ELECTRONIC JOURNAL OF DIFFERENTIAL EQUATIONS, Vol. **2000**(2000), No. 50, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu ftp ejde.math.unt.edu (login: ftp)

One-dimensional elliptic equation with concave and convex nonlinearities *

Justino Sánchez & Pedro Ubilla

Abstract

We establish the exact number of positive solutions for the boundary-value problem

$$\begin{gathered} -(|u'|^{m-2}u')' = \lambda u^q + u^p \quad \text{in } (0,1) \\ u(0) = u(1) = 0\,, \end{gathered}$$

where $0 \le q < m-1 < p$ and λ is positive.

1 Introduction

We establish the exact number of positive solutions for the boundary-value problem

$$\begin{gathered} -(|u'|^{m-2}u')' = \lambda u^q + u^p \quad \text{in } (0,1) \\ u(0) = u(1) = 0\,, \end{gathered} \tag{1}$$

where $0 \le q < m-1 < p$ and $\lambda > 0$. Problem (1) with $m = 2$ was suggested by Ambrosetti, Brezis, and Cerami in [2]. Indeed, the equation

$$\begin{gathered} -\Delta u = \lambda u^q + u^p \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered}$$

with $0 < q < 1 < p$ and Ω a bounded domain of $\mathbb{R}^N$ is studied in [2]. There, it was proved that there exists $\Lambda > 0$ such that: if $\lambda \in (0, \Lambda)$, then the latter problem has at least two positive solutions; if $\lambda = \Lambda$, then it has at least one positive solution; finally, if $\lambda > \Lambda$, then it has no positive solution.

* *Mathematics Subject Classifications:* 34B15.
Key words: m-Laplacian, concave-convex nonlinearities, exactness results, time-maps.

Submitted February 1, 2000. Published June 25, 2000.
(J.S.) Partially supported by FONDECYT Grant # 1990183.
(P.U.) Partially supported by FONDECYT Grants # 1990183 and # 1980812, and by a DICYT-USACH grant.

Using shooting methods, the existence and multiplicity of solutions for the quasi-linear problem

$$\begin{gathered} -(|u'|^{p-2}u')' = |u|^p - \mu \quad \text{in } (0,1) \\ u(0) = u(1) = 0 \end{gathered} \tag{2}$$

was studied recently by Addou and Benmezaï [1]. For $\mu > 0$, they determine a lower bound on the number of solutions of the problem (2), and their nodal properties. In the case $\mu \leq 0$, they also obtained the exact number of solutions.

We note that if $\lambda = -\mu$, $m = p$ and $q = 0$, then the equation studied in [1], with $\mu < 0$, turns out to be a particular case of our equation (1). This fact inspired us to apply the techniques developed by Addou and Benmezaï [1]. The strategy is to localize the critical points of a time mapping on a bounded interval J. We point out that this problem was simultaneously and independently studied by I. Addou and A. Benmezaï. We remark that the novelty of our result is that we obtain the exact number of solutions for an equation with concave-convex nonlinearity, as well as their asymptotic behavior for small parameter λ. Finally, we should note that in [2, 3, 4], the problem of determining the exact number of solution is not solved.

2 Results and Methods Employed

We first state our main result.

Theorem 1 *There exists a number $\lambda^* > 0$ such that:*

(a) If $\lambda > \lambda^*$, then (1) has no solutions.

(b) If $\lambda = \lambda^*$, then (1) has exactly one positive solution.

(c) If $0 < \lambda < \lambda^*$, then (1) has exactly two positive solutions, u_λ and v_λ.

(d) The solutions u_λ and v_λ satisfy $\lim_{\lambda \to 0} \|u_\lambda\|_\infty = 0$ and

$$\Big(\frac{2^m}{m'} I_p^m (p+1)\Big)^{1/(p-m+1)} \leq \lim_{\lambda \to 0} \|v_\lambda\|_\infty \leq \Big(\frac{2^m}{m'} I_q^m (p+1)\Big)^{1/(p-m+1)},$$

where $I_r = \int_0^1 (1 - t^{r+1})^{-1/m}\, dt$.

In this article we use a shooting method. More precisely, we study the ordinary differential equation

$$\begin{gathered} -(|u'|^{m-2}u')' = \lambda |u|^{q-1}u + |u|^{p-1}u \quad \text{in } (0,1) \\ u(0) = 0, \quad u'(0) = E > 0\,. \end{gathered} \tag{3}$$

The solution to this problem is $4T$-periodic function, with

$$T = T(\lambda, S) = (m')^{-1/m} S^{\frac{m-1-q}{m}} \int_0^1 \Big(S^{p-q} \frac{(1 - t^{p+1})}{p+1} + \lambda \frac{(1 - t^{q+1})}{q+1}\Big)^{-1/m} dt\,,$$

where $S = S(\lambda, E)$ is the first zero of the function $E^m - m'G(\lambda, \cdot)$,

$$G(\lambda, u) = \frac{\lambda u^{q+1}}{q+1} + \frac{u^{p+1}}{p+1},$$

and m' is defined by $1/m + 1/m' = 1$. See, e.g. [3, 4].

The solution u to Problem (3) satisfies the following conditions

- $u(2kT) = 0$, with $k \in \mathbb{Z}$.
- $u(x) > 0$, for $x \in (0, 2T)$ and $u(x) < 0$, for $x \in (2T, 4T)$.
- Every hump of u is symmetrical about the center of the interval of its definition, where we call hump of u its restriction to the open interval $I = (x_1, x_2)$, with x_1 and x_2 two consecutive zeros of u.
- Every positive (resp. negative) hump of u may be obtained by translating the first positive (resp. negative) hump.
- The derivative of each hump of u vanishes once and only once.

Thus, when $T = 1/2$, we obtain positive solutions of Problem (1). In order to prove Theorem 1, we need the following technical lemma.

Lemma 1 *(a) $S(\lambda, \cdot)$ is an increasing function,*

$$S(\lambda, 0) = 0 \quad \textit{and} \quad \lim_{E \to +\infty} S(\lambda, E) = +\infty\,.$$

(b) $\lim_{E \to +\infty} T(\lambda, S(\lambda, E)) = \lim_{E \to 0^+} T(\lambda, S(\lambda, E)) = 0\,.$

(c) $T(\lambda, S(\lambda, \cdot))$ has a unique maximum for each $\lambda > 0$.

(d) $\lambda \to T(\lambda, S^*_\lambda)$ is a decreasing function that satisfies

$$\lim_{\lambda \to 0^+} T(\lambda, S^*_\lambda) = +\infty \qquad \textit{and} \qquad \lim_{\lambda \to +\infty} T(\lambda, S^*_\lambda) = 0$$

where $S^*_\lambda = S(\lambda, E^*(\lambda))$ and $E^*(\lambda)$ is the unique critical point of the function $T(\lambda, S(\lambda, \cdot))$.

(e) For each λ sufficiently small, the solutions S_1 and S_2 of $T(\lambda, S) = 1/2$ satisfy

$$S_1 \le \Big(\lambda\big(\frac{m-1-q}{p-m+1}\big)\frac{(p+1)}{(q+1)}\Big)^{1/(p-q)},$$

$$\frac{2^m}{m'} I_p^m (p+1) \le S_2^{p-m+1} \le \frac{2^m}{m'} I_q^m (p+1)\,.$$

3 Proof of the Main Results

Proof of Theorem 1. By Lemma 1 and the continuity of the function $\lambda \to T(\lambda, S^*_\lambda)$, there exists λ^* which satisfies $T(\lambda^*, S^*_{\lambda^*}) = 1/2$ and such that:

- If $\lambda > \lambda^*$, then $T(\lambda, S(\lambda, E)) < \frac{1}{2}$, for each $E > 0$.
- If $\lambda = \lambda^*$, then $\max_{E>0} T(\lambda, S(\lambda, E)) = 1/2$.
- If $\lambda < \lambda^*$, then $\max_{E>0} T(\lambda, S(\lambda, E)) > 1/2$ and for each λ sufficiently small we have that

$$\|u_\lambda\|_\infty \leq \left(\lambda \left(\frac{m-1-q}{p-m+1}\right) \frac{(p+1)}{(q+1)}\right)^{1/(p-q)}$$

and

$$\left(\frac{2^m}{m'} I_p^m (p+1)\right)^{1/(p-m+1)} \leq \|v_\lambda\|_\infty \leq \left(\frac{2^m}{m'} I_q^m (p+1)\right)^{1/(p-m+1)} .$$

From these three statements and Lemma 1, Theorem 1 follows.

Proof of Lemma 1. The proof of (a) and (b) can be found in [3] and [4]. Concerning (c), using statements (a) and (b) it suffices to show that the function $S \to T(\lambda, S)$ has a unique critical point for each $\lambda > 0$.
On the other hand, it is easy to prove that

$$\frac{\partial T}{\partial S} = (m')^{-\frac{1}{m}} \int_0^S \frac{A(\lambda, S) - A(\lambda, \eta)}{mS(G(\lambda, S) - G(\lambda, \eta))^{\frac{m+1}{m}}}\, d\eta$$

where $A(\lambda, u) = \left(\frac{m-1-q}{q+1}\right)\lambda u^{q+1} - \left(\frac{p-m+1}{p+1}\right) u^{p+1}$. Direct computations show that

$$\frac{\partial T}{\partial S} > 0\,, \quad \text{for } S \in [0, \rho_1] \quad \text{and} \quad \frac{\partial T}{\partial S} < 0\,, \quad \text{for } S \in [\rho_2, +\infty) \tag{4}$$

where

$$\rho_1 = \left(\lambda \left(\frac{m-1-q}{p-m+1}\right)\right)^{1/(p-q)} \quad \text{and} \quad \rho_2 = \left(\lambda \frac{(m-1-q)(p+1)}{(p-m+1)(q+1)}\right)^{1/(p-q)} . \tag{5}$$

Moreover, $A(\lambda, 0) = A(\lambda, \rho_2) = 0$ and $\rho_2 > \rho_1$. Thus the critical points of $T(\lambda, \cdot)$ belong to the interval $J := [\rho_1, \rho_2]$.

By the same arguments as in Lemma 7 of [1], it is not difficult to verify that

$$\frac{\partial^2 T}{\partial S^2} = (m')^{-1/m} \int_0^1 \frac{S(1-\eta^{p+1})^2 P(x(\eta))}{m^2(G(\lambda, S) - G(\lambda, \eta S))^{(2m+1)/m}}\, d\eta\,,$$

where

$$x(\eta) = \begin{cases} \frac{q+1}{p+1} & \text{if } \eta = 1 \\ \frac{1-\eta^{q+1}}{1-\eta^{p+1}} & \text{if } \eta \in [0,1) \end{cases}$$

and

$$P(x) = \frac{(q-m+1)}{q+1}\lambda^2 S^{2q} x^2 - C(m,p,q)\lambda S^{p+q} x + \frac{(p-m+1)}{p+1} S^{2p}\,.$$

Since $0 \le q < m-1 < p$, there exists $x_1 < 0 < x_2$ such that:

- $P(x_1) = P(x_2) = 0$;
- $P(x) > 0$, $x \in (x_1, x_2)$;
- $P(x) < 0$, $x \in (-\infty, x_1) \cup (x_2, +\infty)$.

Indeed,

$$\begin{aligned} x_2 &= \frac{S^{p-q}}{2\lambda(p+1)(m-1-q)} \\ &\quad \times\left(\sqrt{\mu} - (m(p^2+q^2) - 2(m+1)pq + (m-2)(p+q) + 2(m-1))\right), \end{aligned}$$

where

$$\begin{aligned} \mu &= (m(p^2+q^2) - 2(m+1)pq + (m-2)(p+q) + 2(m-1))^2 \\ &\quad +4(p+1)(q+1)(p-m+1)(m-1-q)\,. \end{aligned}$$

On the other hand, using that

$$\alpha(x) = (p+1)x^{q+1} - (q+1)x^{p+1}, \quad \text{for } 0 \le x \le 1,$$

is an increasing function, it is easy to see that

$$x(\eta) \in \left[\frac{q+1}{p+1}, 1\right], \quad \text{for all } \eta \in [0,1].$$

We note that the function $S \to x_2(S)$ is an increasing function that satisfies

$$x_2(\rho_2) = \frac{\sqrt{\mu} - (m(p^2+q^2) - 2(m+1)pq + (m-2)(p+q) + 2(m-1))}{2(q+1)(p-m+1)}\,.$$

We now claim that

$$x_2(\rho_2) < \frac{q+1}{p+1}\,.$$

For this, we consider the function

$$\begin{aligned} F(x) &= ((m+1)q+1)x^3 - (2(m+1)q^2 + q + (m-1))x^2 \\ &\quad +q((m+1)q^2 - q + 2(m-1))x - q^2(m-1-q)\,. \end{aligned}$$

Let us show that the function F satisfies

$$F(x) > 0, \quad \text{for } x > m-1\,.$$

This statement will follow, if we prove that $F(m-1) \geq 0$, $F'(m-1) > 0$ and F is convex in $[m-1, +\infty)$. In fact,

$$\begin{aligned} F(m-1) &= (m-1)\big((m+1)q^3 - (2m^2-1)q^2 + m^2(m-1)q\big) - q^2(m-1-q) \\ &= m^2q^3 - 2m^2(m-1)q^2 + m^2(m-1)^2q \\ &= m^2q(q-m+1)^2 \geq 0 \end{aligned}$$

On the other hand,

$$F'(x) = 3((m+1)q+1)x^2 - 2(2(m+1)q^2+q+m-1)x + q((m+1)q^2-q+2(m-1))\,.$$

Thus

$$\begin{aligned} F'(m-1) &= (m-1)(3(m-1)((m+1)q+1) - 2(2(m+1)q^2+q+m-1)) \\ &\quad +(m+1)q^3 - q^2 + 2(m-1)q \\ &= (m-1)(-4(m+1)q^2 + (3m^2-5)q + m - 1) + (m+1)q^3 \\ &\quad -q^2 + 2(m-1)q \\ &= (m+1)q^3 - (4m^2-3)q^2 + 3(m-1)^2(m+1)q + (m-1)^2. \end{aligned}$$

We note that the point $x = m-1$ is a zero of the algebraic equation

$$(m+1)x^3 - (4m^2-3)x^2 + 3(m-1)^2(m+1)x + (m-1)^2 = 0\,.$$

Thus

$$\begin{aligned} F'(m-1) &= (q-m+1)((m+1)q^2 - (3m^2-2)q - m + 1) \\ &= (q-m+1)((q-m+1)((m+1)q+1) - 2m^2q) > 0\,. \end{aligned}$$

We now prove that F is convex in $[m-1, +\infty)$. Note that

$$F(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$$

where

$$\alpha = (m+1)q+1\,, \quad \beta = -(2(m+1)q^2+q+m-1)\,,$$
$$\gamma = q((m+1)q^2 - q + 2(m-1))\,, \quad \delta = -q^2(m-1-q)\,.$$

Then it easily follows that $F''(x) > 0$ if and only if $x > -\beta/(3\alpha)$. Thus, F is convex in $[m-1, +\infty)$ when

$$m-1 \geq -\frac{\beta}{3\alpha} = \frac{2(m+1)q^2+q+m-1}{3((m+1)q+1)}$$

which is equivalent to saying that

$$G(q) = 2(m+1)q^2 + (4-3m^2)q - 2(m-1) \leq 0$$

But $G(q) \le 0$, when $q \in (r_1, r_2)$, where

$$r_1 = \frac{3m^2 - 4 - m\sqrt{9m^2-8}}{4(m+1)} \quad \text{and} \quad r_2 = \frac{3m^2 - 4 + m\sqrt{9m^2-8}}{4(m+1)}.$$

Since $r_1 < 0 < m-1 < r_2$, we obtain that $G(q) \le 0$, for each $q \in [0, m-1)$. Hence F is convex in $[m-1, +\infty)$. Thus

$$\left[\frac{q+1}{p+1}, 1\right] \subseteq (x_2(S), +\infty), \quad \text{for each } S \in [\rho_1, \rho_2].$$

Now using that $P(x) < 0$ for each $x \in (-\infty, x_1) \cup (x_2, +\infty)$, we have that $P(x(\eta)) < 0$, for all $\eta \in [0, 1]$. Hence

$$\frac{\partial^2 T}{\partial S^2} < 0, \quad \text{for each } S \in [\rho_1, \rho_2],$$

which proves (c), that is, the uniqueness of the critical point of the function $T(\lambda, S(\lambda, \cdot))$.

To prove (d), we note that

$$\frac{d}{dE}(T(\lambda, S(\lambda, E)) = (m')^{-1/m} \frac{\partial S}{\partial E} \int_0^S \frac{A(\lambda, S) - A(\lambda, \eta)}{mS(G(\lambda, S) - G(\lambda, \eta))^{\frac{m+1}{m}}}\, d\eta$$

and

$$\begin{aligned} \frac{\partial T}{\partial \lambda} &= (m')^{-\frac{1}{m}} \frac{\partial S}{\partial \lambda} \int_0^S \frac{A(\lambda, S) - A(\lambda, \eta)}{mS(G(\lambda, S) - G(\lambda, \eta))^{\frac{m+1}{m}}}\, d\eta \\ &\quad -(m')^{-\frac{1}{m}} \int_0^S \frac{S^{q+1} - \eta^{q+1}}{m(q+1)(G(\lambda, S) - G(\lambda, \eta))^{\frac{m+1}{m}}}\, d\eta. \end{aligned}$$

Let $\Delta = -\frac{\partial S}{\partial \lambda} \frac{d}{dE}(T(\lambda, S(\lambda, E)) + \frac{\partial S}{\partial E} \frac{\partial T}{\partial \lambda}$. Thus

$$\Delta = -(m')^{-\frac{1}{m}} \frac{\partial S}{\partial E} \int_0^S \frac{S^{q+1} - \eta^{q+1}}{m(q+1)(G(\lambda, S) - G(\lambda, \eta))^{\frac{m+1}{m}}}\, d\eta.$$

Hence $\Delta < 0$, for each $E > 0$ and $\lambda > 0$.

Using that $\frac{d}{dE}(T(\lambda, S(\lambda, E^*(\lambda))) = 0$ and by part (a), we see that

$$\frac{\partial T}{\partial \lambda}(\lambda, S(\lambda, E^*(\lambda))) < 0, \text{ for each } \lambda > 0.$$

Therefore $T(\lambda, S^*_\lambda)$ is a decreasing function of λ.

On the other hand, we know that the function $T(\lambda, S(\lambda, \cdot))$ is increasing in $(0, E_1(\lambda))$, where $E_1(\lambda) = (m'G(\lambda, \rho_1))^{1/m}$, hence $E^*(\lambda)$ is the unique maximum of the function $T(\lambda, S(\lambda, \cdot))$. Then $E^*(\lambda) \ge E_1(\lambda)$. By part (a), we have that

$$S(\lambda, E^*(\lambda)) \ge S(\lambda, E_1(\lambda)).$$

Thus

$$T(\lambda, S(\lambda, E^*(\lambda)) \leq (m')^{-1/m} \rho_1^{\frac{m-1-p}{m}} \int_0^1 \left(\frac{1-t^{p+1}}{p+1} \right)^{-1/m} dt.$$

Since $\lim_{\lambda \to +\infty} \rho_1 = \lim_{\lambda \to +\infty} \left(\lambda \frac{(m-1-q)}{(p-m+1)} \right)^{1/(p-q)} = +\infty$, it follows that

$$\lim_{\lambda \to +\infty} T(\lambda, S_\lambda^*) = 0\,.$$

From the fact that $\rho_1 = S(\lambda, E_1(\lambda))$, then after some computations, we obtain that

$$\begin{aligned} T(\lambda, S(\lambda, E_1(\lambda)) \quad &= \quad \lambda^{\frac{m-1-p}{m(p-q)}} (m')^{-\frac{1}{m}} \left(\frac{m-1-q}{p-m+1} \right)^{\frac{m-1-q}{m(p-q)}} \\ &\quad \times \int_0^1 \left(\left(\frac{m-1-q}{p-m+1} \right) \left(\frac{1-t^{p+1}}{p+1} \right) + \frac{1-t^{q+1}}{q+1} \right)^{-\frac{1}{m}} dt\,. \end{aligned}$$

Hence

$$\lim_{\lambda \to 0^+} T(\lambda, S(\lambda, E^*(\lambda)) \geq \lim_{\lambda \to 0^+} T(\lambda, S(\lambda, E_1(\lambda)) = +\infty\,,$$

which proves (d). In order to prove (e), using the concavity of the function $T(\lambda, \cdot)$ and assertions (4) and (5), we obtain that $S_1 < \rho_2$ which prove the first inequality of (e). Using the hypothesis that $T(\lambda, S) = 1/2$, we have

$$\frac{2^m}{m'} I_p^m S^p \leq \frac{S^{p+1}}{p+1} + \lambda \frac{S^{q+1}}{q+1} \leq \frac{2^m}{m'} I_q^m S^m.$$

Since $S_2 > \rho_1$, the above inequalities imply

$$\frac{2^m}{m'} I_p^m (p+1) \leq S_2^{p-m+1} \leq \frac{2^m}{m'} I_q^m (p+1).$$

Hence the proof of Lemma 1 is now complete.

Remark 1. If $\lambda = 0$, then it is easy to prove that the time mapping function is a decreasing function of S. Therefore (1) has a unique positive solution.

Remark 2. Using the methods developed in this article, it is possible to find the exact number of solutions with k nodes in the interval $(0, 1)$ for the problem

$$\begin{gathered} -(|u'|^{m-2}u')' = \lambda |u|^{q-1} u + |u|^{p-1} u, \quad \text{in } (0,1) \\ u(0) = u(1) = 0\,. \end{gathered}$$

In this case the same time mapping $T(\lambda, S)$ may be used.

Remark 3. We conjecture that the same techniques developed in [1] may be applied to determine a lower bound on the number of solutions together with their nodal properties for the problem (1) , with $\lambda < 0$.

Acknowledgments The authors thank the referee for his careful reading of the manuscript and his valuable suggestions.

References

[1] I. Addou and A. Benmezaï, *Boundary Value Problems for the one dimensional p-Laplacian with even super-linear Nonlinearity*, Elect. J. of Differential Equations, Vol. **9** (1999), 1–29.

[2] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. 122, **2** (1994), 519–543.

[3] M. García-Huidobro and P. Ubilla, *Multiplicity of solutions for a class of nonlinear second order equations*, Nonlinear Analysis. T.M.A., Vol. 28, **9** (1997), 1509–1520.

[4] P. Ubilla, *Multiplicity results for the 1-dimensional generalized p-Laplacian*, J. Math. Anal. and Appl. **190** (1995), 611–623.

Justino Sánchez & Pedro Ubilla
Universidad de Santiago de Chile
Casilla 307, Correo 2, Santiago, Chile
e-mail: pubilla@fermat.usach.cl