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# ON A MIXED NONLOCAL PROBLEM FOR A WAVE EQUATION

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Abstract. A nonlocal problem for a wave equation with integral condition is studied in a cylinder. The uniqueness and existence of a generalized solution is proved with the help of an a priori estimate and the Galerkin procedure, respectively.

## 1. Introduction

Certain problems of modern physics and technology can be effectively described in terms of nonlocal problems for partial differential equations. The history of nonlocal problems with integral conditions for partial differential equations goes back to [3]. Cannon studied a problem for a heat equation, and in most papers, devoted to nonlocal problems, parabolic and elliptic equations were studied. Mixed problems with nonlocal integral conditions for one-dimensional hyperbolic equations were considered in [7, 2, 4, 1, 8, 9].

Nonlocal problem for a hyperbolic equation with $n$ space variables with a different integral condition was considered in [5]

In this paper we consider an $n$-dimensional wave equation with weighted integral nonlocal condition. Consider a wave equation

$$u_{tt} - \Delta u + c(x,t)u = f(x,t) \tag{1.1}$$

in a cylinder $Q = \Omega \times (0,T)$, where $\Omega \in R^n$ is a bounded domain with a smooth boundary, with Cauchy conditions

$$u(x,0) = \phi(x),\ \ u_t(x,0) = \psi(x), \tag{1.2}$$

and an integral nonlocal condition

$$\frac{\partial u}{\partial n}\big|_S + \int_0^t \int_\Omega K(x,\xi,\tau)u(\xi,\tau)d\xi d\tau = 0, \quad x \in \partial\Omega, \tag{1.3}$$

where $\phi(x), \psi(x), K(x,\xi,\tau)$ are given functions.

Denote $\widehat{W}_2^1(Q) = \{v(x,t) : v \in W_2^1(Q), v(x,T) = 0\}$. Let $u(x,t)$ be a solution of the problem. Multiply the equation (1.1) by the function $v(x,t) \in \widehat{W}_2^1(Q)$ and

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integrate it over the cylinder $Q$. After integrating by parts, we obtain

$$\begin{aligned}&\int_0^T\int_\Omega (\nabla u\nabla v-u_tv_t+cuv)\,dx\,dt\\&=\int_0^T\int_\Omega fvdx\,dt+\int_0^T\int_{\partial\Omega}\frac{\partial u}{\partial n}vds\,dt+\int_\Omega \psi(x)v(x,0)dx.\end{aligned}$$

After substituting the condition (1.3), we have:

$$\begin{aligned}&\int_0^T\int_\Omega (\nabla u\nabla v-u_tv_t+cuv)\,dx\,dt\\&+\int_0^T\int_{\partial\Omega}v(x,t)\int_0^t\int_\Omega K(x,\xi,\tau)u(\xi,\tau)d\xi d\tau ds\,dt\\&=\int_0^T\int_\Omega fvdx\,dt+\int_\Omega \psi(x)v(x,0)dx.\end{aligned}\tag{1.4}$$

**Definition 1.1.** A function $u(x,t)\in W_2^1(Q)$ is called a generalized solution of (1.1)-(1.2)-(1.3), if it satisfies (1.4) for every $v\in \widehat{W}_2^1(Q)$ and $u(x,0)=\phi(x)$.

The main result of this paper can be formulated as the following theorem.

**Theorem 1.2.** *If $\phi(x)\in W_2^1(\Omega)$, $\psi(x)\in L_2(\Omega)$, $f(x,t)\in L_2(Q)$, $K(x,\xi,t)\in C(\Omega\times\Omega\times(0,T))$, there exist $\frac{\partial K}{\partial \xi_i}$, $i=1\dots n$ and*

$$\max_{\bar Q}|K|\le K_0,\quad \max_{\bar Q}\Big|\frac{\partial K}{\partial\xi_i}\Big|\le K_1$$

*then there exists a unique generalized solution to the problem* (1.1)-(1.2)-(1.3).

The following two sections provide the proof of the above theorem.

## 2. Uniqueness

Let $u_1\neq u_2$ be solutions of (1.1)-(1.2)-(1.3); then $u=u_1-u_2$ is a solution of the same problem with $f=\phi=\psi=0$; so, $u(x,t)$ satisfies

$$\begin{aligned}&\int_0^T\int_\Omega (\nabla u\nabla v-u_tv_t+cuv)\,dx\,dt\\&+\int_0^T\int_{\partial\Omega}v(x,t)\int_0^t\int_\Omega K(x,\xi,\tau)u(\xi,\tau)d\xi d\tau ds\,dt=0.\end{aligned}\tag{2.1}$$

Consider the function

$$v(x,t)=\begin{cases}\int_t^\tau u(x,\eta)d\eta, & 0\le t\le\tau,\\ 0, & \tau\le t\le T.\end{cases}\tag{2.2}$$

Note, that $v(x,t)\in \widehat{W}_2^1(Q)$, and $v_t(x,t)=-u(x,t)$ $\forall t\in[0,\tau]$. Integration by parts in (2.1) will give us:

$$\int_0^T\int_\Omega \nabla u\nabla vdx\,dt=-\int_0^\tau\int_\Omega \nabla v_t\nabla vdx\,dt=\int_0^\tau\int_\Omega \nabla v\nabla v_tdx\,dt-\int_\Omega |\nabla v|^2|_0^\tau dx,$$

hence,

$$\int_0^T \int_\Omega \nabla u \nabla v dx\, dt = -\frac{1}{2} \int_\Omega |\nabla v(x,0)|^2 dx;$$

$$\int_0^T \int_\Omega u_t v_t dx\, dt = -\int_0^\tau \int_\Omega u_t u dx\, dt = \int_0^\tau \int_\Omega u u_t dx\, dt - \int_\Omega u^2|_0^\tau dx,$$

hence,

$$\int_0^T \int_\Omega u_t v_t dx\, dt = \frac{1}{2} \int_\Omega u^2(x,\tau) dx.$$

Thus, (2.1) will take the form:

$$\begin{aligned} &\frac{1}{2} \int_\Omega \left(|\nabla v(x,0)|^2 + u^2(x,\tau)\right) dx\, dt \\ &= \int_0^T \int_{\partial\Omega} v(x,t) \int_0^t \int_\Omega K(x,\xi,\tau) u(\xi,\tau) d\xi\, d\tau\, ds\, dt. \end{aligned}$$

With the help of elementary inequalities we obtain the following estimates:

$$\begin{aligned} &\frac{1}{2} \int_\Omega \left(|\nabla v(x,0)|^2 + u^2(x,\tau)\right) dx\, dt \\ &\le \int_0^T \int_{\partial\Omega} |v(x,t)| \int_0^t \int_\Omega |K(x,\xi,\tau)||u(\xi,\tau)| d\xi d\tau ds\, dt \\ &\le K_0 \int_0^T \int_{\partial\Omega} |v(x,t)| \int_0^t \int_\Omega |u(\xi,\tau)| d\xi d\tau ds\, dt \\ &\le K_0 \int_0^T \int_{\partial\Omega} \Big( v^2(x,t) + T|\Omega| \int_0^t \int_\Omega u^2(\xi,\tau) d\xi d\tau \Big) ds\, dt \\ &= K_0 \int_0^\tau \int_{\partial\Omega} v^2(x,t) ds\, dt + K_0 T^2 |\Omega||\partial\Omega| \int_0^\tau \int_\Omega u^2(x,t) dx\, dt. \end{aligned} \tag{2.3}$$

Using the inequality

$$\int_{\partial\Omega} v^2(x,t) ds \le \int_\Omega \left(\epsilon|\nabla v|^2 + c(\epsilon) v^2\right) dx$$

and denoting $L = T^2|\Omega||\partial\Omega|$, we obtain

$$\int_\Omega \left(|\nabla v(x,0)|^2 + u^2(x,\tau)\right) dx\, dt \le K_0 \int_0^\tau \int_\Omega \left(\epsilon|\nabla v|^2 + c(\epsilon) v^2 + L u^2\right) dx\, dt.$$

It's trivial that

$$v^2 = \Big(\int_t^\tau u(x,\eta) d\eta\Big)^2 \le \tau \int_0^\tau u^2(x,\eta) d\eta,$$

and

$$\int_0^\tau \int_\Omega v^2 dx\, dt \le \int_0^\tau \int_\Omega \Big(\tau \int_0^\tau u^2(x,\eta) d\eta\Big) dx\, dt \le \tau^2 \int_0^\tau \int_\Omega u^2(x,t) dx\, dt.$$

Using this result and denoting $C_0 = K_0 \max\{\epsilon, c(\epsilon), L\}$, we obtain the following inequality:

$$\int_\Omega \left(|\nabla v(x,0)|^2 + u^2(x,\tau)\right) dx\, dt \le C_0 \int_0^\tau \int_\Omega \left(|\nabla v(x,t)|^2 + u^2(x,t)\right) dx\, dt. \tag{2.4}$$

Consider the function

$$w(x,t)=\int_0^t u(x,\eta)d\eta.$$

Then $v(x,t)=w(x,\tau)-w(x,t)$, $\nabla v(x,0)=\nabla w(x,\tau)$, and also

$$|\nabla v|^2=|\nabla w_(x,\tau)-\nabla w(x,t)|^2\le 2|\nabla w(x,\tau)|^2+2|\nabla w(x,t)|^2,$$

so we have

$$\int_0^\tau\int_\Omega|\nabla v|^2dx\,dt\le 2\tau\int_\Omega|\nabla w(x,\tau)|^2dx+2\int_0^\tau\int_\Omega|\nabla w|^2dx\,dt.$$

Thus, the inequality (2.4) takes the form

$$\begin{aligned}&\int_\Omega\big(|\nabla w(x,\tau)|^2+u^2(x,\tau)\big)\,dx\\ &\le 2C_0\tau\int_\Omega|\nabla w(x,\tau)|^2dx+2C_0\int_0^\tau\int_\Omega\big(|\nabla w|^2+u^2\big)\,dx\,dt.\end{aligned}\tag{2.5}$$

Because of arbitrariness of $\tau$, let $\tau$ satisfy $2C_0\tau<1$. Then it follows from (2.5):

$$(1-2C_0\tau)\int_\Omega\big(|\nabla w(x,\tau)|^2+u^2(x,\tau)\big)\,dx\le 2C_0\int_0^\tau\int_\Omega\big(|\nabla w|^2+u^2\big)\,dx\,dt.\tag{2.6}$$

Applying the Gronwall inequality, we conclude that

$$\int_\Omega\big(|\nabla w(x,\tau)|^2+u^2(x,\tau)\big)\,dx\le 0\quad\forall\tau\in[0,\frac{1}{2C_0}],$$

whence it follows that

$$u(x,\tau)=0\quad\forall\tau\in[0,\frac{1}{2C_0}].$$

Obtaining the same inequality for the "slices" $\tau\in\big[\frac{(k-1)}{2C_0},\frac{k}{2C_0}\big]$ to cover the $[0,T]$, we ensure that

$$u(x,\tau)=0\quad\forall\tau\in[0,T].$$

Thus, the uniqueness is proved.

## 3. Existence

To prove the existence, we will apply the Galerkin method. Let $w_k(x)$ be a fundamental system in $W_2^1(\Omega)$ such that $(w_k,w_l)_{L_2(\Omega)}=\delta_{k,l}$.

We will try to find approximate solutions in the form

$$u^m(x,t)=\sum_{k=1}^m d_k(t)w_k(x),\tag{3.1}$$

where coefficients $d_k(t)$ are to be determined, from

$$\begin{aligned}&\int_\Omega\left(u_{tt}^m w_l+\nabla u^m\nabla w_l+cu^m w_l\right)dx\\ &+\int_{\partial\Omega}w_l(x)\int_0^t\int_\Omega K(x,\xi,\tau)u^m(\xi,\tau)d\xi d\tau ds\quad=\int_\Omega fw_l dx.\end{aligned}\tag{3.2}$$

$$d_k(0)=\alpha_k,\quad d_k'(0)=\beta_k,\tag{3.3}$$

where $\alpha_k$ are coefficients of sums $\phi^m(x)=\sum_{k=1}^m\alpha_k w_k$, approximating the function $\phi(x)$ as $m\to\infty$ with respect to the norm of $W_2^1(\Omega)$, and $\beta_k=(\psi,w_k)_{L_2(\Omega)}$.

Substituting (3.1) into (3.2), we obtain

$$\begin{aligned}&\int_\Omega \sum_{k=1}^m (d_k'' w_k w_l + d_k \nabla w_k \nabla w_l + c d_k w_k w_l)\,dx\\&+\int_{\partial\Omega} w_l \int_0^t \int_\Omega K(x,\xi,\tau)\sum_{k=1}^m d_k(\tau) w_k(\xi) d\xi d\tau ds = f_l(t),\end{aligned}$$

where $f_l(t) = (f, w_l)_{L_2(\Omega)}$. Changing the order of summation and integration, we have

$$\begin{aligned}&\sum_{k=1}^m (d_k''(t)(w_k,w_l)_{L_2} + d_k(t)(\nabla w_k, \nabla w_l)_{L_2} + d_k(t)(c w_k, w_l)_{L_2}))\\&+\sum_{k=1}^m \int_0^t \Big( d_k(\tau)\int_{\partial\Omega} w_l(x)\int_\Omega K(x,\xi,\tau) w_k(\xi) d\xi\, ds\Big) d\tau = f_l(t).\end{aligned}$$

Denote, for short,

$$\gamma_{kl}(t) = (\nabla w_k, \nabla w_l) + (c w_k, w_l); \tag{3.4}$$

$$\kappa_{kl}(\tau) = \int_{\partial\Omega} w_l(x) \int_\Omega K(x,\xi,\tau) w_k(\xi) d\xi\, ds. \tag{3.5}$$

and obtain

$$\sum_{k=1}^m d_k''(t)\delta_{kl} + d_k(t)\gamma_{kl}(t) + \int_0^t d_k(\tau)\kappa_{kl}(\tau)d\tau = f_l(t). \tag{3.6}$$

The resulting system of integro–differential equations can be reduced to a system of differential equations of the third order by differentiating (3.6) with respect to $t$,

$$\sum_{k=1}^m d_k'''(t)\delta_{kl} + d_k'(t)\gamma_{kl}(t) + d_k(t)(\kappa_{kl}(t) + \gamma_{kl}'(t)) = f_l'(t). \tag{3.7}$$

Together with the initial conditions

$$d_k(0) = \alpha_k, \quad d_k'(0) = \beta_k, \quad d_k''(0) = f_l(0) - \alpha_k \gamma_k(0), \tag{3.8}$$

we have a Cauchy problem for the system of linear differential equations with smooth coefficients, that is uniquely solvable.

Thus, for every $m$ there exists a unique $u^m(x,t)$, that satisfies (3.2), in other words, the sequence $\{u^m\}$ is defined. We shall investigate it's convergence. Multiply (3.2) by $d_l'(t)$, sum by $l$ from 1 to $m$ and, finally, integrate by $t$ from 0 to $\tau$,

$$\begin{aligned}&\int_0^\tau \int_\Omega (u_{tt}^m u_t^m + \nabla u^m \nabla u_t^m + c u^m u_t^m)\,dx\,dt\\&+\int_0^\tau \int_{\partial\Omega} u_t^m \int_0^t \int_\Omega K(x,\xi,\eta) u^m(\xi,\eta) d\xi\, d\eta ds\, dt\\&= \int_0^\tau \int_\Omega f u_t^m dx\,dt.\end{aligned}$$

Integration by parts on the left will give us

$$\int_0^\tau \int_\Omega u_{tt}^m u_t^m dx\,dt = -\int_0^\tau \int_\Omega u_t^m u_{tt}^m dx\,dt + \int_\Omega (u_t^m)^2 \,|_0^\tau dx, \tag{3.9}$$

hence,

$$\int_0^\tau \int_\Omega u_{tt}^m u_t^m dx\,dt = \frac{1}{2}\int_\Omega \left(u_t^m(x,\tau)\right)^2 dx - \frac{1}{2}\int_\Omega \left(u_t^m(x,0)\right)^2 dx;$$

$$\int_0^\tau \int_\Omega \nabla u^m \nabla u_t^m dx\,dt = -\int_0^\tau \int_\Omega \nabla u^m \nabla u_t^m dx\,dt + \int_\Omega \left(\nabla u^m\right)^2 |_0^\tau dx,$$

hence,

$$\int_0^\tau \int_\Omega \nabla u^m \nabla u_t^m dx\,dt = \frac{1}{2}\int_\Omega \left(\nabla u^m(x,\tau)\right)^2 dx - \frac{1}{2}\int_\Omega \left(\nabla u^m(x,0)\right)^2 dx;$$

$$\begin{aligned}\int_0^\tau \int_\Omega c u^m u_t^m dx\,dt = &-\int_0^\tau \int_\Omega c u_t^m u^m dx\,dt - \int_0^\tau \int_\Omega c_t (u^m)^2 dx\,dt \\ &+ \int_\Omega c\left(u^m(x,t)\right)^2 |_0^\tau dx;\end{aligned}$$

hence,

$$\begin{aligned}\int_0^\tau \int_\Omega c u^m u_t^m dx\,dt = &\frac{1}{2}\int_\Omega c\left(u^m(x,\tau)\right)^2 dx - \frac{1}{2}\int_\Omega c\left(u^m(x,0)\right)^2 dx \\ &- \frac{1}{2}\int_0^\tau \int_\Omega c_t (u^m)^2 dx\,dt;\end{aligned}$$

$$\begin{aligned}&\int_{\partial\Omega}\int_0^\tau u_t^m(x,t)\int_0^t \int_\Omega K(x,\xi,\eta)u^m(\xi,\eta)d\xi\,d\eta\,dt\,ds \\ &= -\int_{\partial\Omega}\int_0^\tau u^m(x,t)\int_\Omega K(x,\xi,t)u^m(\xi,\eta)d\xi\,dt\,ds \\ &\quad + \int_{\partial\Omega}\Big(u^m(x,t)\int_0^t\int_\Omega K(x,\xi,\eta)u^m(\xi,\eta)d\xi\,d\eta\Big)\Big|_0^\tau ds \\ &= -\int_{\partial\Omega}\int_0^\tau u^m(x,t)\int_\Omega K(x,\xi,t)u^m(\xi,\eta)d\xi\,dt\,ds \\ &\quad + \int_{\partial\Omega} u^m(x,\tau)\int_0^\tau\int_\Omega K(x,\xi,\eta)u^m(\xi,\eta)d\xi\,d\eta\,ds.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega \Big(\left(u_t^m(x,\tau)\right)^2 + |\nabla u^m(x,\tau)|^2 + c(x,\tau)\left(u^m(x,\tau)\right)^2\Big)dx \\ &= \frac{1}{2}\int_\Omega \Big(\left(u_t^m(x,0)\right)^2 + |\nabla u^m(x,0)|^2 + c(x,0)\left(u^m(x,0)\right)^2\Big)dx \\ &\quad + \frac{1}{2}\int_0^\tau\int_\Omega c_t\left(u_m\right)^2 dx\,dt + \int_0^\tau\int_\Omega f u_t^m dx\,dt \\ &\quad + \int_0^\tau\int_{\partial\Omega} u^m(x,t)\int_\Omega K(x,\xi,t)u^m(\xi,t)d\xi ds\,dt \\ &\quad - \int_{\partial\Omega} u^m(x,\tau)\int_0^\tau\int_\Omega K(x,\xi,\eta)u^m(\xi,\eta)d\xi\,d\eta ds.\end{aligned} \tag{3.10}$$

Consider the right-most integral in (3.10). Applying the Cauchy inequality, we have

$$\begin{aligned}
&\int_0^\tau \int_{\partial\Omega} u^m(x,t) \int_\Omega K(x,\xi,t) u^m(\xi,t) d\xi\, ds\, dt \\
&\leq \frac{1}{2}\int_0^\tau \int_{\partial\Omega} (u^m(x,t))^2\, ds\, dt + \frac{1}{2}\int_0^\tau \int_{\partial\Omega} \Big(\int_\Omega K(x,\xi,t) u^m(\xi,t) d\xi\Big)^2 ds\, dt;
\end{aligned}$$

next, using an inequality in [6, p.77],

$$\int_{\partial\Omega} v^2(x,t) ds \leq \int_\Omega \big(\epsilon|\nabla v|^2 + c(\epsilon)v^2\big)\, dx \tag{3.11}$$

we have

$$\begin{aligned}
&\frac{1}{2}\int_0^\tau \int_{\partial\Omega} (u^m(x,t))^2\, ds\, dt + \frac{1}{2}\int_0^\tau \int_{\partial\Omega} \Big(\int_\Omega K(x,\xi,t) u^m(\xi,t) d\xi\Big)^2 ds\, dt \\
&\leq \frac{1}{2}\int_0^\tau \int_\Omega \big(\epsilon|\nabla u^m(x,t)|^2 + c(\epsilon)u^2\big) dx\, dt \\
&\quad + \frac{1}{2}\int_0^\tau \int_\Omega \Big(\epsilon \Big(\int_\Omega \nabla K(x,\xi,t) u^m(\xi,t) d\xi\Big)^2 \\
&\quad + c(\epsilon)\Big(\int_\Omega K(x,\xi,t) u^m(\xi,t) d\xi\Big)^2\Big) dx\, dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^\tau \int_{\partial\Omega} u^m(x,t) \int_\Omega K(x,\xi,t) u^m(\xi,t) d\xi ds\, dt \\
&\leq \frac{1}{2}\int_0^\tau \int_\Omega \big(\epsilon|\nabla u^m(x,t)|^2 + c(\epsilon)u^2\big)\, dx\, dt \\
&\quad + \frac{1}{2}\int_0^\tau \int_\Omega \epsilon|\Omega| \int_\Omega |\nabla K(x,\xi,t) u^m(\xi,t)|^2\, d\xi dx\, dt \\
&\quad + \frac{1}{2}\int_0^\tau \int_\Omega c(\epsilon)|\Omega| \int_\Omega (K(x,\xi,t) u^m(\xi,t))^2\, d\xi dx\, dt \\
&\leq \frac{1}{2}\int_0^\tau \int_\Omega \big(\epsilon|\nabla u^m(x,t)|^2 + c(\epsilon)u^2\big)\, dx\, dt + \frac{1}{2}\int_0^\tau \int_\Omega \epsilon|\Omega| K_1 \int_\Omega (u^m(\xi,t))^2\, d\xi dx\, dt \\
&\quad + \frac{1}{2}\int_0^\tau \int_\Omega c(\epsilon)|\Omega| K_0 \int_\Omega (u^m(\xi,t))^2\, d\xi dx\, dt,
\end{aligned}$$

and, finally,

$$\begin{aligned}
&\int_0^\tau \int_{\partial\Omega} u^m(x,t) \int_\Omega K(x,\xi,t) u^m(\xi,t) d\xi ds\, dt \\
&\leq \frac{1}{2}\int_0^\tau \int_\Omega \big(\epsilon|\nabla u^m|^2 + \big(c(\epsilon) + \epsilon|\Omega|^2 K_1 + c(\epsilon)|\Omega|^2 K_0\big)\big)\, dx\, dt.
\end{aligned} \tag{3.12}$$

Applying the modified Cauchy inequality to the second integral in (3.10), we have

$$\begin{aligned}
&\int_{\partial\Omega} u^m(x,\tau) \int_0^\tau \int_\Omega K(x,\xi,\eta) u^m(\xi,\eta) d\xi\, d\eta ds \\
&\leq \frac{\epsilon}{2}\int_{\partial\Omega} (u^m(x,\tau))^2\, ds + \frac{1}{2\epsilon}\int_{\partial\Omega} \Big(\int_0^\tau \int_\Omega K(x,\xi,\eta) u^m(\xi,\eta) d\xi\, d\eta\Big)^2 ds.
\end{aligned}$$

Using again the inequality (3.11),

$$\begin{aligned}
&\frac{\epsilon}{2}\int_{\partial\Omega}(u^m(x,\tau))^2\,ds+\frac{1}{2\epsilon}\int_{\partial\Omega}\Big(\int_0^\tau\int_\Omega K(x,\xi,\eta)u^m(\xi,\eta)d\xi\,d\eta\Big)^2ds\\
&\le\frac{\epsilon}{2}\int_\Omega\big(\mu|\nabla u^m(x,\tau)|^2+c(\mu)(u^m(x,\tau))^2\big)\,dx\\
&\quad+\frac{1}{2\epsilon}\int_\Omega\Big(\tau\mu|\Omega|\int_0^\tau\int_\Omega|\nabla K(x,\xi,\eta)|^2\,(u^m(\xi,\eta))^2\,d\xi\,d\eta\\
&\quad+\tau c(\mu)|\Omega|\int_0^\tau\int_\Omega(K(x,\xi,\eta))^2\,(u^m(\xi,\eta))^2\,d\xi\,d\eta\Big)dx.
\end{aligned}\tag{3.13}$$

Choose $\epsilon$ and $\mu$ so that $\epsilon\mu<1,\ \epsilon c(\mu)<c_1$. Inequalities (3.12) and (3.13), give us:

$$\begin{aligned}
&\int_\Omega\big((u_t^m(x,\tau))^2+(1-\epsilon\mu)|\nabla u^m(x,\tau)|^2+(c_1-\epsilon c(\mu))(u^m(x,\tau))^2\big)\,dx\\
&\le\int_\Omega\big((u_t^m(x,0))^2+|\nabla u^m(x,0)|^2+c_2(u^m(x,0))^2\big)\,dx\\
&\quad+\int_0^\tau\int_\Omega f^2dx\,dt+\int_0^\tau\int_\Omega c_3(u^m)^2dx\,dt+\int_0^\tau\int_\Omega(u_t^m)^2dx\,dt\\
&\quad+\int_0^\tau\int_\Omega\big(\epsilon|\nabla u^m|^2+h(\epsilon)(u^m)^2\big)\,dx\,dt\\
&\quad+\frac{\mu}{\epsilon}K_1\tau|\Omega|^2\int_0^\tau\int_\Omega(u^m(x,t))^2dx\,dt+\frac{c(\mu)}{\epsilon}K_0\tau|\Omega|^2\int_0^\tau\int_\Omega(u^m(x,t))^2dx\,dt.
\end{aligned}$$

Thus, denoting

$$\begin{gathered}
m=\min\{1,1-\epsilon\mu,c_1-\epsilon c(\mu)\},\\
M=\max\{c_3+h(\epsilon)+\frac{\mu}{\epsilon}K_1\tau|\Omega|^2+\frac{c(\mu)}{\epsilon}K_0\tau|\Omega|^2\}
\end{gathered}$$

we obtain the inequality

$$\begin{aligned}
&m\int_\Omega\big((u_t^m(x,\tau))^2+|\nabla u^m(x,\tau)|^2+(u^m(x,\tau))^2\big)\,dx\\
&\le\int_\Omega\big((u_t^m(x,0))^2+|\nabla u^m(x,0)|^2+(u^m(x,0))^2\big)\,dx\\
&\quad+M\int_0^\tau\int_\Omega\big((u_t^m)^2+|\nabla u^m|^2+(u^m)^2\big)\,dx\,dt+\int_0^\tau\int_\Omega f^2(x,t)dx\,dt.
\end{aligned}$$

Applying the Gronwall inequality and integrating on $\tau$ from 0 to $\tau$, we finally obtain

$$\|u^m\|_{W_2^1(Q_\tau)}\le C(T)\Big(\|f\|_{L_2(Q)}+\|\phi\|_{W_2^1(\Omega)}+\|\psi\|_{L_2(\Omega)}\Big).\tag{3.14}$$

Since $\|f\|_{L_2(Q)},\|\phi\|_{W_2^1(\Omega)},\|\psi\|_{L_2(\Omega)}$ are bounded, the sequence $\{u^m\}$ is bounded in $W_2^1(Q)$: $\|u^m\|_{W_2^1(Q)}\le\mathcal{D}(T)$; therefore, there exists a weakly converging subsequence (we will denote it $u^m$ again for simplicity); now we shall show, that its limit $u\in W_2^1(Q)$ is a desired solution.

To prove this, we multiply (3.2) by the function $h_l(t)\in W_2^1(0,T),h_l(T)=0$, sum by $l$ from 1 to $m$, and integrate on $t$ from 0 to $T$; denoting $\eta^m(x,t)=$

$\sum_{l=1}^{m} w_l(x)h_l(t)$, we obtain

$$\begin{aligned}&\int_0^T \int_\Omega (-u_t^m \eta_t^m + \nabla u^m \nabla \eta^m + cu^m \eta^m)\,dx\,dt \\ &+ \int_0^T \int_{\partial\Omega} \eta^m \int_0^t \int_\Omega K(x,\xi,\tau)u^m(\xi,\tau)d\xi d\tau ds\,dt \\ &= \int_\Omega u_t^m(x,0)\eta^m(x,0)dx + \int_0^T \int_\Omega f\eta^m dx\,dt.\end{aligned}$$

It is no a problem to go to the limit at $m \to \infty$ in the latter expression but the one addend

$$\int_0^T \int_{\partial\Omega} \eta^m \int_0^t \int_\Omega K(x,\xi,\tau)u^m(\xi,\tau)d\xi d\tau ds\,dt. \tag{3.15}$$

Consider the integral

$$\int_0^T \int_{\partial\Omega} \eta^m \int_0^t \int_\Omega K(x,\xi,\tau)\Big(u^m(\xi,\tau) - u(\xi,\tau)\Big)d\xi d\tau ds\,dt.$$

With the help of the Cauchy-Bunyakovski inequality, we have for the "internal" integral

$$\begin{aligned}&\int_0^t \int_\Omega K(x,\xi,\tau)\Big(u^m(\xi,\tau) - u(\xi,\tau)\Big)d\xi d\tau \\ &\le \Big(\int_0^T \int_\Omega K^2(x,\xi,\tau)d\xi d\tau\Big)^{1/2}\Big(\int_0^T \int_\Omega \big(u^m(\xi,\tau) - u(\xi,\tau)\big)^2 d\xi d\tau\Big)^{1/2} \\ &= \|K\|_{L_2(Q)}\|u^m - u\|_{L_2(Q)}.\end{aligned}$$

However, $\|K\|_{L_2(Q)} \le |Q|K_0$, and $\|u^m - u\|_{L_2(Q)} \to 0$ (?!), therefore, we can go to the limit in (3.15). Hence, the limit function $u$ satisfies (3.2) for every $\eta^m(x,t) = \sum_{l=1}^{m} w_l(x)h_l(t)$.

Denote $\mathcal{N}_m$ a set of the functions of the form $\eta^m(x,t) = \sum_{l=1}^{m} w_l(x)h_l(t)$, $h_l(t) \in W_2^1(0,T), h_l(T) = 0$. While $\bigcup_{m=1}^{\infty} \mathcal{N}_m$ is dense in $\hat{W}_2^1(Q)$, then (3.2) holds true for every function from $\hat{W}_2^1(Q)$. This proves that $u(x,t)$ is a generalized solution of (1.1)-(1.2)-(1.3).

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