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BOUNDARY CONDITION OF THE VOLUME POTENTIAL FOR AN ELLIPTIC-PARABOLIC EQUATION WITH A SCALAR PARAMETER

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Abstract. Using the descent method for the fundamental solution of the heat equation with a scalar parameter, we find the fundamental solution of the multidimensional Helmholtz equation in an explicit form. We also find a boundary condition of the volume potential for an elliptic-parabolic equation with a scalar parameter. In turn, this condition allows us to construct and study a new correct nonlocal (initial) Bitsadze-Samarsky type problem for an elliptic-parabolic equation with a scalar parameter.

1. Introduction

Most of the references in this paper are devoted to systematic study of the boundary conditions of the Newton's potential [1], the heat potential [2] and the surface heat potential [3, 4, 5]. In this paper, we present a boundary condition for an elliptic-parabolic equation with a scalar parameter.

Let $\varepsilon^+_{n+1}(x,t,\lambda)$ be the fundamental solution of the heat equation with a scalar parameter

$$\frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t) + \lambda u(x,t) = f^+(x,t), \quad x \in R^n, t > 0 \tag{1.1}$$

and $\varepsilon^-_{n+1}(x,t,\lambda)$ be the fundamental solution of the Helmholtz equation

$$-\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta_x u(x,t) + \lambda u(x,t) = f^-(x,t), \quad x \in R^n, t < 0 \tag{1.2}$$

where λ is an arbitrary complex number and $Re\lambda \geq 0$.

Let $\Omega \in R^n$ be a bounded domain with smooth boundary $\partial\Omega$ and $D^+ = \Omega \times [0,T]$ be a cylindrical domain. $D^- \subset R^{n+1}$ is the domain at $t < 0$ with smooth surface σ and when $t = 0$ bounded with the domain Ω. We will also use the notation $D = D^+ \cup \Omega \cup D^-$.

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We define an elliptic-parabolic potential as

$$u(x,t) = (L_B^{-1} f)(x,t) \\ = \begin{cases} \int_{D^+} \varepsilon_{n+1}^{+}(x-\xi, t-\eta, \lambda) f^{+}(\xi,\eta) dD^{+} \\ + \int_{\Omega} \varepsilon_{n+1}^{+}(x-\xi, t, \lambda)\tau(\xi) d\xi, & \text{if } t > 0, \\ \int_{D^-} \varepsilon_{n+1}^{-}(x-\xi, t-\eta, \lambda) f^{-}(\xi,\eta) dD^{-}, & \text{if } t < 0 \end{cases} \tag{1.3}$$

where the unknown function $\tau(x)$ is determined from the condition of continuity of the potential $L_B^{-1} f(x,t)$ when $t = 0$.

2. Main results

First we find the fundamental solutions of the heat equation with a scalar parameter and Helmholtz equation.

Lemma 2.1. *The fundamental solution of the heat equation* (1.1) *is a function*

$$\varepsilon_{n+1}^{+}(x,t,\lambda) = \Theta(t) \frac{e^{-\frac{|x|^2}{4t}}}{(2\sqrt{\pi t})^n} e^{-\lambda t}. \tag{2.1}$$

Proof. A direct calculation shows that

$$\begin{aligned} &\Big(\frac{\partial}{\partial t} - \Delta_x + \lambda\Big)\varepsilon_{n+1}^{+}(x,t,\lambda) \\ &= \Big[\big(\frac{\partial}{\partial t} - \Delta_x\big)\Theta(t)\frac{e^{-\frac{|x|^2}{4t}}}{(2\sqrt{\pi t})^n}\Big] e^{\lambda t} + \Big[\big(\frac{\partial}{\partial t} + \lambda\big)e^{-\lambda t}\Big]\Theta(t)\frac{e^{-\frac{|x|^2}{4t}}}{(2\sqrt{\pi t})^n} \\ &= \delta(x,t)e^{-\lambda t} - \lambda e^{-\lambda t}\Theta(t)\frac{e^{-\frac{|x|^2}{4t}}}{(2\sqrt{\pi t})^n} + \lambda e^{-\lambda t}\Theta(t)\frac{e^{-\frac{|x|^2}{4t}}}{(2\sqrt{\pi t})^n} \\ &= \delta(x,t)e^{-\lambda t} = \delta(x,t). \end{aligned}$$

□

Lemma 2.2. *The fundamental solution of the Helmholtz equation* (1.2) *can be represented as*

$$\varepsilon_{n+1}^{-}(\overline{x},\lambda) = \frac{1}{(n-1)\omega_{n+1}|\overline{x}|^{n-1}} \frac{(n-1)}{\Gamma(\frac{n+1}{2})} \Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{n-1}{2}} K_{\frac{1-n}{2}}(\sqrt{\lambda}|\overline{x}|), \tag{2.2}$$

where $|\overline{x}|^2 = x_1^2 + \dots + x_n^2 + t^2$,

$$K_\nu(z) = \frac{1}{2}\big(\frac{z}{2}\big)^{\nu} \int_0^\infty \xi^{-\nu-1} e^{-\xi - \frac{z^2}{4\xi}} d\xi$$

is the Macdonald function [8, p. 183] *and*

$$\omega_{n+1} = \frac{2(\sqrt{\pi})^{n+1}}{\Gamma(\frac{n+1}{2})}$$

is the area of a unit sphere in R^{n+1}.

Proof. We note that when $n = 1, 2, 3$ the fundamental solutions of the Helmholtz equation are given in [9, p. 203-205]:

$$\varepsilon_1^{-}(x,\lambda) = -\frac{1}{2ik} e^{-i\sqrt{\lambda}|x|},$$

$$\varepsilon_2^{-}(x_1,x_2,\lambda)=\pm\frac{i}{4}H_0^{1,2}(\sqrt{\lambda}|x|),\quad |x|=(x_1^2+x_2^2)^{\frac12},$$
$$\varepsilon_3^{-}(x_1,x_2,x_3,\lambda)=\pm\frac{e^{\pm i\sqrt{\lambda}|x|}}{4\pi|x|},\quad |x|=(x_1^2+x_2^2+x_3^2)^{\frac12}.$$

The proof for $n\geq 3$ is based on the descent method for the fundamental solution of the heat equation with a scalar parameter

$$(\frac{\partial}{\partial\eta}-\Delta_{\overline{x}}+\lambda)\varepsilon_{n+2}^{+}(\overline{x},\eta,\lambda)=\delta(\overline{x},\eta),\quad \overline{x}\in R^{n+1},\ \eta>0,$$

where $\overline{x}=(x_1,x_2,\dots,x_n,t)$.

Putting the function

$$\varepsilon_{n+2}^{+}(\overline{x},\eta,\lambda)=\Theta(\eta)\frac{e^{-\frac{|\overline{x}|^2}{4\eta}}}{(2\sqrt{\pi\eta})^{n+1}}e^{-\lambda\eta}$$

in the formula of the descent method

$$\varepsilon_{n+1}^{-}(\overline{x},\lambda)=\int_0^\infty\varepsilon_{n+2}^{+}(\overline{x},\eta,\lambda)d\eta \tag{2.3}$$

and after replacing

$$\xi=\frac{|\overline{x}|^2}{4\eta},\quad \eta=\frac{|\overline{x}|^2}{4\xi},\quad d\eta=-\frac{|\overline{x}|^2}{4\xi^2}d\xi,$$

we find that

$$\begin{aligned}\varepsilon_{n+1}^{-}(\overline{x},\lambda)=&\frac{1}{(2\sqrt{\pi})^{n+1}}\int_0^\infty e^{-\xi-\lambda\frac{|\overline{x}|^2}{4\xi}}\Big(\frac{|\overline{x}|^2}{4\xi}\Big)^{-\frac{n+1}{2}}\frac{|\overline{x}|^2}{4\xi^2}d\xi\\ =&\frac{4^{\frac{n-1}{2}}|\overline{x}|^{1-n}}{(2\sqrt{\pi})^{n+1}}\int_0^\infty e^{-\xi-\lambda\frac{|\overline{x}|^2}{4\xi}}\xi^{\frac{n}{2}+\frac12-2}d\xi.\end{aligned}$$

The obtained fundamental solution $\varepsilon_{n+1}^{-}(\overline{x},\lambda)$ we will be expressed in terms of the MacDonald function

$$\begin{aligned}&\varepsilon_{n+1}^{-}(\overline{x},\lambda)\\ &=\frac{1}{2(\sqrt{\pi})^{n+1}|\overline{x}|^{n-1}}\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{n-1}{2}}\frac12\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{1-n}{2}}\int_0^\infty e^{-\xi-\frac{(\sqrt{\lambda}|\overline{x}|)^2}{4\xi}}\xi^{-(\frac{1-n}{2})-1}d\xi\\ &=\frac{1}{2(\sqrt{\pi})^{n+1}|\overline{x}|^{n-1}}\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{n-1}{2}}K_{\frac{1-n}{2}}(\sqrt{\lambda}|\overline{x}|),\quad \operatorname{Re}\sqrt{\lambda}\geq 0.\end{aligned}$$

Then we will express it with the fundamental solution $\varepsilon_\Delta^{n+1}(\overline{x})=\frac{1}{(n-1)\omega_{n+1}|\overline{x}|^{n-1}}$ of the Laplace equation

$$\begin{aligned}\varepsilon_{n+1}^{-}(\overline{x},\lambda)=&\frac{1}{2(\sqrt{\pi})^{n+1}|\overline{x}|^{n-1}}\frac{\omega_{n+1}(n-1)}{\omega_{n+1}(n-1)}\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{n-1}{2}}K_{\frac{1-n}{2}}(\sqrt{\lambda}|\overline{x}|)=\\ &=\frac{1}{(n-1)\omega_{n+1}|\overline{x}|^{n-1}}\frac{(n-1)}{\Gamma(\frac{n+1}{2})}\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{n-1}{2}}K_{\frac{1-n}{2}}(\sqrt{\lambda}|\overline{x}|)=\\ &=\varepsilon_\Delta^{n+1}(\overline{x})\frac{(n-1)}{\Gamma(\frac{n+1}{2})}\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{n-1}{2}}K_{\frac{1-n}{2}}(\sqrt{\lambda}|\overline{x}|).\end{aligned}$$

We can show that when $\lambda \to 0$ we obtain the fundamental solution of the Laplace equation.

It is well known that Macdonald defined the function $K_\nu(z)$ for arbitrary numbers ν on the basis of equality

$$K_\nu(z) = \frac{\pi}{2\sin\nu\pi}\left(I_{-\nu}(z) - I_\nu(z)\right),$$

where

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{\nu+2m}}{m!\Gamma(\nu+m+1)}.$$

So, when $\nu = (1-n)/2$ and $m = 0$ we get

$$\begin{aligned}
&\lim_{\lambda\to 0} \varepsilon_{n+1}^{-}(\overline{x},\lambda)\\
&= \lim_{\lambda\to 0} \varepsilon_{\Delta}^{n+1}(\overline{x}) \frac{(n-1)}{\Gamma(\frac{n+1}{2})}\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{n-1}{2}} K_{\frac{1-n}{2}}(\sqrt{\lambda}|\overline{x}|)\\
&= \lim_{\lambda\longrightarrow 0} \varepsilon_{\Delta}^{n+1}(\overline{x}) \frac{(n-1)}{\Gamma(\frac{n+1}{2})}\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{n-1}{2}} \frac{\pi}{2\sin\frac{1-n}{2}\pi}\Big(\sum_{m=0}^{\infty} \frac{(\frac{\sqrt{\lambda}|\overline{x}|}{2})^{-\frac{1-n}{2}+2m}}{m!\Gamma(-\frac{1-n}{2}+m+1)}\\
&\quad - \sum_{m=0}^{\infty} \frac{(\frac{\sqrt{\lambda}|\overline{x}|}{2})^{\frac{1-n}{2}+2m}}{m!\Gamma(\frac{1-n}{2}+m+1)}\Big)\\
&= \lim_{\lambda\longrightarrow 0} \varepsilon_{\Delta}^{n+1}(\overline{x}) \frac{(n-1)}{\Gamma(\frac{n+1}{2})}\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{n-1}{2}} \frac{\pi}{2\sin\frac{1-n}{2}\pi}\Big(\frac{(\frac{\sqrt{\lambda}|\overline{x}|}{2})^{-\frac{1-n}{2}}}{\Gamma(-\frac{1-n}{2}+1)} - \frac{(\frac{\sqrt{\lambda}|\overline{x}|}{2})^{\frac{1-n}{2}}}{\Gamma(\frac{1-n}{2}+1)}\Big)\\
&= \lim_{\lambda\longrightarrow 0} \varepsilon_{\Delta}^{n+1}(\overline{x}) \frac{(n-1)}{\Gamma(\frac{n+1}{2})}\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{n-1} \frac{\pi}{2\sin\frac{1-n}{2}\pi} \frac{1}{\Gamma(-\frac{1-n}{2}+1)}\\
&\quad - \varepsilon_{\Delta}^{n+1}(\overline{x}) \frac{(n-1)}{\Gamma(\frac{n+1}{2})} \frac{\pi}{2\sin\frac{1-n}{2}\pi} \frac{1}{\Gamma(\frac{1-n}{2}+1)}\\
&= J_1 - J_2 = 0 - J_2.
\end{aligned}$$

If we use the the formula of the Gamma function $\frac{\pi}{sin\pi z} = \Gamma(z)\Gamma(1-z)$ and $\Gamma(z+1) = z\Gamma(z)$,

$$\begin{aligned}
-J_2 &= -\varepsilon_{\Delta}^{n+1}(\overline{x}) \frac{(n-1)}{2\Gamma(\frac{n+1}{2})} \frac{\pi}{sin\frac{1-n}{2}\pi} \frac{1}{\Gamma(\frac{1-n}{2}+1)}\\
&= -\varepsilon_{\Delta}^{n+1}(\overline{x}) \frac{(n-1)}{2\Gamma(\frac{n+1}{2})} \Gamma(\frac{1-n}{2})\Gamma(1-\frac{1-n}{2}) \frac{1}{\Gamma(\frac{1-n}{2}+1)}\\
&= -\varepsilon_{\Delta}^{n+1}(\overline{x}) \frac{(n-1)}{2\Gamma(\frac{n+1}{2})} \Gamma(\frac{1-n}{2})\Gamma(\frac{1+n}{2}) \frac{1}{\Gamma(\frac{1-n}{2})(\frac{1-n}{2})}\\
&= \varepsilon_{\Delta}^{n+1}(\overline{x}) = \frac{1}{(n-1)\omega_{n+1}|\overline{x}|^{n-1}}.
\end{aligned}$$

It should be noted that the function

$$\widetilde{K}_{\frac{1-n}{2}}(\sqrt{\lambda}|\overline{x}|) = \frac{(n-1)}{\Gamma(\frac{n+1}{2})}\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{n-1}{2}} K_{\frac{1-n}{2}}(\sqrt{\lambda}|\overline{x}|)$$

does not have singularity.

By the property of analytic continuation we obtained for all complex λ,

$$\varepsilon_{n+1}^{-}(\overline{x},\lambda)=\frac{1}{(n-1)\omega_{n+1}|\overline{x}|^{n-1}}\frac{(n-1)}{\Gamma(\frac{n+1}{2})}\Big(\frac{\sqrt{\lambda}|\overline{x}|}{2}\Big)^{\frac{n-1}{2}}K_{\frac{1-n}{2}}(\sqrt{\lambda}|\overline{x}|). \tag{2.4}$$

This completes the proof. □

Lemma 2.3. *Let $f(x,t)\in C^{\alpha}(\overline{D})$. Then*

$$u(x,t)=(L_B^{-1}f)(x,t)\in C^{\alpha}(\overline{D})\cap C^{2+\alpha}(\overline{D^-})\cap C_{x,t}^{2+\alpha,1+\alpha}(\overline{D^+}). \tag{2.5}$$

Proof. Taking into account the conditions $u(x,0-)=u(x,0+)=\tau(x)$, from the formula

$$u(x,t)=\int_{D^-}\varepsilon_{n+1}^{-}(x-\xi,t-\eta,\lambda)f^{-}(\xi,\eta)dD^{-},\quad t<0$$

it follows that

$$\tau(x)=u(x,0-)=\int_{D^-}\varepsilon_{n+1}^{-}(x-\xi,-\eta)f^{-}(\xi,\eta)d\xi d\eta\quad\in C^{2+\alpha}(\overline{\Omega}), \tag{2.6}$$

$$\|\tau(x)\|_{C^{2+\alpha}(\overline{\Omega})}\le d_1\|f^{-}(x,t)\|_{C^{\alpha}(\overline{D^-})}. \tag{2.7}$$

From this and by properties of the heat potential and surface heat potential we obtain

$$\begin{aligned}\|(L_B^{-1}f)(x,t)\|_{C_{x,t}^{2+\alpha,1+\alpha}(\overline{D^+})}&=\|\int_0^t d\eta\int_{\Omega}\varepsilon_{n+1}^{+}(x-\xi,t-\eta,\lambda)f^{+}(\xi,\eta)d\xi\\&\quad+\int_{\Omega}\varepsilon_{n+1}^{+}(x-\xi,t,\lambda)\tau(\xi)d\xi\|_{C_{x,t}^{2+\alpha,1+\alpha}(\overline{D^+})}\\&\le d_2\Big(\|f^{+}(x,t)\|_{C^{\alpha}(\overline{D^+})}+\|\tau(x)\|_{C^{2+\alpha}(\overline{\Omega})}\Big)\\&\le d_3\Big(\|f^{+}(x,t)\|_{C^{\alpha}(\overline{D^+})}+\|f^{-}(x,t)\|_{C^{\alpha}(\overline{D^-})}\Big).\end{aligned} \tag{2.8}$$

Using the properties of the Newton's potential we obtain

$$\begin{aligned}\|(L_B^{-1}f)(x,t)\|_{C^{2+\alpha}(\overline{D^-})}&=\|\int_{D^-}\varepsilon_{n+1}^{-}(x-\xi,t-\eta,\lambda)f^{-}(\xi,\eta)d\xi d\eta\|_{C^{2+\alpha}(\overline{D^-})}\\&\le d_4\|f^{-}(x,t)\|_{C^{\alpha}(\overline{D^-})}.\end{aligned} \tag{2.9}$$

Comparing inequalities (2.8)-(2.9) we have

$$u(x,t)=(L_B^{-1}f)(x,t)\in C^{\alpha}(\overline{D})\cap C^{2+\alpha}(\overline{D^-})\cap C_{x,t}^{2+\alpha,1+\alpha}(\overline{D^+})$$

and

$$\begin{aligned}\|u(x,t)\|&=\|(L_B^{-1}f)(x,t)\|_{C^{\alpha}(\overline{D})\cap C^{2+\alpha}(\overline{D^-})\cap C_{x,t}^{2+\alpha,1+\alpha}(\overline{D^+})}\\&\le d_5\|f(x,t)\|_{C^{\alpha}(\overline{D})}.\end{aligned} \tag{2.10}$$

The proof is complete. □

As in [2, 3], it can be proved that the volume heat potential is

$$u_f(x,t)=\int_{D^+}\varepsilon_{n+1}^{+}(x-\xi,t-\eta,\lambda)f^{+}(\xi,\eta)d\xi d\eta \tag{2.11}$$

which satisfies the inhomogeneous heat equation with a scalar parameter

$$\Diamond u_f(x,t)=\frac{\partial u_f(x,t)}{\partial t}-\Delta_x u_f(x,t)+\lambda u_f(x,t)=f^{+}(x,t),\quad x\in R^n,\ t>0 \tag{2.12}$$

and satisfies the homogeneous initial condition

$$u_f(x,t)|_{t=0} = 0\,. \tag{2.13}$$

Also the surface heat potential is

$$u_\tau(x,t) = \int_\Omega \varepsilon^+_{n+1}(x-\xi,t,\lambda)\tau(\xi)d\xi \tag{2.14}$$

which satisfies the homogeneous heat equation with a scalar parameter

$$\Diamond u_\tau(x,t) = \frac{\partial u_\tau(x,t)}{\partial t} - \Delta_x u_\tau(x,t) + \lambda u_\tau(x,t) = 0, \quad x \in R^n,\ t > 0 \tag{2.15}$$

and satisfies the nonhomogeneous initial condition

$$u_\tau(x,t)|_{t=0} = u_0(x) \tag{2.16}$$

satisfy the same lateral potential boundary condition, i.e. the condition

$$\begin{aligned} &-\frac{u_f(x,t)+u_\tau(x,t)}{2} + \int_0^t d\eta \int_{\partial\Omega} \Big(\frac{\partial \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)}{\partial n_{xi}} (u_f+u_\tau)(\xi,\eta) \\ &- \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda) \frac{\partial (u_f+u_\tau)(\xi,\eta)}{\partial n_\xi} \Big) d\xi = 0, \end{aligned} \tag{2.17}$$

for all $(x,t) \in \partial\Omega \cap (0,T)$, where $\frac{\partial}{\partial n_\xi}$ is the normal derivative.

Lemma 2.4. *For any function $f(x,t) \in C^\alpha(\overline{D})$ the volume heat potential* (2.11) *satisfies the inhomogeneous heat equation with a scalar parameter* (2.12)*, the homogeneous initial condition* (2.13)*, and the lateral boundary condition*

$$\begin{aligned} &-\frac{u_f(x,t)}{2} + \int_0^t d\eta \int_{\partial\Omega} \Big(\frac{\partial \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)}{\partial n_\xi} u_f(\xi,\eta) \\ &- \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda) \frac{\partial u_f(\xi,\eta)}{\partial n_\xi} \Big) d\xi = 0, \end{aligned} \tag{2.18}$$

for all $(x,t) \in \partial\Omega \cap (0,T)$.

Conversely, if $u(x,t) \in W_2^{2,1}(D)$ is a solution of the inhomogeneous heat equation with a scalar parameter (2.12)*, which satisfies the homogeneous initial condition* (2.13) *and the lateral boundary condition* (2.18)*, then it coincides with the volume heat potential* (2.11).

Proof. We consider the heat potential

$$u_f(x,t) = \int_{D^+} \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda) f^+(\xi,\eta)d\xi d\eta =$$

$$= \int_{D^+} \frac{e^{-\frac{|x-\xi|^2}{4(t-\eta)}} e^{-\lambda(t-\eta)}}{(2\sqrt{\pi(t-\eta)})^n} f^+(\xi,\eta)d\xi d\eta = \int_{D^+} \frac{e^{-\frac{|x-\xi|^2}{4(t-\eta)}} e^{-\lambda(t-\eta)}}{(2\sqrt{\pi(t-\eta)})^n} \Diamond u_f(\xi,\eta)d\xi d\eta.$$

Since the integral

$$\int_0^t d\eta \int_\Omega \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda) \Diamond u_f(\xi,\eta)d\xi$$

the improper integral as an integral of the function $\varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)$ with a singularity at $t=\eta$ then we understand it as

$$\lim_{\delta\to 0} u_\delta(x,t) = \lim_{\delta\to 0} \int_0^{t-\delta} d\eta \int_\Omega \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda) \Diamond u_f(\xi,\eta)d\xi.$$

By a direct calculation and integration by parts for $x \in \Omega$ and $0 < \delta < t$ it can be verified that

$$\begin{aligned}
&u_f(x,t)\\
&= \lim_{\delta\to 0} u_\delta(x,t)\\
&= \lim_{\delta\to 0}\int_0^{t-\delta} d\eta \int_\Omega \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)\big(\frac{\partial}{\partial\eta} - \Delta_\xi + \lambda\big)u_f(\xi,\eta)d\xi\\
&= \lim_{\delta\to 0}\int_\Omega \varepsilon^+_{n+1}(x-\xi,\delta,\lambda)u_f(\xi,t-\delta)d\xi\\
&\quad - \lim_{\delta\to 0}\int_\Omega \varepsilon^+_{n+1}(x-\xi,t,\lambda)u_f(\xi,0)d\xi\\
&\quad - \lim_{\delta\to 0}\int_0^{t-\delta} d\eta \int_{\partial\Omega} \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)\frac{\partial u_f(\xi,\eta)}{\partial n_\xi}d\xi\\
&\quad + \lim_{\delta\to 0}\int_0^{t-\delta} d\eta \int_{\partial\Omega} \frac{\partial\varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)}{\partial n_\xi}u_f(\xi,\eta)d\xi\\
&\quad + \lim_{\delta\to 0}\int_0^{t-\delta} d\eta \int_\Omega u_f(\xi,\eta)\big(-\frac{\partial}{\partial\eta} - \Delta_\xi + \lambda\big)\varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)d\xi\\
&= I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned} \tag{2.19}$$

for all $(x,t) \in \Omega \times (0,T)$.

To calculate the value of the integral I_1, we use the explicit form of the fundamental solution and the change of variables

$$\begin{aligned}
I_1 &= \lim_{\delta\to 0}\int_\Omega \varepsilon^+_{n+1}(x-\xi,\delta,\lambda)u_f(\xi,t-\delta)d\xi\\
&= \lim_{\delta\to 0}\int_\Omega \frac{1}{(2\sqrt{\pi\delta})^n}e^{-\frac{|x-\xi|^2}{4\delta}}e^{-\lambda\delta}u_f(\xi,t-\delta)d\xi\\
&= \lim_{\delta\to 0}\int_{R^n} \frac{1}{(2\sqrt{\pi\delta})^n}e^{-\frac{|x-\xi|^2}{4\delta}}e^{-\lambda\delta}\widehat{u}_f(\xi,t-\delta)d\xi = \Big|\frac{|x-\xi|}{2\sqrt{\delta}} = z,
\end{aligned}$$

$$\begin{aligned}
d\xi = -2\sqrt{\delta}dz\Big| &= \lim_{\delta\to 0}\frac{1}{(\sqrt{\pi})^n}\int_{\frac{|x-a|}{2\sqrt{\delta}}}^{\frac{|x+a|}{2\sqrt{\delta}}} u_f(x-2\sqrt{\delta}z,t-\delta)e^{-z^2}e^{-\lambda\delta}dz\\
&= \lim_{\delta\to 0}\frac{1}{(\sqrt{\pi})^n}u_f(x-2\sqrt{\delta}z,t-\delta)e^{-\lambda\delta}\int_{\frac{|x-a|}{2\sqrt{\delta}}}^{\frac{|x+a|}{2\sqrt{\delta}}} e^{-z^2}dz\\
&= u(x,t)\frac{1}{(\sqrt{\pi})^n}\int_{-\infty}^{+\infty} e^{-z^2}dz = u(x,t),
\end{aligned}$$

where $\widehat{u}(x,t)$ is the extension by a zero the function $u(x,t)$ to the cube $-a < \xi < a$ from the R^n containing the domain Ω.

Since $u(x,0) = 0$ then the integral is $I_2 = 0$. The integrals I_3 and I_4 have a limit at $\delta \to 0$ and equals to

$$\begin{aligned}
I_3 &= \int_0^t d\eta \int_{\partial\Omega} \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)\frac{\partial u_f(\xi,\eta)}{\partial n_\xi}d\xi,\\
I_4 &= \int_0^t d\eta \int_{\partial\Omega} \frac{\partial\varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)}{\partial n_\xi}u_f(\xi,\eta)d\xi.
\end{aligned}$$

In that $\eta \le t-\delta < t$, then

$$(-\frac{\partial}{\partial \eta} - \Delta_\xi + \lambda)\varepsilon_{n+1}^{+}(x-\xi, t-\eta, \lambda) \equiv 0,$$

therefore $I_5 = 0$.

Taking into account (2.19), we get that for all $(x,t) \in \Omega \times (0,T)$,

$$\begin{aligned} I_u(x,t) = \int_0^t d\eta \int_{\partial\Omega} \Big(& \frac{\partial \varepsilon_{n+1}^{+}(x-\xi, t-\eta, \lambda)}{\partial n_\xi} u_f(\xi,\eta) \\ & - \varepsilon_{n+1}^{+}(x-\xi, t-\eta, \lambda) \frac{\partial u_f(\xi,\eta)}{\partial n_\xi} \Big) d\xi = 0. \end{aligned}$$

When $x \to \partial\Omega$, using the properties of the double layer potential, we obtain

$$\begin{aligned} I_u(x,t) = -\frac{u_f(x,t)}{2} + \int_0^t d\eta \int_{\partial\Omega} \Big(& \frac{\partial \varepsilon_{n+1}^{+}(x-\xi, t-\eta, \lambda)}{\partial n_\xi} u_f(\xi,\eta) \\ & - \varepsilon_{n+1}^{+}(x-\xi, t-\eta, \lambda) \frac{\partial u_f(\xi,\eta)}{\partial n_\xi} \Big) d\xi = 0. \end{aligned} \tag{2.20}$$

for all $(x,t) \in \partial\Omega \times (0,T)$.

When $x \ne \xi$ and $t \ne \eta$, we have

$$(\frac{\partial}{\partial t} - \Delta_x + \lambda)\varepsilon_{n+1}^{+}(x-\xi, t-\eta, \lambda) \equiv 0,$$
$$(\frac{\partial}{\partial t} - \Delta_x + \lambda)\frac{\partial}{\partial n_\xi}\varepsilon_{n+1}^{+}(x-\xi, t-\eta, \lambda) \equiv 0,$$

so we obtain

$$(\frac{\partial}{\partial t} - \Delta_x + \lambda) I_u(x,t) \equiv 0. \tag{2.21}$$

Since $I_u(x,t)$ is the solution of the homogeneous heat equation (2.21), by the uniqueness of the mixed Cauchy problem, the identity

$$I_u(x,t) \equiv 0$$

is equivalent to (2.20), i.e.

$$I_u(x,t)|_{x \in \partial\Omega} = 0$$

is the lateral boundary condition of the volume heat potential (2.11) for the heat equation with a scalar parameter.

Now we prove the converse statement. If $u_1(x,t)$ is an arbitrary solution of the inhomogeneous heat equation with a scalar parameter (2.12), which satisfies the homogeneous initial condition (2.13) and the lateral boundary condition (2.18), then it coincides with the volume heat potential $u_f(x,t)$, i.e. $u_1(x,t) = u_f(x,t)$.

If not, then the function

$$\vartheta(x,t) = u_1(x,t) - u_f(x,t)$$

satisfies the homogeneous heat equation with a scalar parameter

$$\Diamond \vartheta(x,t) = \Diamond u_1(x,t) - \Diamond u_f(x,t) = 0$$

and the homogeneous initial condition

$$\vartheta(x,0) = u_1(x,0) - u_f(x,0) = 0$$

and the lateral boundary condition

$$I_\vartheta(x,t) = I_{u_1}(x,t) - I_{u_f}(x,t) = 0.$$

As above, by direct calculation we obtain

$$
\begin{aligned}
0 &= \lim_{\delta\to 0}\int_0^{t-\delta} d\eta \int_\Omega \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)\Diamond\vartheta(\xi,\eta)d\xi \\
&= \vartheta(x,t) + \int_0^t d\eta\int_{\partial\Omega}\Big(\frac{\partial\varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)}{\partial n_\xi}\vartheta(\xi,\eta) \\
&\quad - \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)\frac{\partial\vartheta(\xi,\eta)}{\partial n_\xi}\Big)d\xi
\end{aligned}
$$

for all $(x,t)\in\Omega\times(0,T)$. Therefore,

$$
\begin{gathered}
(\vartheta(x,t)+I_\vartheta(x,t))\,|_{x\in\partial\Omega} = 0,\\
\vartheta(x,t)|_{x\in\partial\Omega} = 0.
\end{gathered}
$$

By the uniqueness of the mixed Cauchy problem for the homogeneous heat equation with a scalar parameter, according to the maximum principle, we have

$$
\begin{gathered}
\vartheta(x,t)\equiv 0,\\
u_1(x,t) = u_f(x,t)
\end{gathered}
$$

for all $(x,t)\in\Omega\times(0,T)$.

Thus, the lateral boundary condition (2.18) and the initial condition (2.13) for the heat equation with a scalar parameter (2.12) generates a volume heat potential uniquely. The proof is complete. $\square$

Lemma 2.5. *For any function $u_0(x)\in W_2^2(\Omega)$ the surface heat potential* (2) *satisfies the homogeneous heat equation with a scalar parameter* (2.15)*, the nonhomogeneous initial condition* (2.16)*, and the following lateral boundary condition:*

$$
\begin{aligned}
&-\frac{u_\tau(x,t)}{2} + \int_0^t d\eta\int_{\partial\Omega}\Big(\frac{\partial\varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)}{\partial n_\xi}u_\tau(\xi,\eta) \\
&- \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)\frac{\partial u_\tau(\xi,\eta)}{\partial n_\xi}\Big)d\xi = 0,
\end{aligned}
\tag{2.22}
$$

for all $(x,t)\in\partial\Omega\cap(0,T)$.

Conversely, if $u(x,t)\in W_2^{2,1}(D^+)$ is a solution of the homogeneous heat equation with a scalar parameter (2.15)*, which satisfies the nonhomogeneous initial condition* (2.16) *and the lateral boundary condition* (2.22)*, then it coincides with the surface heat potential* (2.11)*.*

Proof. In this case formula (2.19) becomes

$$\begin{aligned}
0 &= \lim_{\delta\to 0}\int_0^{t-\delta} d\eta \int_\Omega \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)(\frac{\partial}{\partial\eta} - \Delta_\xi + \lambda)u_\tau(\xi,\eta)d\xi \\
&= \lim_{\delta\to 0}\int_\Omega \varepsilon^+_{n+1}(x-\xi,\delta,\lambda)u_\tau(\xi,t-\delta)d\xi \\
&\quad - \lim_{\delta\to 0}\int_\Omega \varepsilon^+_{n+1}(x-\xi,t,\lambda)u_\tau(\xi,0)d\xi \\
&\quad - \lim_{\delta\to 0}\int_0^{t-\delta} d\eta \int_{\partial\Omega} \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)\frac{\partial u_\tau(\xi,\eta)}{\partial n_\xi}d\xi \\
&\quad + \lim_{\delta\to 0}\int_0^{t-\delta} d\eta \int_{\partial\Omega} \frac{\partial \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)}{\partial n_\xi}u_\tau(\xi,\eta)d\xi \\
&\quad + \lim_{\delta\to 0}\int_0^{t-\delta} d\eta \int_\Omega u_\tau(\xi,\eta)(-\frac{\partial}{\partial\eta} - \Delta_\xi + \lambda)\varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)d\xi \\
&= I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \tag{2.23}$$

for all $(x,t)\in\Omega\times(0,T)$.

The integral I_1 is calculated as in the case of the heat potential and $I_1 = u_\tau(x,t)$. By definition of the surface heat potential the integral I_2 coincides with the surface heat potential

$$u_\tau(x,t) = \int_\Omega \varepsilon^+_{n+1}(x-\xi,t,\lambda)\tau(\xi)d\xi = \int_\Omega \varepsilon^+_{n+1}(x-\xi,t,\lambda)u_\tau(\xi,0)d\xi = I_2. \tag{2.24}$$

As above

$$\begin{gathered}
I_3 = \int_0^t d\eta \int_{\partial\Omega} \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)\frac{\partial u_\tau(\xi,\eta)}{\partial n_\xi}d\xi, \\
I_4 = \int_0^t d\eta \int_{\partial\Omega} \frac{\partial \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)}{\partial n_\xi}u_\tau(\xi,\eta)d\xi, \\
I_5 = \lim_{\delta\to 0}\int_0^{t-\delta} d\eta \int_\Omega u_\tau(\xi,\eta)(-\frac{\partial}{\partial\eta} - \Delta_\xi + \lambda)\varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)d\xi = 0.
\end{gathered}$$

We obtain that for all $(x,t)\in\Omega\times(0,T)$,

$$\begin{aligned}
I_u(x,t) = \int_0^t d\eta \int_{\partial\Omega} \Big(&\frac{\partial \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)}{\partial n_\xi}u_\tau(\xi,\eta) \\
&- \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)\frac{\partial u_\tau(\xi,\eta)}{\partial n_\xi}\Big)d\xi = 0.
\end{aligned}$$

When $x\to\partial\Omega$, once again using the properties of the double layer potential, we obtain the lateral boundary condition for the surface heat potential

$$\begin{aligned}
I_u(x,t) = -\frac{u_\tau(x,t)}{2} + \int_0^t d\eta \int_{\partial\Omega} \Big(&\frac{\partial \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)}{\partial n_\xi}u_\tau(\xi,\eta) \\
&- \varepsilon^+_{n+1}(x-\xi,t-\eta,\lambda)\frac{\partial u_\tau(\xi,\eta)}{\partial n_\xi}\Big)d\xi = 0.
\end{aligned} \tag{2.25}$$

for all $(x,t)\in\partial\Omega\times(0,T)$ and

$$(\frac{\partial}{\partial t} - \Delta_x + \lambda)I_u(x,t) \equiv 0. \tag{2.26}$$

As above, since $I_u(x,t)$ is the solution of the homogeneous heat equation (2.26), then, by virtue of the uniqueness of the mixed Cauchy problem, the identity

$$I_u(x,t) \equiv 0$$

is equivalent to (2.25), i.e.

$$I_u(x,t)|_{x\in\partial\Omega} = 0$$

is the lateral boundary condition of the surface heat potential (2) for the homogeneous heat equation with a scalar parameter.

The converse statement is proved as in the case of the volume heat potential. This completes the proof. ☐

As in [1] we can show that Newton's potential (volume potential)

$$u(x,t) = (L_B^{-1} f)(x,t) = \int_{D^-} \varepsilon_{n+1}^-(x-\xi, t-\eta, \lambda) f^-(\xi,\eta) dD^-, \ t < 0, \tag{2.27}$$

satisfies Helmholtz equation

$$-\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta_x u(x,t) + \lambda u(x,t) = f^-(x,t), \quad x \in R^n, \ t < 0 \tag{2.28}$$

satisfies the potential boundary condition

$$\begin{aligned} &-\frac{u(x,t)}{2} + \int_{\partial D^-} \Big(\frac{\partial \varepsilon_{n+1}^-(x-\xi, t-\eta, \lambda)}{\partial n_\xi} u(\xi,\eta) \\ &- \varepsilon_{n+1}^-(x-\xi, t-\eta, \lambda) \frac{\partial u(\xi,\eta)}{\partial n_\xi} \Big) dD^- = 0, \end{aligned} \tag{2.29}$$

for all $(x,t) \in \partial D^- = \sigma \cup \Omega$.

It should be noted that the boundary ∂D^- includes a domain Ω, which is an internal subset of the domain D.

Lemma 2.6. *For any function $f(x,t) \in C^\alpha(\overline{D})$ the Newton's potential (2.27) satisfies the inhomogeneous Helmholtz equation (2.28) and the potential boundary condition*

$$\begin{aligned} &-\frac{u(x,t)}{2} + \int_{\partial D^-} \Big(\frac{\partial \varepsilon_{n+1}^-(x-\xi, t-\eta, \lambda)}{\partial n_\xi} u(\xi,\eta) \\ &- \varepsilon_{n+1}^-(x-\xi, t-\eta, \lambda) \frac{\partial u(\xi,\eta)}{\partial n_\xi} \Big) dD^- = 0 \end{aligned} \tag{2.30}$$

for all $(x,t) \in \partial D^- = \sigma \cup \Omega$.

Conversely, if $u(x,t) \in W_2^2(D^-)$ is a solution of the inhomogeneous Helmholtz equation (2.28), which satisfies the potential boundary condition (2.30), then it coincides with Newton's potential (2.27), where

$$\begin{aligned} \varepsilon_{n+1}^-(\overline{x}, \lambda) &= \frac{1}{(n-1)\omega_{n+1}|\overline{x}|^{n-1}} \frac{(n-1)}{\Gamma(\frac{n+1}{2})} \Big(\frac{\sqrt{\lambda}|\overline{x}|}{2} \Big)^{\frac{n-1}{2}} K_{\frac{1-n}{2}}(\sqrt{\lambda}|\overline{x}|) \\ &= \varepsilon_\Delta^{n+1}(\overline{x}) \widetilde{K}_{\frac{1-n}{2}}(\sqrt{\lambda}|\overline{x}|) \end{aligned} \tag{2.31}$$

is the fundamental solution of the Helmholtz equation.

Proof. Assuming that $u(x,t) \in C^2(D^-) \cap C^1(\overline{D^-})$, by direct calculation and using Green's formula for any $(x,t) \in (D^-)$, we have

$$u(x,t) = \int_{D^-} \varepsilon_{n+1}^-(x-\xi, t-\eta, \lambda) f(\xi,\eta) dD^-$$

$$
\begin{aligned}
&= \int_{D^-} \varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda)(-\frac{\partial^2}{\partial\eta^2} - \Delta_\xi + \lambda)u(\xi,\eta)dD^- \\
&= \int_{\partial D^-} \Big(\frac{\partial\varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda)}{\partial n}u(\xi,\eta) - \varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda)\frac{\partial u(\xi,\eta)}{\partial n}\Big)dS \\
&\quad - \int_{D^-} u(\xi,\eta)(-\frac{\partial^2}{\partial\eta^2} - \Delta_\xi + \lambda)\varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda)dD^- \\
&= u(x,t) + \int_{\partial D^-} \Big(\frac{\partial\varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda)}{\partial n}u(\xi,\eta) - \varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda)\frac{\partial u(\xi,\eta)}{\partial n}\Big)dS.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_u(x,t) = \int_{\partial D^-} \Big(&\frac{\partial\varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda)}{\partial n}u(\xi,\eta) \\
&- \varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda)\frac{\partial u(\xi,\eta)}{\partial n}\Big)dS = 0
\end{aligned}
$$

for all $(x,t) \in D^-$.

When $x \to \partial\Omega$ using the properties of the double layer potential for (2.31), we obtain the potential boundary condition of the Newton's potential,

$$
\begin{aligned}
I_u(x,t) = -\frac{u(x,t)}{2} + \int_{\partial D^-} \Big(&\frac{\partial\varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda)}{\partial n}u(\xi,\eta) \\
&- \varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda)\frac{\partial u(\xi,\eta)}{\partial n}\Big)dS = 0
\end{aligned} \tag{2.32}
$$

for all $(x,t) \in \partial D^- = \sigma \cup \Omega$.

When $x \neq \xi$ and $t \neq \eta$,

$$
\begin{gathered}
(-\frac{\partial^2}{\partial t^2} - \Delta_x + \lambda)\varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda) \equiv 0, \\
(-\frac{\partial^2}{\partial t^2} - \Delta_x + \lambda)\frac{\partial}{\partial n}\varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda) \equiv 0.
\end{gathered}
$$

Then we have

$$
(-\frac{\partial^2}{\partial t^2} - \Delta_x + \lambda)I_u(x,t) \equiv 0. \tag{2.33}
$$

Since $I_u(x,t)$ is the solution of the homogeneous Helmholtz equation (2.33), by the uniqueness of the Dirichle problem, it follows that

$$
I_u(x,t) \equiv 0
$$

is equivalent to (2.32), i.e.

$$
I_u(x,t)|_{(x,t)\in\partial D^-} = 0
$$

is the potential boundary condition of the Newton's potential (2.27) for the inhomogeneous Helmholtz equation.

The converse statement is proved as in the case of the volume heat potential and surface heat potential. This completes the proof. □

We have proved that an elliptic-parabolic potential $u(x,t) = (L_B^{-1}f)(x,t)$ (1.3) satisfies the boundary conditions (2.17) and (2.29), now we prove the converse statement. If

$$
u(x,t) = (L_B^{-1}f)(x,t) \in C^\alpha(\overline{D}) \cap C^{2+\alpha}(\overline{D^-}) \cap C^{2+\alpha,1+\alpha}_{x,t}(\overline{D^+})
$$

is a solution of

$$\begin{gathered}\Big(\frac{\partial}{\partial t}-\Delta_x+\lambda\Big)u(x,t)=f^+(x,t),\quad (x,t)\in D^+\\ \Big(-\frac{\partial^2}{\partial t^2}-\Delta_x+\lambda\Big)u(x,t)=f^-(x,t),\quad (x,t)\in D^-\end{gathered}\tag{2.34}$$

and satisfies conditions (2.17) and (2.29), then $u(x,t)$ coincides with the elliptic-parabolic potential $u(x,t)=(L_B^{-1}f)(x,t)$ (1.3).

From the continuity of solution $u(x,t)$ when $t=0$ we can find

$$\tau(x)=u(x,0-)=\int_{D^-}\varepsilon_{n+1}^-(x-\xi,t-\eta,\lambda)f^-(\xi,\eta)d\xi d\eta.\tag{2.35}$$

The general solution of (1.1) in the domain D^+ satisfying condition (2.35) and $u(x,t)|_{t=0}=\tau(x)$, so we can represent the general solution of(1.1) in the following form

$$\begin{aligned}u(x,t)&=(L_B^{-1}f)(x,t)\\ &=\int_{D^+}\varepsilon_{n+1}^+(x-\xi,t-\eta,\lambda)f^+(\xi,\eta)d\xi d\eta\\ &\quad+\int_\Omega\varepsilon_{n+1}^+(x-\xi,t,\lambda)\Big(\int_{D^-}\varepsilon_{n+1}^-(\xi-\bar{\xi},t-\eta,\lambda)f^-(\bar{\xi},\eta)d\bar{\xi}d\eta\Big)d\xi.\end{aligned}\tag{2.36}$$

Theorem 2.7. *For any $f(x,t)\in C^\alpha(\overline{D})$ the elliptic-parabolic potential*

$$u(x,t)=(L_B^{-1}f)(x,t)\in C^\alpha(\overline{D})\cap C^{2+\alpha}(\overline{D^-})\cap C_{x,t}^{2+\alpha,1+\alpha}(\overline{D^+})$$

which represented as (1.3) *satisfies the Bitsadze-Samarsky boundary conditions* (2.17) *and* (2.29).

Conversely, if

$$u(x,t)=(L_B^{-1}f)(x,t)\in C^\alpha(\overline{D})\cap C^{2+\alpha}(\overline{D^-})\cap C_{x,t}^{2+\alpha,1+\alpha}(\overline{D^+})$$

is a solution of the equations (1.1)-(1.2) *and it satisfies the Bitsadze-Samarsky boundary conditions* (2.17) *and* (2.29)*, then it coincides with the elliptic-parabolic potential* (1.3).

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