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AMBROSETTI-PRODI PROBLEM WITH DEGENERATE POTENTIAL AND NEUMANN BOUNDARY CONDITION

DUŠAN D. REPOVŠ

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ABSTRACT. We study the degenerate elliptic equation

$$-\operatorname{div}(|x|^{\alpha}\nabla u) = f(u) + t\phi(x) + h(x)$$

in a bounded open set Ω with homogeneous Neumann boundary condition, where $\alpha \in (0,2)$ and f has a linear growth. The main result establishes the existence of real numbers t_* and t^* such that the problem has at least two solutions if $t \le t_*$, there is at least one solution if $t_* < t \le t^*$, and no solution exists for all $t > t^*$. The proof combines *a priori* estimates with topological degree arguments.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary. In their seminal paper [1], Ambrosetti and Prodi studied the semilinear elliptic problem

$$\begin{gathered} \Delta u + f(u) = v(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where the nonlinearity f is a function whose derivative crosses the first (principal) eigenvalue λ_1 of the Laplace operator in $H_0^1(\Omega)$, in the sense that

$$0 < \lim_{t\to-\infty} \frac{f(t)}{t} < \lambda_1 < \lim_{t\to+\infty} \frac{f(t)}{t} < \lambda_2.$$

By using the abstract approach developed in [1], Ambrosetti and Prodi have been able to describe the exact number of solutions of (1.1) in terms of v, provided that $f'' > 0$ in $\mathbb{R}$. More precisely, they proved that there exists a closed connected manifold $A_1 \subset C^{0,\alpha}(\overline{\Omega})$ of codimension 1 such that $C^{0,\alpha}(\overline{\Omega}) \setminus A_1 = A_0 \cup A_2$ and problem (1.1) has exactly zero, one or two solutions according as v is in A_0, A_1, or A_2. The proof of this pioneering result is based upon an extension of Caccioppoli's mapping theorem to some singular case.

A cartesian representation of A_1 is due to Berger and Podolak [6], who observed that it is convenient to write problem (1.1) in an equivalent way, as follows. Let

$$Lu := \Delta u + \lambda_1 u, \qquad g(u) := f(u) - \lambda_1 u$$

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and

$$v(x) := t\phi(x) + h(x) \quad \text{with} \quad \int_\Omega h(x)\phi(x)\,dx = 0.$$

In such a way, problem (1.2) is equivalent to

$$\begin{gathered} Lu + g(u) = t\phi(x) + h(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

with $g'' > 0$ in $\mathbb{R}$ and

$$-\lambda_1 < \lim_{t\to-\infty} \frac{g(t)}{t} < 0 < \lim_{t\to+\infty} \frac{g(t)}{t} < \lambda_2 - \lambda_1.$$

Under these assumptions, Berger and Podolak [6] proved that there exists t_1 such that problem (1.2) has exactly zero, one or two solutions according as $t < t_1$, $t = t_1$, or $t > t_1$. The proof of this result is based on a global Lyapunov-Schmidt reduction method.

For related developments on Ambrosetti-Prodi problems we refer to Amann and Hess [2], Arcoya and Ruiz [3], Dancer [11], Hess [17], Kazdan and Warner [18], Mawhin [20, 21].

The present paper is concerned with the Ambrosetti-Prodi problem in relationship with the contributions of Caldiroli and Musina [8], who initiated the study of Dirichlet elliptic problems driven by the differential operator $\operatorname{div}(|x|^\alpha \nabla u)$, where $\alpha \in (0, 2)$. This operator is a model for equations of the type

$$-\operatorname{div}(a(x)\nabla u) = f(x, u) \quad x \in \Omega, \tag{1.3}$$

where the weight a is a non-negative measurable function that is allowed to have "essential" zeros at some points or even to be unbounded. According to Dautray and Lions [12, p. 79], equations like (1.3) are introduced as models for several physical phenomena related to equilibrium of anisotropic continuous media which possibly are somewhere "perfect" insulators or "perfect" conductors. We also refer to the works by Murthy and Stampacchia [16], by Baouendi and Goulaouic [4] concerning degenerate elliptic operators (regularity of solutions and spectral theory). Problem (1.3) also has some interest in the framework of optimization and G-convergence, cf. Franchi, Serapioni, and Serra Cassano [16]. For degenerate phenomena in nonlinear PDEs we also refer to Fragnelli and Mugnai [15], and Nursultanov and Rozenblum [23].

This article studies of the Ambrosetti-Prodi problem in the framework of the degenerate elliptic operator studied in [8]. A feature of this work is that the analysis is developed in the framework of Neumann boundary conditions.

2. Main result and abstract setting

Let $\alpha \in (0, 2)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary. Consider the nonlinear problem

$$\begin{gathered} -\operatorname{div}(|x|^\alpha \nabla u) = f(u) + t\phi(x) + h(x) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

We assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\limsup_{t\to-\infty} \frac{f(t)}{t} < 0 < \liminf_{t\to+\infty} \frac{f(t)}{t} \tag{2.2}$$

and there exists $C_f > 0$ such that

$$|f(t)| \le C_f(1+|t|) \quad \text{for all } t \in \mathbb{R}. \tag{2.3}$$

Since the first eigenvalue of the Laplace operator with respect to the Neumann boundary condition is zero, condition (2.2) asserts that the nonlinear term f crosses this eigenvalue.

Next, we assume that ϕ, $h \in L^\infty(\Omega)$ and

$$\phi \ge 0, \quad \phi \not\equiv 0 \quad \text{in } \Omega. \tag{2.4}$$

Since $\alpha > 0$, the weight $|x|^\alpha$ breaks the invariance under translations and can give rise to an abundance of existence results, according to the geometry of the open set Ω.

For $\zeta \in C_c^\infty(\Omega)$ we define

$$\|\zeta\|_\alpha^2 := \int_\Omega (|x|^\alpha |\nabla\zeta|^2 + \zeta^2)\, dx$$

and we consider the function space

$$H^1(\Omega; |x|^\alpha) := \text{closure of } C_c^\infty(\overline{\Omega}) \text{ with respect to the } \|\cdot\|_\alpha\text{-norm.}$$

It follows that $H^1(\Omega; |x|^\alpha)$ is a Hilbert space with respect to the scalar product

$$\langle u, v\rangle_\alpha := \int_\Omega (|x|^\alpha \nabla u \cdot \nabla v + uv)\, dx, \quad \text{for all } u.v \in H^1(\Omega; |x|^\alpha).$$

Moreover, by the Caffarelli-Kohn-Nirenberg inequality (see [8, Lemma 1.2]), the space $H^1(\Omega; |x|^\alpha)$ is continuously embedded in $L^{2_\alpha^*}(\Omega)$, where 2_α^* denotes the corresponding critical Sobolev exponent, that is, $2_\alpha^* = 2N/(N-2+\alpha)$.

We say that u is a solution of problem (2.1) if $u \in H^1(\Omega; |x|^\alpha)$ and for all $v \in H^1(\Omega; |x|^\alpha)$

$$\int_\Omega |x|^\alpha \nabla u \cdot \nabla v\, dx = \int_\Omega f(u)v\, dx + t\int_\Omega \phi v\, dx + \int_\Omega hv\, dx.$$

Since the operator $Lu := -\operatorname{div}(|x|^\alpha \nabla u)$ is uniformly elliptic on any strict subdomain ω of Ω with $0 \notin \overline{\omega}$, the standard regularity theory can be applied in ω. Hence, a solution $u \in H^1(\Omega; |x|^\alpha)$ of problem (2.1) is of class C^∞ on $\Omega \setminus \{0\}$. We refer to Brezis [7, Theorem IX.26] for more details.

The main result of this paper extends to the degenerate setting formulated in problem (2.1) the abstract approach developed by Hess [17] and de Paiva and Montenegro [14]. For related properties on Ambrosetti-Prodi problems with Neumann boundary condition, we refer to Presoto and de Paiva [24], Sovrano [25], Vélez-Santiago [26, 27].

Theorem 2.1. *Assume that hypotheses* (2.2), (2.3) *and* (2.4) *are fulfilled. Then there exist real numbers t_* and t^* with $t_* \le t^*$ such that the following properties hold:*

(a) *problem* (2.1) *has at least two solutions solution, provided that $t \le t_*$;*
(b) *problem* (2.1) *has at least one solution, provided that $t_* < t \le t^*$;*
(c) *problem* (2.1) *has no solution, provided that $t > t^*$.*

Strategy of the proof. Let C_f be the positive constant defined in hypothesis (2.3) and assume that $v \in L^2(\Omega)$. Consider the linear Neumann problem

$$\begin{gathered} -\operatorname{div}(|x|^\alpha \nabla w) + C_f w = v \quad \text{in } \Omega \\ \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.5}$$

With the same arguments as in [7, Chapter IX, Exemple 4], problem (2.5) has a unique solution $w \in H^1(\Omega; |x|^\alpha)$. This defines a linear map

$$L^2(\Omega) \ni v \mapsto w \in H^1(\Omega; |x|^\alpha).$$

It follows that the linear operator $T : L^\infty(\Omega) \to H^1(\Omega; |x|^\alpha)$ defined by $Tv := w$ is compact. We also point out that if $v \geq 0$ then $w \geq 0$, hence T is a positive operator.

We observe that u is a solution of problem (2.1) if and only if u is a fixed point of the nonlinear operator

$$S_t(v) := T(f(v) + C_f v + t\phi + h).$$

Thus, solving problem (2.1) reduces to finding the critical points of S_t.

3. Proof of the main result

We split the proof into several steps.

3.1. **Non-existence of solutions if t is big.** In fact, we show that a necessary condition for the existence of solutions of problem (2.1) is that the parameter t should be small enough.

We first observe that hypothesis (2.2) implies that there are positive constants C_1 and C_2 such that

$$f(t) \geq C_1|t| - C_2 \quad \text{for all } t \in \mathbb{R}.$$

Assuming that u is a solution of problem (2.1), we obtain by integration

$$\begin{aligned} 0 &= \int_\Omega f(u)\,dx + t\int_\Omega \phi\,dx + \int_\Omega h\,dx \\ &\geq C_1 \int_\Omega |u|\,dx - C_2|\Omega| + t\int_\Omega \phi\,dx + \int_\Omega h\,dx \\ &\geq -C_2|\Omega| + t\int_\Omega \phi\,dx + \int_\Omega h\,dx. \end{aligned}$$

It follows that a necessary condition for the existence of solutions of problem (2.1) is

$$t \leq \frac{C_2|\Omega| - \int_\Omega h\,dx}{\int_\Omega \phi\,dx}\,.$$

3.2. **Problem (2.1) has solutions for small t: a preliminary step.** In this subsection, we prove that for any $\rho > 0$ there exists $t_\rho \in \mathbb{R}$ such that for all $t \leq t_\rho$ and all $s \in [0,1]$ we have

$$v \neq sS_t(v) \quad \text{for all } v \in L^\infty(\Omega),\ \|v^+\|_\infty = \rho. \tag{3.1}$$

Our argument is by contradiction. Thus, there exist three sequences $(s_n) \subset [0,1]$, $(t_n) \subset \mathbb{R}$ and $(v_n) \subset L^\infty(\Omega)$ such that $\lim_{n\to\infty} t_n = -\infty$, $\|v_n^+\|_\infty = \rho$ and

$$v_n = s_n S_{t_n}(v_n) \quad \text{for all } n \geq 1. \tag{3.2}$$

By hypothesis (2.3) we have

$$\begin{aligned} f(v_n) + C_f v_n &\le C_f + C_f|v_n| + C_f v_n \\ &= C_f + 2C_f v_n^+ \le C_f + 2C_f\rho. \end{aligned} \tag{3.3}$$

Using the definition of S and the fact that T is a positive operator, relations (3.2) and (3.3) yield

$$\begin{aligned} v_n = s_n S_{t_n}(v_n) &= s_n T(f(v_n) + C_f v_n + t_n\phi + h) \\ &\le s_n T(C_f + 2C_f\rho + t_n\phi + h), \end{aligned}$$

hence

$$v_n^+ \le s_n[T(C_f + 2C_f\rho + t_n\phi + h)]^+.$$

Let

$$w_n := C_f + 2C_f\rho + t_n\phi + h.$$

It follows that w_n is the unique solution of the problem

$$\begin{gathered} -\operatorname{div}(|x|^\alpha w_n) + C_f w_n = C_f + 2C_f\rho + t_n\phi + h \quad \text{in } \Omega \\ \frac{\partial w_n}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Dividing by t_n (recall that $\lim_{n\to\infty} t_n = -\infty$) we obtain

$$\begin{gathered} -\operatorname{div}\Big(|x|^\alpha \frac{w_n}{t_n}\Big) + C_f\,\frac{w_n}{t_n} = \phi + \frac{C_f + 2C_f\rho + h}{t_n} \quad \text{in } \Omega \\ \frac{\partial}{\partial\nu}\Big(\frac{w_n}{t_n}\Big) = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

However,

$$\lim_{n\to\infty} \frac{C_f + 2C_f\rho + h}{t_n} = 0.$$

So, by elliptic regularity (see [7, Theorem IX.26]),

$$\frac{w_n}{t_n} \to T\phi \quad \text{in } C^{1,\beta}(\overline{\Omega}\setminus\{0\}) \text{ as } n\to\infty.$$

Next, by the strong maximum principle, we have $T\phi > 0$ in Ω and

$$\frac{\partial T\phi}{\partial\nu}(x) < 0 \quad \text{for all } x \in \partial\Omega \text{ with } T\phi(x) = 0.$$

We deduce that for all n sufficiently large

$$\frac{w_n}{t_n} > 0 \quad \text{in } \Omega,$$

which forces $w_n^+ = 0$ for all n large enough. But

$$v_n^+ \le s_n w_n^+ \le w_n^+,$$

hence

$$\rho = \|v_n^+\|_\infty \le \|w_n^+\|_\infty = 0,$$

a contradiction. This shows that our claim (3.1) is true.

3.3. **Problem (2.1) has solutions for small t: an intermediary step.** In this subsection, we prove that for any $t \in \mathbb{R}$ there exists $\rho_t > 0$ such that for all $s \in [0,1]$ we have

$$v \neq sS_t(v) \quad \text{for all } v \in L^\infty(\Omega),\ \|v^-\|_\infty = \rho_t. \tag{3.4}$$

Fix arbitrarily $t \in \mathbb{R}$. Assume that there exist $s \in [0,1]$ and a function v (depending on s) such that $v = sS_t(v)$. It follows that v is the unique solution of the problem

$$\begin{gathered} -\operatorname{div}(|x|^\alpha \nabla v) + C_f v = s(f(v) + C_f v + t\phi + h) \quad \text{in } \Omega \\ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.5}$$

By hypotheses (2.2) and (2.3), there exist positive constants C_3 and C_4 with $C_3 < C_f$ such that

$$f(t) \geq -C_3 t - C_4 \quad \text{for all } t \in \mathbb{R}. \tag{3.6}$$

Returning to (3.5) we deduce that

$$\begin{aligned} -\operatorname{div}(|x|^\alpha \nabla v) + C_f v &\geq s(-C_3 v - C_4 + C_f v + t\phi + h) \\ &= s[(C_f - C_3)v + t\phi + h - C_4]. \end{aligned}$$

Therefore,

$$\begin{gathered} -\operatorname{div}(|x|^\alpha \nabla v) + [sC_3 + (1-s)C_f]v \geq s(t\phi + h - C_4) \quad \text{in } \Omega \\ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.7}$$

where

$$0 < C_3 \leq sC_3 + (1-s)C_f \leq C_f.$$

Let w denote the unique solution of the Neumann problem

$$\begin{gathered} -\operatorname{div}(|x|^\alpha \nabla w) + [sC_3 + (1-s)C_f]w = s(t\phi + h - C_4) \quad \text{in } \Omega \\ \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.8}$$

By (3.7), (3.8) and the maximum principle, we deduce that

$$w \leq v \quad \text{in } \Omega. \tag{3.9}$$

Moreover, since $C_3 \leq C_3 \leq sC_3 + (1-s)C_f \leq C_f$ for all $s \in [0,1]$, we deduce that the solutions $w = w(s)$ of problem (3.8) are uniformly bounded. Thus, there exists $C_0 = C_0(t) > 0$ such that

$$\|w\|_\infty \leq C_0 \quad \text{for all } s \in [0,1]. \tag{3.10}$$

Next, relation (3.9) yields

$$v^- = \max\{-v, 0\} \leq \max\{-w, 0\} = w^- \quad \text{in } \Omega.$$

Using now the uniform bound established in (3.10), we conclude that our claim (3.4) follows if we choose $\rho_t = C_0 + 1$.

3.4. **Problem (2.1) has a solution for small t.** Let $\rho > 0$ and let t_ρ be as defined in subsection 3.2 such that relation (3.1) holds. We prove that problem (2.1) has at least one solution, provided that $t \le t_\rho$.

Fix $t \le t_\rho$ and let ρ_t be the positive number defined in subsection 3.3. Consider the open set

$$G = G_t := \{v \in L^\infty(\Omega) : \|v^+\|_\infty < \rho,\ \|v^-\|_\infty < \rho_t\}.$$

It follows that

$$v \ne sS_t(v) \quad \text{for all } v \in \partial G, \text{ all } s \in [0,1].$$

So, we can apply the homotopy invariance property of the topological degree, see Denkowski, Migórski and Papageorgiou [13, Theorem 2.2.12]. It follows that

$$\deg(I - S_t, G, 0) = \deg(I, G, 0) = 1.$$

We conclude that S_t has at least one fixed point for all $t \le t_\rho$, hence problem (2.1) has at least one solution.

3.5. **Proof of Theorem 2.1 concluded.** We first show that problem (2.1) has a subsolution for all t. Fix a positive real number t. By (3.6), we have

$$f(u) + t\phi + h \ge -C_3 u - C_4 - |t|\,\|\phi\|_\infty - \|h\|_\infty \quad \text{for all } u \in \mathbb{R}.$$

It follows that the function

$$u \equiv -\frac{|t|\,\|\phi\|_\infty + \|h\|_\infty + C_4}{C_3}$$

is a subsolution of problem (2.1).

Next, with the same arguments as in the proof of [14, Lemma 2.1], we obtain that if t belongs to a bounded interval I then the set of corresponding solutions of problem (2.1) is uniformly bounded in $L^\infty(\Omega)$. Thus, there exists $C = C(I) > 0$ such that for every solution of (2.1) corresponding to some $t \in I$ we have $\|u\|_\infty \le C$. Since weak solutions of problem (2.1) are bounded, the nonlinear regularity theory of G. Lieberman [19] implies that for every $\omega \subset\subset \Omega$ with $0 \notin \overline{\omega}$, the set of all solutions corresponding to I is bounded in $C^{1,\beta}(\overline{\omega})$.

We already know (subsection 3.1) that problem (2.1) does not have any solution for large values of t and solutions exist if t is small enough (section 3.4). Let

$$\mathcal{S} := \{t \in \mathbb{R} : \text{problem (2.1) has a solution}\}.$$

It follows that $\mathcal{S} \ne \emptyset$. Let

$$t^* := \sup \mathcal{S} < +\infty.$$

We prove in what follows that problem (2.1) has a solution if $t = t^*$. Indeed, by the definition of t^*, there is an increasing sequence $(t_n) \subset \mathcal{S}$ that converges to t^*. Let u_n be a solution of (2.1) corresponding to $t = t_n$. Since (t_n) is a bounded sequence, we deduce that the sequence (u_n) is bounded in $C^{1,\beta}(\overline{\omega})$ for all $\omega \subset\subset \Omega$ with $0 \notin \overline{\omega}$. By the Arzela-Ascoli theorem, the sequence (u_n) is convergent to some u_* in $C^1(\overline{\omega})$, which is a solution of problem (2.1) for $t = t^*$.

Fix arbitrarily $t_0 < t^*$. We prove that problem (2.1) has a solution for $t = t_0$. We already know that problem (2.1) considered for $t = t_0$ has a subsolution $\underline{U}_{t_0}$. Let u_{t^*} denote the solution of problem (2.1) for $t = t^*$. Then u_{t^*} is a supersolution of problem (2.1) for $t = t_0$. Since $\underline{U}_{t_0}$ (which is a constant) can be chosen even smaller, it follows that we can assume that

$$\underline{U}_{t_0} \le u_{t^*} \quad \text{in } \Omega.$$

Using the method of lower and upper solutions, we conclude that problem (2.1) has at least one solution for $t = t_0$.

Returning to subsection 3.4, we know that for all $\rho > 0$ there exists a real number t_ρ such that problem (2.1) has at least one solution, provided that $t \leq t_\rho$. Let

$$t_* := \sup\{t_\rho : \rho > 0\}.$$

We already know that (2.1) has at least one solution for all $t < t_*$. We show that, in fact, problem (2.1) has at least two solutions, provided that $t < t_*$.

Fix $t_0 < t_*$ and let ρ_{t_0} be the positive number defined in subsection 3.3. Consider the bounded open set

$$G_{t_0} := \{v \in L^\infty(\Omega) : \|v^+\|_\infty < \rho,\ \|v^-\|_\infty < \rho_{t_0}\}.$$

Since G_{t_0} is bounded, we can assume that

$$\overline{G}_{t_0} \subset \{u \in L^\infty(\Omega) : \|u\|_\infty < R\} =: B(0,R),$$

for some $R > 0$.

Recall that if t belongs to a bounded interval I then the set of corresponding solutions of problem (2.1) is uniformly bounded in $L^\infty(\Omega)$. So, choosing eventually a bigger R, we can assume that $\|u\|_\infty < R$ for any solution of problem (2.1) corresponding to $t \in [t_0, t^* + 1]$.

Since problem (2.1) does not have any solution for $t = t^* + 1$, it follows that

$$\deg(I - S_{t^*+1}, B(0,R), 0) = 0.$$

So, using the homotopy invariance property of the topological degree we obtain

$$\deg(I - S_{t_0}, B(0,R), 0) = \deg(I - S_{t^*+1}, B(0,R), 0) = 0.$$

Next, using the excision property of the topological degree (see Denkowski, Migórski and Papageorgiou [13, Proposition 2.2.19]) we have

$$\deg(I - S_{t_0}, B(0,R) \setminus G_{t_0}, 0) = \deg(I - S_{t^*+1}, B(0,R) \setminus G_{t_0}, 0) = -1.$$

We conclude that problem (2.1) has at least two solutions for all $t < t^*$. □

Perspectives and open problems. The result established in the present paper can be extended if problem (2.1) is driven by degenerate operators of the type $\operatorname{div}(a(x)\nabla u)$, where a is a measurable and non-negative weight in Ω, which can have at most a finite number of (essential) zeros. Such a behavior holds if there exists an exponent $\alpha \in (0,2)$ such that a decreases more slowly than $|x - z|^\alpha$ near every point $z \in a^{-1}\{0\}$. According to Caldiroli and Musina [9], such an hypothesis can be formulated as follows: $a \in L^1(\Omega)$ and there exists $\alpha \in (0,2)$ such that

$$\liminf_{x \to z} |x - z|^{-\alpha} a(x) > 0 \quad \text{for every } z \in \overline{\Omega}.$$

Under this assumption, the weight a could be nonsmooth, as the Taylor expansion formula can easily show. For example, the function a cannot be of class C^2 and it cannot have bounded derivatives if $\alpha \in (0,1)$. As established in [9, Lemma 2.2, Remark 2.3] a function a satisfying the above hypothesis has a finite number of zeros in $\overline{\Omega}$. Notice that in such we can allow degeneracy also at some point of its boundary.

To the best of our knowledge, no results are known for degenerate "double-phase" Ambrosetti-Prodi problems, namely for equations driven by differential operators like

$$\operatorname{div}(|x|^\alpha \nabla u) + \operatorname{div}(|x|^\beta |\nabla u|^{p-2} \nabla u) \tag{3.11}$$

or

$$\operatorname{div}(|x|^\alpha \nabla u) + \operatorname{div}(|x|^\beta \log(e+|x|)|\nabla u|^{p-2}\nabla u), \tag{3.12}$$

where $\alpha \neq \beta$ are positive numbers and $1 < p \neq 2$.

Problems of this type correspond to "double-phase variational integrals" studied by Mingione et al. [5, 10]. The cases covered by the differential operators defined in (3.11) and (3.12) correspond to a degenerate behavior both at the origin and on the zero set of the gradient. That is why it is natural to study what happens if the associated integrands are modified in such a way that, also if $|\nabla u|$ is small, there exists an imbalance between the two terms of the corresponding integrand.

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Dušan D. Repovš
Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, SI-1000 Ljubljana, Slovenia
E-mail address: dusan.repovs@guest.arnes.si