oscillation criteria for second order differential equations. Of course there is also the discussion of the existence of eventually positive solutions, such as [10, 6, 13, 14]. But there are relatively few which guarantee the existence of throughout positive solutions. The paper [9] studies the positive solutions of the following second order non-neutral ordinary differential equation

$$y''(t) + F(t, y(t)) = 0, \quad t \geq a$$

where $F : [a, \infty) \times R \to R$ is continuous and nonnegative. We have studied further and extended the results of Erik Wahlén [9] to the self-conjugate and neutral functional differential equations. We obtain the existence of throughout positive solutions by introducing a weighted norm (see [2, 9]) and using the Banach contraction mapping principle (see [2]).

2000 *Mathematics Subject Classification.* 34C10, 34K11.
Key words and phrases. Nonlinear differential equations; neutral term;
eventually positive solution; throughout positive solution.

Submitted December 13, 2004. Published May 19, 2005.
Supported by the Natural Science Foundation of Hebei Province and by the Main Foundation of Hebei Normal University.

In this paper, we are concerned with existence of throughout positive solutions for the following self-conjugate nonlinear differential equations

$$(a(t)x'(t))' + f(t, x(g(t))) = 0, \quad t \geq t_0 \tag{1.1}$$

$$(a(t)(x(t) - p(t)x(t-\tau))')' + f(t, x(g(t))) = 0, \quad t \geq t_0 \tag{1.2}$$

where $a(t) > 0$ is continuous; $f(t,x)$ is continuous and satisfies $f(t,x)x > 0$ for $x \neq 0$; $g(t)$ is continuous, increasing and satisfies $g(t) \leq t$, $\lim_{t\to\infty} g(t) = \infty$.

1.1. **Definitions.** A solution of differential equation is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be non-oscillatory.

A solution of differential equation is said to be eventually positive solution if there exists some $T \geq t_0$ such that $x(t) > 0$ for all $t \geq T$.

A solution of differential equation is said to be throughout positive solution if $x(t) > 0$ for all $t \geq t_0$.

Related Lemmas. To obtain our main results, we need the following lemma.

Lemma 1.1. *Assume $x(t)$ is bounded, $\lim_{t\to\infty} p(t) = p$, $p \neq \pm 1$,*

$$z(t) = x(t) - p(t)x(t-\tau), \quad \lim_{t\to\infty} z(t) = l,$$

then $\lim_{t\to\infty} x(t)$ exists and $\lim_{t\to\infty} x(t) = l/(1-p)$.

Proof. **(1)** $p \in (-\infty, -1)$. Since $x(t)$ is bounded, we get that $\limsup_{t\to\infty} x(t) = M$ and $\liminf_{t\to\infty} x(t) = m$ exist. Then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} x(t_n - \tau) = M$ and

$$l = \limsup_{n\to\infty} z(t_n) = \limsup_{n\to\infty} (x(t_n) - p(t_n)x(t_n - \tau)) \geq m - pM\,.$$

Similarly there exists a sequence $\{t'_n\}$ such that $\lim_{n\to\infty} x(t'_n - \tau) = m$ and

$$l = \liminf_{n\to\infty} z(t'_n) = \liminf_{n\to\infty} (x(t'_n) - p(t'_n)x(t'_n - \tau)) \leq M - pm\,.$$

So we have $M - pm \geq m - pM$, that is, $(1+p)M \geq (1+p)m$. In view of $1 + p < 0$, we get $M \leq m$. Hence $M = m$ and $\lim_{t\to\infty} x(t)$ exists. By the assumption, we obtain $\lim_{t\to\infty} x(t) = 1/(1-p)$.

(2) $p \in (-1, 0)$. Similarly, there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} x(t_n) = M$. Then there exists a sequence $\{t'_n\}$ such that $\lim_{n\to\infty} x(t'_n) = m$ and

$$l = \limsup_{n\to\infty} z(t_n) = \limsup_{n\to\infty} (x(t_n) - p(t_n)x(t_n - \tau)) \geq M - pm,$$

$$l = \liminf_{n\to\infty} z(t'_n) = \liminf_{n\to\infty} (x(t'_n) - p(t'_n)x(t'_n - \tau)) \leq m - pM\,.$$

Therefore, $M - pm \leq m - pM$, that is, $(1+p)M \leq (1+p)m$. In view of $1 + p > 0$, we get $M \leq m$. Hence $M = m$ and $\lim_{t\to\infty} x(t)$ exists. By the assumption, we obtain $\lim_{t\to\infty} x(t) = 1/(1-p)$.

(3) $p \in [0, 1)$. Similarly, there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} x(t_n) = M$. Then there exists a sequence $\{t'_n\}$ such that $\lim_{n\to\infty} x(t'_n) = m$ and

$$l = \limsup_{n\to\infty} z(t_n) = \limsup_{n\to\infty} (x(t_n) - p(t_n)x(t_n - \tau)) \geq M(1-p),$$

$$l = \liminf_{n\to\infty} z(t'_n) = \liminf_{n\to\infty} (x(t'_n) - p(t'_n)x(t'_n - \tau)) \leq m(1-p)\,.$$

Therefore, $M(1-p) \leq m(1-p)$. In view of $1-p > 0$ we get $M \leq m$. Hence $M = m$ and $\lim_{t\to\infty} x(t)$ exists. By the assumption, we obtain $\lim_{t\to\infty} x(t) = 1/(1-p)$.

(4) $p \in (1, +\infty)$. Similarly, there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} x(t_n - \tau) = M$. Then there exists a sequence $\{t'_n\}$ such that $\lim_{n\to\infty} x(t'_n - \tau) = m$ and

$$\begin{gathered}
l = \limsup_{n\to\infty} z(t_n) = \limsup_{n\to\infty}(x(t_n) - p(t_n)x(t_n - \tau)) \le M(1-p), \\
l = \liminf_{n\to\infty} z(t'_n) = \liminf_{n\to\infty}(x(t'_n) - p(t'_n)x(t'_n - \tau)) \ge m(1-p)\,.
\end{gathered}$$

Therefore, $M(1-p) \ge m(1-p)$. In view of $1 - p < 0$ we get $M \le m$. Hence $M = m$ and $\lim_{t\to\infty} x(t)$ exists. By the assumption, $\lim_{t\to\infty} x(t) = l/(1-p)$ which completes the proof. $\square$

2. Main Results

In this section we give existence theorems of throughout positive solutions for equations (1.1) and (1.2). First of all we need the following conditions:

Assume that the nonlinearity f satisfies a Lipschitz condition

$$|f(t,u) - f(t,v)| \le k(t)|u - v|, \quad \text{for } 0 \le u,\ v \le C \text{ and } t \ge t_0, \tag{2.1}$$

where the constant C will be specified in the theorems below, and $k(t) > 0$ is a continuous function satisfying

$$\int_{t_0}^{\infty} \frac{s}{\overline{a}(s)} k(s)ds < \infty, \tag{2.2}$$

where $\overline{a}(s) = \min\{a(\theta) : \min\{t_0 - \tau, g(t_0)\} \le \theta \le s\}$.

Theorem 2.1. *For equation* (1.1)*, we define the set*

$$X = \{u \in C^1[t_0, \infty),\ 0 \le u(t) \le M,\ \textit{for } t \ge t_0; u(t) = u(t_0),\ \textit{for } g(t_0) \le t < t_0\}.$$

Assume that for every $u \in X$*,*

$$\int_{t_0}^{\infty} \frac{ds}{a(s)} \int_s^{\infty} f(\theta, u(g(\theta)))d\theta < M\,. \tag{2.3}$$

Let conditions (2.1) *and* (2.2) *hold for* $0 \le u, v \le M$*. Assume further that there exists a positive integer* $N > 1$ *such that* $0 < \frac{l}{N} < 1$*, where* $l(N) = \max\{\frac{G(g(t))}{G(t)},\ t \ge t_0\}$*,* $G(t) = \exp(N\int_t^{\infty} \frac{s}{\overline{a}(s)} k(s)ds)$*. Then equation* (1.1) *has a throughout positive solution* $x(t)$ *on* $[t_0, \infty)$ *satisfying* $\lim_{t\to\infty} x(t) = M$*.*

Proof. Define a mapping $\mathcal{T}$ on X as follows

$$(\mathcal{T}x)(t) = \begin{cases} M - \int_t^{\infty} \frac{ds}{a(s)} \int_s^{\infty} f(\theta, x(g(\theta)))d\theta & t \ge t_0 \\ (\mathcal{T}x)(t_0) & g(t_0) \le t < t_0\,. \end{cases} \tag{2.4}$$

From (2.3) we have $0 \le (\mathcal{T}x)(t) \le M$, so $\mathcal{T}X \subseteq X$. From the assumption $G(t) = \exp(N\int_t^{\infty} \frac{s}{\overline{a}(s)} k(s)ds)$, we introduce the norm $\|\cdot\|$ on X, $\|x\| = \sup_{t\ge t_0} |x(t)|/G(t)$. Note that X is closed with respect to this norm, and therefore we have a complete metric space.

We now show that $\mathcal{T}$ is a contraction mapping on X. For any $x_1, x_2 \in X$, in view of the assumptions we have

$$\begin{aligned}
\frac{|(\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t)|}{G(t)} &\le \frac{1}{G(t)} \int_t^\infty \frac{ds}{a(s)} \int_s^\infty |f(\theta, x_1(g(\theta))) - f(\theta, x_2(g(\theta)))| d\theta \\
&\le \frac{1}{G(t)} \int_t^\infty \frac{ds}{a(s)} \int_s^\infty \frac{G(g(\theta))k(\theta)|x_1(g(\theta)) - x_2(g(\theta))|}{G(g(\theta))} d\theta \\
&\le \frac{1}{G(t)} \|x_1 - x_2\| \int_t^\infty \frac{ds}{a(s)} \int_s^\infty G(\theta)k(\theta)\frac{G(g(\theta))}{G(\theta)} d\theta \\
&\le \frac{l}{G(t)} \|x_1 - x_2\| \int_t^\infty \frac{(s-t)G(s)k(s)}{\overline{a}(s)} ds \\
&\le \frac{l}{G(t)} \|x_1 - x_2\| \int_t^\infty \frac{sG(s)k(s)}{\overline{a}(s)} ds \\
&= \frac{l}{G(t)} \|x_1 - x_2\| \int_t^\infty (-\frac{1}{N}) G'(s) ds \\
&= l\frac{G(t) - 1}{NG(t)} \|x_1 - x_2\| \\
&\le \frac{l}{N} \|x_1 - x_2\|.
\end{aligned}$$

Since $0 < \frac{l}{N} < 1$, $\mathcal{T}$ is a contraction mapping on X. Finally we use the Banach fixed point theorem to deduce the existence of a unique fixed point in X,

$$x(t) = (\mathcal{T}x)(t) = M - \int_t^\infty \frac{ds}{a(s)} \int_s^\infty f(\theta, x(g(\theta))) d\theta.$$

From (2.3) we know that $x(t) > 0$ for $t \ge t_0$. Clearly $x(t)$ satisfies

$$(a(t)x'(t))' + f(t, x(g(t))) = 0,$$

thus $x(t)$ is a throughout positive solution of (1.1) and $\lim_{t\to\infty} x(t) = M$. The proof is complete. □

Now we discuss the equation (1.2).

Theorem 2.2. *Assume that* $\lim_{t\to\infty} p(t) = p$, *where* $p \in [0, 1)$ *and* $0 < p(t) \le p$. *Define*

$$\begin{aligned}
X = \big\{&u \in C^1[t_0, \infty),\ 0 \le u(t) \le M,\ \textit{for } t \ge t_0;\ u(t) = u(t_0), \\
&\textit{for } \min\{g(t_0), t_0 - \tau\} \le t < t_0\big\}.
\end{aligned}$$

Let condition (2.1) *and* (2.2) *hold for* $0 \le u, v \le M$, *and we replace* (2.3) *by*

$$\int_{t_0}^\infty \frac{ds}{a(s)} \int_s^\infty f(\theta, u(g(\theta))) d\theta < M(1-p). \tag{2.5}$$

Assume further there exists a positive integer $N > 1$ *such that* $0 < (p + \frac{1}{N})l < 1$, *where* $l(N) = \max\{\frac{G(t-\tau)}{G(t)}, \frac{G(g(t))}{G(t)}, t \ge t_0\}$, $G(t) = \exp\ (N\int_t^\infty \frac{s}{\overline{a}(s)} k(s) ds)$. *Then equation* (1.2) *has a throughout positive solution* $x(t)$ *on* $[t_0, \infty)$ *satisfying* $\lim_{t\to\infty} x(t) = M$.

Proof. Define a mapping $\mathcal{T}$ on X as follows

$$(\mathcal{T}x)(t) = \begin{cases} M(1-p) + p(t)x(t-\tau) - \int_t^\infty \frac{ds}{a(s)} \int_s^\infty f(\theta, x(g(\theta)))d\theta & t \geq t_0 \\ (\mathcal{T}x)(t_0) & \min\{t_0 - \tau, g(t_0)\} \leq t \leq t_0 \,. \end{cases}$$

For $t \geq t_0$, from (2.5) and $p(t) \leq p$, we have $0 \leq (\mathcal{T}x)(t) \leq M(1-p) + pM = M$, so $\mathcal{T}X \subseteq X$. We introduce the norm $\|\cdot\|$ on X, $\|x\| = \sup_{t \geq t_0} |x(t)|/G(t)$. Now we show that $\mathcal{T}$ is a contraction mapping on X. For any $x_1, x_2 \in X$, in view of the assumptions we have

$$\begin{aligned}
&\frac{|(\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t)|}{G(t)} \\
&\leq p(t)\frac{|x_1(t-\tau) - x_2(t-\tau)|}{G(t)} \\
&\quad + \frac{1}{G(t)} \int_t^\infty \frac{ds}{a(s)} \int_s^\infty |f(\theta, x_1(g(\theta))) - f(\theta, x_2(g(\theta)))| d\theta \\
&\leq p(t)\frac{G(t-\tau)}{G(t)} \frac{|x_1(t-\tau) - x_2(t-\tau)|}{G(t-\tau)} \\
&\quad + \frac{1}{G(t)} \int_t^\infty \frac{ds}{a(s)} \int_s^\infty |f(\theta, x_1(g(\theta))) - f(\theta, x_2(g(\theta)))| d\theta \\
&\leq p\ l\ \|x_1 - x_2\| + \frac{1}{G(t)} \int_t^\infty \frac{ds}{a(s)} \int_s^\infty \frac{G(g(\theta))k(\theta)|x_1(g(\theta)) - x_2(g(\theta))|}{G(g(\theta))} d\theta \\
&\leq p\ l\ \|x_1 - x_2\| + \frac{1}{G(t)} \|x_1 - x_2\| \int_t^\infty \frac{ds}{a(s)} \int_s^\infty G(\theta)k(\theta)\frac{G(g(\theta))}{G(\theta)} d\theta \\
&\leq p\ l\ \|x_1 - x_2\| + \frac{l}{G(t)} \|x_1 - x_2\| \int_t^\infty \frac{(s-t)G(s)k(s)}{\overline{a}(s)} ds \\
&\leq p\ l\ \|x_1 - x_2\| + \frac{l}{G(t)} \|x_1 - x_2\| \int_t^\infty \frac{sG(s)k(s)}{\overline{a}(s)} ds \\
&= p\ l\ \|x_1 - x_2\| + \frac{l}{G(t)} \|x_1 - x_2\| \int_t^\infty (-\frac{1}{N})G'(s) ds \\
&\leq \big(p + \frac{G(t)-1}{NG(t)}\big)\ l\ \|x_1 - x_2\| \\
&\leq (p + \frac{1}{N})\ l\ \|x_1 - x_2\|.
\end{aligned}$$

Since $0 < (p + \frac{1}{N})l < 1$, $\mathcal{T}$ is a contraction mapping on X. Finally we use the Banach fixed point theorem to deduce the existence of a unique fixed point in X

$$x(t) = (\mathcal{T}x)(t) = M(1-p) + p(t)x(t-\tau) - \int_t^\infty \frac{ds}{a(s)} \int_s^\infty f(\theta, x(g(\theta)))d\theta.$$

From the condition (2.5) and $p(t)x(t-\tau) \geq 0$ we know that $x(t) > 0$ for $t \geq t_0$. Clearly $x(t)$ satisfies

$$(a(t)(x(t) - p(t)x(t-\tau))')' + f(t, x(g(t))) = 0,$$

thus $x(t)$ is a throughout positive solution of (1.2) and

$$\lim_{t\to\infty} (x(t) - p(t)x(t-\tau)) = M(1-p)\,.$$

In view of the Lemma 1.1, $\lim_{t\to\infty} x(t) = M$ which completes the proof. $\square$

Theorem 2.3. *Assume that* $\lim_{t\to\infty} p(t) = p$ *where* $p \in (-1, 0)$ *and* $p \le p(t) < 0$ *and define*

$$\begin{aligned} Y = \{\ & u \in C^1[t_0,\infty),\ 0 \le u(t) \le M(1-p),\ \textit{for} t \ge t_0;\ u(t) = u(t_0), \\ & \textit{for } \min\{g(t_0), t_0-\tau\} \le t < t_0\}. \end{aligned}$$

Let conditions (2.1) *and* (2.2) *hold for* $0 \le u, v \le M(1-p)$*. Assume that for every* $u \in Y$*,*

$$\int_{t_0}^{\infty} \frac{ds}{a(s)} \int_s^{\infty} f(\theta, u(g(\theta)))d\theta < M(1-p^2). \tag{2.6}$$

Assume further that there exists a positive integer $N > 1$ *such that* $0 < (\frac{1}{N} - p)l < 1$*, where* $l(N) = \max\{\frac{G(t-\tau)}{G(t)}, \frac{G(g(t))}{G(t)}, t \ge t_0\}$*,* $G(t) = \exp(N \int_t^{\infty} \frac{s}{\bar{a}(s)}\ k(s)ds)$*. Then equation* (1.2) *has a throughout positive solution* $x(t)$ *on* $[t_0,\infty)$ *satisfying* $\lim_{t\to\infty} x(t) = M$*.*

Proof. Define a mapping $\mathcal{T}$ on Y as follows

$$(\mathcal{T}x)(t) = \begin{cases} M(1-p) + p(t)x(t-\tau) - \int_t^{\infty} \frac{ds}{a(s)} \int_s^{\infty} f(\theta, x(g(\theta)))d\theta & t \ge t_0 \\ (\mathcal{T}x)(t_0) & \min\{t_0-\tau, g(t_0)\} \le t \le t_0\,. \end{cases}$$

Since $p(t) < 0$, we easily know that $0 \le (\mathcal{T}x)(t) \le M(1-p)$. So $\mathcal{T}X \subseteq X$. We introduce the norm $\|\cdot\|$ on Y, $\|x\| = \sup_{t\ge t_0} |x(t)|/G(t)$. We now show that $\mathcal{T}$ is a contraction mapping on Y. Similar to the proof of Theorem 2.2, for any $x_1, x_2 \in Y$, in view of the assumptions we have

$$\begin{aligned} \frac{|(\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t)|}{G(t)} &\le |p(t)| \frac{G(t-\tau)}{G(t)} \frac{|x_1(t-\tau) - x_2(t-\tau)|}{G(t-\tau)} \\ &\quad + \frac{1}{G(t)} \int_t^{\infty} \frac{ds}{a(s)} \int_s^{\infty} |f(\theta, x_1(g(\theta))) - f(\theta, x_2(g(\theta)))| d\theta \\ &\le |p|\ l\ \|x_1 - x_2\| + \frac{l}{N} \|x_1 - x_2\| \\ &= (\frac{1}{N} - p)\ l\ \|x_1 - x_2\|. \end{aligned}$$

Since $0 < (\frac{1}{N} - p)l < 1$, $\mathcal{T}$ is a contraction mapping on Y. Finally we use the Banach fixed point theorem to deduce the existence of a unique fixed point in Y,

$$x(t) = (\mathcal{T}x)(t) = M(1-p) + p(t)x(t-\tau) - \int_t^{\infty} \frac{ds}{a(s)} \int_s^{\infty} f(\theta, x(g(\theta)))d\theta.$$

Since $x \in Y$ and $p \le p(t) < 0$, we have $p(t)x(t-\tau) \ge pM(1-p)$. From the inequality and the condition (2.6), we obtain

$$x(t) > M(1-p) + pM(1-p) - M(1-p^2) = 0.$$

Hence $x(t) > 0$ for $t \ge t_0$. Substituting $x(t)$ into (1.2), we know that $x(t)$ is a throughout positive solution of equation (1.2) and

$$\lim_{t\to\infty} (x(t) - p(t)x(t-\tau)) = M(1-p).$$

In view of the Lemma 1.1, $\lim_{t\to\infty} x(t) = M$ which completes the proof. $\square$

Theorem 2.4. *Assume that* $\lim_{t\to\infty} p(t) = p$ *where* $p \in (-\infty, -1)$ *and* $p(t) \le p$. *Define*

$$Z = \Big\{\ u \in C^1[t_0,\infty),\ 0 \le u(t) \le \frac{M(1+|p|)}{|p|},\ \textit{for } t \ge t_0;\ u(t) = u(t_0),\\ \textit{for } g(t_0) \le t < t_0\Big\}$$

where M *is a positive constant. Let conditions* (2.1) *and* (2.2) *hold for* $0 \le u, v \le \frac{M(1+|p|)}{|p|}$. *Assume that for every* $u \in Z$,

$$\int_{t_0}^{\infty} \frac{ds}{a(s)} \int_s^{\infty} f(\theta, u(g(\theta)))d\theta < \frac{M(p^2-1)}{|p|}. \tag{2.7}$$

Assume further there exists a positive integer $N > 1$ *such that* $0 < \frac{1}{|p|}(1 + \frac{l}{N}) < 1$, *where* $l(N) = \max\{\frac{G(g(t))}{G(t)}, t \ge t_0\}$, $G(t) = \exp\ (N \int_t^{\infty} \frac{s}{\bar{a}(s)}\ k(s)ds)$. *Then equation* (1.2) *has a throughout positive solution* $x(t)$ *on* $[t_0,\infty)$ *satisfying* $\lim_{t\to\infty} x(t) = M$.

Proof. Define a mapping $\mathcal{T}$ on Z as follows

$$(\mathcal{T}x)(t) = \begin{cases} \frac{1}{-p(t+\tau)}\big[M(1-p) - x(t+\tau) - \int_{t+\tau}^{\infty} \frac{ds}{a(s)} \int_s^{\infty} f(\theta, x(g(\theta)))d\theta\big] & t \ge t_0 \\ (\mathcal{T}x)(t_0) & g(t_0) \le t \le t_0. \end{cases}$$

From (2.7), we have $0 \le (\mathcal{T}x)(t) \le \frac{M(1+|p|)}{|p|}$. So $\mathcal{T}Z \subseteq Z$. We introduce the norm $\|\cdot\|$ on Z, $\|x\| = \sup_{t\ge t_0} |x(t)|/G(t)$. We now show that $\mathcal{T}$ is a contraction mapping on Z. For any $x_1, x_2 \in Z$, in view of the assumptions we have

$$\begin{aligned}
&\frac{|(\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t)|}{G(t)}\\
&\le \frac{-1}{G(t+\tau)p(t+\tau)}|x_1(t+\tau) - x_2(t+\tau)|\\
&\quad + \frac{-1}{G(t)p(t+\tau)} \int_{t+\tau}^{\infty} \frac{ds}{a(s)} \int_s^{\infty} |f(\theta, x_1(g(\theta))) - f(\theta, x_2(g(\theta)))|d\theta\\
&\le \frac{1}{|p|}\|x_1 - x_2\| + \frac{1}{G(t)|p|} \int_{t+\tau}^{\infty} \frac{ds}{a(s)} \int_s^{\infty} \frac{G(g(\theta))k(\theta)|x_1(g(\theta)) - x_2(g(\theta))|}{G(g(\theta))}d\theta\\
&\le \frac{1}{|p|}\|x_1 - x_2\| + \frac{1}{G(t)|p|}\|x_1 - x_2\| \int_{t+\tau}^{\infty} \frac{ds}{a(s)} \int_s^{\infty} G(\theta)k(\theta)\frac{G(g(\theta))}{G(\theta)}d\theta\\
&\le \frac{1}{|p|}\|x_1 - x_2\| + \frac{l}{G(t)|p|}\|x_1 - x_2\| \int_{t+\tau}^{\infty} \frac{(s-t-\tau)G(s)k(s)}{\bar{a}(s)}ds\\
&\le \frac{1}{|p|}\|x_1 - x_2\| + \frac{l}{G(t)|p|}\|x_1 - x_2\| \int_{t+\tau}^{\infty} \frac{sG(s)k(s)}{\bar{a}(s)}ds\\
&= \frac{1}{|p|}\|x_1 - x_2\| + \frac{l}{G(t)|p|}\|x_1 - x_2\| \int_{t+\tau}^{\infty} (-\frac{1}{N})G'(s)ds\\
&\le \frac{1}{|p|}\left(1 + l\ \frac{G(t+\tau) - 1}{NG(t)}\right)\|x_1 - x_2\|\\
&\le \frac{1}{|p|}(1 + \frac{l}{N})\|x_1 - x_2\|.
\end{aligned}$$

Since $0 < \frac{1}{|p|}(1 + \frac{l}{N}) < 1$, $\mathcal{T}$ is a contraction mapping on Z. Finally we use the Banach fixed point theorem to deduce the existence of a unique fixed point in Z,

$$\begin{aligned} x(t) &= (\mathcal{T}x)(t) \\ &= \frac{1}{-p(t+\tau)}\Big[M(1-p) - x(t+\tau) - \int_{t+\tau}^{\infty} \frac{ds}{a(s)} \int_{s}^{\infty} f(\theta, x(g(\theta)))d\theta\Big]. \end{aligned}$$

Since $x \in Z$, we have $x(t+\tau) \le \frac{M(1+|p|)}{|p|}$. From the inequality and the condition (2.7), we obtain

$$x(t) > \frac{1}{-p(t+\tau)}\big[M(1-p) - \frac{M(1+|p|)}{|p|} - \frac{M(p^2-1)}{|p|}\big] = 0.$$

Hence $x(t) > 0$ for $t \ge t_0$. Substituting $x(t)$ into (1.2), we know that $x(t)$ is a throughout positive solution of (1.2) and

$$\lim_{t\to\infty}(x(t) - p(t)x(t-\tau)) = M(1-p).$$

In view of Lemma 1.1, we have $\lim_{t\to\infty} x(t) = M$. The proof is complete. $\square$

Theorem 2.5. *Assume that* $\lim_{t\to\infty} p(t) = p$ *where* $p \in (1, +\infty)$ *and* $p(t) \ge p$. *Define*

$$\begin{aligned} \Omega = \big\{&u \in C^1[t_0, \infty),\ 0 \le u(t) \le \frac{M(1+p)}{p},\ \text{for } t \ge t_0;\ u(t) = u(t_0), \\ &\text{for } g(t_0) \le t < t_0\big\} \end{aligned}$$

where M *is a positive constant. Let conditions* (2.1) *and* (2.2) *hold for* $0 \le u, v \le \frac{M(1+p)}{p}$. *We assume that for every* $u \in \Omega$

$$\int_{t_0}^{\infty} \frac{ds}{a(s)} \int_{s}^{\infty} f(\theta, u(g(\theta)))d\theta \le \frac{p-1}{p}M. \tag{2.8}$$

Assume further that there exists a positive integer $N > 1$ *such that* $0 < \frac{1}{p}(1 + \frac{l}{N}) < 1$, *where* $l(N) = \max\{\frac{G(g(t))}{G(t)}, t \ge t_0\}$, $G(t) = \exp(N\int_t^{\infty} \frac{s}{a(s)}\ k(s)ds)$. *Then* (1.2) *has a throughout positive solution* $x(t)$ *on* $[t_0, \infty)$ *satisfying* $\lim_{t\to\infty} x(t) = M$.

Proof. Define a mapping $\mathcal{T}$ on Ω as follows

$$(\mathcal{T}x)(t) = \begin{cases} \frac{1}{p(t+\tau)}\big[M(p-1) + x(t+\tau) + \int_{t+\tau}^{\infty} \frac{ds}{a(s)} \int_{s}^{\infty} f(\theta, x(g(\theta)))d\theta\big] & t \ge t_0 \\ (\mathcal{T}x)(t_0) & g(t_0) \le t \le t_0\,. \end{cases}$$

From (2.8), we have $0 \le (\mathcal{T}x)(t) \le \frac{p+1}{p}M$. So $\mathcal{T}\Omega \subseteq \Omega$. We introduce the norm $\|\cdot\|$ on Ω, $\|x\| = \sup_{t \ge t_0} |x(t)|/G(t)$. We now show that $\mathcal{T}$ is a contraction mapping on Ω. Similar to the proof of Theorem 2.4, for any $x_1, x_2 \in \Omega$, in view of the assumptions we have

$$
\begin{aligned}
&\frac{|(\mathcal{T}x_1)(t)-(\mathcal{T}x_2)(t)|}{G(t)}\\
&\le \frac{1}{G(t+\tau)p(t+\tau)}|x_1(t+\tau)-x_2(t+\tau)|\\
&\quad+\frac{1}{G(t)p(t+\tau)}\int_{t+\tau}^\infty\frac{ds}{a(s)}\int_s^\infty|f(\theta,x_1(g(\theta)))-f(\theta,x_2(g(\theta)))|d\theta\\
&\le \frac{1}{p}(1+\frac{l}{N})\|x_1-x_2\|.
\end{aligned}
$$

Since $0<\frac{1}{p}(1+\frac{l}{N})<1$, $\mathcal{T}$ is a contraction mapping on Ω. Finally we use the Banach fixed point theorem to deduce the existence of a unique fixed point in Ω,

$$
\begin{aligned}
x(t)&=(\mathcal{T}x)(t)\\
&=\frac{1}{p(t+\tau)}\big[M(p-1)+x(t+\tau)+\int_{t+\tau}^\infty\frac{ds}{a(s)}\int_s^\infty f(\theta,x(g(\theta)))d\theta\big].
\end{aligned}
$$

Because $p>1$, that is $M(p-1)>0$, and all the other terms which are in the expression of $x(t)$ are nonnegative, we easily know that $x(t)>0$ for $t\ge t_0$. Substituting $x(t)$ into (1.2), we know that $x(t)$ is a throughout positive solution of equation (1.2) and

$$
\lim_{t\to\infty}(x(t)-p(t)x(t-\tau))=M(1-p).
$$

In view of the Lemma 1.1 we have $\lim_{t\to\infty}x(t)=M$. The proof is complete. $\square$

3. Examples

Example 3.1. Consider the second order self-conjugate differential equation

$$
(tx'(t))'+\frac{4(t-1)^6}{t^6(t-2)^3}\quad x^3(t-1)=0,\quad t\ge t_0=6. \tag{3.1}
$$

In our notation, $a(t)=t$, $\overline{a}(s)=5$, $g(t)=t-1$, $f(t,u)=\frac{4(t-1)^6}{t^6(t-2)^3}\ u^3$. We choose $M=1$, $k(t)=\frac{12(t-1)^6}{t^6(t-2)^3}$, $N=3$. We know that for any $0\le u,v\le 1$,

$$
|f(t,u)-f(t,v)| \quad = \quad |\ \frac{4(t-1)^6}{t^6(t-2)^3}\ (u^3-v^3)|\le\frac{12(t-1)^6}{t^6(t-2)^3}\ |u-v|.
$$

For any $u,v\in X$

$$
\int_{t_0}^\infty\frac{s}{\overline{a}(s)}\ k(s)\ ds=\frac{1}{5}\int_6^\infty\frac{12(s-1)^6}{s^5(s-2)^3}\ ds<\infty
$$

$$
\begin{aligned}
\int_{t_0}^\infty\frac{ds}{a(s)}\int_s^\infty f(\theta,u(g(\theta)))\,d\theta&=\int_6^\infty\frac{4}{s}\ ds\int_s^\infty\frac{(\theta-1)^6(u(\theta-1))^3}{\theta^6(\theta-2)^3}\ d\theta\\
&\le\int_6^\infty\frac{4}{s}\,ds\int_s^\infty\frac{d\theta}{(\theta-2)^3}\\
&=\frac{1}{4}+\frac{1}{2}\ln\frac{4}{6}\le\frac{1}{4}<1\,,
\end{aligned}
$$

$$l = \exp\Big(N\int_{t_0-1}^{t_0}\frac{s}{t_0-1}\frac{12(s-1)^6}{s^6(s-2)^3}ds\Big) = \exp\Big(3\int_5^6\frac{s}{5}\frac{12(s-1)^6}{s^6(s-2)^3}ds\Big) < 3.$$

Thus the conditions in Theorem 2.1 are satisfied. So (3.1) has a throughout positive solution $x(t)$ on $[t_0,\infty)$ and $\lim_{t\to\infty} x(t) = 1$. In fact, $x(t) = 1-\frac{1}{t^2}$ is such a solution.

Example 3.2. Consider the second-order neutral differential equation

$$(x(t) - \frac{1}{2}x(t-1))'' + \frac{2(t-1)^3 - t^3}{(t-1)^3(t-2)^3}x^3(t-1) = 0, \quad t \ge t_0 = 13. \tag{3.2}$$

Here $a(t) = 1$, $\overline{a}(s) = 1$, $p(t) = \dfrac{1}{2}$, $g(t) = t-1$, $f(t,u) = \dfrac{[2(t-1)^3 - t^3]u^3}{(t-1)^3(t-2)^3}$. We choose $M = 1$, $k(t) = \dfrac{3[2(t-1)^3 - t^3]}{(t-1)^3(t-2)^3}$, $N = 4$. It is easy to show that for any $0 \le u, v \le 1$,

$$|f(t,u) - f(t,v)| = |\frac{2(t-1)^3 - t^3}{(t-1)^3(t-2)^3}(u^3 - v^3)| \le \frac{3[2(t-1)^3 - t^3]}{(t-1)^3(t-2)^3}|u-v|.$$

For any $u, v \in X$

$$\int_{t_0}^\infty \frac{s}{\overline{a}(s)}k(s)\,ds = \int_{13}^\infty 3s\frac{2(s-1)^3 - s^3}{(s-1)^3(s-2)^3}\,ds < \infty\,,$$

$$\begin{aligned}
\int_{t_0}^\infty \frac{ds}{a(s)}\int_s^\infty f(\theta, u(g(\theta)))d\theta &= \int_{13}^\infty\int_s^\infty \frac{2(\theta-1)^3-\theta^3}{(\theta-1)^3(\theta-2)^3}(u(\theta-1))^3 d\theta ds\\
&= \int_{13}^\infty (\theta - t)\frac{2(\theta-1)^3-\theta^3}{(\theta-1)^3(\theta-2)^3}(u(\theta-1))^3 d\theta\\
&\le \int_{13}^\infty \frac{2\theta}{(\theta-2)^3}\ d\theta\\
&= \frac{24}{121} < \frac{1}{2}\,,
\end{aligned}$$

$$(p + \frac{1}{N})l = (\frac{1}{2} + \frac{1}{4})\ \exp\ \Big(4\int_{12}^{13} s\frac{3[2(s-1)^3 - s^3]}{(s-1)^3(s-2)^3}\ ds\Big) < 1.$$

Thus the conditions in Theorem 2.2 are satisfied. So (3.2) has a throughout positive solution $x(t)$ on $[t_0,\infty)$ and $\lim_{t\to\infty} x(t) = 1$. In fact, $x(t) = 1-\frac{1}{t}$ is such a solution.

Example 3.3. Consider the second-order self-conjugate neutral differential equation

$$\Big[\frac{t^3(t-1)}{4((t-1)^3 + 2t^3)}(x(t) + 2x(t-1))'\Big]' + \frac{(t-1)^3}{t^3(t-2)^3}\ x^3(t-1) = 0,\ t \ge t_0 = 9. \tag{3.3}$$

In our notation, $p(t) = -2$, $g(t) = t-1$, $\tau = 1$, $a(t) = \dfrac{t^3(t-1)}{4((t-1)^3 + 2t^3)}$, $\overline{a}(s) = \frac{896}{1367}$, $f(t,u) = \dfrac{(t-1)^3}{t^3(t-2)^3}\ u^3$. We choose that $M = 1$, $k(t) = \dfrac{27(t-1)^3}{4t^3(t-2)^3}$, $N = 3$. Here we define $Z = \{u \in C^1[t_0,\infty) : 0 \le u(t) \le \frac{3}{2},\ t \ge t_0\}$. It is easy to show that for any $0 \le u, v \le \frac{3}{2}$,

$$|f(t,u) - f(t,v)| = |\frac{(t-1)^3}{t^3(t-2)^3}(u^3 - v^3)| \le \frac{27(t-1)^3}{4t^3(t-2)^3}|u-v|.$$

For any $u, v \in Z$,
$$\int_{t_0}^{\infty} \frac{s}{\overline{a}(s)} k(s)\, ds = \frac{1367}{896} \int_9^{\infty} \frac{27(s-1)^3}{4s^2(s-2)^3}\, ds < \infty\,,$$
$$\begin{aligned}
&\int_{t_0}^{\infty} \frac{ds}{a(s)} \int_s^{\infty} f(\theta, u(g(\theta)))d\theta \\
&\leq \frac{4((t_0-1)^3 + 2t_0^3)}{t_0^3(t_0-1)} \int_9^{\infty} ds \int_s^{\infty} \frac{(\theta-1)^3 (u(\theta-1))^3}{\theta^3(\theta-2)^3}\, d\theta \\
&\leq \frac{12}{t_0-1} \int_9^{\infty} ds \int_s^{\infty} \frac{d\theta}{(\theta-2)^3} \\
&= \frac{3}{28} < \frac{3}{2}\,,
\end{aligned}$$
$$\frac{1}{|p|}(1 + \frac{l}{N}) = \frac{1}{2}\Big[1 + \frac{1}{3} \exp\Big(3 \int_8^9 s \frac{4((9-2)^3 + 2(9-1)^3)}{(9-1)^3(9-2)} \frac{27(s-1)^3}{4s^2(s-2)^3}\, ds\Big)\Big] < 1.$$
Thus the conditions in Theorem 2.4 are satisfied. So (3.3) has a throughout positive solution $x(t)$ on $[t_0, \infty)$ and $\lim_{t\to\infty} x(t) = 1$. In fact, $x(t) = 1 - \frac{1}{t^2}$ is such a solution.

References

[1] M. M. A. El-Sheikh; *Oscillation and nonoscillation criteria for second order nonlinear differential equations.* J. Math. Anal. Appl. 179(1993) 14-27.
[2] L. H. Erbe; Qingkai. Kong; B. G. Zhang; Oscillation theory for functional differential equations. New York. Basel. Hong Kong, 1995.
[3] W. T. Li and J. R. Yan; *Oscillation criteria for second order superlinear differential equations.* Indina. J. Pure. Appl. Math. 28(6)(1997) 735-740.
[4] W. T. Li; *Oscillation of certain second order nonlinear differential equations.* J. Math. Anal. Appl. 217(1998) 1-14.
[5] A. Lomtatidze; *Oscillation and nonoscillation criteria for second order linear differential equations.* Georgian. Math. Journal. Vol4. No.2 (1997) 129-138.
[6] K. Y. Liu, Z. Q. Zhang; *Existence of positive solutions in neutral differential and differnce equations with delays.* Acta Sci. Nat. Univ. Norm. Hunan . Vol21. (1998) 12-18.
[7] S. H. Saker; *Oscillation of second order nonlinear delay difference equations.* Bull. Korean. Math. Soc. 40(2003) No.3 489-501.
[8] E. Thandapani, K. Ravi; *Oscillation of second order half-linear difference equations.* Applied Math. Letters. 13(2000) 43-49.
[9] E. Wahlén; *Positive solutions of second order differential equations.* Nonlinear Analysis. 58(2004) 359-366.
[10] Jun Yang, Xin Ping Guan; *Positive solution of a class of neutral delay difference equations.* Acta Mathematic Sinica. Vol. 44. No.3 (2001) 409-416.
[11] J. R. Yan; *Oscillation of second order neutral functional differential equations.* Appl. Math. Comp. 83(1997) 27-41.
[12] J. R. Yan; *The oscillation properties for the solution of the second order differential equation with "integral small" coefficient.* Acta. Math. Sinica. 30(2)(1987) 206-215.
[13] X. J. Yang; *Nonoscillation criteria for second order nonlinear differential equations.* Appl. Math. Comp. 131(2002) 125-131.
[14] J. S. Yu, M. P. Chen, H. Zhang; *Oscillation and nonoscillation in neutral equations with integrable coefficients.* Comp. Math. Applic. Vol35. No.6 (1998) 65-71.
[15] Z. G. Zhang, J. L. Zhang; *Oscillation criteria for second order functional difference equations with "summation small" coefficient.* Comp. Math. Appl. 38(1999) 25-31.

Zhenguo Zhang
College of Mathematics and Information Science, Hebei normal University, Heibei, Shijiazhuang, 050016, China
E-mail address: Zhangzhg@mail.hebtu.edu.cn

Chunjiao Wang
College of Mathematics and Information Science, Hebei normal University, Heibei, Shijiazhuang, 050016, China
E-mail address: ccjjj0601@sina.com.cn

Qiaoluan Li
College of Mathematics and Information Science, Hebei normal University, Heibei, Shijiazhuang, 050016, China
E-mail address: qll71125@163.com

Fang Li
College of Mathematics and Information Science, Hebei normal University, Heibei, Shijiazhuang, 050016, China
E-mail address: lifanglucky@126.com