Electronic Journal of Differential Equations, Vol. 2007(2007), No. 63, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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POSITIVE SOLUTIONS FOR SINGULAR THREE-POINT BOUNDARY-VALUE PROBLEMS WITH SIGN CHANGING NONLINEARITIES DEPENDING ON x'

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Abstract. Using a fixed point theorem in cones, this paper shows the existence of positive solutions for the singular three-point boundary-value problem

$$x''(t) + a(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < 1,$$
$$x'(0) = 0, \quad x(1) = \alpha x(\eta),$$

where $0 < \alpha < 1$, $0 < \eta < 1$, and f may change sign and may be singular at $x = 0$ and $x' = 0$.

1. Introduction

The study of multi-point boundary value problem (BVP) for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [3, 4]. Since then, many authors studied more general nonlinear multi-point BVPs, for example [2, 5, 6], and references therein. Recently, Liu [5] proved the existence of positive solutions for the three-point BVP

$$y''(t) + a(t)f(y(t)) = 0, \quad 0 < t < 1,$$
$$y'(0) = 0, \quad y(1) = \beta y(\eta),$$

where $0 < \beta < 1$, $0 < \eta < 1$ and $f : [0, +\infty) \to [0, +\infty)$ has no singularity at $y = 0$. Guo and Ge [2] presented the existence of positive solutions for the three-point BVP

$$x''(t) + f(t, x, x') = 0, \quad 0 < t < 1,$$
$$x(0) = 0, \quad x(1) = \beta x(\eta),$$

where $\beta\eta \in (0, 1)$, $0 < \eta < 1$ and $f \in C([0, 1] \times [0, +\infty) \times R, [0, +\infty))$ has no singularity at $t = 0$, $x = 0$ and $x' = 0$.

2000 *Mathematics Subject Classification.* 34B15, 34B10.
Key words and phrases. Three-point boundary value problem; singularity; positive solutions; fixed point theorem.

Submitted September 15, 2006. Published April 25, 2007.
Supported by grants 10571111 from the National Natural Science, and Y2005A07 from the Natural Science of Shandong Province.

Motivated by the works of [4, 5], in this paper, we discuss the equation

$$\begin{gathered} x''(t)+a(t)f(t,x(t),x'(t))=0, \quad 0<t<1, \\ x'(0)=0, \quad x(1)=\alpha x(\eta), \end{gathered} \tag{1.1}$$

where $0<\alpha<1$, $0<\eta<1$, f may change sign and may be singular at $x=0$ and $x'=0$.

The features in this article, that different from those in [2, 5], are as follows: First, the nonlinearity $a(t)f(t,x,x')$ may be singular at $t=0$, $t=1$, $x=0$ and $x'=0$; also the degree of singularity in x and x' may be arbitrary; i. e., if f contains $\frac{1}{x^\alpha}$ and $\frac{1}{(-x')^\gamma}$, α and γ may be big enough). Second, f is allowed to change sign.

The paper is organized as follows. In the next section, we present some preliminaries. Section 3 is devoted to our main result, Theorem 3.1. An example is also given to illustrate the main result. Some of the idea used here come from [6, 7].

2. Preliminaries

In this paper, we assume the following conditions

- (P1) $f(t,x,y)\in C((0,1)\times(0,+\infty)\times(-\infty,0),(-\infty,+\infty))$;
- (P2) $\beta(t),a(t),k(t)\in C((0,1),(0,+\infty))$, $F(x)\in C((0,+\infty),(0,+\infty))$, $G(y)\in C((-\infty,0),(0,+\infty)), a(t)k(t)\in L[0,1]$;
- (P3) $0<\alpha<1$, $0<\eta<1$ and $|f(t,x,y)|\le k(t)F(x)G(y)$;
- (H1) There exists $\delta>0$ such that $f(t,x,y)\ge\beta(t), y\in(-\delta,0)$;
- (H2) $\sup F[z,+\infty)=\sup\{F(x),z\le x<+\infty\}<+\infty$ for all fixed $z\in(0,+\infty)$;
- (H3) $\frac{1}{G(y)}\notin L(-\infty,-1]$;

Lemma 2.1 ([1]). *Let E be a Banach space, K a cone of E, and $B_R=\{x\in E:\|x\|<R\}$, where $0<r<R$. Suppose that $F:K\cap\overline{B_R\backslash B_r}=K_{R,r}\to K$ is a completely continuous operator and the following two conditions are satisfied*

1. *$\|F(x)\|\ge\|x\|$ for any $x\in K$ with $\|x\|=r$.*
2. *If $x\ne\lambda F(x)$ for any $x\in K$ with $\|x\|=R$ and $0<\lambda<1$.*

Then F has a fixed point in $K_{R,r}$.

Lemma 2.2. *For each natural number $n>0$, there exists $y_n(t)\in C[0,1]$ with $y_n(t)\le-\frac{1}{n}$ such that*

$$y_n(t)=-\frac{1}{n}+\min\{0,-\int_0^t a(s)f(s,Ay_n(s)+\frac{1}{n},y_n(s))ds\}, \quad t\in[0,1]. \tag{2.1}$$

Proof. For $y(t)\in P=\{y(t):y(t)\le0,y(t)\in C[0,1]\}$, define the operator

$$\begin{gathered} Ty(t)=-\frac{1}{n}+\min\{0,-\int_0^t a(s)f(s,Ay(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds\}, \\ Ay(s)=\frac{1}{1-\alpha}\int_0^1 -y(\tau)d\tau-\frac{\alpha}{1-\alpha}\int_0^\eta -y(\tau)d\tau-\int_0^s -y(\tau)d\tau, \end{gathered}$$

where $n>0$ is a natural number. Using the equality $\min\{c,0\}=\frac{c-|c|}{2}$ and

$$c(y(t))=-\int_0^t a(s)f(s,Ay(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds,$$

it is easy to know that

$$Ty(t)=-\frac{1}{n}+\frac{c(y(t))-|c(y(t)|}{2}.$$

Let $y_k(t), y(t) \in P, \|y_k - y\| \to 0$, then there exists a constant $h > 0$, such that $\|y_k\| \le h$ and $\|y\| \le h$, and let

$$c(y_k(t)) = -\int_0^t a(s)f(s, Ay_k(s) + \frac{1}{n}, \min\{y_k(s), -\frac{1}{n}\})ds,$$

which yields

$$\begin{aligned} |Ty_k(t) - Ty(t)| &= \frac{1}{2}\big|c(y_k(t)) - c(y(t)) - |c(y_k(t))| + |c(y(t))|\big| \\ &\le \frac{1}{2}\big|c(y_k(t)) - c(y(t)) + |c(y_k(t)) - c(y(t))|\big|. \end{aligned}$$

Assumption (P1) implies that $\{a(s)f(s, Ay_k(s) + \frac{1}{n}, \min\{y_k(s), -\frac{1}{n}\}$ converges to $\{a(s)f(s, Ay(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\}$, for $s \in (0,1)$. By the Lebesgue dominated convergence theorem (the dominated function $a(s)k(s)F[\frac{1}{n}, +\infty)G[-h - \frac{1}{n}, -\frac{1}{n}]$), $|Ty_k(t) - Ty(t)| \to 0$, T is a continuous operator in P.

Let C be a bounded set in P, i.e., there exists $h_1 > 0$ such that $\|y\| \le h_1$, for any $y(t) \in C, t_1, t_2 \in [0,1]$, $t_1 < t_2$, $y(t) \in P$,

$$\begin{aligned} |Ty(t_2) - Ty(t_1)| &= \frac{1}{2}\Big| -\int_{t_1}^{t_2} a(s)f(s, Ay(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\} \\ &\quad + \int_{t_1}^{t_2} a(s)f(s, Ay(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}|)ds\Big| \\ &\le \Big| -\int_{t_1}^{t_2} a(s)k(s)dsF[\frac{1}{n} + \infty)G[-h_1 - \frac{1}{n}, -\frac{1}{n}] \\ &\quad + \int_{t_1}^{t_2} a(s)k(s)dsF[\frac{1}{n}, +\infty)G[-h_1 - \frac{1}{n}, -\frac{1}{n}]|\Big|. \end{aligned}$$

According to the absolute continuity of the Lebesgue integral, for any $\epsilon > 0$, there exists $\delta > 0$ such that, when $|t_2 - t_1| < \delta, |\int_{t_1}^{t_2} a(s)k(s)ds < \epsilon$ holds. Therefore, $\{Ty(t), y(t) \in P\}$ is equicontinuous. Hence T is a completely continuous operator in P.

By (H3), we may choose a sufficiently large $R_n > 1$ to fit

$$\int_{-R_n}^{-1} \frac{dy}{G(y)} \ge \int_0^t a(s)k(s)ds \sup F[\frac{1}{n}, +\infty).$$

For any fixed n, we prove that

$$y(t) \ne \lambda Ty(t) = \frac{-\lambda}{n} + \lambda \min\{0, -\int_0^t a(s)f(s, Ay(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds\} \tag{2.2}$$

for any $y(t) \in P$ with $\|y\| = R_n$ and $0 < \lambda < 1$.

In fact, if there exist $y(t) \in P$ with $\|y\| = R_n$ and $0 < \lambda < 1$, such that

$$y(t) = \frac{-\lambda}{n} + \lambda \min\{0, -\int_0^t a(s)f(s, Ay(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds\}. \tag{2.3}$$

First, we prove an important fact: for $t, z \in [0,1]$, $t > z$, $y(t) < y(z) \le -\frac{1}{n}$,

$$\int_{y(t)}^{y(z)} \frac{dy}{G(y)} \le \int_z^t a(s)k(s)ds \sup F[Ay(t) + \frac{1}{n}, +\infty). \tag{2.4}$$

Let $t' \in (0,t]$ such that $y(t') = y(t), y(s) \geq y(t'), s \in (0,t']$. We may choose $\{t_i\}(i = 1,2,\dots,2m)$ to fit

1) $t' = t_1 > t_2 \geq t_3 > t_4 \geq t_5 > \dots \geq t_{2m-1} > t_{2m} = z \geq 0$;

(1) $y(t_1) = y(t')$, $y(t_{2i}) = y(t_{2i+1})$, $i = 1,2,\dots m-1$, $y(t_{2m}) = y(z)$;

(2) $y(t)$ is decreasing in $[t_{2i}, t_{2i-1}]$, $i = 1,2,\dots m$. (if $y(t)$ is decreasing in $[0,t']$. Let $m = 1$, i.e. $[t_2,t_1] = [0,t']$.)

Note that $y(t) < -\frac{1}{n}$, $t \in (t_{2i}, t_{2i-1}]$, which implies

$$-\int_0^t a(s)f(s, Ay(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds < 0, \quad t \in (t_{2i}, t_{2i-1}].$$

Differentiating (2.3) and using (H2), we obtain

$$-y'(t) = \lambda a(t)f(t, Ay(t) + \frac{1}{n}, y(t))$$

$$\frac{-y'(t)}{G(y(t))} \leq a(t)k(t)\sup F[Ay(t) + \frac{1}{n}, +\infty) \leq a(t)k(t)\sup F[\frac{1}{n}, +\infty),$$

for $t \in (t_{2i}, t_{2i-1}]$, $i = 1,2,\dots m$. Integrating from t_{2i} to t_{2i-1}, we have

$$\int_{y(t_{2i-1})}^{y(t_{2i})} \frac{dy}{G(y)} \leq \int_{t_{2i}}^{t_{2i-1}} a(s)k(s)ds \sup F[\frac{1}{n}, +\infty), \quad i = 1,2,\dots m.$$

Summing from m to 1, we have

$$\int_{y(t)}^{y(z)} \frac{dy}{G(y)} \leq \int_z^t a(s)k(s)ds \sup F[\frac{1}{n}, +\infty).$$

Set $y(z) = -\frac{1}{n}$, $y(t) = -R_n$ in (2.4), we have

$$\int_{-R_n}^{-1} \frac{dy}{G(y)} \leq \int_{-R_n}^{-\frac{1}{n}} \frac{dy}{G(y)} \leq \int_0^t a(s)k(s)ds \sup F[\frac{1}{n}, +\infty),$$

which contradicts

$$\int_{-R_n}^{-1} \frac{dy}{G(y)} \geq \int_0^t a(s)k(s)ds \sup F[\frac{1}{n}, +\infty).$$

Hence (2.2) holds. Put $r = \frac{1}{n}$, Lemma 2.1 leads to the desired result. $\square$

3. Main results

Main result in this paper is as follows.

Theorem 3.1. *Let* (H1)–(H3) *hold. Then the three-point boundary-value problem* (1.1) *has at least one positive solution.*

Proof. Put $M_n = \min\{y_n(t) : t \in [0,\eta]\}$. (H1) implies $\gamma = \sup\{M_n\} < 0$. Set $\tau = \max\{\gamma, -\delta\}, n > -\frac{1}{\tau}$.

(1) First, we prove that

$$y_n(t) = -\frac{1}{n} - \int_0^t a(s)f(s, Ay_n(s) + \frac{1}{n}, y_n(s))ds, \quad t \in [0,1]. \tag{3.1}$$

Set $y_n(t_n) = \tau, t_n \in (0,\eta], y_n(t) \geq \tau, t \in [0,t_n]$. We easily check that $y_n(t)$ is decreasing in $(0,t_n]$. We only need to prove that

$$y_n(t) \leq \tau, \quad t \in [t_n, 1]. \tag{3.2}$$

If there exist $t \in (t_n, 1]$ such that $y_n(t) > \tau$, then we may choose $t', t'' \in [t_n, 1], t' < t''$ to fit $y_n(t') = \tau, \tau < y_n(t) < -\frac{1}{n}, t \in (t', t'']$, we have from (2.1)

$$0 < \int_{t'}^{t''} a(s)f(s, Ay_n(s) + \frac{1}{n}, y_n(s))ds = y_n(t') - y_n(t'') < 0.$$

This contradiction implies (3.2).

Using $y_n(t), 1$ and 0 in place of $y(t), \lambda$ and z in (2.3) in Lemma 2.2, we notice that

$$\begin{aligned} Ay_n(t) + \frac{1}{n} &= \frac{1}{1-\alpha}\int_0^1 -y_n(\tau)d\tau - \frac{\alpha}{1-\alpha}\int_0^\eta -y_n(\tau)d\tau - \int_0^t -y_n(\tau)d\tau + \frac{1}{n} \\ &> \frac{\alpha}{1-\alpha}\int_\eta^1 -y_n(\tau)d\tau \\ &\geq \frac{\alpha}{1-\alpha}(-\tau)(1-\eta), \quad t \in [0,1]. \end{aligned}$$

From (2.4), putting $t = t_n$, we know that

$$\int_{y_n(t_n)}^{-\frac{1}{n}} \frac{dy_n}{G(y_n)} \leq \int_0^{t_n} a(s)k(s)ds \sup F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty). \tag{3.3}$$

Equation (3.3) shows $t_0 = \inf\{t_n\} > 0$. Also, $y_n(t)$ is decreasing for $t \in (0, t_0]$ and (H1) imply that $W(t) = \sup\{y_n(t)\} < 0$, $t \in (0, t_0]$.

(2) We show that $\{y_n(t)\}$ is equicontinuous on $[\frac{1}{3k}, 1 - \frac{1}{3k}]$, for a natural number $k \geq 1$, and uniformly bounded on $[0, 1]$.

Using $y_n(t)$, 1 and 0 instead of $y_(t)$, λ and z in (2.3) in Lemma 2.2, we notice that

$$Ay_n(t) + \frac{1}{n} \geq \frac{\alpha}{1-\alpha}(-\tau)(1-\eta), \quad t \in [0,1].$$

We know from (2.4),

$$\int_{y_n(t)}^{-\frac{1}{n}} \frac{dy_n}{G(y_n)} \leq \int_0^t a(s)k(s)ds \sup F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty), \quad t \in [0,1]. \tag{3.4}$$

Now we use (H3) and (3.4) show that $\omega(t) = \inf\{y_n(t)\} > -\infty$ is bounded on $[0, 1]$. On the other hand, it follows from (3.1) and (3.2) that

$$|y_n'(t)| \leq k(t)a(t) \sup F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty) \sup G[\omega_k, \max\{\tau, W(\frac{1}{k})\}], \ (n \geq k). \tag{3.5}$$

Where $\omega_k = \inf\{\omega(t),\ t \in [\frac{1}{3k}, 1 - \frac{1}{3k}]\}$. Thus (3.5) and the absolute continuity of Lebesgue integral show that $\{y_n(t)\}$ is equicontinuous on $[\frac{1}{3k}, 1 - \frac{1}{3k}]$. Now the Arzela-Ascoli theorem guarantees that there exists a subsequence of $\{y_n(t)\}$, which converges uniformly on $[\frac{1}{3k}, 1 - \frac{1}{3k}]$. When $k = 1$, there exists a subsequence $\{y_n^{(1)}(t)\}$ of $\{y_n(t)\}$, which converges uniformly on $[\frac{1}{3}, \frac{2}{3}]$. When $k = 2$, there exists a subsequence $\{y_n^{(2)}(t)\}$ of $\{y_n^{(1)}(t)\}$, which converges uniformly on $[\frac{1}{6}, \frac{5}{6}]$. In general, there exists a subsequence $\{y_n^{(k+1)}(t)\}$ of $\{y_n^{(k)}(t)\}$, which converges uniformly on $[\frac{1}{3(k+1)}, 1 - \frac{1}{3(k+1)}]$. Then the diagonal sequence $\{y_k^{(k)}(t)\}$ converges everywhere in $(0, 1)$ and it is easy to verify that $\{y_k^{(k)}(t)\}$ converges uniformly on any interval $[c, d] \subseteq (0, 1)$. Without loss of generality, let $\{y_k^{(k)}(t)\}$ be itself of $\{y_n(t)\}$ in the rest. Put $y(t) = \lim_{n\to\infty} y_n(t), t \in (0,1)$. Then $y(t)$ is continuous in $(0,1)$ and $y(t) < 0, t \in (0,1)$.

(3) Now (3.4) shows that

$$\sup\{\max\{-y_n(t), t \in [0,1]\}\} < +\infty.$$

We have

$$\lim_{t\to 0+} \sup\{\int_0^t -y_n(s)ds\} = 0, \quad \lim_{t\to 1-} \sup\{\int_t^1 -y_n(s)ds\} = 0, \tag{3.6}$$

and we obtain

$$\begin{aligned} Ay_n(t) &= \frac{1}{1-\alpha}\int_0^1 -y_n(\tau)d\tau - \frac{\alpha}{1-\alpha}\int_0^\eta -y_n(\tau)d\tau - \int_0^t -y_n(\tau)d\tau \\ &< \frac{1}{1-\alpha}\int_0^1 -y_n(\tau)d\tau < +\infty, \quad t \in [0,1]. \end{aligned} \tag{3.7}$$

Since (3.6) and (3.7) hold, Fatou's theorem of the Lebesgue integral implies $Ay(t) < +\infty$, for any fixed $t \in (0,1)$.

(4) $y(t)$ satisfies

$$y(t) = -\int_0^t a(s)f(s, Ay(s), y(s))ds, \quad t \in (0,1).$$

Since $y_n(t)$ converges uniformly on $[a,b] \subset (0,1)$, (3.6) leads that $Ay_n(s)$ converges to $Ay(s)$ for any $s \in (0,1)$. For each fixed $t \in (0,1)$, thee exists $d > 0$ such that $0 < d < t$, then

$$y_n(t) - y_n(d) = -\int_d^t a(s)f(s, Ay_n(s) + \frac{1}{n}, y_n(s))ds.$$

for all $n > k$. Since $y_n(s) \le \max\{\tau, W(d)\}$, $Ay_n(s) + \frac{1}{n} \ge \frac{\alpha}{1-\alpha}(-\tau)(1-\eta)$, $s \in [d,t]$, the set $\{Ay_n(s)\}$ or $\{y_n(s)\}$ is bounded and equicontinuous on $[d,t]$. Let $n \to \infty$

$$y(t) - y(d) = -\int_d^t a(s)f(s, Ay(s), y(s))ds. \tag{3.8}$$

Putting $t = d$ in (3.4), we have

$$\int_{y_n(d)}^{-\frac{1}{n}} \frac{dy_n}{G(y_n)} \le \int_0^d a(s)k(s)ds \sup F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty).$$

Let $n \to \infty$ and $d \to 0+$, we obtain

$$y(0+) = \lim_{d\to 0+} y(d) = 0.$$

Letting $d \to 0+$ in (3.8), we have

$$y(t) = -\int_0^t a(s)f(s, Ay(s), y(s))ds, \quad t \in (0,1), \tag{3.9}$$

and $Ay(1) = \alpha Ay(\eta)$. Hence $x(t) = Ay(t)$ is a positive solution of (1.1). $\square$

Corollary 3.2. *Suppose that* (H1)-(H3) *hold , then the set of positive solutions of* (1.1) *is compact.*

Proof. Let $M = \{y \in C[0,1] : Ay(t)$is a positive solution of (1.1)$\}$. First we show that M is compact. Note that (1) M is not empty; (2) M is relatively compact(bounded, equicontinuous). (3) M is closed.

Obviously Theorem 3.1 implies M is not empty.

First we show that $M \in C[0,1]$ is relatively compact. For any $y(t) \in M$, differentiating (3.9) and using (H2), we obtain

$$\begin{gathered}
-y'(t) = \lambda a(t) f(t, Ay(t), y(t)) \\
\begin{aligned}
\frac{-y'(t)}{G(y(t))} &\le a(t)k(t) \sup F[Ay(t), +\infty) \\
&\le a(t)k(t) \sup F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty), \quad t \in [0,1].
\end{aligned}
\end{gathered}$$

Integrating from 0 to t, we have

$$\int_{y(t)}^{0} \frac{dy}{G(y)} \le \int_0^1 a(s)k(s)ds \sup F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty), \quad t \in [0,1]. \tag{3.10}$$

Now (H3) and (3.10) show that for any $y(t) \in M$, there exists $K > 0$ such that $|y(t)| < K$, for all $t \in [0,1]$. Then M is bounded.

For each $y(t) \in M$, we obtain from (3.9),

$$\begin{aligned}
-y'(t) &= a(t) f(t, Ay(t), y(t)) \\
&\le a(t)|f(t, Ay(t), y(t))| \\
&\le a(t)k(t)F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty)G(y(t)), \quad t \in (0,1),
\end{aligned}$$

and

$$\begin{aligned}
y'(t) &= -a(t) f(t, Ay(t), y(t)) \\
&\le a(t)|f(t, Ay(t), y(t))| \\
&\le a(t)k(t)F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty)G(y(t)), \quad t \in (0,1),
\end{aligned}$$

which yields

$$\frac{-y'(t)}{G(y(t))+1} \le a(t)k(t) \sup F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty), \quad t \in (0,1), \tag{3.11}$$

$$\frac{y'(t)}{G(y(t))+1} \le a(t)k(t) \sup F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty), \quad t \in (0,1). \tag{3.12}$$

Note that the right-hand sides of the above inequalities are always positive. Let $I(y(t)) = \int_0^{y(t)} \frac{dy}{G(y)+1}$, for any $t_1, t_2 \in [0,1]$. Integration from t_1 to t_2 in (3.11) and (3.12) yields

$$|I(y(t_1)) - I(y(t_2))| \le \int_{t_1}^{t_2} a(t)k(t)F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty)dt. \tag{3.13}$$

Since I^{-1} is uniformly continuous on $[I(-K), 0]$, for any $\bar{\epsilon} > 0$, there is a $\epsilon' > 0$ such that

$$|I^{-1}(s_1) - I^{-1}(s_2)| < \bar{\epsilon}, \forall |s_1 - s_2| < \epsilon', s_1, s_2 \in [I(-K), 0]. \tag{3.14}$$

Inequality (3.13) guarantees that for $\epsilon' > 0$, there is a $\delta' > 0$ such that

$$|I(y(t_1)) - I(y(t_2))| < \epsilon', \forall |t_1 - t_2| < \delta', \quad t_1, t_2 \in [0,1].$$

This inequality and (3.14)imply

$$|y(t_1) - y(t_2)| = |I^{-1}(I(y(t_1)) - I^{-1}(I(y(t_2))| < \bar{\epsilon}, \quad t_1, t_2 \in [0, 1],$$

which means that M is equicontinuous. So M is relatively compact.

Second, we show that M is closed. Suppose that $\{y_n\} \subseteq M$ and

$$\lim_{n\to+\infty} \max_{t\in[0,1]} |y_n(t) - y_0(t)| = 0.$$

Obviously $y_0 \in C[0, 1]$ and $\lim_{n\to+\infty} Ay_n(t) = Ay_0(t)$, $t \in [0, 1]$. Moreover,

$$\begin{aligned} Ay_n(t) &= \frac{1}{1-\alpha} \int_0^1 -y_n(\tau)d\tau - \frac{\alpha}{1-\alpha} \int_0^\eta -y_n(\tau)d\tau - \int_0^t -y_n(\tau)d\tau \\ &< \frac{1}{1-\alpha} \int_0^1 -y_n(\tau)d\tau \\ &< \frac{K}{1-\alpha}, \quad t \in [0, 1]. \end{aligned}$$

For $y_n(t) \in M$, from (3.9) we obtain

$$y_n(t) = -\int_0^t a(s)f(s, Ay_n(s), y_n(s))ds, \quad t \in (0, 1).$$

For fixed $t \in (0, 1)$, there exists $d > 0$ such that $0 < d < t$, then

$$y_n(t) - y_n(d) = -\int_d^t a(s)f(s, Ay_n(s), y_n(s))ds.$$

Since $y_n(s) \le \max\{\tau, W(d)\}$, $Ay_n(s) \ge \frac{\alpha}{1-\alpha}(-\tau)(1-\eta)$, $s \in [d, t]$, the Lebesgue dominated convergence theorem yields

$$y_0(t) - y_0(d) = -\int_d^t a(s)f(s, Ay_0(s), y_0(s))ds, \quad t \in (0, 1). \tag{3.15}$$

From (3.9), we have

$$-y_n'(t) = a(t)f(t, Ay_n(s), y_n(s)) \le a(t)k(t)F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty)G(y_n(t)),$$

which yields

$$\frac{-y_n'(t)}{G(y_n(t))} \le a(t)k(t)ds \sup F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty), \quad t \in (0, 1),$$

integrating from 0 to d,

$$\int_{y_n(d)}^0 \frac{dy_n}{G(y_n)} \le \int_0^d a(s)k(s)ds \sup F[\frac{\alpha}{1-\alpha}(-\tau)(1-\eta), +\infty).$$

Let $n \to \infty$ and $d \to 0+$, we obtain $y_0(0+) = \lim_{d\to 0+} y_0(d) = 0$. Letting $d \to 0+$ in (3.15), we have

$$y_0(t) = -\int_0^t a(s)f(s, Ay_0(s), y_0(s))ds, t \in (0, 1), Ay_0(1) = \alpha Ay_0(\eta),$$

then $x_0(t) = Ay_0(t)$ is a positive solution of (1.1). So $y_0(t) \in M$ and M is a closed set. Hence $\{Ay(t), y(t) \in M\} \in C^1[0, 1]$ is compact. $\square$

Example 3.3. In (1.1), let

$$f(t,x,y)=k(t)[1+x^{-\gamma}+(-y)^{-\sigma}-(-y)\ln(-y)], a(t)=t^{-\frac{1}{3}},$$

and $k(t)=t^{-\frac{1}{2}}$, $0<t<1$, where $\gamma>0$, $\sigma\geq 0$, and let $F(x)=1+x^{-\gamma}, G(y)=1+(-y)^{-\sigma}+(-y)\ln(-y)$. Then

$$f(t,x,y)\leq k(t)F(x)G(y), \quad \delta=-1, \quad \beta(t)=k(t),$$
$$\int_{-\infty}^{-1}\frac{dy}{G(y)}=+\infty.$$

By Theorem 3.1, equation (1.1) has at least a positive solution and Corollary 3.2 implies the set of solutions is compact.

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