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EXISTENCE OF SOLUTIONS FOR SUBLINEAR EQUATIONS ON EXTERIOR DOMAINS

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Abstract. In this article we prove the existence of an infinite number of radial solutions of $\Delta u + K(r)f(u) = 0$, one with exactly n zeros for each nonnegative integer n on the exterior of the ball of radius $R > 0$, B_R, centered at the origin in $\mathbb{R}^N$ with $u = 0$ on ∂B_R and $\lim_{r\to\infty} u(r) = 0$ where $N > 2$, f is odd with $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞), $f(u) \sim u^p$ with $0 < p < 1$ for large u and $K(r) \sim r^{-\alpha}$ with $0 < \alpha < 2$ for large r.

1. Introduction

In this article we study radial solutions of

$$\Delta u + K(r)f(u) = 0 \quad \text{in } \mathbb{R}^N \backslash B_R, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial B_R, \tag{1.2}$$

$$u \to 0 \quad \text{as } |x| \to \infty \tag{1.3}$$

where B_R is the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ and $K(r) > 0$. We assume:

- (H1) f is odd and locally Lipschitz, $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞), $f'(\beta) > 0$, and $f'(0) < 0$.
- (H2) there exists p with $0 < p < 1$ such that $f(u) = |u|^{p-1}u + g(u)$ where $\lim_{u\to\infty} \frac{|g(u)|}{|u|^p} = 0$.

We let $F(u) = \int_0^u f(s)\, ds$. Since f is odd it follows that F is even and from (H1) it follows that F is bounded below by $-F_0 < 0$, F has a unique positive zero, γ, with $0 < \beta < \gamma$, and

- (H3) $-F_0 < F < 0$ on $(0, \gamma)$, $F > 0$ on (γ, ∞).

Interest in the topic for this paper comes from recent papers [5, 6, 15, 16, 18] about solutions of differential equations on exterior domains. When f grows superlinearly at infinity - i.e. $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, $\Omega = \mathbb{R}^N$, and $K(r) \equiv 1$ then the problem (1.1), (1.3) has been extensively studied [1, 2, 3, 7, 8, 13, 17, 19, 20]. In [11, 12] equations (1.1)-(1.3) were studied with $K(r) \sim r^{-\alpha}$, f superlinear, and $\Omega = \mathbb{R}^N \backslash B_R$ with $R > 0$ with various values for α. In those papers we proved

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existence of an infinite number of solutions - one with exactly n zeros for each nonnegative integer n such that $u \to 0$ as $|x| \to \infty$ for all $R > 0$. In [9] we studied (1.1)-(1.3) with $K(r) \sim r^{-\alpha}$, f bounded, and $\Omega = \mathbb{R}^N \backslash B_R$.

In this article we consider the case where f grows sublinearly at infinity - i.e. $\lim_{u\to\infty} \frac{f(u)}{u^p} = c_0 > 0$ with $0 < p < 1$ and $K(r) \sim r^{-\alpha}$ with $0 < \alpha < 2$. In earlier papers [10, 14] the case where f is sublinear and $\alpha > N - p(N-2)$ was investigated.

Since we are interested in radial solutions of (1.1)-(1.3) we assume that $u(x) = u(|x|) = u(r)$ where $x \in \mathbb{R}^N$ and $r = |x| = \sqrt{x_1^2 + \cdots + x_N^2}$ so that u solves

$$u''(r) + \frac{N-1}{r} u'(r) + K(r) f(u(r)) = 0 \quad \text{on } (R, \infty), \text{ where } R > 0, \tag{1.4}$$

$$u(R) = 0, u'(R) = b \in \mathbb{R}. \tag{1.5}$$

We will also assume that there exist constants $k_1 > 0$, $k_2 > 0$, and α with $0 < \alpha < 2$ such that

(H4) $k_1 r^{-\alpha} \le K(r) \le k_2 r^{-\alpha}$ on $[R, \infty)$.

(H5) K is differentiable, on $[R, \infty)$, $\lim_{r\to\infty} \frac{rK'}{K} = -\alpha$ where $0 < \alpha < 2$, and $\frac{rK'}{K} + 2(N-1) > 0$ on $[R, \infty)$.

Note that (H5) implies $r^{2(N-1)} K(r)$ is increasing.

In this paper we prove the following result.

Theorem 1.1. *Let $N > 2$, $0 < p < 1$, and $0 < \alpha < 2$. Assuming* (H1)–(H5) *then given a nonnegative integer n then there exists a solution of* (1.4)-(1.5) *with n zeros on (R, ∞) and $\lim_{r\to\infty} u(r) = 0$.*

It is interesting to compare this theorem with the case $\alpha > 2$. When $\alpha > 2$ and R is sufficiently large then it was shown in [10, 14] that there are *no* solutions of (1.1)-(1.3) with $\lim_{r\to\infty} u(r) = 0$. On the other hand, it was also shown in [10, 14] that if $R > 0$ is sufficiently small then solutions of (1.1)-(1.3) exist for $\alpha > N - p(N-2)$. We note in Theorem 1.1 that existence of solutions is established *for all* $R > 0$. Also to the best of our knowledge existence of solutions of (1.1)-(1.3) is still unknown when $2 < \alpha < N - p(N-2)$, $0 < p < 1$, and $R > 0$ sufficiently small.

2. Preliminaries

From the standard existence-uniqueness theorem for ordinary differential equations [4] it follows there is a unique solution of (1.4)-(1.5) on $[R, R+\epsilon)$ for some $\epsilon > 0$. We then define

$$E = \frac{1}{2} \frac{u'^2}{K} + F(u). \tag{2.1}$$

Using (H5) we see that

$$E' = -\frac{u'^2}{2rK}\Big(2(N-1) + \frac{rK'}{K}\Big) \le 0 \quad \text{for } 0 < \alpha < 2(N-1). \tag{2.2}$$

Thus E is nonincreasing. Hence it follows that

$$\frac{1}{2}\frac{u'^2}{K} + F(u) = E(r) \le E(R) = \frac{1}{2}\frac{b^2}{K(R)} \quad \text{for } r \ge R \tag{2.3}$$

and so we see from (H2)–(H4) that u and u' are uniformly bounded wherever they are defined from which it follows that the solution of (1.4)-(1.5) is defined on $[R, \infty)$.

Lemma 2.1. *Let u satisfy* (1.4)-(1.5) *and suppose* (H1)–(H5) *hold. Let $N > 2$, $0 < p < 1$, and $0 < \alpha < 2$. If $\lim_{r\to\infty} u(r) = L$ then $f(L) = 0$.*

Proof. Multiplying (1.4) by r^{N-1} and integrating on (r_0, r) where $r_0 > R$ gives
$$r^{N-1}u' = r_0^{N-1}u'(r_0) - \int_{r_0}^{r} t^{N-1}Kf(u)\,dt.$$
Dividing by $r^N K$ gives
$$\frac{u'}{rK} = \frac{r_0^{N-1}u'(r_0)}{r^N K} - \frac{\int_{r_0}^{r} t^{N-1}Kf(u)\,dt}{r^N K}. \tag{2.4}$$
Using (H4) and that $0 < \alpha < 2 < N$ then $r^N K \geq k_1 r^{N-\alpha} \to \infty$ as $r \to \infty$. Also if $f(L) \neq 0$ and r_0, r are sufficiently large then it follows from (H4) that $|\int_{r_0}^{r} t^{N-1}Kf(u)\,dt| \geq \frac{|f(L)|k_1}{2(N-\alpha)}(r^{N-\alpha} - r_0^{N-\alpha}) \to \infty$ as $r \to \infty$ and so by L'Hôpital's rule and (2.4) we see
$$\lim_{r\to\infty} \frac{u'}{rK} = -\lim_{r\to\infty} \frac{\int_{r_0}^{r} t^{N-1}Kf(u)\,dt}{r^N K} = -\lim_{r\to\infty} \frac{f(u)}{N + \frac{rK'}{K}} = -\frac{f(L)}{N-\alpha}. \tag{2.5}$$
Thus by (H4) and (2.5) there exists an $r_0 > R$ such that
$$|u'| \geq \frac{|f(L)|k_1}{2(N-\alpha)} r^{1-\alpha} > 0 \quad \text{for } r > r_0. \tag{2.6}$$
Integrating (2.6) on (r_0, r) then gives
$$|u(r) - u(r_0)| \geq \frac{|f(L)|k_1}{2(N-\alpha)(2-\alpha)}(r^{2-\alpha} - r_0^{2-\alpha}). \tag{2.7}$$
Since $0 < \alpha < 2$ we see the right-hand side of (2.7) goes to $+\infty$ but the left-hand side goes to $|L - u(r_0)|$ - a contradiction. Thus it must be that $f(L) = 0$. ☐

Lemma 2.2. *Let u satisfy* (1.4)-(1.5) *with $b > 0$ and suppose* (H1)–(H5) *hold. Let $N > 2$, $0 < p < 1$, and $0 < \alpha < 2$. Let $0 < \epsilon < \beta$. Then there exists $t_{\epsilon,b} > R$ such that $u(t_{\epsilon,b}) = \beta - \epsilon$ and $u' > 0$ on $[R, t_{\epsilon,b}]$.*

Proof. From (1.5) and since $b > 0$ by assumption we see that u is initially increasing and positive. Now if u has a first critical point, M, with $u' > 0$ on $[R, M)$ then $u'(M) = 0$ and $u''(M) \leq 0$ from which it follows that $f(u(M)) \geq 0$. In addition, by uniqueness of solutions of initial value problems it follows that $u''(M) < 0$ and so $f(u(M)) > 0$ and thus $u(M) > \beta$. Since $u(R) = 0$ the lemma then follows in this case by the intermediate value theorem. Otherwise suppose the lemma does not hold. Then $u' > 0$ and $0 < u < \beta - \epsilon$ for all $r > R$ for some $\epsilon > 0$ and so by (H1) there exists a constant $\epsilon_1 > 0$ and $r_0 > R$ such that $f(u) \leq -\epsilon_1 < 0$ for $r > r_0 > R$. Next multiplying (1.4) by r^{N-1}, integrating on (r_0, r), and using (H4) gives
$$\begin{aligned} -r^{N-1}u' &= -r_0^{N-1}u'(r_0) + \int_{r_0}^{r} t^{N-1}Kf(u)\,dt \\ &\leq -r_0^{N-1}u'(r_0) - \frac{\epsilon_1 k_1}{N-\alpha}(r^{N-\alpha} - r_0^{N-\alpha}). \end{aligned}$$
Thus for some constant C_1,
$$u' \geq C_1 r^{1-N} + \frac{\epsilon_1 k_1}{N-\alpha} r^{1-\alpha}. \tag{2.8}$$

Integrating on (r_0, r) gives:

$$u(r) \geq u(r_0) + \frac{C_1}{2-N}(r^{2-N} - r_0^{2-N}) + \frac{\epsilon_1 k_1}{(N-\alpha)(2-\alpha)}(r^{2-\alpha} - r_0^{2-\alpha}). \tag{2.9}$$

Now the left-hand side of (2.9) is bounded above by β but the right-hand side goes to $+\infty$ as $r \to \infty$ since $0 < \alpha < 2 < N$ - a contradiction. Hence the lemma holds. $\square$

Lemma 2.3. *Let u satisfy* (1.4)-(1.5) *and suppose* (H1)–(H5) *hold. Let $N > 2$, $0 < p < 1$, and $0 < \alpha < 2$. Then there exists a $t_b > R$ such that $u(t_b) = \beta$ and $u' > 0$ on $[R, t_b]$.*

Proof. We rewrite (1.4) as

$$u'' + \frac{N-1}{r}u' + K(r)\frac{f(u)}{u-\beta}(u-\beta) = 0$$

and then make the change of variables

$$u - \beta = r^{\frac{1-N}{2}} v. \tag{2.10}$$

Thus v satisfies

$$v'' + \Big(K(r)\frac{f(u)}{u-\beta} - \frac{(N-1)(N-3)}{4r^2}\Big)v = 0.$$

Suppose now that the lemma does not hold. Then by Lemma 2.2 we see for some sufficiently small $\epsilon > 0$ we have $u' > 0$, $\beta - \epsilon < u < \beta$, and $\frac{f(u)}{u-\beta} > \frac{f'(\beta)}{2}$ (by (H1)) for $r > t_{\epsilon,b}$. Also for some $r_0 > R$ sufficiently large then by (H4) and since $0 < \alpha < 2$,

$$K(r)\frac{f(u)}{u-\beta} - \frac{(N-1)(N-3)}{4r^2} \geq \frac{k_1 f'(\beta)}{2r^\alpha} - \frac{(N-1)(N-3)}{4r^2} \geq \frac{1}{r^2} \text{ for } r > r_0.$$

Next we consider a nontrivial solution w of

$$w'' + \frac{1}{r^2}w = 0 \text{ for } r > r_0.$$

It is straightforward to show $w = C_2 e^{r/2} \sin\big(\frac{\sqrt{3}}{2}\ln(r) + C_3\big)$ for constants $C_2 \neq 0$ and C_3. Hence w has an infinite number of zeros on (r_0, ∞). It follows by the Sturm comparison theorem [4] that between any two zeros of w then v must have a zero and from (2.10) we see that u must equal β. Hence there exists a smallest value of r, denoted t_b, such that $u(t_b) = \beta$ and $0 < u < \beta$ on (R, t_b). Thus $u'(t_b) \geq 0$ and by uniqueness of solutions of initial value problems $u'(t_b) > 0$. Also from Lemma 2.2 we have $u' > 0$ on $[R, t_{\epsilon,b}]$ for all $\epsilon > 0$ and since $u'(t_b) > 0$ it follows that $u' > 0$ on $[R, t_b]$. This completes the proof. $\square$

Lemma 2.4. *Let u satisfy* (1.4)-(1.5) *and suppose* (H1)–(H5) *hold. Let $N > 2$, $0 < p < 1$, and $0 < \alpha < 2$. Then $\lim_{b \to 0^+} t_b = \infty$.*

Proof. First we rewrite (1.4) as

$$(r^{N-1}u')' = -r^{N-1}Kf(u). \tag{2.11}$$

From (H1) we have

$$\text{there exists an } \epsilon_2 > 0 \text{ such that } -f(u) \leq \epsilon_2 u \text{ on } [0, \beta/2]. \tag{2.12}$$

Next we define $t_{b_0} < t_b$ to be the smallest value of $t > 0$ such that $u(t_{b_0}) = \frac{\beta}{2}$. The existence of t_{b_0} follows from Lemma 2.3, since $u(R) = 0$, and the intermediate value theorem. Combining (2.12) with (H4) gives

$$-r^{N-1}Kf(u) \le \epsilon_2 k_2 r^{N-1-\alpha} u \text{ on } [R, t_{b_0}]. \tag{2.13}$$

Integrating (2.11) on $[R, t_{b_0}]$, using (2.13) and that u is increasing on $[R, t_{b_0}]$ (by Lemma 2.3) gives

$$\begin{aligned} r^{N-1}u' &\le R^{N-1}b + \int_R^r \epsilon_2 k_2 t^{N-1-\alpha} u(t)\, dt \\ &\le R^{N-1}b + \epsilon_2 k_2 u(r) \int_R^r t^{N-1-\alpha}\, dt \\ &\le R^{N-1}b + \frac{\epsilon_2 k_2}{N-\alpha} r^{N-\alpha} u. \end{aligned} \tag{2.14}$$

Rewriting this inequality gives

$$u' - \frac{\epsilon_2 k_2}{N-\alpha} r^{1-\alpha} u \le R^{N-1} b r^{1-N}. \tag{2.15}$$

Now let $\epsilon_3 = \frac{\epsilon_2 k_2}{(2-\alpha)(N-\alpha)} > 0$ and denote

$$\mu(r) = e^{-\epsilon_3(r^{2-\alpha} - R^{2-\alpha})} \le 1 \quad \text{for } R \le r \le t_{b_0}. \tag{2.16}$$

Multiplying (2.15) by $\mu(r)$, using (2.16), and integrating on $[R, r] \subset [R, t_{b_0}]$ gives

$$u \le \frac{R^{N-1}b}{N-2}(R^{2-N} - r^{2-N}) e^{\epsilon_3(r^{2-\alpha} - R^{2-\alpha})}. \tag{2.17}$$

Now evaluating (2.17) at t_{b_0} gives

$$\frac{\beta}{2} \le \frac{R^{N-1}b}{N-2}(R^{2-N} - t_{b_0}^{2-N}) e^{\epsilon_3(t_{b_0}^{2-\alpha} - R^{2-\alpha})}. \tag{2.18}$$

Since $0 < \alpha < 2$ it follows from (2.18) that $\lim_{b\to 0^+} t_{b_0} = \infty$ and since $t_{b_0} < t_b$ it follows that

$$\lim_{b\to 0^+} t_b = \infty.$$

This completes the proof. □

Lemma 2.5. *Let u satisfy* (1.4)-(1.5) *and suppose* (H1)–(H5) *hold. Let $N > 2$, $0 < p < 1$, and $0 < \alpha < 2$. Then u has a local maximum, M_b, and $u' > 0$ on $[R, M_b)$.*

Proof. From Lemma 2.3 we know $u(t_b) = \beta$ and $u'(t_b) > 0$ so if the lemma does not hold then it follows from Lemma 2.3 that $u' > 0$ for $r \ge R$. Since u is bounded by (2.3) then it follows from (H2) and (H3) that there exists an L such that $u \to L > \beta$ with L finite. We see then by Lemma 2.1 that $f(L) = 0$ and so (H1) implies $|L| \le \beta$ contradicting that $L > \beta$. Thus u has a local maximum and so there is a smallest value of t, denoted M_b, such that $u'(M_b) = 0$ and $u' > 0$ on $[R, M_b)$. This completes the proof. □

Lemma 2.6. *Let u satisfy* (1.4)-(1.5) *and suppose [(H1)–(H5)] hold. Let $N > 2$, $0 < p < 1$, and $0 < \alpha < 2$. Then $u(r) > 0$ if $b > 0$ is sufficiently small.*

Proof. We use a similar argument as in [12]. First, if $u' > 0$ for $r \geq R$ then $u > 0$ for all $r > R$ and so we are done in this case. Thus we suppose that u has a first critical point M_b. Then $u'(M_b) = 0$, $u''(M_b) \leq 0$, and $u' > 0$ on $[R, M_b)$. By uniqueness of solutions of initial value problems it follows that $u''(M_b) < 0$ and thus M_b is a local maximum. Now if $0 < u(M_b) < \gamma$ then it follows that $E(M_b) = F(u(M_b)) < 0$ (by (H3)). Since E is nonincreasing by (2.2) it follows that u cannot be zero for $r > M_b$ for at such a zero, z_b, of u we would have $0 \leq \frac{1}{2}\frac{u'^2(z_b)}{K(z_b)} = E(z_b) \leq E(M_b) < 0$ a contradiction. So we suppose now that $u(M_b) \geq \gamma$. Then there exists t_{b_1} with $t_b < t_{b_1} < M_b$ so that $u(t_{b_1}) = \frac{\beta+\gamma}{2}$ and $u' > 0$ on $[R, M_b)$.

Next we have the following identity which follows from (1.4) and (2.2),

$$(r^{2(N-1)}KE)' = (r^{2(N-1)}K)'F(u). \tag{2.19}$$

Integrating this on $[R, r]$ gives

$$r^{2(N-1)}KE = \frac{1}{2}R^{2(N-1)}b^2 + \int_R^r (t^{2(N-1)}K)'F(u)\,dt. \tag{2.20}$$

By (H3) we have $F(u) \leq 0$ on $[R, t_b]$ and by (H5) we have $(r^{2(N-1)}K)' > 0$ so for $R < t_b < r$ we have

$$\int_R^r (t^{2(N-1)}K)'F(u)\,dt \leq \int_{t_b}^r (t^{2(N-1)}K)'F(u)\,dt. \tag{2.21}$$

Next on $[\beta, \frac{\beta+\gamma}{2}]$ it follows that there exists an $\epsilon_4 > 0$ such that $F(u) \leq -\epsilon_4 < 0$. Also from (H5) we see there is a $k_0 > 0$ such that

$$2(N-1) + \frac{rK'}{K} \geq k_0 \text{ for } r \geq R. \tag{2.22}$$

Then it follows from (2.22) and (H4) that

$$(t^{2(N-1)}K)' = t^{2N-3}K[2(N-1) + \frac{rK'}{K}] \geq k_0k_1t^{2N-3-\alpha} \text{ for } t \geq R. \tag{2.23}$$

Thus from (2.20)-(2.23) we see

$$t_{b_1}^{2(N-1)}K(t_{b_1})E(t_{b_1}) \leq \frac{1}{2}R^{2(N-1)}b^2 - \frac{\epsilon_4k_0k_1}{2N-2-\alpha}[t_{b_1}^{2N-2-\alpha} - t_b^{2N-2-\alpha}]. \tag{2.24}$$

Next solving (2.3) for u', using (H4), and integrating on $[t_b, t_{b_1}]$ where $u' > 0$ gives

$$\begin{aligned} \int_\beta^{\frac{\beta+\gamma}{2}} \frac{dt}{\sqrt{\frac{b^2}{K(R)} - 2F(t)}} &= \int_{t_b}^{t_{b_1}} \frac{u'(r)\,dr}{\sqrt{\frac{b^2}{K(R)} - 2F(u(r))}} \leq \int_{t_b}^{t_{b_1}} \sqrt{K}\,dr \\ &= \frac{\sqrt{k_2}}{1-\frac{\alpha}{2}}(t_{b_1}^{1-\frac{\alpha}{2}} - t_b^{1-\frac{\alpha}{2}}) \end{aligned} \tag{2.25}$$

and so by (H4) we see from (2.25) that for small $b > 0$ we have

$$0 < \frac{1}{2}\int_\beta^{\frac{\beta+\gamma}{2}} \frac{dt}{\sqrt{-2F(t)}} \leq \int_\beta^{\frac{\beta+\gamma}{2}} \frac{dt}{\sqrt{\frac{b^2}{K(R)} - 2F(t)}} \leq \frac{\sqrt{k_2}}{1-\frac{\alpha}{2}}(t_{b_1}^{1-\frac{\alpha}{2}} - t_b^{1-\frac{\alpha}{2}}). \tag{2.26}$$

It follows then from (2.26) and since $0 < \alpha < 2$ that there exists an $\epsilon_5 > 0$ such that

$$t_{b_1}^{1-\frac{\alpha}{2}} \geq t_b^{1-\frac{\alpha}{2}} + \epsilon_5. \tag{2.27}$$

From the inequality

$$(x+y)^l \geq x^l + y^l \tag{2.28}$$

which holds if $l \geq 1$, $x \geq 0$, and $y \geq 0$, it follows from (2.27) and since $\frac{2}{2-\alpha} \geq 1$ that

$$t_{b_1} \geq t_b + \epsilon_6 \tag{2.29}$$

where $\epsilon_6 = \epsilon_5^{\frac{2}{2-\alpha}}$. Next from (2.27)-(2.29) we see

$$\begin{aligned} t_{b_1}^{2N-2-\alpha} - t_b^{2N-2-\alpha} &= [t_{b_1}^{N-1-\frac{\alpha}{2}} - t_b^{N-1-\frac{\alpha}{2}}][t_{b_1}^{N-1-\frac{\alpha}{2}} + t_b^{N-1-\frac{\alpha}{2}}] \\ &\geq [(t_b+\epsilon_6)^{N-1-\frac{\alpha}{2}} - t_b^{N-1-\frac{\alpha}{2}}]t_b^{N-1-\frac{\alpha}{2}} \\ &\geq \epsilon_7 t_b^{N-1-\frac{\alpha}{2}} \end{aligned} \tag{2.30}$$

where $\epsilon_7 = \epsilon_6^{N-1-\frac{\alpha}{2}} > 0$ and since $N-1-\frac{\alpha}{2} \geq 1$ by (H5).

Thus we see it follows from (2.24), (2.30), and Lemma 2.4 that

$$t_{b_1}^{2(N-1)} K(t_{b_1}) E(t_{b_1}) \leq \frac{1}{2} R^{2N-2} b^2 - \frac{\epsilon_4 \epsilon_7 k_0 k_1}{2N-2-\alpha} t_b^{N-1-\frac{\alpha}{2}} \to -\infty \text{ as } b \to 0^+.$$

Therefore, $E(t_{b_1}) < 0$ if $b > 0$ is sufficiently small. It then follows that $u(t) > 0$ for $t > t_{b_1}$ for if there were a $z_b > t_{b_1}$ such that $u(z_b) = 0$ then since E is nonincreasing we would have $0 \leq E(z_b) \leq E(t_{b_1}) < 0$ - a contradiction. In addition we know from earlier that $u > 0$ on $(R, M_b]$ and $R < t_{b_1} < M_b$. Thus we see $u > 0$ on (R, ∞). This completes the proof. □

Lemma 2.7. *Let u satisfy* (1.4)-(1.5) *and suppose (H1)–(H5) hold. Let $N > 2$, $0 < p < 1$, and $0 < \alpha < 2$. Then $M_b \to \infty$ as $b \to \infty$.*

Proof. If the M_b are bounded then there exists $M_0 > R$ such that $M_b \leq M_0$ for all large b. Now let $v_b = \frac{u}{b}$. Then $v_b(R) = 0$, $v_b'(R) = 1$ and v_b satisfies

$$v_b'' + \frac{N-1}{r} v_b' + \frac{K(r) f(b v_b)}{b} = 0 \quad \text{for } r \geq R. \tag{2.31}$$

As in (2.1)-(2.2),

$$\Big(\frac{1}{2} \frac{v_b'^2}{K(r)} + \frac{F(b v_b)}{b^2}\Big)' \leq 0 \quad \text{for } r \geq R$$

and therefore

$$\frac{1}{2} \frac{v_b'^2}{K(r)} + \frac{F(b v_b)}{b^2} \leq \frac{1}{2K(R)} \quad \text{for } r \geq R.$$

It follows from this that the v_b' are uniformly bounded on $[R, \infty)$ and since $|v_b(r)| \leq \int_R^r |v_b'(s)|\, ds$ it follows that the v_b are uniformly bounded on $[R, M_0 + 1]$. Since f is sublinear we now show that $\frac{K(r) f(b v_b)}{b} \to 0$ on $[R, M_0+1]$ as $b \to \infty$. To see this note that from (H2) we have $\frac{|g(u)|}{|u|^p} \leq 1$ if $|u| \geq u_0 > 0$ and since g is continuous on $[0, u_0]$ then $|g(u)| \leq C_4$ for $|u| \leq u_0$ for some constant C_4. Combining these we see:

$$|g(u)| \leq C_4 + |u|^p \text{ for all } u. \tag{2.32}$$

Therefore since the v_b are uniformly bounded on $[R, M_0+1]$ and $0 < p < 1$ then

$$\begin{aligned} \Big|\frac{K(r) f(b v_b)}{b}\Big| &= K(r) \Big|\frac{|v_b|^{p-1} v_b}{b^{1-p}} + \frac{g(b v_b)}{b}\Big| \\ &\leq K(r) \Big(\frac{|v_b|^p}{b^{1-p}} + \frac{C_4}{b} + \frac{|v_b|^p}{b^{1-p}}\Big) \to 0 \quad \text{as } b \to \infty. \end{aligned} \tag{2.33}$$

Thus from (2.31), (2.33), and since the v_b' are uniformly bounded it follows that the v_b'' are also uniformly bounded on $[R, M_0+1]$. Then by the Arzela-Ascoli theorem there exists a subsequence of the v_b and v_b' (still denoted v_b and v_b') such that $v_b \to v$ uniformly and $v_b' \to v'$ uniformly on $[R, M_0+1]$. In addition, $v'' + \frac{N-1}{r}v' = 0$, $v(R) = 0$, and $v'(R) = 1$. Thus $v = \frac{R}{N-2}(1 - (\frac{R}{r})^{N-2})$. In particular $v' > 0$. On the other hand, $v_b'(M_b) = 0$ and since the M_b are bounded by M_0 then there is a subsequence (still labeled M_b) such that $M_b \to M$ and since $v_b' \to v'$ uniformly on $[R, M_0+1]$ then $0 < v'(M) = \lim_{b\to\infty} v_b'(M_b) = 0$ - a contradiction. Thus it must be that $M_b \to \infty$ as $b \to \infty$. This completes the proof. □

Lemma 2.8. - *Let u satisfy* (1.4)-(1.5) *and suppose* (H1)-(H5) *hold. Let $N > 2$, $0 < p < 1$, and $0 < \alpha < 2$. Then $u(M_b) \to \infty$ as $b \to \infty$. In addition, there exists a constant $\epsilon_5 > 0$ such that*

$$[u(M_b)]^{\frac{1-p}{2}} \geq \epsilon_5 M_b^{1-\frac{\alpha}{2}}.$$

Proof. It follows from Lemma 2.6 that

$$\frac{u}{b} \to \frac{R}{N-2}\Big(1 - \Big(\frac{R}{r}\Big)^{N-2}\Big) \quad \text{uniformly on } [R, 2R].$$

Hence

$$\frac{u(2R)}{b} \to \frac{R}{N-2}\left(1 - 2^{2-N}\right) \quad \text{as } b \to \infty.$$

Thus $u(2R) \geq \frac{R}{2(N-2)}\left(1 - 2^{2-N}\right) b$ for sufficiently large b, and therefore $u(2R) \to \infty$ as $b \to \infty$. Since $M_b \to \infty$ as $b \to \infty$ (by Lemma 2.7), it follows that $M_b > 2R$ for large b, and since u is increasing on $[R, M_b)$ it follows that $u(M_b) \geq u(2R) \to \infty$ from which it follows that $u(M_b) \to \infty$ as $b \to \infty$. This completes the first part of the proof.

Next, from (2.1)-(2.2) we have

$$\frac{1}{2}\frac{u'^2}{K} + F(u) \geq F(u(M_b)) \quad \text{on } [R, M_b].$$

Rewriting this, integrating on $[R, M_b]$ and using (H4) gives

$$\begin{aligned}
\int_R^{M_b} \frac{u'}{\sqrt{2}\sqrt{F(u(M_b)) - F(u(t))}} &\geq \int_R^{M_b} \sqrt{K}\, dr \\
&\geq \int_R^{M_b} \sqrt{k_1} r^{-\frac{\alpha}{2}}\, dr \\
&= \frac{\sqrt{k_1}(M_b^{1-\frac{\alpha}{2}} - R^{1-\frac{\alpha}{2}})}{1-\frac{\alpha}{2}}.
\end{aligned} \tag{2.34}$$

Changing variables on the left-hand side, rewriting, and changing variables again gives

$$\begin{aligned}
\int_R^{M_b} \frac{u'}{\sqrt{2}\sqrt{F(u(M_b)) - F(u(t))}} &= \int_0^{u(M_b)} \frac{dt}{\sqrt{2}\sqrt{F(u(M_b)) - F(t)}} \\
&= \frac{u(M_b)}{\sqrt{2}\sqrt{F(u(M_b))}} \int_0^1 \frac{ds}{\sqrt{1 - \frac{F(u(M_b)s)}{F(u(M_b))}}}.
\end{aligned} \tag{2.35}$$

From the first part of the theorem we know that $u(M_b) \to \infty$ as $b \to \infty$. Then from (H2) it follows that $F(u) = \frac{u^{p+1}}{p+1} + G(u)$ where $G(u) = \int_0^u g(s)\,ds$. In a similar way to (2.32) it follows that

$$|G(u)| \le C_5 + \frac{1}{2(p+1)}|u|^{p+1} \quad \text{for all } u \text{ for some constant } C_5. \tag{2.36}$$

This along with (H2) and that $0 < p < 1$ gives

$$\lim_{b\to\infty} \int_0^1 \frac{ds}{\sqrt{1 - \frac{F(u(M_b)s)}{F(u(M_b))}}} = \int_0^1 \frac{ds}{\sqrt{1-s^{p+1}}} < \int_0^1 \frac{ds}{\sqrt{1-s}} = 2. \tag{2.37}$$

In addition we see that

$$\sqrt{F(u(M_b))} = [u(M_b)]^{\frac{p+1}{2}} \sqrt{\frac{1}{p+1} + \frac{G(u(M_b))}{[u(M_b)]^{p+1}}}. \tag{2.38}$$

Since $u(M_b) \to \infty$ as $b \to \infty$ and $\frac{G(u)}{|u|^{p+1}} \to 0$ as $u \to \infty$ it follows from (2.38) that

$$\lim_{b\to\infty} \frac{[u(M_b)]^{\frac{p+1}{2}}}{\sqrt{F(u(M_b))}} = \sqrt{p+1}. \tag{2.39}$$

Therefore from (2.39) for large b we have

$$\frac{u(M_b)}{\sqrt{F(u(M_b))}} \le 2\sqrt{p+1}\,[u(M_b)]^{\frac{1-p}{2}}. \tag{2.40}$$

Combining (2.34)-(2.40) we obtain for large b,

$$[u(M_b)]^{\frac{1-p}{2}} \ge \frac{\sqrt{k_1}}{2(2-\alpha)\sqrt{p+1}} \left(M_b^{1-\frac{\alpha}{2}} - R^{1-\frac{\alpha}{2}}\right). \tag{2.41}$$

Finally since $M_b \to \infty$ as $b \to \infty$ (by Lemma 2.7) we obtain

$$[u(M_b)]^{\frac{1-p}{2}} \ge \epsilon_5 M_b^{1-\frac{\alpha}{2}} \tag{2.42}$$

with $\epsilon_5 = \frac{\sqrt{k_1}}{4(2-\alpha)\sqrt{p+1}} > 0$. This completes the proof. $\square$

Lemma 2.9. *Let u satisfy* (1.4)-(1.5) *and suppose (H1)–(H5) hold. Let $N > 2$, $0 < p < 1$, and $0 < \alpha < 2$. Then for sufficiently large b there exists a $z_b > M_b$ such that $u' < 0$ on $(M_b, z_b]$ and $u(z_b) = 0$. In addition, given a positive integer n then if b is sufficiently large then u has n zeros on (R, ∞).*

Proof. First let $v(r) = u(r + M_b)$. Then $v(0) = u(M_b)$, $v'(0) = u'(M_b) = 0$, and (1.4) becomes

$$v'' + \frac{N-1}{r+M_b}v' + K(r+M_b)\left(|v|^{p-1}v + g(v)\right) = 0. \tag{2.43}$$

Next let

$$w_\lambda(r) = \lambda^{-\frac{2-\alpha}{1-p}} v(\lambda r) = \lambda^{-\frac{2-\alpha}{1-p}} u(\lambda r + M_b) \quad \text{where } \lambda^{\frac{2-\alpha}{1-p}} = u(M_b). \tag{2.44}$$

Then $w_\lambda(0) = \lambda^{-\frac{2-\alpha}{1-p}} u(M_b) = 1$ and $w_\lambda'(0) = 0$. Now recall from Lemmas 2.7 and 2.8 that $M_b \to \infty$ and $u(M_b) \to \infty$ as $b \to \infty$. Thus $[u(M_b)]^{\frac{2-\alpha}{1-p}} = \lambda \to \infty$ as $b \to \infty$. In addition we see from (2.43)-(2.44) that w_λ solves

$$w_\lambda'' + \frac{N-1}{r + \frac{M_b}{\lambda}} w_\lambda' + \lambda^\alpha K(\lambda r + M_b)\left[|w_\lambda|^{p-1}w_\lambda + \lambda^{-\frac{(2-\alpha)p}{1-p}} g(\lambda^{\frac{2-\alpha}{1-p}} w_\lambda)\right] = 0. \tag{2.45}$$

We now define

$$E_\lambda = \frac{1}{2}\frac{w_\lambda'^2}{\lambda^\alpha K(\lambda r + M_b)} + \frac{1}{p+1}|w_\lambda|^{p+1} + \lambda^{\frac{-(2-\alpha)(1+p)}{1-p}} G(\lambda^{\frac{2-\alpha}{1-p}} w_\lambda). \tag{2.46}$$

Using (2.45) and (H5) we obtain

$$\begin{aligned} E_\lambda' &= -\frac{\lambda^{1-\alpha} w_\lambda'^2}{2(\lambda r + M_b)K(\lambda r + M_b)}\Big(2(N-1) + \frac{(\lambda r + M_b)K'(\lambda r + M_b)}{K(\lambda r + M_b)}\Big) \\ &\le 0 \quad \text{for } r \ge 0. \end{aligned} \tag{2.47}$$

Thus

$$E_\lambda(r) \le E_\lambda(0) = \frac{1}{p+1} + \lambda^{\frac{-(2-\alpha)(1+p)}{1-p}} G(\lambda^{\frac{2-\alpha}{1-p}}). \tag{2.48}$$

From (H2) it follows that $\lambda^{\frac{-(2-\alpha)(1+p)}{1-p}} G(\lambda^{\frac{2-\alpha}{1-p}}) \to 0$ as $\lambda \to \infty$. In addition it follows from (2.36), (2.46), and (2.48) that for large λ,

$$\frac{1}{2}\frac{w_\lambda'^2}{\lambda^\alpha K(\lambda r + M_b)} + \frac{|w_\lambda|^{p+1}}{2(p+1)} \le \frac{2}{p+1}. \tag{2.49}$$

Hence the w_λ are uniformly bounded on $[0,\infty)$. In addition, it follows from (H4) and (2.49) that the w_λ' are uniformly bounded by $\sqrt{\frac{4k_2}{p+1}} r^{-\alpha/2}$ on (r,∞). Then from (2.45) it follows that the w_λ'' are uniformly bounded by $C_6 r^{-(\frac{\alpha}{2}+1)}$ for some constant C_6. Thus w_λ, w_λ', and w_λ'' are uniformly bounded on compact subsets of $(0,\infty)$ and so by the Arzela-Ascoli theorem a subsequence (still labeled w_λ and w_λ') converges uniformly on compact subsets of $(0,\infty)$ to some w and w'. In addition, by the fundamental theorem of calculus with $0 \le r_1 < r_2$ we have

$$\begin{aligned} |w_\lambda(r_1) - w_\lambda(r_2)| &\le \int_{r_1}^{r_2} |w_\lambda'(s)|\,ds \\ &\le \int_{r_1}^{r_2} \sqrt{\frac{4k_2}{p+1}} s^{-\alpha/2}\,ds \\ &= \sqrt{\frac{4k_2}{p+1}}[r_2^{1-\alpha/2} - r_1^{1-\alpha/2}] \end{aligned} \tag{2.50}$$

and so we see from (2.50) and since $0 < \alpha < 2$ that the w_λ are equicontinuous on compact subsets of $[0,\infty)$. Thus it follows that $w(r)$ is continuous on $[0,\infty)$ and in particular $w(0) = 1$.

Next we show that w_λ has a large number of zeros for large λ and hence u has a large number of zeros for large b.

So suppose $w > 0$ on $[0,\infty)$. We see then from (2.45) and (H4) that

$$\begin{aligned} &-(r + \frac{M_b}{\lambda})^{N-1} w_\lambda' \\ &= \int_0^r \lambda^\alpha K(\lambda(r + \frac{M_b}{\lambda}))(r + \frac{M_b}{\lambda})^{N-1}\Big(|w_\lambda|^{p-1}w_\lambda + \lambda^{-\frac{(2-\alpha)p}{1-p}} g(\lambda^{\frac{2-\alpha}{1-p}} w_\lambda)\Big). \end{aligned} \tag{2.51}$$

We claim now that

$$\lim_{\lambda\to\infty} \int_0^r \lambda^\alpha K(\lambda(r + \frac{M_b}{\lambda}))(r + \frac{M_b}{\lambda})^{N-1}\Big(\lambda^{-\frac{(2-\alpha)p}{1-p}} g(\lambda^{\frac{2-\alpha}{1-p}} w_\lambda)\Big) = 0 \tag{2.52}$$

on any fixed compact subset of $[0,\infty)$.

To see this note as in (2.32) we can similarly obtain the inequality $|g(u)| \leq C_7 + \epsilon|u|^p$ for all u for some constant C_7. Therefore using this and (H4) in (2.51) we see that

$$\begin{aligned} &\Big|\int_0^r \lambda^\alpha K(\lambda(t+\frac{M_b}{\lambda}))(t+\frac{M_b}{\lambda})^{N-1}\left(\lambda^{-\frac{(2-\alpha)p}{1-p}} g(\lambda^{\frac{2-\alpha}{1-p}} w_\lambda)\right)\,dt\Big| \\ &\leq \int_0^r k_2(t+\frac{M_b}{\lambda})^{N-1-\alpha}\left(C_7\lambda^{-\frac{(2-\alpha)p}{1-p}} + \epsilon|w_\lambda|^p\right)\,dt. \end{aligned} \tag{2.53}$$

Now it follows from (2.42) and (2.44) that $M_b \leq \epsilon_6[u(M_b)]^{\frac{1-p}{2-\alpha}} = \epsilon_6\lambda$ where $\epsilon_6 = \epsilon_5^{-\frac{2}{2-\alpha}}$ so that for some subsequence $\frac{M_b}{\lambda} \to A$ with $0 \leq A \leq \epsilon_6$ and thus for large λ we obtain from (2.53),

$$\begin{aligned} &\int_0^r k_2(t+\frac{M_b}{\lambda})^{N-1-\alpha}\left(C_7\lambda^{-\frac{(2-\alpha)p}{1-p}} + \epsilon|w_\lambda|^p\right)\,dt \\ &\leq C_7k_2\lambda^{-\frac{(2-\alpha)p}{1-p}}\int_0^r (t+2\epsilon_6)^{N-1-\alpha} + \epsilon k_2\int_0^r (t+2\epsilon_6)^{N-1-\alpha}|w_\lambda|^p. \end{aligned}$$

Both of these terms are small on any compact subset of $[0,\infty)$ (the first since $\lambda \to \infty$ and the second term by (2.49)) and so both of these limit to zero as $\lambda \to \infty$. This establishes (2.52).

Therefore we see by using (H4) and taking limits in (2.51) we obtain

$$-(r+A)^{N-1}w' \geq k_1\int_0^r (t+A)^{N-1-\alpha}w^p\,dt \quad \text{on } (0,\infty). \tag{2.54}$$

Since $w > 0$ on $[0,\infty)$ it follows from (2.54) that w is decreasing so that

$$-(r+A)^{N-1}w' \geq k_1w^p\frac{(r+A)^{N-\alpha} - A^{N-\alpha}}{N-\alpha} \quad \text{on } (0,\infty). \tag{2.55}$$

Rewriting (2.55) gives

$$-w'w^{-p} \geq \frac{k_1}{N-\alpha}(r+A)^{1-\alpha} - \frac{k_1A^{N-\alpha}}{N-\alpha}(r+A)^{1-N} \text{ on } (0,\infty). \tag{2.56}$$

Next we analyze the two cases $A = 0$ and $A \neq 0$ separately.

Case 1: $A \neq 0$. Integrating (2.56) on $(0,r)$ gives

$$\begin{aligned} &-\Big(\frac{w^{1-p}-1}{1-p}\Big) \\ &\geq \frac{k_1}{N-\alpha}\Big(\frac{(r+A)^{2-\alpha}}{2-\alpha} - \frac{A^{2-\alpha}}{2-\alpha}\Big) - \frac{k_1A^{N-\alpha}}{N-\alpha}\Big(\frac{(r+A)^{2-N}}{2-N} - \frac{A^{2-N}}{2-N}\Big). \end{aligned}$$

Thus for some constant C_8 we obtain

$$\frac{w^{1-p}-1}{1-p} \leq -\frac{k_1(r+A)^{2-\alpha}}{(N-\alpha)(2-\alpha)} - \frac{k_1(r+A)^{2-N}A^{N-\alpha}}{(N-2)(N-\alpha)} + C_8. \tag{2.57}$$

The right-hand side of (2.57) goes to $-\infty$ as $r \to \infty$ since $0 < \alpha < 2$, $0 < p < 1$, $N > 2$ and so we see that w becomes negative which is a contradiction because we assumed $w > 0$ and so w and hence u must have a zero for sufficiently large b.

Case 2: $A = 0$. In this case we see that (2.56) becomes

$$-w'w^{-p} \geq \frac{k_1}{N-\alpha}r^{1-\alpha} \quad \text{on } (0,\infty). \tag{2.58}$$

Integrating (2.58) on $(0, r]$ gives

$$\frac{w^{1-p}-1}{1-p} \leq -\frac{k_1 r^{2-\alpha}}{(N-\alpha)(2-\alpha)}$$

and therefore we see that w becomes negative. Thus we again obtain a contradiction and so w and hence u has a zero if b is sufficiently large.

Thus there exists a $z_b > R$ such that $u(z_b) = 0$ and $u > 0$ on (R, z_b). In addition by uniqueness of solutions of initial value problems it follows that $u'(z_b) < 0$ and then we can similarly show as in Lemma 2.5 that u has a local minimum $m_b > z_b$ for large enough $b > 0$ and also that w has a second zero $z_{2,b}$ (and hence u has a second zero) if b is sufficiently large. In a similar way, given any positive integer n we can show for large enough b that u has n zeros on (R, ∞). Since $w_\lambda \to w$ uniformly on compact sets it follows then that if λ is sufficiently large then w_λ will have n zeros on $(0, \infty)$ and hence $u(r, b)$ will have n zeros on (R, ∞) if $b > 0$ is sufficiently large. This completes the proof. $\square$

3. Proof of Theorem 1.1

We consider the set

$$\{b > 0 |\ u(r, b) > 0 \text{ for all } r > R\}.$$

This set is nonempty by Lemma 2.6 and is bounded from above by Lemma 2.9 so there exists a $b_0 > 0$ such that

$$b_0 = \sup\{b > 0 |\ u(r, b) > 0 \text{ for all } r > R\}.$$

We show now that $u(r, b_0) > 0$ for $r > R$. If $u(r_0, b_0) = 0$ and $u(r, b_0) > 0$ on (R, r_0) then $u'(r_0, b_0) \leq 0$. By uniqueness of solutions of initial value problems it follows that $u'(r_0, b_0) < 0$. Thus for $r_1 > r_0$ and r_1 sufficiently close to r_0 we have $u(r_1, b_0) < 0$. Then for b close to b_0 with $b < b_0$ then $u(r_1, b) < 0$ contradicting the definition of b_0. Hence $u(r, b_0) > 0$ for $r > R$. Now by Lemma 2.3 we know that $u(r, b_0)$ must get larger than β. If $u' > 0$ for all $r \geq R$ then since u is bounded it follows that u would has a limit which by Lemma 2.1 would have to be less than or equal to β. Thus we see that $u(r, b_0)$ must have a local maximum $M_{b_0} > R$ and $u' > 0$ on $[R, M_{b_0})$. Next we show $E(r, b_0) \geq 0$ for all $r \geq R$. If $E(r_0, b_0) < 0$ then $E(r_0, b) < 0$ for $b > b_0$ and b close to b_0. On the other hand, since $b > b_0$ it follows that there exists a z_b such that $u(z_b, b) = 0$. Thus $E(z_b, b) \geq 0$. Since E is nonincreasing this implies $z_b < r_0$ for all $b > b_0$. However $z_b \to \infty$ as $b \to b_0^+$ for if the z_b were bounded then this would force a subsequence of the z_b to converge to some z_0 and then $u(z_0, b_0) = 0$ contradicting that $u(r, b_0) > 0$. Thus $E(r, b_0) \geq 0$ for all $r \geq R$. It now follows that $u(r, b_0)$ cannot have a positive local minimum, $m_{b_0} > M_{b_0}$ for at such a point $u'(m, b_0) = 0$, $u''(m, b_0) \geq 0$ and so $f(u(m, b_0)) \leq 0$. Since $u(m, b_0) > 0$ this then forces $0 < u(m, b_0) \leq \beta$ and thus $E(m, b_0) = F(u(m, b_0)) < 0$ contradicting that $E(r, b_0) \geq 0$. Thus $u'(r, b_0) < 0$ for $r > M_{b_0}$ and so $\lim_{r\to\infty} u(r, b_0)$ exists. Denoting this limit as L then $L \geq 0$ since $u(r, b_0) > 0$ for $r > R$ and by Lemma 2.1 we have $f(L) = 0$ so that $L = 0$ or $L = \beta$. Then a similar argument as in Lemma 2.2 shows that $u(r, b_0)$ gets less than β and so it follows that $L = 0$ and thus $\lim_{r\to\infty} u(r, b_0) = 0$. Hence $u(r, b_0)$ is a positive solution on (R, ∞) and $\lim_{r\to\infty} u(r, b_0) = 0$.

Next from a lemma in [9] if $b > b_n$ is sufficiently close to b_n where $u(r, b_n)$ has n zeros on (R, ∞) and

$$\lim_{r\to\infty} u(r, b) = 0$$

then $u(r, b)$ has at most $n + 1$ zeros on (R, ∞). From this lemma it then follows that

$$\{b > b_0 | u(r, b) \text{ has exactly one zero on } (R, \infty)\}$$

is nonempty and again from Lemma 2.9 this set is bounded from above. Thus there exists a $b_1 > b_0$ such that

$$b_1 = \sup\{b > b_0 | u(r, b) \text{ has exactly one zero on } (R, \infty)\}.$$

As above we can show $u(r, b_1)$ has exactly one zero on (R, ∞) and

$$\lim_{r\to\infty} u(r, b_1) = 0.$$

Similarly we can find $b_n > b_{n-1}$ such that $u(r, b_n)$ has exactly n zeros on (R, ∞) and

$$\lim_{r\to\infty} u(r, b_n) = 0.$$

This completes the proof of Theorem 1.1.

References

[1] H. Berestycki, P.L. Lions; Non-linear scalar field equations I, *Arch. Rational Mech. Anal.*, Volume 82, 313-347, 1983.

[2] H. Berestycki, P.L. Lions; Non-linear scalar field equations II, *Arch. Rational Mech. Anal.*, Volume 82 (1983), 347-375.

[3] M. Berger; Nonlinearity and functional analysis, Academic Free Press, New York, 1977.

[4] G. Birkhoff, G. C. Rota; *Ordinary differential equations*, Ginn and Company, 1962.

[5] A. Castro, L. Sankar, R. Shivaji; Uniqueness of nonnegative solutions for semipositone problems on exterior domains, *Journal of Mathematical Analysis and Applications*, Volume 394, Issue 1 (2012), 432-437.

[6] M. Chhetri, L. Sankar, R. Shivaji; Positive solutions for a class of superlinear semipositone systems on exterior domains, *Boundary Value Problems,* 2014 (2014), 198-207.

[7] C. Cortazar, J. Dolbeault, M. Garcia-Huidobro, R. Manasevich; Existence of sign changing solutions for an equation with weighted p-Laplace operator, *Nonlinear Analysis: Theory, Methods, Applications,* Vol 110 (2014), 1-22.

[8] C. Cortazar, M. Garcia-Huidobro, C. Yarur; On the existence of sign changing bound state solutions of a quasilinear equation, *Journal of Differential Equations,* Vol 254 Issue 16 (2013), 2603-2625.

[9] J. Iaia; Existence and nonexistence for semilinear equations on exterior domains, submitted to the *Journal of Partial Differential Equations*, 2016.

[10] J. Iaia; Existence and nonexistence of solutions for sublinear equations on exterior domains, submitted to the *Electronic Journal of Differential Equations*, Vol. 2017 (2017), No. 214, pp. 1-13.

[11] J. Iaia; Existence for semilinear equations on exterior domains, *Electronic Journal of the Qualitative Theory of Differential Equations*, No. 108 (2016), 1-12.

[12] J. Iaia; Existence of solutions for semilinear problems with prescribed number of zeros on exterior domains, to appear in *Journal of Mathematical Analysis and Applications*, 446 (2017), 591-604.

[13] C. K. R. T. Jones, T. Kupper; On the infinitely many solutions of a semi-linear equation, *SIAM J. Math. Anal.*, Volume 17 (1986), 803-835.

[14] J. Joshi; Existence and nonexistence of solutions of sublinear problems with prescribed number of zeros on exterior domains, *Electronic Journal of Differential Equations*, Vol. 2017 (2017), No. 133, 1-10.

[15] E. K. Lee, R. Shivaji, B. Son; Positive radial solutions to classes of singular problems on the exterior of a ball, *Journal of Mathematical Analysis and Applications,* 434, No. 2 (2016), 1597-1611.

[16] E. Lee, L. Sankar, R. Shivaji; Positive solutions for infinite semipositone problems on exterior domains, *Differential and Integral Equations*, Volume 24, Number 9/10 (2011), 861-875.

[17] K. McLeod, W. C. Troy, F. B. Weissler; Radial solutions of $\Delta u + f(u) = 0$ with prescribed numbers of zeros, *Journal of Differential Equations*, Volume 83, Issue 2 (1990), 368-373.

[18] L. Sankar, S. Sasi, R. Shivaji; Semipositone problems with falling zeros on exterior domains, *Journal of Mathematical Analysis and Applications*, Volume 401, Issue 1 (2013), 146-153.

[19] J. Serrin, M. Tang; Uniqueness of ground states for quasilinear elliptic equations, *Indiana University Mathematics Journal,* Vol 49 No 3 (2000), 897-923.

[20] W. Strauss; Existence of solitary waves in higher dimensions, *Comm. Math. Phys.*, Volume 55 (1977), 149-162.

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