Electronic Journal of Differential Equations, Vol. 2005(2005), No. 80, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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SELF SIMILAR SOLUTIONS OF GENERALIZED BURGERS EQUATION

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Abstract. In this paper, we study the initial-value problem

$$(|u'|^{p-2}u')' + \beta r u' + \alpha u - \gamma |u|^{q-1} u |u'|^{p-2} u' = 0, \quad r > 0,$$
$$u(0) = A, \quad u'(0) = 0,$$

where $A > 0$, $p > 2$, $q > 1$, $\alpha > 0$, $\beta > 0$ and $\gamma \in \mathbb{R}$. Existence and complete classification of solutions are established. Asymptotic behavior for nonnegative solutions is also presented.

1. Introduction

This paper concerns the nonlinear parabolic equation

$$U_t - (|U_x|^{p-2} U_x)_x = -\frac{k}{t} U + |U|^{q-1} U |U_x|^{p-2} U_x \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \tag{1.1}$$

where $p > 2$, $q > 1$ and $k > 0$. As is often the case in nonlinear PDE's of parabolic type the characteristic properties of an equation, are displayed by means of the existence of so-called self similar solutions; this is our main interest. It is worth mentioning that if $p = 2$, $q = 1$ and $k = 0$, we get the classical one dimensional Burgers equation

$$U_t = U_{xx} + U_x U, \tag{1.2}$$

which is originally proposed as a simplified model of Navier-Stokes Turbulence (see [3] and [4]) .

By design, Burgers equation is the simplest model of hydrodynamic flow that captures the interaction of nonlinear wave propagation and viscosity. Burgers turbulence is often viewed as a pared-down model of acoustic turbulence (see [8] and [14]).

The importance and popularity of equation (1.2) lie in its simplicity and in the fact that the well known Hopf-cole substitution $w = \frac{U_x}{U}$ reduces it to the linear heat equation. This nonlinear change of variables permits an explicit description of solutions of (1.2) and explains their essentially nonlinear first order asymptotic as t goes to infinity.

2000 *Mathematics Subject Classification.* 35k55, 35k65.
Key words and phrases. Burgers equation; self similar; classification; asymptotic behavior.

Submitted December 6, 2004. Published July 15, 2005.

If $p = 2$ and $k \neq 0$, equation (1.1) becomes

$$U_t = U_{xx} + |U|^{q-1}UU_x - \frac{k}{t}U \tag{1.3}$$

which is studied by [13]. Note that, if $q = 1$, equation (1.3) describes the propagation of weakly nonlinear longitudinal waves in gases or liquids from a non planar source (see [9] and [10]).

If $p > 2$, equations (1.1) appears in the description of ice sheet dynamics (see [5]) where the reaction term $-\frac{k}{t}U$ can be considered as a turbulent term. In this case the selfsimilar solutions of problem (1.1) take the form

$$U(x,t) = t^{\sigma} f(y), \quad \text{where} \quad y = xt^{\eta},$$

with

$$\sigma = \frac{-1}{pq+p-2} \quad \text{and} \quad \eta = \frac{-q}{pq+p-2}.$$

Then the profile f is determined as a solution in of the ODE

$$(|f'|^{p-2}f')' + \frac{q}{pq+p-2}yf' - (k - \frac{1}{pq+p-2})f + |f|^{q-1}f|f'|^{p-2}f' = 0, \quad y \in \mathbb{R}.$$

Where the prime denotes the differentiation with respect to y. If we set

$$g(y) = \begin{cases} f(y) & \forall \text{ y} \in \mathbb{R}^+, \\ f(-y) & \forall \text{ y} \in \mathbb{R}^-, \end{cases}$$

then, g satisfies

$$(|g'|^{p-2}g')' + \alpha g + \beta y g' + \gamma |g|^{q-1}g|g'|^{p-2}g' = 0,$$

in $\mathbb{R}$; with $\alpha = -k + \frac{1}{pq+p-2}$, $\beta = \frac{q}{pq+p-2}$

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and $\gamma = -1$, if $y > 0$; $\gamma = 1$, if $y < 0$. Consequently, we have just to focus on the study of the initial-value problem

$$\begin{gathered} (|u'|^{p-2}u')' + \beta r u' + \alpha u - \gamma |u|^{q-1}u|u'|^{p-2}u' = 0, \quad r > 0 \\ u(0) = A, \quad u'(0) = 0, \end{gathered} \tag{1.4}$$

when $\alpha > 0$, $\beta > 0$ and $\gamma \in \mathbb{R}$. We will mainly discuss: (i) The existence and uniqueness of solutions for (1.4); (ii) the asymptotic behavior of positive solutions, and (iii) a classification of solutions.

The main results of this paper are the following.

Theorem 1.1. *Assume $p > 2$, $q > 1$, $\alpha > 0$, $\beta > 0$, and $\gamma \in \mathbb{R}$. Then for each $A > 0$, there exists a real $R_{\max} > 0$ such that* (1.4) *has a unique solution $u \equiv u(.,A)$ defined in the right open interval $[0, R_{\max}[$, meaning that u and $|u'|^{p-2}u'$ are a C^1 functions in $[0, R_{\max}[$, satisfying* (1.4)*.*

The following result gives the monotonicity of solutions of problem (1.4) with respect to initial data.

Theorem 1.2. *Assume $\alpha > 0$, $\beta > 0$ and $\gamma < 0$. Let $u(.,A)$ and $u(.,B)$ be two solutions of problem* (1.4) *with $u(0,A) = A$, $u(0,B) = B$ and $A \neq B$. Then $u(.,A)$ and $u(.,B)$ can not intersect each other before their first zero.*

Concerning the asymptotic behavior, we have the following results.

Theorem 1.3. *Let u be a strictly positive solution of* (1.4). *Then*
$$\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} u'(r) = 0.$$
Furthermore, if $\alpha > 0$, $\beta > 0$ and $\gamma \le 0$, then
$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}} u(r) = L$$
exists and lies in $[0, +\infty[$. Moreover this limit L is strictly positive for $0 < \alpha \le \beta$ and $\gamma < 0$.

Finally, the structure of solutions of problem (1.4) consists of three families: The set of strictly positive solutions, the set of changing sign solutions and finally solutions with compact support. This classification depends strongly on the sign of $\gamma(\alpha - \beta)$.

Theorem 1.4. *Assume $p > 2, q > 1$, and $\gamma < 0$. Then we have*

- (i) *For $\alpha > \beta$ there exist two constants A_1 and A_2 such that for any $A > A_1$, the solution $u(., A)$ is strictly positive and for $A < A_2$, $u(., A)$ changes sign. Moreover, there exists at least one solution with compact support.*
- (ii) *For $0 < \alpha \le \beta$, any solution of* (1.4) *is strictly positive.*

The situation for $\gamma > 0$ is quite the opposite.

Theorem 1.5. *Assume $p > 2$, $q > 1$ and $\gamma > 0$. Then*

- (i) *If $\alpha \ge \beta$ any solution of* (1.4) *change sign.*
- (ii) *If $\alpha < \beta$, there exist two constants A_1 and A_2 such that for any $A < A_2$, the solution $u(., A)$ is strictly positive and for $A > A_1, u(., A)$ changes sign.*

The organization of this paper is as follows. Theorems 1.1, 1.2 and 1.3 are proved in section 2. In section 3 a classification of solutions is investigated and then Theorems 1.4 and 1.5 are established.

2. Existence and asymptotic behavior of solutions

In this section, we investigate existence, uniqueness and asymptotic behavior of solutions of the problem (1.4). We start with a local existence and uniqueness result.

Proposition 2.1. *Assume $p > 2$, $q > 1$, $\alpha > 0$, $\beta > 0$ and $\gamma \in \mathbb{R}$. Then for each $A > 0$, there exists a right open interval $I = [0, R_{\max}[$ and a unique function u such that, u and $|u'|^{p-2}u'$ lie in $C^1(I)$ and satisfy* (1.4).

First of all, we note that, for a fixed α, β and γ, it easy to see that $u(., \gamma, A) = -u(., \gamma, -A)$. Therefore, in the sequel we restrict ourselves to the case of $A > 0$.

Remark 2.2. The first equation in (1.4) can be reduced to the first order system
$$\begin{gathered} X' = |Y|^{-\frac{p-2}{p-1}} Y \\ Y' = -\alpha X - \beta |Y|^{-\frac{p-2}{p-1}} Y + \gamma |X|^{q-1} XY. \end{gathered} \tag{2.1}$$
Since the mapping
$$(X, Y) \mapsto \Big(|Y|^{-\frac{p-2}{p-1}} Y, -\alpha X - \beta |Y|^{-\frac{p-2}{p-1}} Y + \gamma |X|^{q-1} XY \Big)$$

is a locally Lipschitz continuous function in the set $\{(X, Y) \in \mathbb{R} \times \mathbb{R}^*\}$, we deduce that, for any $r_0 > 0, A \geq 0$ and $B \neq 0$, there exists a unique solution of (1.4) in a neighborhood of r_0 such that $u(r_0) = A$ and $u'(r_0) = B$.

Because of the presence of the term $|Y|^{-\frac{p-2}{p-1}}Y$, the above function is not locally Lipschitz continuous near r_0 whenever $u'(r_0) = 0$. Consequently, for our problem (1.4) the above argument does not work. To avoid this difficulty, we make use an idea from [7]. Then the proof becomes similar to that of [1, proposition 1.1] and [6, proposition 2.1]. We present it here for the convenience of the reader.

Proof proposition 2.1. The idea of the proof is to convert our initial value problem (1.4) to a fixed point problem of some operator. This will be done in two steps.
Step 1. Local existence and uniqueness. It is clear that to solve problem (1.4) is equivalent to find a function $u \in C^1(I)$ defined in some interval $I = [0, R[$ with $R > 0$ such that $|u'|^{p-2}u' \in C^1(I)$ and satisfies the integral equation

$$u(r) = A - \int_0^r G(F_u)(s)ds, \tag{2.2}$$

where $G(s) = |s|^{(2-p)/(p-1}s$, for all $s \in \mathbb{R}$, and

$$F_u(s) = \beta s u(s) + (\alpha - \beta)\int_0^s u(\tau)d\tau - \gamma \int_0^s |u|^{q-1}u(\tau)|u'|^{p-2}u'(\tau)d\tau. \tag{2.3}$$

Now, let us define on $[0, A]$ the following two functions

$$f_1(X) = \begin{cases} \alpha(A - X) - |\gamma|X^{p-1}(A + X)^q & \text{if } \alpha \geq \beta, \\ \alpha(A + X) - 2\beta X - |\gamma|X^{p-1}(A + X)^q & \text{if } \alpha < \beta, \end{cases} \tag{2.4}$$

and

$$f_2(X) = \begin{cases} (A + X)\left\{\alpha + |\gamma|X^{p-1}(A + X)^{q-1}\right\} & \text{if } \alpha \geq \beta, \\ \alpha(A - X) + |\gamma|X^{p-1}(A + X)^q & \text{if } \alpha < \beta. \end{cases} \tag{2.5}$$

Since f_1 is continuous and $f_1(0) = \alpha A > 0$, then there exists some interval $[0, A_0] \subset [0, A]$ such that

$$f_1(X) > 0 \quad \forall X \in [0, A_0].$$

Let us introduce some useful notation for the proof:

$$f_1(A_0) = K_1, \quad f_2(A_0) = K_2, \quad R_0 = \inf\{1, \frac{A_0^{p-1}}{2\Gamma A}, \frac{K_1^{p-2}}{(2\Gamma)^{p-1}}\}, \tag{2.6}$$

where

$$\Gamma = \beta + |\alpha - \beta| + (2p - 2 + q)2^{q-1}|\gamma|A^{q+p-2}. \tag{2.7}$$

It is easy to see that the function f_2 satisfies the estimate

$$f_2(X) \leq 2A\Gamma, \quad \forall X \in [0, A]. \tag{2.8}$$

Now, we consider the complete metric space

$$X = \{\varphi \in C^1([0, R_0]) : \|\varphi - A\|_X \leq A_0\} \tag{2.9}$$

where

$$\|\varphi\|_X = \max(\|\varphi\|_0, \|\varphi'\|_0). \tag{2.10}$$

and $\|.\|_0$ denotes the sup norm. Next we define the mapping $\mathcal{T}$ on X, by

$$T(\varphi) = A - \int_0^r G(F_\varphi)(s)ds, \quad \forall r \in [0, R_0] \tag{2.11}$$

Claim 1. $\mathcal{T}$ maps X into itself. In fact, take $\varphi \in X$. First, it is easy to see that $\mathcal{T}(\varphi) \in C^1([0,R_0])$. Also by a simple calculation we get

$$0 < K_1 s < F_\varphi(s) < K_2 s \quad \forall s \in [0,R_0]. \tag{2.12}$$

And thereby, $\mathcal{T}(\varphi)$ satisfies the following estimates

$$|\mathcal{T}(\varphi)(r) - A| \le \int_0^r F_\varphi^{1/(p-1)}(s)ds \le \frac{p-1}{p} K_2^{1/(p-1)} r^{p/(p-1)}, \tag{2.13}$$

$$|\mathcal{T}'(\varphi)(r)| \le F_\varphi^{1/(p-1)}(r) \le K_2^{1/(p-1)} r^{1/(p-1)}, \tag{2.14}$$

for any $r \in]0,R_0]$. These last two equations, combined with the expression of R_0 given by (2.6) imply that $\mathcal{T}(\varphi) \in X$.

Claim 2. $\mathcal{T}$ is a contraction. To prove this, take φ and $\psi \in X$. Then

$$|\mathcal{T}(\varphi)(r) - T(\psi)(r)| \le \int_0^r |G(F_\varphi(s)) - G(F_\psi(s))|ds, \tag{2.15}$$

for any $r \in [0,R_0]$, where F_φ is given by (2.3). In view of estimate (2.12) (which is also valid for F_ψ), we get

$$\begin{aligned} |G(F_\varphi)(s) - G(F_\psi)(s)| &\le |F_\varphi^{1/(p-1)}(s) - F_\psi^{1/(p-1)}(s)| \\ &\le \frac{1}{p-1} K_1^{(2-p)/(p-1)} |F_\varphi(s) - F_\psi(s)| s^{(2-p)/(p-1)}. \end{aligned}$$

Recalling the expression of F_φ and F_ψ, we deduce

$$|F_\varphi(s) - F_\psi(s)| \le [\beta + |\alpha - \beta|] \|\varphi - \psi\|_0 s + |\gamma| I \tag{2.16}$$

where

$$I = \int_0^s |\varphi^q(\tau)|\varphi'|^{p-2}\varphi'(\tau) - \psi^q(\tau)|\psi'|^{p-2}\psi'(\tau)|d\tau. \tag{2.17}$$

But

$$I \le \int_0^s |\varphi\prime(\tau)|^{p-1}|\varphi^q(\tau) - \psi^q(\tau)|d\tau + \int_0^s \psi^q(\tau)\big||\varphi'|^{p-2}\varphi' - |\psi'|^{p-2}\psi'\big|d\tau. \tag{2.18}$$

Using the fact that φ and ψ are elements of X, we get

$$I \le (q + 2p - 2)2^{q-1} A^{q+p-2} \|\varphi - \psi\|_X s. \tag{2.19}$$

Combining this last equation with (2.16), we get

$$|F_\varphi(s) - F_\psi(s)| \le \Gamma \|\varphi - \psi\|_X s, \quad \forall s \in [0,R_0] \tag{2.20}$$

where Γ is given by (2.7). Therefore

$$|\mathcal{T}(\varphi)(r) - T(\psi)(r)| \le \frac{\Gamma}{p} K_1^{(2-p)/(p-1)} \|\varphi - \psi\|_X r^{p/(p-1)}. \tag{2.21}$$

Similarly, one can easily obtain

$$|\mathcal{T}'(\varphi)(r) - \mathcal{T}'(\psi(r)| \le \frac{\Gamma}{p-1} K_1^{(2-p)/(p-1)} \|\varphi - \psi\|_X r^{1/(p-1)}. \tag{2.22}$$

¿From the choice of R_0, these last two equations imply that $\mathcal{T}$ is a contraction. The use of the Banach's Contraction theorem leads to the existence of a unique function u solving problem (1.4) in $(0,R_0)$.

Step 2. $|u'|^{p-2}u' \in C^1([0,R_0))$. We have just to prove the regularity at $r=0$. For this purpose, note that the first equation in (1.4) gives

$$\lim_{r\to 0}(|u'|^{p-2}u')'(r) = -\alpha A. \tag{2.23}$$

Integrating equation (1.4) from 0 to r, and letting r go to 0, we obtain

$$\lim_{r\to 0} \frac{|u'|^{p-2}u'(r)}{r} = -\alpha A. \tag{2.24}$$

Hence $|u'|^{p-2}u' \in C^1(([0,R_0[)$. This completes the proof of Proposition 2.1. □

Remark 2.3. It is not difficult to see that the solution u of (1.4) is a C^∞ function at any $r > 0$ whenever $u'(r) \neq 0$.

The remaining of this section is devoted to the proof of the Theorem 1.2. For this purpose we start with the following lemma.

Lemma 2.4. *Let $A > 0$ and u be the corresponding solution of* (1.4)*. Then as long as u is strictly positive we have $0 < u(r) < A$ and $u'(r) < 0$.*

Proof. Let

$$E(r) = \frac{p-1}{p}|u'|^p(r) + \frac{\alpha}{2}u^2(r), \quad \forall r \in [0, R_{\max}[, \tag{2.25}$$

be the induced energy function. We have

$$E'(r) = -\beta r|u'|^2 + \gamma|u|^{q-1}u|u'|^p, \quad \forall r \in [0, R_{\max}[. \tag{2.26}$$

Then if $\gamma \le 0$ the energy function is decreasing as long as u is positive. Particularly $u(r) \le A$. On the other hand since

$$(|u'|^{p-2}u')'(0) = -\alpha A < 0, \tag{2.27}$$

the lemma follows easily when $\gamma > 0$. □

The next result gives the monotonicity of solutions of the problem (1.4) with respect to initial data. More exactly, we have

Proposition 2.5. *Assume $\alpha > 0$, $\beta > 0$ and $\gamma < 0$. Let $0 < A_1 < A_2$. Then $u(.,A_1)$ and $u(.,A_2)$ can not intersect each other before their first zero.*

Proof. For ease of notation, we write $u(.,A_1) = u(.)$ and $u(.,A_2) = v(.)$ and we denote by R_1 (respectively R_2) the first zero of u (respectively v). The proof will be done by contradiction: it is based on the idea of [11, Lemma 2.4]. We assume that there exists some point $R_0 \in [0, \min\{R_1,R_2\}[$ such that

$$u(r) < v(r) \quad \text{for} \quad r \in [0,R_0[\quad \text{and} \quad u(R_0) = v(R_0). \tag{2.28}$$

Now, for any $k > 0$, we set

$$u_k(r) = k^{-p/(p-2)}u(kr), \quad r \in [0, \frac{R_1}{k}]. \tag{2.29}$$

Then u_k satisfy the equation

$$(|u_k'|^{p-2}u_k'(r))' + \beta r u_k'(r) + \alpha u_k - \gamma k^\mu u_k^q(s)|u_k'|^{p-2}u_k'(r) = 0, \tag{2.30}$$

with $\mu = 1 + \frac{pq}{p-2}$. Since u is strictly positive and decreasing on $[0,R_1[$, the function $k \mapsto u_k$ is strictly increasing. Moreover for any $r \in [0,R_0]$ $\lim_{k\to 0} u_k(r) = +\infty$. Then there exists a small k_0 >0 such that

$$u_k(r) > v(r) \quad \text{for } r \in [0,R_0] \quad \text{and} k \in [0,k_0]$$

Therefore, the set

$$\Omega \equiv \{k \in]0,k_0[; u_k(r) > v(r) \text{for} \ \ r \in [0,R_0]\} \tag{2.31}$$

is not empty and open. In particular if we denote by K the supremum of Ω, the real $K \notin \Omega$ and thereby, necessarily there exists $r_0 \in [0, R_0]$ such that $u_K(r_0) = v(r_0)$. As k_0 is small without loss of generality we assume

$$k_0 = (\frac{A_1}{2A_2})^{(p-2)/p}. \tag{2.32}$$

If $r_0 = R_0$, then

$$u_K(R_0) = K^{-p/(p-2)} u(KR_0) = v(R_0). \tag{2.33}$$

But $u(R_0) = v(R_0)$, then using again the strictly increasing of the function $k \mapsto u_k$ is strictly increasing we deduce necessarily $K = 1$. This is a contradiction with the choice of the real k_0. If $r_0 = 0$ we get

$$u_K(0) = K^{-p/p-2)} u(0) = K^{-p/(p-2)} A_1 = A_2$$

which contradicts (2.32). Consequently we deduce that there exists some point $r_0 \in [0, R_0[$ such that

$$u_K > v \quad \text{on }]0, R_0[\text{ and } u_K(r_0) = v(r_0). \tag{2.34}$$

So $u_K - v$ has a local minimum at the point r_0, where the graphs of u_K and $v$are tangent. Moreover, as v' and u' are strictly negative, the equation satisfied by v (respectively by u_K) can be written in the form

$$(p-1)|v'|^{p-2} v'' + \beta r v' + \alpha v - \gamma v^q |v'|^{p-2} v' = 0, \tag{2.35}$$

and respectively

$$(p-1)|u_K'|^{p-2} u_K'' + \beta r u_k' + \alpha u_K - \gamma K^{\mu} u_K^q |u_K'|^{p-2} u_K' = 0. \tag{2.36}$$

Subtract (2.35) from (2.36), we obtain at point r_0,

$$(p-1)|v'|^{p-2} (u_K - v)''(x) = \gamma (K^{\mu} - 1) v^q |v'|^{p-2} v'(r_0). \tag{2.37}$$

Since $\gamma < 0$, $v' < 0, K < 1$ and $\mu > 0$, we get

$$(p-1)|v'|^{p-2} (u_K - v)''(r_0) = \gamma (k^{\mu} - 1) v^q |v'|^{p-2} v'(r_0) < 0. \tag{2.38}$$

This is impossible because $(u_K - v)$ has a local minimum at x and then the proposition is proved. □

In the next result, we investigate the asymptotic behavior of positive solutions.

Proposition 2.6. *Let u be a positive solution of* (1.4) *defined on* $[0, +\infty[$. *Then*

$$\lim_{r \to +\infty} u(r) = \lim_{r \to +\infty} u'(r) = 0.$$

The proof of this result depends strongly on the sign of γ. In fact, if $\gamma \le 0$, the result follows from the energy function. However, for $\gamma > 0$ we need some information about the monotonicity of u' this is given in the following lemma.

Lemma 2.7. *Assume $\gamma > 0$. Let a real $A > 0$ and $u(., A)$ be a strictly positive solution of* (1.4) *defined in* $[0, +\infty[$. *Then there exists a unique real number $R(A) > 0$ such that*

$$u'' < 0 \quad \text{on } [0, R(A)[\quad \text{and} \quad u'' \ge 0 \quad \text{on} [R(A), +\infty[.$$

Proof. First, note that from Lemma 2.4, the solution $u = u(., A)$ is decreasing and converges to some nonnegative constant. On the other hand (2.27), implies that $(|u'|^{p-2}u')'$ must change sign. Let $R(A) > 0$ its first zero. For simplicity we set $R = R(A)$. Then $(|u'|^{p-2}u')'(r) < 0$ for any r in $[0, R[$. Furthermore,

$$\alpha u(R) = -[\beta R - \gamma u^q |u'|^{p-2}(R)]u'(R). \tag{2.39}$$

As u is strictly positive we deduce that $u'(R) \neq 0$, and then u is a C^∞ function at the point R. So, the first equation in (1.4) can be written in some neighborhood of R, say $]R - \varepsilon, R + \varepsilon[$ $(\varepsilon > 0)$, in the form

$$(p-1)|u'|^{p-2}u'' + \beta r u' + \alpha u - \gamma u^q |u'|^{p-2}u' = 0. \tag{2.40}$$

Differentiating this last equality and taking $r = R$, we obtain

$$(p-1)|u'|^{p-2}u^{(3)}(R) = -(\alpha + \beta)u'(R) + \gamma q u^{q-1}|u'|^p(R). \tag{2.41}$$

But, since $\gamma > 0$, $\alpha + \beta \geq 0$ and $u'(R) < 0$, then the left hand side of the last equation is strictly positive. By continuity of $u^{(3)}$ we get

$$u^{(3)}(r) > 0 \quad \text{for any } r \in [R, R + \varepsilon[.$$

Hence u'' is non-negative on $[R, R + \varepsilon[$. Finally, using (1.4) we deduce

$$u''(r) \geq 0 \quad \text{for any } r \text{ in } [R, +\infty[,$$

which completes the proof. □

Remark 2.8. Note that the right hand side of (2.41) satisfies

$$-(\alpha + \beta)u'(R) + \gamma q u^{q-1}|u'|^p(R) = \frac{u'(R)}{u(R)}\{\gamma q u^q |u'|^{p-2}u'(R) - (\alpha + \beta)u(R)\}.$$

Using (2.39), the relation (2.41) becomes

$$(p-1)|u'|^{p-2}u^{(3)}(R) = \beta(1 + \frac{\beta}{\alpha})R\frac{|u'(R)|^2}{u(R)} - \gamma(1 + \frac{\beta}{\alpha} - q)u^{q-1}|u'|^p(R).$$

Hence, if $\gamma < 0$ and $q \leq 1 + \frac{\beta}{\alpha}$, we get $u^{(3)}(R) > 0$ and thereby the Lemma 2.7 also holds in this case.

Proof of Proposition 2.6. By Lemma 2.4 $\lim_{r\to+\infty} u(r) = l$ exists and lies in $[0, A[$. We start by establishing the proposition when $\gamma \leq 0$, in this case the energy function given by (2.25) is positive and decreasing. It then converges, and $\lim_{r\to+\infty} u'(r) = 0$. Moreover integrating equation (1.4) between 0 and r, we get

$$|u'|^{p-2}u'(r) + \beta r u(r) + \int_0^r \left\{(\alpha - \beta)u(s) - \gamma u^q(s)|u'|^{p-2}u'(s)\right\} ds = 0. \tag{2.42}$$

Therefore,

$$\lim_{r\to+\infty} \frac{1}{r}\int_0^r \{(\alpha - \beta)u(s) - \gamma u^q(s)|u'|^{p-2}u'(s)\}ds = -\beta l. \tag{2.43}$$

On the other hand, if $l \neq 0$ the L'Hôpital rule implies that

$$\lim_{r\to+\infty} \frac{1}{r}\int_0^r \{(\alpha - \beta)u(s) - \gamma u^q(s)|u'|^{p-2}u'(s)\}ds = (\alpha - \beta)l.$$

This contradicts (2.43) and therefore $l = 0$.

To handle the case $\gamma > 0$, we use the above Lemma 2.7. Assume that $l \neq 0$ and integrate equation (1.4) on $(r, 2r)$ for some $r > 0$. We obtain

$$\begin{aligned} |u'|^{p-2}u'(2r) =&|u'|^{p-2}u'(r) + \beta r u(r) - 2\beta r u(2r) \\ &+ (\beta - \alpha)\int_r^{2r} u(s)ds + \gamma \int_r^{2r} u^q(s)|u'|^{p-2}u'(s)ds. \end{aligned} \tag{2.44}$$

Since $\gamma > 0$ and $u' \leq 0$, for i$\beta \geq \alpha$, we obtain

$$\frac{|u'|^{p-2}u'(2r)}{2r} \leq (\beta - \frac{\alpha}{2})(u(r) - u(2r)) - \frac{\alpha}{2}u(2r)\,. \tag{2.45}$$

On the other hand if $\beta < \alpha$,

$$\frac{|u'|^{p-2}u'(2r)}{r} \leq \beta(u(r) - u(2r)) - \alpha u(2r). \tag{2.46}$$

Now, observe that

$$\lim_{r\to+\infty}[u(r) - u(2r)] = 0 \quad \text{and} \quad \lim_{r\to+\infty} u(2r) = l \neq 0;$$

therefore, for any $\alpha, \beta \geq 0$, we get

$$\frac{|u'|^{p-2}u'(2r)}{2r} \leq -\frac{\alpha}{2}u(2r) \quad \text{for large } r. \tag{2.47}$$

This gives

$$(u^{(p-2)/(p-1)})'(r) \leq -\frac{p-2}{p-1}(\frac{\alpha}{2})^{1/(p-1)}r^{1/(p-1)}, \tag{2.48}$$

which contradicts that $u$is strictly positive. Consequently $l = 0$ and the proof is complete. $\square$

Now, we pass to the asymptotic behavior of positive solutions.

Proposition 2.9. *Assume $\alpha > 0$, $\beta > 0$ and $\gamma \leq 0$. Let u be a strictly positive solution of* (1.4)*. Then $\lim_{r\to+\infty} r^{\alpha/\beta}u(r) = L$ exists and lies in $[0, +\infty[$.*

Some preliminary results are needed for the proof of this proposition.

Lemma 2.10. *Assume $\alpha > 0$, $\beta > 0$ and $\gamma \leq 0$. Let u be a strictly positive solution of* (1.4) *such that*

$$u(r) \leq K(1 + r)^{-\sigma} \quad \textit{for } r \geq 0. \tag{2.49}$$

Then, there exists a constant M depending on K and σ such that

$$|u'(r)| \leq M(1 + r)^{-\sigma-1} \quad \textit{for } r \geq 0. \tag{2.50}$$

Proof. Without loss of generality we have just to prove (2.50) for $r > 2$. In fact, as u' is a continuous function, it is bounded in $[0, 2]$. So there exists some constant $C > 0$ such that

$$|u'(r)| \leq C, \quad \text{for } r \text{ in } [0, 2]. \tag{2.51}$$

Hence, if we take $M \geq C3^{\sigma+1}$, then (2.50) holds for r in $[0, 2]$. For any $r > 2$, we set

$$F(r) = \exp\big[\frac{-\gamma}{p-1}\int_0^r u^q(s)ds\big] \tag{2.52}$$

and consider the function

$$G(r) = \exp\big[\frac{\beta}{p-1}\int_0^r s|u'(s)|^{2-p}ds\big], \quad r > 2. \tag{2.53}$$

In view of (2.27) we have

$$u'(r) \sim -(\alpha A)^{1/(p-1)} r^{1/(p-1)} \quad \text{as } r \to 0. \tag{2.54}$$

Recalling that u' is strictly negative, we deduce that the function G is well posed.

Now, we write equation (1.4) in the form

$$(FGu')'(r) + \frac{\alpha}{\beta}\frac{F(r)}{r} u(r)G'(r) = 0. \tag{2.55}$$

Integrating the above equation, we obtain

$$|u'(r)| = \frac{\alpha}{\beta F(r)G(r)} \int_0^r \frac{F(s)}{s} u(s)G'(s)ds. \tag{2.56}$$

Since $\gamma \le 0$ the function F is increasing and

$$|u'(r)| \le \frac{\alpha}{\beta G(r)} \int_0^r \frac{u(s)}{s} G'(s)ds. \tag{2.57}$$

Next, we find a bound the right-hand side of the above inequality. We set

$$I_1 = \int_0^1 \frac{u(s)}{s} G'(s)ds, \quad I_2 = \int_1^{r/2} \frac{u(s)}{s} G'(s)ds, \quad I_3 = \int_{r/2}^r \frac{u(s)}{s} G'(s)ds, \tag{2.58}$$

so that

$$\int_0^r \frac{u(s)}{s} G'(s)ds = I_1 + I_2 + I_3\,. \tag{2.59}$$

First, note that (2.54) implies easily that I_1 is bounded. On the other hand, in view of Proposition 2.6, there exists a constant $K > 0$ such that

$$|u'(r)|^{2-p} \ge K \quad \text{for } r \ge 0. \tag{2.60}$$

Then

$$G(r) \ge \exp(Kr^2) \quad \text{for } r > 2. \tag{2.61}$$

To estimate I_2, we use (2.52) to obtain

$$I_2 \le C\int_1^{r/2} \frac{(s+1)^{-\sigma}}{s} G'(s)ds \le C\int_1^{r/2} G'(s)ds \le CG(\frac{r}{2}). \tag{2.62}$$

Or

$$\frac{1}{G(r)} I_2 \le C\frac{G(r/2)}{G(r)} \le C\exp[\frac{-\beta}{p-1}\int_{\frac{r}{2}}^r s|u'(s)|^{2-p}ds]. \tag{2.63}$$

Now recalling (2.60), we get

$$\frac{1}{G(r)} I_2 \le C\exp(-K_1 r^2), \tag{2.64}$$

with $K_1 = \frac{3\beta}{8(p-1)}K$. But as the solution u is decreasing, then

$$\frac{1}{G(r)} I_3 = \frac{1}{G(r)}\int_{r/2}^r \frac{u(s)}{s} G'(s)ds \le \frac{2}{r} u(r/2).$$

Using again the estimate (2.49), we obtain

$$\frac{1}{G(r)} I_3 \le C(r+1)^{-\sigma-1} \quad \text{for } r > 2. \tag{2.65}$$

Finally, putting together (2.57), (2.64) and (2.65) the desired estimate (2.53) follows. This completes the proof of the lemma. $\square$

Lemma 2.11. *Assume $\alpha > 0$, $\beta > 0$ and $\gamma \le 0$. Let u be a strictly positive solution of (1.4). Then*

$$u(r) \le C r^{-\frac{\alpha}{\beta}}, \quad \text{for large } r. \tag{2.66}$$

Proof. Using the first equation in (1.4), the function $u(r)$ satisfies

$$\begin{aligned}\frac{\alpha}{2}\frac{u^2(r)}{r} =& \frac{|u'|^p}{2r} - \frac{\beta}{2}uu' - \frac{1}{2r^2}u|u'|^{p-2}u' \\ &+ \frac{\gamma}{2r}\ u^{q+1}|u'|^{p-2}u'(r) - \frac{1}{2}[\frac{u|u'|^{p-2}u'}{r}]'.\end{aligned} \tag{2.67}$$

Recalling the expression of the energy function given by (2.25) we deduce

$$\begin{aligned}\frac{E(r)}{r} =& \frac{3p-2}{2p}\frac{|u'|^p}{r} - \frac{\beta}{4}(u^2)' - \frac{1}{2r^2}u|u'|^{p-2}u' \\ &- \frac{1}{2}[\frac{u|u'|^{p-2}u'}{r}]' + \frac{\gamma}{2r}\ u^{q+1}|u'|^{p-2}u'(r).\end{aligned} \tag{2.68}$$

Integrating the above inequality on the interval (r, R) we obtain

$$\begin{aligned}\int_r^R \frac{E(s)}{s}ds =& \frac{3p-2}{2p}\int_r^R \frac{|u'(s)|^p}{s}ds + \frac{u(r)|u'|^{p-2}u'(r)}{2r} \\ &- \frac{u(R)|u'|^{p-2}u'(R)}{2R} + \frac{\beta}{4}u^2(r) - \frac{\beta}{4}u^2(R) \\ &- \frac{1}{2}\int_r^R \frac{u(s)|u'(s)|^{p-2}u'(s)}{s^2}ds + \frac{\gamma}{2}\int_r^R \frac{u^{q+1}(s)|u'(s)|^{p-2}u'(s)}{s}ds.\end{aligned}$$

Since u' is negative, $\beta \ge 0$ and $\gamma \le 0$ we get

$$\begin{aligned}\int_r^R \frac{E(s)}{s}\,ds \le& \frac{3p-2}{2p}\int_r^R \frac{|u'(s)|^p}{s}ds + \frac{u(R)|u'(R)|^{p-1}}{2R} + \frac{\beta}{4}u^2(r) \\ &+ \frac{1}{2}\int_r^R \frac{u(s)|u'(s)|^{p-1}}{s^2}ds + \frac{|\gamma|}{2}\int_r^R \frac{u^{q+1}(s)|u'(s)|^{p-1}}{s}ds.\end{aligned} \tag{2.69}$$

Since E is strictly decreasing and converges to zero when r approaches to infinity, we deduce that $E' \in L^1(]0,\infty[)$. In particular $r|u'|^2$ and $u^q|u'|^p$ lie in $L^1(]0,\infty[)$. Letting $R \to \infty$,

$$\begin{aligned}\int_r^\infty \frac{E(s)}{s}ds \le& \frac{\beta}{4}u^2(r) + \frac{3p-2}{2p}\int_r^\infty \frac{|u'(s)|^p}{s}\,ds \\ &+ \frac{1}{2}\int_r^\infty \frac{u(s)|u'(s)|^{p-1}}{s^2}ds + \frac{|\gamma|}{2}\int_r^\infty \frac{u^{q+1}(s)|u'(s)|^{p-1}}{s}\,ds.\end{aligned} \tag{2.70}$$

Now, we set

$$H(r) = \int_r^\infty \frac{E(s)}{s}ds. \tag{2.71}$$

First, using the fact that $u^2(r) \le \frac{2}{\alpha}E(r)$, we obtain

$$H(r) \ge \int_r^{2r} \frac{E(s)}{s}ds \ge \frac{E(2r)}{2} \ge \frac{\alpha}{4}u^2(2r). \tag{2.72}$$

On the other hand, inequality (2.70) gives

$$\begin{aligned} H(r)+\frac{\beta}{2\alpha}rH'(r) \leq &\frac{3p-2}{2p}\int_r^\infty \frac{|u'(s)|^p}{s}ds+\frac{|\gamma|}{2}\int_r^\infty \frac{u^{q+1}(s)|u'(s)|^{p-1}}{s}\,ds \\ &+\frac{1}{2}\int_r^\infty \frac{u(s)}{s^2}|u'(s)|^{p-1}ds. \end{aligned} \tag{2.73}$$

Assume that the function u satisfies

$$u(r) \leq Cr^{-\sigma} \quad \text{for } r \geq 1, \tag{2.74}$$

for some fixed $\sigma \geq 0$ and some constant C (this is possible because $u(r) \leq A$ for all $r \geq 0$). If $\sigma \geq (2\alpha)\beta$ we have obviously (2.66). Assume now that $\sigma < (2\alpha)\beta$. Lemma 2.10 implies $|u'(r)| \leq Cr^{-\sigma-1}$ for any $r \geq 1$. Consequently

$$[r^{2\alpha/\beta}H(r)]' \leq Cr^{2\alpha/\beta-1-p(\sigma+1)}[1+r^{1-q\sigma}] \quad \text{for } r \geq 1. \tag{2.75}$$

By a simple integration, we obtain

$$H(r) \leq Cr^{-2\alpha/\beta}+Cr^{-p(1+\sigma)}+Cr^{1-p-(p+q)\sigma} \tag{2.76}$$

when $[p(1+\sigma)-2\alpha/\beta][2\alpha/\beta-p+1-(p+q)\sigma] \neq 0$. Otherwise if $[p(1+\sigma)-2\alpha/\beta][2\alpha/\beta-p+1-(p+q)\sigma]=0$, we have

$$H(r) \leq Cr^{-2\alpha/\beta}+Cr^{-2\alpha/\beta}\ln r+Cr^{-(1-q\sigma+2\alpha/\beta)}. \tag{2.77}$$

Combining (2.72), (2.76), (2.77) and using the fact that $\sigma < (2\alpha)\beta$, we deduce that there exists $m > \sigma$ such that

$$u(r) \leq Cr^{-m} \quad \text{for all } r \geq 1. \tag{2.78}$$

If $m = \alpha/\beta$ we have exactly the estimate (2.66). Otherwise if $m \neq \alpha/\beta$, the desired estimate (2.66) follows by induction starting with $\sigma = m$. This completes the proof. □

Proof of Proposition 2.9. Set

$$I(r) = r^{\alpha/\beta}\big[u+\frac{1}{\beta r}|u'|^{p-2}u'\big]\,. \tag{2.79}$$

Then we have

$$I'(r) = \frac{1}{\beta}r^{\alpha/\beta-1}\big[(\frac{\alpha}{\beta}-1)\frac{|u'|^{p-2}u'(r)}{r}+\gamma u^q|u'|^{p-2}u'\big]. \tag{2.80}$$

In view of Lemma 2.10 and Lemma 2.11, the functions $r \mapsto r^{\alpha/\beta-1}u^q|u'(r)|^{p-1}$ and $r \mapsto r^{\alpha/\beta-2}|u'(r)|^{p-1}$ are in $L^1(]0,\infty[)$. Consequently, $I'(r) \in L^1(]0,\infty[)$. Moreover (2.54) implies I(0)=0, and therefore

$$\lim_{r\to+\infty} I(r) = \int_0^\infty I'(s)ds$$

exists. Since $\lim_{r\to+\infty} r^{\alpha/\beta-1}|u'|^{p-2}u' = 0$, we deduce that

$$\lim_{r\to+\infty} r^{\alpha/\beta}u(r) = L \in [0,\infty[.$$

This completes the proof. □

Proposition 2.12. *Assume $\alpha > 0$, $\beta > 0$, $\gamma < 0$ and $L = 0$ in Proposition 2.9. Then $r^m u(r) \to 0$ and $r^m u'(r) \to 0$ as $r \to +\infty$ for all positive integers m.*

Proof. ¿From the proof of the previous proposition, $\lim_{r\to+\infty} I(r) = 0$. Thus, $I(r) = -\int_r^\infty I'(s)ds$. Therefore ,(2.79) gives

$$u(r) = \frac{-1}{\beta r}|u'|^{p-2}u'(r) - r^{-\alpha/\beta}\int_r^\infty I'(s)ds. \tag{2.81}$$

Since $\gamma < 0$ and $u' < 0$, we deduce from (2.84) that

$$u(r) \le \frac{1}{\beta r}|u'|^{p-1} + \frac{1}{\beta}|\frac{\alpha}{\beta} - 1|r^{-\alpha/\beta}\int_r^\infty s^{\alpha/\beta-2}|u'|^{p-1}ds. \tag{2.82}$$

Then in view of Lemma 2.10, we get

$$u(r) \le Cr^{-(p+(p-1)\alpha)/\beta}. \tag{2.83}$$

Define the sequence $(m_k)_{k\in\mathbb{N}}$ by

$$\begin{cases} m_0 = \alpha/\beta, \\ m_k = p + (p-1)m_{k-1}. \end{cases} \tag{2.84}$$

Then $\lim_{k\to+\infty} m_k = +\infty$. Consequently, the proposition follows by induction starting with $m_0 = \alpha/\beta$. This completes the proof. ☐

Proposition 2.13. *Assume $\alpha \le \beta$ and $\gamma < 0$. Let u be a strictly positive solution of* (1.4)*. Then $\lim_{r\to+\infty} r^{\alpha/\beta}u(r) > 0$.*

Proof. By Proposition 2.9,

$$\lim_{r\to+\infty} r^{\alpha/\beta}u(r) \in [0,\infty[.$$

Suppose that $\lim_{r\to+\infty} r^{\alpha/\beta}u(r) = 0$. Then Proposition 2.12 implies

$$\lim_{r\to+\infty} r^{\alpha/\beta-1}|u'|^{p-2}u' = 0,$$

and therefore, $\lim_{r\to+\infty} I(r) = 0$. On the other hand, (2.80) implies that the function I given by (2.79) is strictly increasing; this is a contradiction which completes the proof. ☐

3. Classification of solutions

In this section we give a classification of solutions of (1.4). For this purpose we Set

$$P = \{A > 0 : u(r,A) > 0, \forall r > 0\},$$
$$N = \{A > 0 : \exists\, r_0 > 0; u(r,A) > 0 \text{for } r \in [0,r_0[, u(r_0,A) = 0 \text{ and } u'(r_0,A) < 0\},$$
$$C = \{A > 0; \exists\, r_0 > 0; u(r_0,A) = u'(r_0,A) = 0\}.$$

This classification depends strongly on the sign ofγ and $\alpha - \beta$. First, we start with the following result.

Proposition 3.1. *Assume $\gamma > 0$ and $\alpha \ge \beta$. Then any solution of* (1.4) *changes sign.*

Proof. Let u be a solution of (1.4). Then

$$|u'|^{p-2}u'(r) = -\beta ru(r) - (\alpha-\beta)\int_0^r u(s)ds + \gamma\int_0^r u^q(s)|u'|^{p-2}u'(s)ds. \tag{3.1}$$

for any $r \in [0, R_{\max}$. If the set C is not empty, there exists a finite $r_0 > 0$ such that $u(r_0) = u'(r_0) = 0$ and $u(r) > 0$ on $]0, r_0[$. Taking $r = r_0$ in (3.1) and using again Lemma 2.4, we get

$$(\alpha - \beta) \int_0^{r_0} u(s)ds + \gamma \int_0^{r_0} u^q |u'|^{p-1} ds = 0. \tag{3.2}$$

This contradicts $\gamma > 0$ and $\alpha - \beta \geq 0$. Hence $C = \emptyset$.

If the set $P$is not empty, without loss of generality we can assume that equation (3.1) holds with u strictly positive and u' negative. Then since $\alpha - \beta \geq 0$ and $\gamma > 0$ we deduce

$$|u'|^{p-2} u'(r) \leq -\beta r u(r). \tag{3.3}$$

By integrating this last inequality we get a contradiction. □

Proposition 3.2. *Assume $\gamma > 0$ and $\alpha < \beta$. Then*

(i) *$u(., A)$ is strictly positive for any $A \in]0, A_0[$ with*

$$A_0 = [\frac{q-1}{\beta} (\frac{\beta - \alpha}{\gamma})^{p/(p-1)}]^{(p-1)/(p(q+1)-2)}$$

(ii) *$u(., A)$ changes sign for large A.*

For the proof we need some preliminary results. Let u be a solution of problem (1.4) defined in $[0, R_{\max}[$. Set

$$h(r) = (\beta - \alpha) u(r) + \gamma |u|^{q-1} u |u'|^{p-2} u'(r) \quad \forall r \in [0, R_{\max}[. \tag{3.4}$$

Then the following result holds.

Lemma 3.3. *Assume $\alpha < \beta$. Let u be a solution of* (1.4). *Then u cannot vanish before the first zero of h.*

Proof. On the contrary, suppose that u vanishes before$h$and let r_0 be the first zero of u. As $h(0) = (\beta - \alpha) u(0) > 0$, then $h(r_0) \geq 0$ and $h(r) > 0$ for $r \in [0, r_0[$. Integrating (1.4), we obtain

$$|u'|^{p-2} u'(r_0) = \int_0^{r_0} h(s) ds > 0. \tag{3.5}$$

This contradicts$u'(r_0) \leq 0$ and then, the Lemma is proved. □

Now, assume that there exists some initial data $A > 0$ such that $u(., A) (= u)$ is a strictly positive solution of (1.4) and set

$$g(r) = \beta r - \gamma u^q |u'|^{p-2}(r), \quad r > 0. \tag{3.6}$$

Lemma 3.4. *There exists $\rho (= \rho(A)) > 0$ such that*

$$g(\rho) = 0 \quad \textit{and} \quad g(r) < 0 \textit{ for } r \textit{ in }]0, \rho[. \tag{3.7}$$

Furthermore,

$$\lim_{A \to +\infty} u(\rho, A) = 0 \quad \textit{and} \quad \lim_{A \to +\infty} u'(\rho, A) = -\infty. \tag{3.8}$$

Proof. First, we observe that the function g satisfies

$$(|u'|^{p-2} u'(r))' = -\alpha u(r) - u'(r) g(r) \quad \text{for all } r > 0. \tag{3.9}$$

The proof is divided in 3 steps.

Step 1. $g(\rho) = 0$ and $g(r) < 0$ for $r \in [0, \rho[$. Recalling (2.24) we get

$$|u'|^{p-2}u'(r) \sim -\alpha A r, \quad \text{as } r \to 0. \tag{3.10}$$

Hence,

$$g(r) \sim -\gamma A^q (\alpha A r)^{(p-2)/(p-1)}, \quad \text{as } r \to 0. \tag{3.11}$$

and therefore g starts with a negative value. If g has a constant sign for all $r > 0$, equation (3.9) gives

$$(|u'|^{p-2}u')'(r) < -\alpha u(r) < 0, \quad \text{for } r > 0.$$

and then the solution $u(., A)$ must change sign; this is a contradiction and then (3.7) follows.

Step 2. We claim that $\lim_{A\to+\infty} u'(\rho, A) = -\infty$. In fact, equation (3.9) implies that the solution u satisfies

$$\big[\frac{p-1}{p}|u'|^p + \frac{\alpha}{2}u^2\ \big]'(r) = -(u')^2 g(r), \quad \text{for } r > 0. \tag{3.12}$$

Integrating this last equality on $[0, R] \subset [0, \rho[$ and using the fact that g is negative on $[0, \rho[$, we get

$$|u'|^p(r) \geq \frac{p\alpha}{2(p-1)}[A^2 - u^2(r)], \quad \forall r \in [0, \rho[. \tag{3.13}$$

Hence, if $u(\rho, A)$ is bounded, by letting A approach ∞, we deduce

$$\lim_{A\to+\infty} u'(\rho(A), A) = -\infty.$$

Otherwise if $u(\rho, A)$ is not bounded, then there exists a subsequence, denoted also $\rho(A)$ such that

$$\lim_{A\to+\infty} u(\rho(A), A) = +\infty.$$

Now, recalling (3.7) and (3.9) we get

$$(|u'|^{p-2}u'(r))' \leq -\alpha u(r) < 0, \quad \text{for any } r \in [0, \rho[. \tag{3.14}$$

In particularly, we deduce that u is concave in $[0, \rho[$ and therefore

$$u(r) \geq A + \frac{u(\rho) - A}{\rho} r, \quad \text{for any } r \in [0, \rho[. \tag{3.15}$$

Thus integrating (3.14) on $(0, \rho)$, we obtain

$$|u'|^{p-2}u'(\rho) \leq -\alpha \int_0^\rho \big(A + \frac{u(\rho) - A}{\rho} r\big) dr. \tag{3.16}$$

Hence,

$$|u'|^{p-2}u'(\rho) \leq -\frac{\alpha}{2}\rho[A + u(\rho)]. \tag{3.17}$$

But $g(\rho) = 0$, then $u^q|u'|^{p-2}(\rho) = \frac{\beta}{\gamma}\rho$. Inserting this last equality in (3.17) the following estimate holds

$$|u'(\rho)| \geq \frac{\gamma\alpha}{2\beta} u^q(\rho)[A + u(\rho)]. \tag{3.18}$$

Consequently, $\lim_{A\to+\infty} u'(\rho, A) = -\infty$.

Step3. We assert that $\lim_{A\to+\infty} u(\rho, A) = 0$. In fact, integrating (3.14) on an interval $]0, r[\subset]0, \rho[$ and using (3.15) we obtain

$$|u'|^{p-2}u'(r) \leq -\alpha r[A + \frac{u(\rho) - A}{2\rho} r] \text{ for any } 0 < r < \rho. \tag{3.19}$$

On the other hand (3.14) implies that u' is decreasing in $]0,\rho[$, so (3.19) gives

$$|u'(\rho)|^{p-2}u'(r) \leq -\alpha r[A + \frac{u(\rho)-A}{2\rho}r] \quad \text{for any } 0<r<\rho. \tag{3.20}$$

Integrating this last inequality on $]0,\rho[$ we get

$$A[|u'(\rho)|^{p-2} - \frac{\alpha}{3}\rho^2] \geq u(\rho)[\frac{\alpha}{6}\rho^2 + |u'(\rho)|^{p-2}]. \tag{3.21}$$

Therefore,

$$|u'(\rho)|^{p-2} - \frac{\alpha}{3}\rho^2 \geq 0. \tag{3.22}$$

Recalling that $g(\rho)=0$, this means

$$\rho = \frac{\gamma}{\beta}u^q(\rho)|u'(\rho)|^{p-2}. \tag{3.23}$$

Combining (3.22) and (3.23) we obtain

$$u(\rho) \leq [\frac{3}{\alpha}(\frac{\beta}{\gamma})^2]^{1/2q}|u'(\rho)|^{(2-p)/2q}, \tag{3.24}$$

Since $\lim_{A\to+\infty} u'(\rho,A) = -\infty$, we deduce that $\lim_{A\to+\infty} u(\rho,A)=0$. The proof is complete. $\square$

Lemma 3.5. *Let $u(.,A)$ be a strictly positive solution and let $R(A)$ given by Lemma 2.7. Then,*

$$\lim_{A\to+\infty} u(R(A),A) = 0 \quad \textit{and} \quad \lim_{A\to+\infty} u'(R(A),A) = -\infty. \tag{3.25}$$

Proof. First, note that Lemma 2.7 implies that $u''<0$ on $[0,R(A)[$ and then from step1 of the proof of Lemma 3.4 we get

$$\rho(A) \leq R(A) \quad \text{and} \quad u'(R(A),A) \leq u'(\rho(A),A). \tag{3.26}$$

On the other hand, since the function $r \to u(r,A)$ is decreasing, we deduce

$$u(R(A),A) \leq u(\rho(A),A). \tag{3.27}$$

Letting $A\to\infty$ in there two inequalities, (3.25) holds. $\square$

Proof of Proposition 3.2. The proof is divided in two steps.
Step 1. the proof of part (i). Set

$$R_0 = \sup\{r>0; h(s)>0 \text{ on } [0,r[\}. \tag{3.28}$$

Since $h(0)=(\beta-\alpha)u(0)>0$, the set $\{r>0 : h(r)>0 \text{ on } [0,r[\}$ is not empty.

We claim that R_0 is infinite. To the contrary, assume that R_0 is a real number. Then $h(R_0)=0$ and $h'(R_0)\leq 0$, so from Lemma 3.3, $u(R_0)>0$. Moreover, by continuity, $u(r)\neq 0$ for r $\in]R_0-\varepsilon, R_0+\varepsilon[$ (with some $\varepsilon>0$). Thus, we can write $h(r)$ in the form

$$h(r) = u^q(r)\widetilde{h}(r), \tag{3.29}$$

for any $r\in]R_0-\varepsilon, R_0+\varepsilon[$, with

$$\widetilde{h}(r) = (\beta-\alpha)u^{1-q}(r) + \gamma|u'|^{p-2}u'(r). \tag{3.30}$$

We clearly have

$$\begin{aligned}\widetilde{h}'(R_0) =&u(R_0)[-\gamma\beta+\gamma\beta R_0(\frac{\beta-\alpha}{\gamma})^{1/(p-1)}u^{-(q-1)(p-1)-1}(R_0)\\&+(q-1)(\beta-\alpha)(\frac{\beta-\alpha}{\gamma})^{1/(p-1)}u^{(2-p(q+1))/(p-1)}(R_0)].\end{aligned}\tag{3.31}$$

Since u is decreasing, $\gamma>0$ and $\beta-\alpha\geq 0$,

$$\widetilde{h}'(R_0)>u(R_0)[-\gamma\beta+(q-1)(\beta-\alpha)(\frac{\beta-\alpha}{\gamma})^{1/(p-1)}A^{(2-p(q+1))/(p-1)}].\tag{3.32}$$

Consequently, for any A such that

$$A^{(p(q+1)-2)/(p-1)}<\frac{q-1}{\beta}(\frac{\beta-\alpha}{\gamma})^{p/(p-1)},\tag{3.33}$$

we obtain $\widetilde{h}'(R_0)>0$. This contradicts $\widetilde{h}(r)>0$ for r$\in[0,R_0[$ and $\widetilde{h}(R_0)=0$. Hence R_0 is infinite; meaning that the function h is strictly positive and therefore by Lemma 3.3, u is also strictly positive.

Step 2. The proof of part (ii). Assume for contradiction that u is positive for all A. Since $u'(R(A),A)<0$ and $u''(R(A),A)=0$; then by putting $r=R(A)$ in (1.4) we get

$$\alpha\frac{u(R(A))}{|u'(R(A)|^{p-2}}=u'(R(A))\big\{-\beta\frac{R(A)}{|u'(R(A)|^{p-2}}+\gamma u^q(R(A))\big\}.\tag{3.34}$$

Invoking Lemma 3.5, we deduce

$$\lim_{A\to+\infty}\frac{R(A)}{|u'(R(A)|^{p-2}}=0.\tag{3.35}$$

Integrating equation (1.4) on $]R(A),r[$, we obtain

$$\begin{aligned}&|u'(r)|^{p-2}u'(r)-|u'(R(A)|^{p-2}u'(R(A))+\beta ru(r)\\&-\beta R(A)u(R(A))+(\alpha-\beta)\int_{R(A)}^r u(s)ds-\gamma\int_{R(A)}^r u^q|u'|^{p-2}u'(s)ds=0.\end{aligned}\tag{3.36}$$

Since u' is negative and strictly increasing in $[R(A),\infty[$ we get

$$|u'(r)|^{p-2}u'(r)>|u'(R(A)|^{p-2}u'(r),\tag{3.37}$$

for any $r>R(A)$. Hence, equation (3.36) gives

$$\begin{aligned}|u'(R(A))|^{p-2}u'(r)<&|u'(R(A))|^{p-2}u'(R(A))-\beta ru(r)+\beta R(A)u(R(A))\\&+(\beta-\alpha)\int_{R(A)}^r u(s)ds.\end{aligned}\tag{3.38}$$

Now using the fact that u is decreasing and that $\beta>\alpha>0$, we obtain

$$u'(r)<u'(R(A))+\beta\frac{R(A)u(R(A))}{|u'(R(A))|^{p-2}}+(\beta-\alpha)\frac{u(R(A))}{|u'(R(A))|^{p-2}}(r-R(A)).\tag{3.39}$$

Integrating this last inequality on $]R(A),R(A)+1[$ we get

$$\begin{aligned}&u(R(A)+1)\\&<u(R(A))+u'(R(A))+\beta\frac{R(A)}{|u'(R(A))|^{p-2}}u(R(A))+\frac{\beta-\alpha}{2}\frac{u(R(A))}{|u'(R(A))|^{p-2}}.\end{aligned}\tag{3.40}$$

Putting together formula (3.35) and Lemma 3.5 we arrive at $\lim_{A\to+\infty}u(R(A)+1)=-\infty$ which is not possible. $\square$

The remaining of the paper is devoted to the case $\gamma < 0$.

Proposition 3.6. *Assume $\gamma < 0$ and $\alpha \leq \beta$. Then all solutions of* (1.4) *are strictly positive.*

Proof. Assume that there exists a real $r_0 > 0$such that $u(r_0) = 0$ and $u(r) > 0$ in $[0, r_0[$. Then from Lemma 2.4, $u'(r) < 0$ in $]0, r_0[$ and $u'(r_0) \leq 0$. Now integrating equation (1.4) on $]0, r_0[$ we get

$$|u'|^{p-2}u'(r_0) = (\beta - \alpha) \int_0^{r_0} u(s)ds + \gamma \int_0^{r_0} u^q(s)|u'|^{p-2}u'(s)ds. \tag{3.41}$$

This is a contradiction with $\gamma < 0$ *and* $\alpha \leq \beta$. □

In the case of $\gamma < 0$ and $\alpha > \beta > 0$ we will prove that the three sets P, N and C are not empty. More precisely we have the following statement.

Proposition 3.7. *Assume $\gamma < 0$ and $\beta < \alpha$. Then there exist two constants A_- and A_+ such that $u(., A)$ is strictly positive for any $A \geq A_+$ and $u(., A)$ changes sign for any $0 < A \leq A_-$; this implies $]A_+, +\infty[\subset P$ and $]0, A_-[\subset N$.*

Proof. The proof is divided in two steps.
Step 1. Let A,a large real positive. We scale the variables and set

$$u(r) = Av(x), \quad x = A^q r, \tag{3.42}$$

for any $r \in [0, R_{\max}[$. Then in terms of the new variables, problem (1.4) becomes

$$\begin{gathered} (|v'|^{p-2}v')' + \beta x A^{2-p(q+1)}v' + \alpha A^{2-p(q+1)}v - \gamma|v|^{q-1}v|v'|^{p-2}v' = 0 \\ v(0) = 1, \quad v'(0) = 0. \end{gathered} \tag{3.43}$$

Since $\gamma < 0$, the energy function E given by (2.25) is decreasing. In particular for any $r \in [0, R_{\max}[$,

$$0 < u(r) < A \quad \text{and} \quad |u'(r)|^p \leq \frac{p\alpha}{2(p-1)}A^2. \tag{3.44}$$

Therefore, v and v' are bounded for all $A \geq 1$. In fact for any $x \in [0, A^q R_{\max}[$,

$$0 < v(x) < 1 \quad \text{and} \quad |v'(x)| \leq [\frac{p\alpha}{2(p-1)}]^{1/p} A^{2/p-(1+q)}.$$

Let $A \to +\infty$, it follows from standard O.D.E. arguments that $v(x)$ converges to the solution of the problem

$$\begin{gathered} (|V'|^{p-2}V')' - \gamma|V|^{q-1}V|V'|^{p-2}V' = 0, \\ V(0) = 1, \quad V'(0) = 0. \end{gathered} \tag{3.45}$$

The first equation of this problem can be written as

$$(|V'|^{p-2}V' exp(-\gamma \int_0^r |V|^{q-1}V(s)ds))' = 0. \tag{3.46}$$

Hence $V \equiv 1$. Consequently, $v(., A)$ converges to 1 when A approaches $+\infty$; in particular $u(., A)$ is strictly positive.
Step 2. As for (i), we introduce new variables. We set

$$u(r) = Aw(x) \quad x = rA^{-(p-2)/p}. \tag{3.47}$$

Then w satisfies

$$\begin{gathered}(|w'|^{p-2}w')' + \beta r w' + \alpha w - \gamma A^{q-1+(2(p-1))/p}|w|^{q-1}w|w'|^{p-2}w' = 0,\\ w(0) = 1, \quad w'(0) = 0.\end{gathered} \tag{3.48}$$

By letting A approach 0, the function $w(x)$ converges to the solution of the problem

$$\begin{gathered}(|W'|^{p-2}W')' + \beta x W' + \alpha W = 0,\\ W(0) = 1, \quad W'(0) = 0.\end{gathered} \tag{3.49}$$

We claim that W changes sign. In fact we have

$$|W'|^{p-2}W'(x) + \beta x W(x) = (\beta - \alpha)\int_0^x W(s)ds. \tag{3.50}$$

If W is strictly positive for all x, by using $\beta - \alpha < 0$, we get

$$(W^{(p-2)/(p-1)})'(x) \le -\frac{p-2}{p-1}\beta^{\frac{1}{p-1}}x^{\frac{1}{p-1}}, \quad \forall x > 0; \tag{3.51}$$

This is a contradiction. As if there exists some $x_0 > 0$ such that $W(x_0) = W'(x_0) = 0$ and $W(x) > 0$ in $]0, x_0[$ we obtain

$$(\beta - \alpha)\int_0^{x_0} W(s)ds = 0,$$

this is also a contradiction because $\alpha \neq \beta$.Thereby W is non positive and consequently $u(., A)$ changes sign for small A. This completes the proof. □

We have also the following result.

Proposition 3.8. *Assume $\gamma < 0$ and $0 < \beta < \alpha$. Then N and P are non-empty open sets.*

Before to start the proof we introduce the function

$$\Gamma(r) = u(r) + |u'|^{p-2}u'(r). \tag{3.52}$$

Lemma 3.9. *Assume $\gamma < 0$, $\alpha > 0$, and $\beta > 0$. Let u be a strictly positive solution of* (1.4). *Then the function $\Gamma(r)$ is strictly positive for large r.*

Proof. Since $u(r) > 0$, Proposition 2.9 implies

$$\lim_{r\to+\infty} r^{\alpha/\beta}u(r) = L \in [0, +\infty[.$$

If $L > 0$, $u(r) \approx Lr^{-\alpha/\beta}$ for large r and then Lemma 2.10 implies

$$\lim_{r\to+\infty} r^{\alpha/\beta}|u'|^{p-2}u' = 0.$$

Thus, the function $\Gamma(r)$ behaves like $Lr^{-\alpha/\beta}$, as $r \to \infty$ and therefore, Γ is strictly positive.

For the case $L = 0$, the proof will be done into two steps.

Step 1. $\Gamma(r)$ is monotone for large r. For this purpose we set

$$J(r) = \beta r u'(r) + \alpha u(r). \tag{3.53}$$

We assert that $J(r)$has a constant sign for large r. In fact, assume that there exists a large r_0 such that $J(r_0) = 0$. According to equation (1.4), we obtain

$$(p-1)|u'(r_0)|^{p-2}J'(r_0) = -\beta(\frac{\alpha}{\beta})^{p-1}\frac{u^{p-1}(r_0)}{r_0^{p-1}}\left\{(p-1)(\alpha/\beta + 1) + \gamma r_0 u^q(r_0)\right\}. \tag{3.54}$$

Since $\lim_{r\to+\infty} r^{\alpha/\beta}u(r) = 0$, we deduce that $J'(r_0) < 0$. Consequently $J(r)$ has the same sign for large r. Now, note that for any $r > 0$,

$$\Gamma'(r) = u'(r) - \beta r u' - \alpha u(r) + \gamma u^q |u'|^{p-2}u'(r). \tag{3.55}$$

Hence $\Gamma'(r)$ and $J(r)$ have the opposite signs, in particular the function J is monotone for large r.

Step 2. We claim that Γ is not negative for large r. In fact if not,using the step1 we deduce that there exists a large R_1 such that $\Gamma(r) = u(r) + |u'|^{p-2}u' \le 0$ for any $r \ge R_1$. Integrating this last inequality on (R_1, r) we get

$$u^{(p-2)/(p-1)}(r) \le u^{(p-2)/(p-1)}(R_1) - (p-2)/(p-1)r + (p-2)/(p-1)R_1. \tag{3.56}$$

By letting $r \to +\infty$, we obtain a contradiction.

Combining step 1 and step 2 we deduce $\Gamma(r) > 0$ for large r. This completes the proof. □

Now we use step by step the idea introduced by Brezis et al [2], for studying a very singular solution of the heat equation with absorption. Ever since, this idea was used in many papers, see for example [12]. In order to do this, we write (1.4) as the system

$$\begin{gathered} u' = |v|^{-(p-2)/(p-1)}v, \\ v' = -\beta r|v|^{-(p-2)/(p-1)}v - \alpha u + \gamma |u|^{q-1}uv. \end{gathered} \tag{3.57}$$

For each $\lambda > 0$, we define the set

$$L_\lambda = \{(f_1, f_2) : 0 < f_1 < 1, -\lambda f_1 < f_2 < 0\} \tag{3.58}$$

Lemma 3.10. *For any $\lambda > 0$ there exists $r_\lambda = [\lambda + \alpha\lambda^{-1/(p-1)}]/\beta$; such that L_λ is positively invariant for $r \ge r_\lambda$. That is, if $(u_0, v_0) \in L_\lambda$ and $(u(r), v(r))$ is the solution of (3.57) which satisfies $(u(r_0), v(r_0)) = (u_0, v_0)$ for some $r_0 > r_\lambda$, then for any $r \ge r_\lambda$ the orbit $(u(r), v(r))$ lies in L_λ for all $r \ge r_0$.*

Proof. We shall show that, given $\lambda > 0$, there exists $r_\lambda > 0$ such that if $r > r_\lambda$, then the vector field determined by (3.57) points into L_λ, except at the critical point $(0, 0)$. On the top $(f_2 = 0)$,

$$v' = -\alpha u < 0 \quad \text{for all } r > 0.$$

While on the right side $(u = 1)$,

$$v' = -\beta r|v|^{-(p-2)(p-1)}v < 0 \quad \text{for all } r > 0.$$

On the line $f_2 = -\lambda f_1$ we must prove that $\frac{v'}{u'} < -\lambda$ for large r. This is true because

$$\frac{v'}{u'} = \frac{v'|v|^{(p-2)/(p-1)}}{v} = -\beta r + \alpha\lambda^{-1/(p-1)}u^{(p-2)/(p-1)} + \gamma\lambda^{(p-2)/(p-1)}u^{q+(p-2)/(p-1)}.$$

Since $\gamma < 0$, if $r \ge (\lambda + \alpha\lambda^{-1/(p-1)})/\beta = r_\lambda$ we obtain $\frac{v'}{u'} < -\lambda$. □

Proof of Proposition 3.8. . First, note that from Proposition 3.7, the sets P and N are not empty. On the other hand, the continuous dependence of solutions on the initial value implies that N is an open set. To prove that P is open, take $A_0 \in P$ and let a large $r_0 > 0$ be fixed. Then by continuous dependence of solutions on the initial data, there is a neighborhood O of A_0 such that $u(r, A) > 0$ for any $(r, A) \in [0, r_0] \times O$. In particular $u(r_0, A) > 0$ for any $A \in O$ and then from Lemma 3.9

$$\Gamma(r_0) = u(r_0, A) + |u'|^{p-2}u'(r_0, A) > 0. \tag{3.59}$$

Since r_0 is large then $u(r_0, A) < 1$ and consequently $(u(r_0, A), |u'|^{p-2}u'(r_0, A))$ is in L_1. Recalling Lemma 3.10, we deduce that the trajectory remains in L_1 for any $r \geq r_0$, which implies in particular that $u(r, A) > 0$ for any $r \geq r_0$ and $A \in O$. Therefore, $u(r, A) > 0$ for any $r \geq 0$ and there by P is open. The proof is complete. $\square$

The rest of the paper is devoted to the study of solution with compact support

Proposition 3.11. *Assume that $\gamma < 0$ and $\beta < \alpha$, then there exists at least one solution with compact support.*

Proof. As P and N are open disjoint sets, then there exists $A \in \mathbb{R}^+ - (P \cup N)$; that is, $u(., A)$ has a compact support. $\square$

We conclude this paper with a study of the behavior of solution with a compact support.

Lemma 3.12. *Assume $\gamma < 0$.Let $u$be a solution with compact support $[0, R]$. Then*

$$(u^{(p-2)/(p-1)})'(R) = -\frac{p-2}{p-1}\beta^{1/(p-1)}R^{1/(p-1)}. \tag{3.60}$$

Proof. Take r close to R. and integrate (1.4) between $r$and R; using the fact that $u$is decreasing, we get

$$|u'|^{p-2}u'(r) = \beta r u(r) - (\alpha - \beta)\int_r^R u(s)ds + \gamma \int_r^R u^q(s)|u'|^{p-2}u'(s)ds.$$

Dividing by $u(r)$, we have

$$\frac{|u'|^{p-1}(r)}{u(r)} = \beta r - \frac{\alpha - \beta}{u(r)}\int_r^R u(s)ds + \frac{\gamma}{u(r)}\int_r^R u^q(s)|u'|^{p-1}u'(s)ds.$$

First, note that

$$0 \leq \int_r^R u(s)ds \leq u(r)(R - r).$$

Hence

$$\lim_{r \to R} \frac{\alpha - \beta}{u(r)}\int_r^R u(s)ds = 0.$$

On the other hand, since the function $u'(s)$is negative and also $|u'|$ decreasing near R, then

$$\begin{aligned}
\frac{\gamma}{u(r)}\int_r^R u^q(s)|u'|^{p-2}u'(s)ds =& \frac{|\gamma|}{u(r)}\int_r^R u^q(s)|u'|^{p-1}(s)ds \\
\leq& \frac{-|\gamma|}{u(r)}|u'|^{p-2}(r)\int_r^R u^q(s)u'(s)ds \\
\leq& \frac{|\gamma|}{q+1}u^q(r)|u'(r)|^{p-2}.
\end{aligned}$$

The last term of this inequality approaches zero as $r \to R$ and then we get

$$\lim_{r \to R} \frac{|u'|^{p-1}(r)}{u(r)} = \beta R.$$

This is equivalent to (3.60); thus the proof of the lemma is complete. $\square$

Acknowledgments. The authors would like to express their deep gratitude to professor M. Kirane for his assistance, and also thank the anonymous referee for his/her helpful comments. This work was supported financially by Centre National de Coordination et de Planification de La Recherche Scientifique et Technique PARS MI 29.

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