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A SEMILINEAR PARABOLIC BOUNDARY-VALUE PROBLEM IN BIOREACTORS THEORY

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Abstract. In this paper, we analyze a dynamical model describing the behavior of bioreactors with diffusion. We obtain a convergence result for solutions of asymptotically autonomous semilinear parabolic equations to steady state solutions of the limiting equations. This allows us to establish the convergence of solutions of the initial value problem that describes the dynamics of the bioreactor.

1. Introduction

We consider a Plug Flow bioreactor with diffusion in which occurs a simple growth reaction (one biomass/one substrate). The dynamics of this bioreactor are described by the following system of partial differential equations

$$\begin{gathered} \frac{\partial S}{\partial t} = -q\frac{\partial S}{\partial x} + d\frac{\partial^2 S}{\partial x^2} - \mu(S)X, \quad (t,x) \in]0,\infty[\times]0,l[\\ \frac{\partial X}{\partial t} = -q\frac{\partial X}{\partial x} + d\frac{\partial^2 X}{\partial x^2} + \mu(S)X, \quad (t,x) \in]0,\infty[\times]0,l[\\ S(0,x) = S_0(x), \quad X(0,x) = X_0(x), \quad x \in]0,l[\,, \end{gathered} \tag{1.1}$$

with the boundary conditions

$$\begin{gathered} d\frac{\partial S}{\partial x}(t,0) - qS(t,0) = -qS_{\rm in}\,, \quad \frac{\partial S}{\partial x}(t,l) = 0, \quad t \in]0,\infty[, \\ d\frac{\partial X}{\partial x}(t,0) - qX(t,0) = -qX_{\rm in}\,, \quad \frac{\partial X}{\partial x}(t,l) = 0, \quad t \in]0,\infty[\,. \end{gathered} \tag{1.2}$$

In (1.1)-(1.2), S, X, $S_{\rm in}$, $X_{\rm in}$, q, d, l and μ denote substrate and biomass concentrations in the bioreactor, feed substrate and biomass concentrations, the flow rate, the diffusion rate, the length of the bioreactor and the kinetic function, respectively. Basically the first equation of (1.1) contains a yield coefficient Y, but it is convenient to rescale X to $\frac{X}{Y}$ in order to reduce the number of parameters. For further details on the modeling, refer to [4] or [24]. This paper is devoted to the analysis of (1.1)-(1.2): we aim at proving uniform boundedness of the solutions and describing their omega-limit sets.

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To ease the analysis, we will perform in Section 2 a linear change of state variables which transforms (1.1) into two equations; one of them is nonlinear, but the other one is linear. Next, in the same section, we will show that the operator associated to this linear equation is the infinitesimal generator of a strongly continuous semigroup on $C[0,l]$ (the Banach space of the continuous real-valued functions on $[0,l]$) which is exponentially stable. As a consequence of this, the unique steady state solution of the linear equation is globally exponentially stable in $C[0,l]$. Following this, we will rewrite (1.1)-(1.2) as a nonautonomous semilinear parabolic equation

$$\begin{gathered}\frac{du}{dt} = Au(t) + f(t,u),\\ u(0) = u_0,\end{gathered} \tag{1.3}$$

where A is a linear operator in the Banach space $C[0,l]$ with domain $D(A)$ and (1.3) is asymptotically autonomous with limiting equation

$$\begin{gathered}\frac{du}{dt} = Au(t) + g(u),\\ u(0) = u_0\end{gathered} \tag{1.4}$$

in the sense that:

(i) (1.3) and (1.4) have a unique mild solution in $C[0,l]$, respectively,
(ii) $\lim_{t\to\infty} f(t,u) = g(u)$ uniformly in u on bounded subsets of $C[0,l]$.

Many works available in the literature are devoted to the study of the asymptotic behavior of solutions of equations of type (1.3) and/or (1.4) (see [1, 2, 10, 11, 14, 15, 16, 17, 18, 19, 24], etc.). In the earlier works of N. Chafee [1] and H. Matano [10, 11], the authors dealt with equations of type (1.4) with Neumann and Robin boundary conditions. In [1], one-dimensional equation was considered and the author used the energy function as a Lyapunov function of (1.4) to prove that the omega-limit sets of solutions consist of steady state solutions of (1.4). Observe that this result is proved under the strong assumption that the initial value is continuously differentiable. In [11], Matano proved a more general result. He considered (1.4) in $C(D)$, where D is a bounded domain of $\mathbb{R}^N$, $N \geq 1$. He established that omega-limit sets of bounded solutions of (1.4) consist of its steady state solutions. In [10], he considered one-dimensional equation and proved that the omega-limit sets contain at most one element, that is, each solution of (1.4) either blows up or converges to steady state solution. More recently, Polàčik et al. investigated the asymptotic behavior of solutions of (1.4) with Dirichlet, Neumann and Robin conditions (see [14, 15, 16, 17, 18, 19]). They established that the omega-limit set of bounded solutions of (1.4) can be a set of continuum of steady state solutions ([14, 16, 17, 18]).

However, the knowledge of the behavior of solutions of (1.4) does not give any a priori information on the structure of the omega-limit sets of solutions of (1.3). In [2] the one-dimensional case was considered. It is proved therein that if f is periodic then any bounded solution of (1.3) converges to a periodic solution of (1.3). In [24], the system of type (1.1)-(1.2) has been studied by Smith for a class of monotonic kinetic functions. In this case, the limiting equation (1.4) generates a monotone dynamical system. However, the author does not establish any result on the behavior of solutions of the nonautonomous equation (equivalently (1.1)-(1.2)), as it is mentioned in his remarks section. His result on the asymptotic behavior of the solutions of the limiting equation are valid only for monotonic kinetic functions.

In this paper, we extend the earlier result of [11] to asymptotically autonomous nonlinear equations. In Theorem 3.4, we prove that the ω-limit set of any bounded solution of the nonautonomous equation (1.3) is nonempty and it is contained in a set of steady state solutions of (1.4). This result relies neither on a particular form of f deduced from the reduction of (1.1) nor on the one-dimensional aspect of the equations. It is also established for equations in abstract Banach spaces with more general properties on f (see remarks following the proof of Theorem 3.4). On the other hand, Theorem 3.4 can be applied to many models in practical applications since we do not consider a particular class of kinetic functions. Based on Theorem 3.4 and [10, Theorem A], in Theorem 3.5 we show that every solution of (1.3) that starts in a certain given set, is bounded and converges to a unique steady state solution of (1.4). We finally apply Theorem 3.4 to the limiting equation although it is autonomous.

We introduce the following assumptions. Observe that they are often fullfiled by kinetic models in practical applications.

- A1 $\mu(s) > 0$ for $s > 0$, $\mu(s) = 0$ for $s \leq 0$, μ is bounded as $s \to +\infty$.
- A2 The function $s \to \mu(s)$ is twice continuously differentiable. Moreover, μ and μ' are Holder continuous in $\mathbb{R}$ (of exponent γ).

2. Preliminaries

Let us consider the new function $U(t,x) = S(t,x) + X(t,x)$ and let us introduce the notation $M = S_{\text{in}} + X_{\text{in}}$. Then $U(t,x)$ satisfies:

$$\begin{gathered} \frac{\partial U}{\partial t} = d\frac{\partial^2 U}{\partial x^2} - q\frac{\partial U}{\partial x}, \quad (t,x) \in]0,\infty[\times]0,l[, \\ U(0,x) = U_0(x), \quad x \in]0,l[, \\ d\frac{\partial U}{\partial x}(t,0) = q(U(t,0) - M), \quad \frac{\partial U}{\partial x}(t,l) = 0, \quad t \in]0,\infty[, \end{gathered} \tag{2.1}$$

with $U_0(x) = S_0(x) + X_0(x)$. It is easy to see that (2.1) has a unique steady state solution $\bar{U}$ and $\bar{U}(x) = M$, for all $x \in [0,l]$.

Let $Z = C[0,l]$. We define the linear operator

$$\begin{gathered} D(A) = \{v \in C^2[0,l] : d\frac{\partial v}{\partial x}(0) - \frac{q}{2}v(0) = 0,\ d\frac{\partial v}{\partial x}(l) + \frac{q}{2}v(l) = 0\}, \\ Av = d\frac{\partial^2 v}{\partial x^2} - \frac{q^2}{4d}v, \quad \forall\, v \in D(A). \end{gathered}$$

Note that if $u(t,x) = e^{-\frac{q}{2d}x}(U(t,x) - M)$, where $U(t,x)$ is a solution of (2.1), then we have $u(t) \in D(A)$ as long as $U(t,x)$ is defined and $t > 0$. Moreover,

$$\begin{gathered} \frac{du}{dt} = Au(t), \\ u(0) = u_0. \end{gathered} \tag{2.2}$$

The linear operator A is closed, densely defined and $A + \delta I$ is dissipative in Z, where $\delta = \frac{q^2}{4d}$. Moreover, for any $\lambda > 0$ and $f \in Z$, the ordinary differential equation $\lambda u - Au = f$ has a unique solution $u \in D(A)$. Then, $\lambda - A$ is surjective for $\lambda > 0$. It follows that A is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$ on Z (see [5, Theorem 3.15] or [12, Theorem 4.3]) and

$$\|T(t)\|_{L(Z)} \leq e^{-\delta t}, \quad \forall\, t \geq 0. \tag{2.3}$$

Further, if $\Gamma(x, y, t)$ denotes the fundamental solution of

$$\frac{\partial v}{\partial t} = d\frac{\partial^2 v}{\partial x^2} - \delta v, \quad (t, x) \in]0, \infty[\times]0, l[$$

and

$$d\frac{\partial v}{\partial x}(t, 0) = \frac{q}{2}v(t, 0);\ d\frac{\partial v}{\partial x}(t, l) = -\frac{q}{2}v(t, l), \quad t > 0,$$

then the semigroup $T(t)$ is given by

$$(T(t)v)(x) = \int_0^l \Gamma(x, y, t)v(y)dy, \quad \forall\, t > 0, \quad \forall\, v \in Z. \tag{2.4}$$

(see [11]). Let us recall [11, Lemma 2.2].

Lemma 2.1. *The functions Γ and $\frac{\partial \Gamma}{\partial t}$ are continuous in $[0, l] \times [0, l] \times]0, \infty[$. Moreover, given any $t_0 > 0$, there exists a constant $C_0 > 0$ such that*

$$\sup_{0 \le x \le l} \int_0^l |\frac{\partial \Gamma}{\partial t}(x, y, t)|dy \le \frac{C_0}{t}, \quad \forall\, 0 < t \le t_0. \tag{2.5}$$

We deduce from the lemma above the following result.

Lemma 2.2. *The semigroup $T(t)$ is continuously differentiable and compact on Z for $t > 0$; i.e: $T(t) : Z \to Z$ is compact and for any $v \in Z$, the map $t \to T(t)v$ is continuously differentiable for $t > 0$. Moreover, for any given $t_0 > 0$, there exists $C_0 > 0$ such that*

$$\|AT(t)\|_{L(Z)} \le \frac{C_0}{t}, \quad \forall\, 0 < t \le t_0. \tag{2.6}$$

Proof. The continuous differentiability of $T(t)$ follows from the continuity of $\frac{\partial \Gamma}{\partial t}$ on $[0, l] \times [0, l] \times]0, \infty[$. Then, $T(t)$ maps Z into $D(A)$ for $t > 0$, $AT(t) \in L(Z)$ for $t > 0$ and $AT(t)v = \frac{d}{dt}T(t)v$ for all $t > 0$ and all $v \in Z$. Hence, (2.6) follows from (2.4) and (2.5). Since Γ is continuous on the compact $[0, l] \times [0, l]$ for any fixed $t > 0$, the compactness of $T(t)$ follows from Ascoli-Arzelà's Theorem (see [25, P. 85]). $\square$

Remarks: Indeed, $T(t)$ defines an analytic semigroup (see [24, P. 121]. However, it is more interesting to consider the properties stated in Lemma 2.2 since the condition of continuous differentiability and (2.6) is weaker than analyticity condition. Moreover, the condition in Lemma 2.2 is sufficient to establish the main result in this paper and it is satisfied in much more situations if one thinks of generalization (see remarks in Section 3).

As a consequence of (2.3), the steady state solution $\bar{U} \equiv M$ of (2.1) is globally exponentially stable in Z. Following this, it can be seen that (1.1)-(1.2) is equivalent to the following semilinear parabolic equation

$$\begin{gathered}
\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} - q\frac{\partial u}{\partial x} + \tilde{f}(t, u), \quad (t, x) \in]0, \infty[\times]0, l[\,, \\
u(0, x) = u_0(x), \quad x \in]0, l[\,, \\
d\frac{\partial u}{\partial x}(t, 0) = q(u(t, 0) - S_{\text{in}}); \quad \frac{\partial u}{\partial x}(t, l) = 0, \quad t \in]0, \infty[\,,
\end{gathered} \tag{2.7}$$

where $\tilde{f}(t, u) = -\mu(u)(U(t) - u)$ and $U(t)$ is the solution of the linear equation (2.1). We have that $\tilde{f}$ is continuous in t and locally Lipschitz continuous in u, uniformly in t and $\lim_{t\to\infty} \tilde{f}(t, u) = \tilde{g}(u) = -\mu(u)(M - u)$ uniformly in u on bounded subsets of Z under assumptions (A1)-(A2). Equation (2.7) is then asymptotically autonomous according to the previous definition and its limiting equation is

$$\begin{gathered}\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} - q\frac{\partial u}{\partial x} - \mu(u)(M-u), \quad (t,x) \in]0,\infty[\times]0,l[\,,\\ u(0,x) = u_0(x), \quad x \in]0,l[\,,\\ d\frac{\partial u}{\partial x}(t,0) = q(u(t,0) - S_{\rm in}); \quad \frac{\partial u}{\partial x}(t,l) = 0, \quad t \in]0,\infty[\,.\end{gathered} \tag{2.8}$$

3. Main results

We give here our main result on the asymptotic behavior of solutions of the nonautonomous equation (2.7) (and equivalently the system (1.1)-(1.2)). Equation (2.8) is also analyzed.

3.1. **The nonautonomous equation.** Instead of (2.7) and (2.8), we consider the following equations

$$\begin{gathered}\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} - \frac{q^2}{4d}u + f(t,u), \quad (t,x) \in]0,\infty[\times]0,l[\,,\\ u(0,x) = u_0(x), \quad x \in]0,l[\,,\\ d\frac{\partial u}{\partial x}(t,0) = \frac{q}{2}u(t,0); \quad d\frac{\partial u}{\partial x}(t,l) = -\frac{q}{2}u(t,l), \quad t \in]0,\infty[\end{gathered} \tag{3.1}$$

and

$$\begin{gathered}\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} - \frac{q^2}{4d}u + g(u), \quad (t,x) \in]0,\infty[\times]0,l[\,,\\ u(0,x) = u_0(x), \quad x \in]0,l[,\\ d\frac{\partial u}{\partial x}(t,0) = \frac{q}{2}u(t,0), \quad d\frac{\partial u}{\partial x}(t,l) = -\frac{q}{2}u(t,l), \quad t \in]0,\infty[\,,\end{gathered} \tag{3.2}$$

where $f : [0,\infty[\times Z \to Z$ is continuous and $f :]0,\infty[\times Z \to Z$, $g : Z \to Z$ are continuously differentiable and $\lim_{t\to\infty} f(t,u) = g(u)$ uniformly in u on bounded subsets of Z. These equations are deduced from (2.7) and (2.8) respectively by introducing $u(t,x) = e^{-\frac{q}{2d}x}(v(t,x) - S_{\rm in})$ for any solution v of (2.7) (respectively (2.8)) as in Section 2. So, it is equivalent to study (3.1) in order to understand the behavior of solutions of (2.7). Note that for any $u_0 \in Z$, (3.1) (resp. (3.2)) has a unique mild solution on some interval $[0,t_u[$, that is: $u \in C([0,,t_u[;Z)$ and is solution of the integral equation $u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds$ (resp. $u(t) = T(t)u_0 + \int_0^t T(t-s)g(u(s))ds$) on $[0,t_u[$.

Lemma 3.1. *Assume that (A1)-(A2) hold. Then*

(i) *For any $u_0 \in Z$, the mild solution $u(t)$ of* (3.1) *(resp. of* (3.2)*) is a classical solution; i.e., $u \in C([0,t_u[;Z) \cap C^1(]0,t_u[;Z)$, $u(t) \in D(A)$, for all $0 < t < t_u$ and $u(t)$ satisfies* (3.1) *(resp.* (3.2)*), where $[0,t_u[$ is the maximum interval of existence of $u(t)$.*

(ii) *If $u(t)$ is bounded in Z then, for any $t_0 > 0$ the subsets $\{Au(t),\, t \geq t_0\}$ and $\{\frac{\partial u(t)}{\partial t}, t \geq t_0\}$ are bounded in Z.*

Proof. We give the proof only for solutions of (3.1) since the other case is similar.
(i) The mild solution $u(t)$ of (3.1) is given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds, \quad 0 < t < t_u.$$

Since $T(t)$ is continuously differentiable, we have $T(t)u_0 \in D(A)$, for $0 < t < t_u$ and $AT(t) \in L(Z)$, for $t > 0$. Let ε, T_0 and T_1 be such that $0 < \varepsilon < T_0 \leq T_1 < t_u$ and rewrite the equality above as follows

$$u(t) = T(t-\varepsilon)u(\varepsilon) + \int_{\varepsilon}^{t} T(t-s)f(s,u(s))ds, \quad \varepsilon \leq t \leq T_1.$$

The map $t \to T(t-\varepsilon)u(\varepsilon)$ is continuously differentiable on $]\varepsilon, T_1]$ and $T(t-\varepsilon)u(\varepsilon) \in D(A)$ fo rall $t \in]\varepsilon, T_1]$. Let

$$v(t) = \int_{\varepsilon}^{t} T(t-s)f(s,u(s))ds, \quad \varepsilon \leq t \leq T_1.$$

Since $f : [\varepsilon, T_1] \times Z \to Z$ is continuously differentiable, by [12, Theorem 1.5], v is continuously differentiable on $]\varepsilon, T_1]$ and if $w(t)$ denotes the solution of the integral equation

$$w(t) = T(t-\varepsilon)f(\varepsilon, u(\varepsilon)) + \int_{\varepsilon}^{t} T(t-s)\frac{\partial}{\partial s}f(s,u(s))ds + \int_{\varepsilon}^{t} T(t-s)\frac{\partial}{\partial u}f(s,u(s))w(s)ds$$

on $[\varepsilon, T_1]$. Then

$$\frac{dv}{dt}(t) = w(t) + \int_{\varepsilon}^{t} AT(t-\varepsilon)\frac{\partial}{\partial u}f(s,u(s))u(\varepsilon)ds, \quad \forall\, t \in]\varepsilon, T_1].$$

Therefore, $v(t) \in D(A)$ for all $t \in]\varepsilon, T_1]$. Hence, $u(t) = T(t-\varepsilon)u(\varepsilon) + v(t) \in D(A)$ for all $t \in [T_0, T_1]$ and $\frac{\partial u}{\partial t} \in C([T_0, T_1];\ Z)$. Since T_0 and T_1 are any given numbers in $]0, t_u[$, we have $u \in C([0, t_u[;\ Z) \cap C^1(]0, t_u[;\ Z)$ and $u(t) \in D(A)$ for all $0 < t < t_u$. Moreover, $u(t)$ satisfies (3.1) on $[0, t_u[$.

(ii) Let $0 < a < t_0$ and $\|u(t)\|_Z \leq N_0$, $\|f(t,u(t))\|_Z \leq N_1$, for all $t \geq 0$. We have

$$\begin{aligned} Au(t_0+t) = AT(t_0)u(t) &+ \int_0^{t_0-a} AT(t_0-s)f(s+t, u(s+t))ds \\ &+ \int_{t_0-a}^{t_0} AT(t_0-s)f(s+t,u(s+t))ds. \end{aligned}$$

By (2.6), we have

$$\begin{aligned} &\|AT(t_0)u(t)\|_Z + \int_0^{t_0-a} \|AT(t_0-s)\|\, \|f(s+t,u(s+t))\|_Z ds \\ &\leq \frac{C_0N_0}{t_0} + C_0N_1 \ln(\frac{t_0}{a}), \end{aligned} \tag{3.3}$$

where $\|AT(t)\|$ denotes the norm of $AT(t)$ in $L(Z)$. Moreover, one can check readily that

$$\begin{aligned} &\int_{t_0-a}^{t_0} AT(t_0-s)f(s+t,u(s+t))ds \\ &= \int_{t_0-a}^{t_0} AT(t_0-s)\left(f(s+t,u(s+t)) - f(t_0+t,u(s+t))\right)ds \\ &\quad + \int_{t_0-a}^{t_0} AT(t_0-s)\left(f(t_0+t,u(s+t)) - f(t_0+t,u(t_0+t))\right)ds \\ &\quad + \int_{t_0-a}^{t_0} AT(t_0-s)f(t_0+t,u(t_0+t))ds. \end{aligned}$$

Under Hypotheses (A1) and (A2), $\mu(r)$ is bounded as $r \to \infty$ and f is locally Lipschitz continuous in u, uniformly in t. Then, let μ_0 be a constant such that $|\mu(r)| \le \mu_0$ for all $r \in \mathbb{R}$ and let L_0 be the (local) Lipschitz constant of f with respect to the second variable since $u(t)$ is bounded. Using (2.3) and (2.6) we have, for all $s \in [t_0 - a, t_0]$,

$$\begin{aligned}\|f(s+t, u(s+t)) - f(t_0+t, u(s+t))\|_Z &\le \mu_0 \|T(s+t)V_0 - T(t_0+t)V_0\|_Z \\ &\le \mu_0 (t_0 - s)\|T(t)\| \, \|AT(s)\| \, \|V_0\|_Z \\ &\le \mu_0 C_0 \frac{(t_0 - s)}{t_0 - a} \|V_0\|_Z,\end{aligned}$$

where $V_0(x) = e^{-\frac{q}{2d}x}(U_0(x) - M)$ for all $x \in [0, l]$ (and $T(\tau)V_0$ is a solution of (2.2)). Moreover,

$$\begin{aligned}&\|\int_{t_0-a}^{t_0} AT(t_0 - s) f(t_0 + t, u(t_0 + t)) ds\|_Z \\ &= \|(I - T(a)) f(t_0 + t, u(t_0 + t))\|_Z \le 2N_1.\end{aligned}$$

It follows that

$$\begin{aligned}&\|\int_{t_0-a}^{t_0} AT(t_0 - s) f(s+t, u(s+t)) ds\|_Z \\ &\le \frac{aC_0^2 \mu_0 \|V_0\|_Z}{t_0 - a} + 2N_1 + L_0 \int_{t_0-a}^{t_0} \|AT(t_0 - s)\| \, \|u(s+t)) - u(t_0 + t)\|_Z.\end{aligned}$$

Let $\Delta s = t_0 - s$, for all $s \in [t_0 - a, t_0]$. We have $\Delta s \ge 0$ and

$$\begin{aligned}&\|u(s+t) - u(t_0+t)\|_Z \\ &\le \|(T(t_0) - T(s))u(t)\|_Z + \int_{\max(s-\Delta s, 0)}^{t_0} \|T(t_0 - \tau) f(\tau + t, u(\tau + t))\|_Z d\tau \\ &\quad + \int_{\max(s-\Delta s, 0)}^{s} \|T(s - \tau) f(\tau + t, u(\tau + t))\|_Z d\tau \\ &\quad + \int_0^{\max(s-\Delta s, 0)} \| \left(T(t_0 - \tau) - T(s - \tau)\right) f(\tau + t, u(\tau + t))\|_Z d\tau.\end{aligned}$$

Then,

$$\begin{aligned}&\|u(s+t) - u(t_0+t)\|_Z \\ &\le \frac{C_0 N_0}{t_0 - a} \Delta s + 3N_1 \Delta s \\ &\quad + \int_{\min(s, \Delta s)}^{s} \| \left(T(\tau + \Delta s) - T(\tau)\right) f(s - \tau + t, u(s - \tau + t))\|_Z d\tau \\ &\le \frac{C_0 N_0}{t_0 - a} \Delta s + 3N_1 \Delta s \\ &\quad + \int_{\min(s, \Delta s)}^{s} \int_0^{\Delta s} \|T(\sigma)\| \, \|AT(\tau)\| \, \|f(s - \tau + t, u(s - \tau + t))\|_Z d\sigma d\tau.\end{aligned}$$

Using (2.3) and (2.6) again and the estimate on f, we have

$$\begin{aligned}\|u(s+t)-u(t_0+t)\|_Z &\le \frac{C_0N_0}{t_0-a}\Delta s+3N_1\Delta s+\int_{\min(s,\Delta s)}^{s}\frac{C_0N_1}{\tau}\Delta s d\tau\\ &\le \frac{C_0N_0}{t_0-a}\Delta s+3N_1\Delta s+C_0N_1\ln(\frac{s}{\min(s,\Delta s)})\Delta s\\ &\le \Big(\frac{C_0N_0}{t_0-a}+3N_1+C_0N_1\max\big(\ln(\frac{s}{t_0-a}),\ \ln(\frac{s}{a})\big)\Big)\Delta s\\ &\le \Big(\frac{C_0N_0}{t_0-a}+3N_1+C_0N_1N_2\Big)(t_0-s),\end{aligned}$$

where $N_2=\max\Big(\ln(\frac{t_0}{t_0-a}),\ \ln(\frac{t_0}{a})\Big)$. Then, using (2.6) once again, we have

$$\begin{aligned}&\|\int_{t_0-a}^{t_0}AT(t_0-s)f(s+t,u(s+t))ds\|_Z\\ &\le \frac{aC_0^2\mu_0\|V_0\|_Z}{t_0-a}+2N_1+aL_0C_0\big(\frac{C_0N_0}{t_0-a}+3N_1+C_0N_1N_2\big).\end{aligned}\tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned}\|Au(t_0+t)\|_Z \le \frac{C_0N_0}{t_0}+C_0N_1\ln(\frac{t_0}{a})+\frac{aC_0^2\mu_0\|V_0\|_Z}{t_0-a}+2N_1\\ +aL_0C_0\big(\frac{C_0N_0}{t_0-a}+3N_1+C_0N_1N_2\big),\end{aligned}$$

for any $t\ge 0$. Hence, $Au(t_0+t)$ remains bounded in Z for $t\ge 0$. Since $\|f(t,u(t))\|_Z\le N_1$ and $u(t)$ is a classical solution of (3.1) then, $\frac{\partial u}{\partial t}(t_0+t)$ also remains bounded for $t\ge 0$ and Lemma 3.1 is proved. □

Lemma 3.2. *Assume that (A1) and (A2) hold. Let $u(t)$ be a bounded solution of (3.1) (resp. of (3.2)) then, $K=\overline{\{u(t),\ t\ge 0\}}$ is compact in Z, where $\overline{E}$ denotes the closure of E.*

Proof. By Lemma 2.2, $T(t)$ is compact for $t>0$. As $u(t)$ is bounded in Z, we have $\|f(t,u(t))\|_Z\le N$, for $t\ge 0$ where $N>0$. The compactness of K follows from [12, Lemma 2.4]. □

Let us define the functional

$$J(t,v)=\int_0^l\Big(\frac{d}{2}(\frac{\partial v}{\partial x})^2-\int_0^v F(t,x,w)dw\Big)dx+\frac{q}{4}(v^2(0)+v^2(l)),$$

where $F(t,x,w)=-\Big[\frac{q^2}{4d}w+e^{-\alpha x}\mu(e^{\alpha x}w+S_{\rm in})(U(t,x)-e^{\alpha x}w-S_{\rm in})\Big]$, $\alpha=\frac{q}{2d}$. For any solution $u(t)$ of (3.1), $J(t,u(t))$ is defined and the following statement holds.

Lemma 3.3. *If $u(t)$ is a solution of (3.1), then*

$$\frac{d}{dt}\left(J(t,u(t))\right)=\int_0^l-(\frac{\partial u}{\partial t})^2dx-\int_0^l\Big(\int_0^{u(t,x)}\frac{\partial F}{\partial t}(t,x,w)\,dw\Big)dx$$

for $0<t<t_u$.

Proof. First, we deduce from Lemma 3.1 (i) and [6, Chap 3 Theorem 10] that for any $u_0 \in Z$ the solution $u(t)$ of (3.1) has continuous partial derivatives $\frac{\partial^3 u}{\partial x^3}$ and $\frac{\partial^2 u}{\partial t \partial x}$ on $]0, t_u[\times]0, l[$. Then the following calculation is well founded. By deriving and integrating by parts, we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_0^l \Big(\frac{d}{2}\Big(\frac{\partial u}{\partial x}\Big)^2 - \int_0^u F(t,x,w)dw\Big)dx \\
&= \int_0^l \left(d\frac{\partial^2 u}{\partial t \partial x}\frac{\partial u}{\partial x} - F(t,x,u)\frac{\partial u}{\partial t}\right)dx - \int_0^l \int_0^{u(t,x)} \frac{\partial F}{\partial t}(t,x,w)dwdx \\
&= \int_0^l -\left(\frac{\partial u}{\partial t}\right)^2 dx + d\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}|_{x=l} - d\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}|_{x=0} - \int_0^l \int_0^{u(t,x)} \frac{\partial F}{\partial t}(t,x,w)dwdx\,.
\end{aligned}
$$

Since $u(t)$ satisfies the boundary conditions in (3.1),

$$
d\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}|_{x=l} - d\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}|_{x=0} = -\frac{q}{4}\frac{\partial}{\partial t}\Big(v^2(t,0)+v^2(t,l)\Big).
$$

Hence,

$$
\frac{d}{dt}\left(J(t,u(t))\right) = \int_0^l -\Big(\frac{\partial u}{\partial t}\Big)^2 dx - \int_0^l \int_0^{u(t,x)} \frac{\partial F}{\partial t}(t,x,w)dwdx,
$$

for $0 < t < t_u$ and for any solution $u(t)$ of (3.1). $\square$

Now we can state the main result dealing with the asymptotic behavior of solutions of (3.1).

Theorem 3.4. *Assume that (A1) and (A2) hold and let $u_0 \in Z$ be such that $u(t)$ is a bounded solution of* (3.1)*. Then, the omega limit set $\omega(u_0)$ of $u(t)$ is nonempty, it is contained in $C^2[0,l]$ and it consists of steady state solutions of* (3.2).

Proof. Let $K = \overline{\{u(t), t \geq 0\}}$. By Lemma 3.2, K is compact in Z. Then, $\omega(u_0)$ is nonempty. Let $\varphi \in \omega(u_0)$, there exists a sequence $(t_n)_{n\geq 0}$ such that $t_n \to +\infty$ and $u(t_n) \to \varphi$ in Z as $n \to +\infty$. Let $u_n = u(t_n)$ and $v_n(t) = u(t+t_n)$ for $n \geq 0$ and $t \geq 0$. We have

$$
\begin{aligned}
v_n(t) &= T(t)u_n + \int_{t_n}^{t+t_n} T(t+t_n-s)f(s,u(s))ds \\
&= T(t)u_n + \int_0^t T(t-s)f(s+t_n, v_n(s))ds.
\end{aligned} \tag{3.5}
$$

The set $B = \{v_n(t), n \geq 0, t \geq 0\}$ is bounded in Z and f is locally Lipschitz continuous in u, uniformly in t. Moreover,

$$
\|f(s+t_n, v_n(s)) - f(s+t_m, v_n(s))\|_Z \leq \mu_0 \|T(t_n)V_0 - T(t_m)V_0\|_Z, \quad \text{for all } s \geq 0,
$$

where μ_0 is a constant such that $|\mu(r)| \leq \mu_0$ for all $r \in \mathbb{R}$ and $V_0(x) = e^{-\alpha x}(U_0(x) - M)$ for all $x \in [0,l]$. Then, by Gronwall's inequality, we have: For all $t_0 > 0$ there exists $C > 0$ such that

$$
\sup_{0\leq t\leq t_0} \|v_m(t) - v_n(t)\|_Z \leq C\left(\|u_m - u_n\|_Z + \mu_0\|T(t_m)V_0 - T(t_n)V_0\|_Z\right). \tag{3.6}
$$

It follows from (3.6) that there exists a continuous function $h : [0, \infty[\to Z$ such that

$$
\lim_{n\to\infty} \sup_{0\leq t\leq t_0} \|v_n(t) - h(t)\|_Z = 0 \quad \text{for any given } t_0 > 0\,.
$$

On the other hand, for all $t > 0$,

$$\lim_{n\to\infty} \|f(t+t_n, v_n(t)) - g(v_n(t))\|_Z \le \lim_{n\to\infty} \sup_{w\in B} \|f(t+t_n, w) - g(w)\|_Z = 0. \quad (3.7)$$

So, rewriting (3.5) as

$$v_n(t) = T(t)u_n + \int_0^t T(t-s)(f(s+t_n, v_n(s)) - g(v_n(s))) + \int_0^t T(t-s)g(v_n(s))ds$$

and passing to the limit when $n \to +\infty$, we have

$$h(t) = T(t)\varphi + \int_0^t T(t-s)g(h(s))ds, \quad t \ge 0. \quad (3.8)$$

It follows from (3.8) that $h(t)$ is a mild solution of (3.2) and by Lemma 3.1 (i), $h(t)$ is a classical solution of (3.2). By Lemma 3.1 (i), we have $v_n(t) \in D(A)$ for $n \ge 0$ and $t > 0$. Moreover,

$$\begin{aligned} Av_n(t) = AT(t)u_n &+ \int_0^t AT(t-s)(f(s+t_n, v_n(s)) - g(v_n(s)))ds \\ &+ \int_0^t AT(t-s)g(v_n(s))ds. \end{aligned}$$

Since $T(t)$ is continuously differentiable, $AT(t) \in L(Z)$ for $t > 0$. Then, using (3.7) and (3.8), we have

$$\lim_{n\to\infty} Av_n(t) = Ah(t) \quad \text{in } Z \quad \text{for } t > 0.$$

Hence,

$$\lim_{n\to\infty} \frac{\partial v_n(t)}{\partial t} = \frac{\partial h(t)}{\partial t} \quad \text{in } Z \quad \text{for } t > 0.$$

Now we aim to prove that $\frac{\partial h}{\partial t} = 0$ in $]0, \infty[$. Let $t_0 > 0$, by Lemma 3.3 we have

$$\int_{t_0}^t \int_0^l \big(\frac{\partial u}{\partial s}\big)^2 \, dx\, ds = J(t_0, u(t_0)) - J(t, u(t)) - \int_{t_0}^t \int_0^l \int_0^{u(s,x)} \frac{\partial F}{\partial s}(s, x, w)\, dw\, dx\, ds$$

for $t \ge t_0$. Since $u(t)$ is bounded in Z, it follows from Lemma 3.1 (ii) that $J(t, u(t))$ remains bounded for $t \ge t_0$. Let

$$\begin{aligned} \xi(t) &= \int_{t_0}^t \int_0^l \int_0^{u(s,x)} \frac{\partial F}{\partial s}(s, x, w)\, dw\, dx\, ds \\ &= \int_{t_0}^t \int_0^l \int_0^{u(s,x)} e^{-\alpha x} \mu(e^{\alpha x} w + S_{\rm in}) \frac{\partial U}{\partial s}(s, x) dw\, dx\, ds \\ &= \int_{t_0}^t \int_0^l \frac{\partial}{\partial s} \big(e^{-\alpha x}(U(s,x) - M)\big)\, k(s, x)\, dx\, ds, \end{aligned}$$

where $k(t, x) = \int_0^{u(t,x)} \mu(e^{\alpha x} w + S_{\rm in}) dw$, $\alpha = \frac{q}{2d}$ and $U(t, x)$ is the solution of the linear equation (2.1). Then,

$$\begin{aligned} \xi(t) = &- \int_0^l \int_{t_0}^t \big(e^{-\alpha x}(U(s,x) - M)\big) \frac{\partial k}{\partial s}(s, x) ds dx \\ &+ \int_0^l e^{-\alpha x}[(U(t,x) - M)k(t, x) - (U(t_0, x) - M)k(t_0, x)]dx \end{aligned}$$

and $\frac{\partial k}{\partial t}(t,x) = \mu(e^{\alpha x}u(t,x) + S_{\rm in})\frac{\partial u}{\partial t}(t,x)$. By Lemma 3.1 (ii), $\frac{\partial u}{\partial t}(t)$ remains bounded in Z for $t \geq t_0$ and therefore $|\frac{\partial k}{\partial t}(t,x)|$ remains also bounded for $t \geq t_0$ and $x \in [0,l]$. Furthermore, by (2.3) we have

$$\sup_{0\leq x\leq l} |e^{-\alpha x}(U(t,x) - M)| \leq \sup_{0\leq x\leq l} |e^{-\alpha x}(U_0(x) - M)|e^{-\delta t}, \quad \forall\, t \geq 0.$$

Since $u(t)$ is bounded in Z, it follows that $\xi(t)$ is bounded for $t \geq t_0$. Hence,

$$\int_{t_0}^{\infty}\int_0^l \big(\frac{\partial u}{\partial t}\big)^2\,dx\,dt < \infty, \quad \forall\ t_0 > 0. \tag{3.9}$$

Let $0 < t_0 < t_1 < \infty$. From (3.9), we have

$$\lim_{n\to\infty}\int_{t_0}^{t_1}\int_0^l \big(\frac{\partial v_n}{\partial t}(t)\big)^2\,dx\,dt = \lim_{n\to\infty}\int_{t_0+t_n}^{t_1+t_n}\int_0^l \big(\frac{\partial u}{\partial t}(t)\big)^2\,dx\,dt = 0.$$

Then, regarding h as a function of (t,x), we have

$$\int_{t_0}^{t_1}\int_0^l \big(\frac{\partial h}{\partial t}\big)^2\,dx\,dt = 0.$$

It follows that $\frac{\partial h}{\partial t} = 0$ on any compact set $[t_0,t_1]\times[0,l]$. Then, $\frac{\partial h}{\partial t} = 0$ in $]0,\infty[\times[0,l]$ and therefore $h(t) = \varphi$ in Z for $t \geq 0$. Hence, $\varphi \in D(A)$ and $A\varphi + g(\varphi) = 0$. This proves that $\omega(u_0) \subset C^2[0,l]$ and for any $\varphi \in \omega(u_0)$, we have

$$\begin{gathered} d\frac{\partial^2\varphi}{\partial x^2} - \frac{q^2}{4d}\varphi + g(\varphi) = 0, \quad x \in]0,l[, \\ d\frac{\partial\varphi}{\partial x}(0) - \frac{q}{2}\varphi(0) = 0, \quad d\frac{\partial\varphi}{\partial x}(l) + \frac{q}{2}\varphi(l) = 0. \end{gathered}$$

□

Remarks: Theorem 3.4 can be stated in a more general form: Consider an asymptotically autonomous nonlinear equation of type (1.3) with limiting equation (1.4) on a Banach space Z. Assume that the linear operator A is the infinitesimal generator of a C_0-semigroup of contractions on Z which is continuously differentiable and satisfies (2.6) and that f is Lipschitz continuous (locally with respect to u) in the sense that for any bounded subset B of Z, there is a constant $C > 0$ such that $\|f(t,u) - f(t',v)\|_Z \leq C(|t-t'| + \|u-v\|_Z)$ for $t,t' \in \mathbb{R}_+$, $u,v \in B$. Let $u(t)$ be a precompact, classical solution of (1.3) satisfying

$$\int_{t_0}^{\infty} \|\frac{\partial u}{\partial t}(t)\|_Z dt < \infty, \quad \text{for some } t_0 > 0.$$

Then, the omega-limit set $\omega(u_0)$ of $u(t)$ is nonempty, it is contained in $D(A)$ and it consists of steady state solutions of (1.4). The proof is almost the same one as above. However, the existence of h is proved by application of Ascoli-Arzela's Theorem to the subset $\{v_n, n \geq 0\}$ of $C(]0,\infty[;\ Z)$ and the equicontinuity is established in the same manner as the estimates of $\|u(s+t) - u(t_0+t)\|_Z$ in the proof of Lemma 3.1(ii).

Now we can apply Theorem 3.4 to prove the convergence of solutions of (2.7). Let

$$\mathcal{K}_0 = \{u \in Z,\ 0 \leq u(x) \leq U_0(x)\}.$$

Theorem 3.5. *Assume that (A1) and (A2) hold. Then, for any $u_0 \in \mathcal{K}_0$, there exists a unique steady state solution $\bar{u}$ of* (2.8) *such that the solution $u(t)$ of* (2.7) *converges to $\bar{u}$ in Z.*

Proof. Let $u_0 \in \mathcal{K}_0$. $U(t,x)$ is then an upper-solution of (2.7) and by the standard comparison Theorem, we have $0 \le u(t,x) \le U(t,x)$, for $t \ge 0$, and $x \in [0,l]$ (see [13, Chap 3 Theorem 8]. As $U(t,x)$ is bounded then $u(t,x)$ is also bounded and by Theorem 3.4 we have that $\omega(u_0)$ is nonempty and consists of steady state solutions of (2.8). Then, it follows from [10, Theorem A] that $\omega(u_0)$ contains exactly one steady state solution (the proof in [10] can be easily extended to the nonautonomous case since as in the autonomous case $\omega(u_0)$ consists of solutions of autonomous ordinary differential equations). $\square$

3.2. **The limiting equation.** Let

$$\mathcal{K}_M = \{u \in Z,\ 0 \le u(x) \le M\}\ .$$

Proposition 3.6. *Assume that(A1) and (A2) hold. For any $u_0 \in \mathcal{K}_M$, the solution $u(t)$ of* (2.8) *remains in $\mathcal{K}_M$ (i.e. for all $t \ge 0$, $u(t) \in \mathcal{K}_M$) and there exists a unique steady state solution $\bar{u}$ of* (2.8) *such that $u(t)$ converges to $\bar{u}$ in Z.*

Proof. Let $h(w) = \mu(w)|M-w|$, for $w \in \mathbb{R}$ and $w_0 = \max(S_{\text{in}},\ \|u_0\|_Z)$. Assumption (A1) implies

$$-\mu(w)(M-w) \le h(w), \quad \forall\, w \in \mathbb{R}.$$

Consider the solution $w(t)$ of the ordinary differential equation

$$\begin{gathered} \frac{dw}{dt} = h(w), \\ w(0) = w_0. \end{gathered}$$

We deduce from the standard comparison theorem that

$$0 \le u(t,x) \le w(t) \le M, \quad \text{for } t \ge 0 \text{ and all } x \in [0,l].$$

The convergence of $u(t)$ to steady state solution of (2.8) follows from Theorem 3.4 above and [10, Theorem A]. To apply Theorem 3.4 to (2.8), we have to consider the functional

$$J_1(u) = \int_0^l \Big(\frac{d}{2}\big(\frac{\partial u}{\partial x}\big)^2 - \int_0^u F(x,w)dw\Big)dx + \frac{q}{4}\left(u^2(0) + u^2(l)\right),$$

where $F(x,w) = -(\frac{q^2}{4d}w + e^{-\alpha x}\mu(e^{\alpha x}w + S_{\text{in}})(X_{\text{in}} - e^{\alpha x}w))$ for $x \in [0,l]$ and $w \in \mathbb{R}$, instead of $J(t,u(t))$. Therefore, $\frac{d}{dt}J_1(u(t)) = -\int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx$ for solutions of the corresponding transformed equation (3.2). $\square$

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References

[1] N. Chafee; Asymptotic behavior for solutions of a one-dimensional parabolic equation with homogeneous Neumann boundary conditions, *J. Differential Equations*, 18 (1975), 111-134.

[2] Xu-Yan Chen and H. Matano; Convergence, Asymptotic Periodicity and Finite-Point Blow-Up in One-Dimensional Semilinear Heat Equations, *Journal of Differential Equations* 78, (1989), 160-190.

[3] A. K. Dramé, J. Harmand, A. Rapaport and C. Lobry; Multiple Steady State Profiles in Interconnected Biological Systems, to appear in *Math. and Computer Modeling of Dynamical Systems*.
[4] A. K. Dramé, C. Lobry, A. Rapaport and J. Harmand; Multiple positive solutions for a two point boundary value problem in bioreactors theory, in preparation.
[5] K. J. Engel and R. Nagel; One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics 194, *Springer*, 2000.
[6] A. Friedman; Partial differential equations of parabolic type, *Prentice-Hall*, 1964.
[7] A. Haraux, Systèmes dynamiques dissipatifs et applications; *Masson*, 1990.
[8] M. Konstantin, Hal Smith and Horst R. Thieme; Asymptotically autonomous semiflows: chain recurrence and Lyapunov functions, *Transactions of the American Math. Society*, 347-5 (1995), 1669-1685.
[9] M. Laabissi, M. E. Achhab, J.J. Winkin and D. Dochain; Trajectory analysis of nonisothermal tubular reactor nonlinear models, *Systems and Control Letters*, 42-3 (2000), 169-184.
[10] H. Matano; Convergence of solutions of one-dimensional semilinear parabolic equations, *J. Math. Kyoto Univ.* 18-2 (1978), 221-227.
[11] H. Matano; Asymptotic behavior and stability of solutions of semilinear diffusion equations, *Publ. RIMS, Kyoto Univ.*, 15 (1979), 401-454.
[12] A. Pazy; Semigroups of Linear Operators and Applications to Partial Differential Equations, *Springer-Verlag*, 1983.
[13] M. H. Protter and H. F. Weinberger; Maximum principles in differential equations, *Prentice-Hall*, 1967.
[14] P. Poláčik, P. Brunovský, X. Mora and J. Sola-Morales; Asymptotic behavior of solutions of semilinear elliptic equations on an unbounded strip, *Acta Math. Univ. Comenianae* LX 2 (1991), 163-183.
[15] P. Poláčik and A. Haraux; Convergence to a positive equilibrium for some nonlinear evolution equations, *Acta Math. Univ. Comenianae*, LXI (1992), 129-141.
[16] P. Poláčik and K. P. Rybakowski; Nonconvergent bounded trajectories in semilinear heat equations, *J. Differential Equations*, 124 (1996), 472-494.
[17] P. Poláčik and H. Matano; Existence of L^1-connections between equilibria of a semilinear parabolic equation, *J. Dynam. Differential Equations* 14 (2002), 463-491.
[18] P. Poláčik and F. Simondon; Nonconvergent bounded solutions of semilinear heat equations on arbitrary domains, *J. Differential Equations* 186 (2002), 586-610.
[19] P. Poláčik; Asymptotic symmetry of positive solutions of parabolic equations, *International conference of Differential Equations. Hasselt-Belgium*, July 22-26, 2003.
[20] H. Robert and J. R. Martin; Nonlinear operators and differential equations in Banach spaces, *Wiley-Interscience Publication*, 1967.
[21] R. Schnaubelt; Asymptotically autonomous parabolic evolution equations, *J. Evol. Equ.* 1 (2001) 19-37.
[22] R. E. Showalter; Monotone Operators in Banach Sapce and Nonlinear Partial Differential Equations, *American Mathematical Society*, 1997.
[23] Hal L. Smith and P. Waltman; The Theory of the Chemostat. Dynamics of Microbial Competition, *Cambridge University Press*, 1995.
[24] Hal L. Smith; Monotone dynamical systems: An introduction to the theory of competitive and cooperative systems, *Math. Surveys Monogr. 41, AMS, Providence, RI*, 1995.
[25] K. Yosida; Functional Analysis, *Springer-Verlag Berlin Heidelberg New-York*, 1968.

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