RELATIONSHIPS BETWEEN INTEGRABLE FUNCTIONS AND THEIR ABSOLUTE VALUES

THESIS

Presented to the Graduate Council of

Southwest Texas State University

in Partial Fulfillment of

the Requirements

For the Degree of

MASTER OF ARTS

By

Terence W. McCabe, B. S.

San Marcos, Texas

May, 1972

ACKNOWLEDGMENTS

The writer wishes to express his gratitude to Dr. John A. Chatfield for his continual efforts in promoting an atmosphere most conducive for creative learning.

Terence W. McCabe

San Marcos, Texas

May, 1972

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION AND DEFINITIONS

The purpose of this paper is to develop several relationships between integrals of the type $\int_a^b fdg$, $\int_a^b |f|dg$, $\int_a^b fd|g|$, $\int_a^b f|dg|$, and $\int_a^b |fdg|$. Chapter II shows that if $\int_a^b fdg$ exists then $\int_a^b f\ dg$ exists. Chapter III shows the equivalency between the existence of $\int_a^b fdg$ and $\int_a^b f|dg|$ with the condition of bounded variation on g. Another theorem allows us to relax this condition while going from $\int_a^b f|dg|$ to $\int_a^b fdg$. All functions used are from numbers to numbers.

<u>DEFINITION 1.1</u>: The statement that $D = (x_i)_{i=0}^n$ is a subdivision of the closed interval (a,b) means that D is a finite subset of (a,b) such that $a = x_o$, $b = x_n$ and for each i, $x_i < x_{i+1}$.

<u>DEFINITION 1.2</u>: The statement that D' is a refinement of a subdivision D of (a,b) means D' is a subdivision of (a,b) and D is a subset of D'.

<u>DEFINITION 1.3</u>: The statement that $(t_i)_{i=1}^n$ is an interpolating sequence for the subdivision $(x_i)_{i=0}^n$ means if $0 < i \leq n$ then $x_{i-1} \leq t_i \leq x_i$.

<u>DEFINITION 1.4</u>: The statement that f is integrable with respect to g means that f and g are functions and there exists a number A such that if $e > 0$ then there is a subdivision D of (a,b) such that if $D' = (x_i)_{i=0}^n$ is a refinement of D and $(t_i)_{i=1}^n$ is an interpolating sequence of D' then

$$\left| \sum_{\substack{i=1 \\ D'}}^{n} f(t_i)[g(x_i) - g(x_{i-1})] - A \right| < e.$$

We denote the number A by $\int_a^b fdg$. We will also denote the numbers $g(x_i) - g(x_{i-1})$ by dg_i and $f(t_i)$ by f_i when no misunderstanding is likely. The symbol $\sum_{D'}$ will be used for $\sum_{\substack{i=1 \\ D'}}^{n}$. As indicated before, (a,b) shall denote the closed interval, containing both a and b.

__DEFINITION 1.5:__ If f and g are functions such that $\int_a^b fdg$ exists and if $D = (x_i)_{i=0}^n$ is a subdivision of (a,b) and $D_1 = (x'_p)_{p=0}^m$ is a refinement of D then

(1) D^+ denotes the set such that x belongs to D^+ if and only if $x = x_i$ for some x_i in D and for each p in (x_{i-1}, x_i), $f(p) \geq 0$.

(2) D^- denotes the set such that x belongs to D^- if and only if $x = x_i$ for some x_i in D and for each p in (x_{i-1}, x_i), $f(p) < 0$.

(3) $D^{\pm}$ denotes the set such $D^{\pm} = D - (D^+ \cup D^-)$.

If $0 < i \leq n$, then

(4) ${}_iD_1$ denotes the set such that x belongs to ${}_iD_1$ if and only if x is in D_1 and $x_{i-1} < x \leq x_i$.

(5) $D \cdot dg \geq o$ denotes the set such that x belongs to $D \cdot dg \geq o$ if and only if $x = x_i$ for some x_i in D and $g(x_i) - g(x_{i-1}) \geq 0$.

(6) $D \cdot dg < o$ denotes the set such that $(D \cdot dg < o) = D - (D \cdot dg \geq o)$.

When no consideration of the sign of f is needed, $D \cdot dg \geq o$ will be denoted by ${}^+D$ and $D \cdot dg < o$ by ${}^-D$.

__DEFINITION 1.6:__ The statement that g is of bounded variation on (a,b) means that there exists a number $M > 0$ such that if $D = (x_i)_{i=0}^n$ is a subdivision of (a,b) then $\sum_D |dg_i| < M$. If S is the set such that p belongs to S if and only if there is a subdivision $(x_q)_{q=0}^m$ of (a,b) such that $p = \sum_{q=1}^m |dg_q|$, then the least upper bound of S is denoted by $V_a^b g$ and is said to be the variation of g on (a,b).

THEOREM 1.7: If $\int_a^b f dg$ exists and $e > 0$ then there is a subdivision $D = (x_i)_{i=0}^n$ of (a,b) such that if $D' = (x_p')_{p=0}^m$ is a refinement of D and $(t_i)_{i=1}^n$ and $(t_p')_{p=1}^m$ are interpolating sequences for the subdivisions D and D', respectively, then [1, p. 304]

$$\sum_{D'} \left| f(t_p')dg_p - \int_{x_{p-1}}^{x_p} fdg \right| < e,$$

$$\sum_D \left| f(t_i)dg_i - \sum_{iD'} f(t_p')dg_p \right| < e,$$

and

$$\sum_D \left| \int_{x_{i-1}}^{x_i} fdg - \sum_{iD'} f(t_p')dg_p \right| < e.$$

CHAPTER II

THE EXISTENCE OF $\int_a^b |f|\, dg$

The first relationship to be considered is that between $\int_a^b fdg$ and $\int_a^b |f|dg$. The following sequence of theorems establish that if $\int_a^b fdg$ exists then $\int_a^b |f|dg$ exists.

THEOREM 2.1: If $\int_a^b fdg$ exists and $e > 0$ then there is a subdivision D of (a,b) such that if $D_1 = (x_i)_{i=0}^n$ is a refinement of D and $(t_i)_{i=1}^n$ is an interpolating sequence for D_1 then $\sum_{D_1\pm} \left| f(t_i)dg_i \right| < e$.

Proof:

Let $e > 0$. Since $\int_a^b fdg$ exists and $\frac{e}{2} > 0$ then, by Theorem 1.7, there is a subdivision D of (a,b) such that if $D_1 = (x_i)_{i=0}^n$ is a refinement of D and $(t_i)_{i=1}^n$ is an interpolating sequence for D_1 then

$$\sum_{D_1} \left| f_i dg_i - \int_{x_{i-1}}^{x_i} fdg \right| < \frac{e}{2}$$

Let $D_1 = (x_i)_{i=0}^n$ be a refinement of D and $(t_i)_{i=1}^n$ be an interpolating sequence for D_1. For each t_i in $D_1\pm$, let q_i be a number such that q_i is in (x_{i-1}, x_i) and if $f(t_i) \geq 0$ then $f(q_i) < 0$ and if $f(t_i) < 0$ then $f(q_i) \geq 0$. Therefore, for each x_i in $D_1\pm$, $\left|f(t_i) - f(q_i)\right| \geq f(t_i)$.

Now, $e = \frac{e}{2} + \frac{e}{2}$

$$> \sum_{D_1\pm} \left| \int_{x_{i-1}}^{x_i} fdg - f(q_i)dg_i \right| + \sum_{D_1\pm} \left| f(t_i)dg_i - \int_{x_{i-1}}^{x_i} fdg \right|$$

$$\geq \sum_{D_1\pm} \left| f(t_i)dg_i - f(q_i)dg_i + \int_{x_{i-1}}^{x_i} fdg - \int_{x_{i-1}}^{x_i} fdg \right|$$

$$= \sum_{D_1\pm} \left| f(t_i) - f(q_i) \right| \cdot \left| dg_i \right|$$

$$\geq \sum_{D_1\pm} \left| f(t_i) \right| \cdot \left| dg_i \right|$$

$$= \sum_{D_1\pm} \left| f(t_i)dg_i \right| .$$

Therefore, $$\sum_{D_1\pm} \left| f(t_i)dg_i \right| < e.$$

THEOREM 2.2: If $\int_a^b fdg$ exists and $e > 0$ then there is a subdivision $D = (x_i)_{i=0}^n$ of (a,b) such that if $D_1 = (x'_p)_{p=0}^m$ is a refinement of D and $(t_i)_{i=1}^n$ and $(t'_p)_{p=1}^m$ are interpolating sequences for D and D_1, respectively, then

$$\sum_{D+ \cup D-} \left| \left| f(t_i) \right| dg_i - \sum_{i^{D_1}} \left| f(t'_p) \right| dg_p \right| < e.$$

Proof:

Let $e > 0$. Since $\int_a^b fdg$ exists and $\frac{e}{2} > 0$ then, by Theorem 1.7, there is a subdivision $D = (x_i)_{i=0}^n$ of (a,b) such that if $D_1 = (x'_p)_{p=0}^m$ is a refinement of D and $(t_i)_{i=1}^n$ and $(t'_p)_{p=1}^m$ are interpolating sequences for D and D_1, respectively, then

$$\sum_{D} \left| f(t_i)dg_i - \sum_{i^{D_1}} f(t'_p)dg_p \right| < \frac{e}{2} .$$

Let $D_1 = (x'_p)_{p=0}^m$ be a refinement of D and $(t_i)_{i=1}^n$ and $(t'_p)_{p=1}^m$ be interpolating sequences for D and D_1, respectively. Hence,

$$\sum_{D+ \cup D-} \left| \left| f(t_i) \right| dg_i - \sum_{i^{D_1}} f(t_p) \; dg_p \right|$$

$$= \sum_{D^+} \left| \quad \cdot \quad \right| + \sum_{D^-} \left| \quad \cdot \quad \right|$$

$$= \sum_{D^+} \left| f(t_i)dg_i - \sum_{{}_iD_1} f(t'_p)dg_p \right| + \sum_{D^-} \left| -f(t_i)dg_i - \sum_{{}_iD_1} -f(t'_p)dg_p \right|$$

$$= \sum_{D^+} \left| \quad \cdot \quad \right| + \sum_{D^-} \left| f(t_i)dg_i - \sum_{{}_iD_1} f(t'_p)dg_p \right|$$

$$\leq \sum_{D} \left| f(t_i)dg_i - \sum_{{}_iD_1} f(t'_p)dg_p \right|$$

$$< \frac{e}{2}$$

$$< e.$$

Therefore, $\displaystyle \sum_{D^+ \cup D^-} \left| |f(t_i)|\, dg_i - \sum_{{}_iD_1} |f(t'_p)|\, dg_p \right| < e$

THEOREM 2.3: If $\int_a^b fdg$ exists and $e > 0$ then there is a subdivision $D = (x_i)_{i=0}^{n}$ of (a,b) such that if $D_1 = (x'_p)_{p=0}^{m}$ is a refinement of D and for each i, $0 < i \leq n$, let ${}_iM = (y_q)_{q=0}^{l_i}$ denote the subdivision of (x_{i-1}, x_i) such that z is in ${}_iM$ if and only if (1) z is x_{i-1} or x_i, or (2) z is x'_p or x'_{p-1}, where x'_p is in $D_1\pm$; and let $(z_p)_{p=1}^{m}$ and $(w_q)_{q=1}^{l_i}$ be interpolating sequences for D_1 and ${}_iM$, respectively, then

(A) $\displaystyle \sum_{D\pm} \left| \sum_{{}_iM^+ \cup {}_iM^-} \sum_{{}_qD_1} |f(z_p)| \left[g(x'_p) - g(x'_{p-1}) \right] \right|$

$\displaystyle < e + \sum_{D\pm} \left| \sum_{{}_iM^+ \cup {}_iM^-} |f(w_q)| \left[g(y_q) - g(y_{q-1}) \right] \right|$

(B) $\displaystyle \sum_{D\pm} \left| \sum_{{}_iM^+ \cdot dg \geq 0} |f(w_q)| \cdot dg_q \right| < e$

(C) $\displaystyle \sum_{D\pm} \left| \sum_{{}_iM^+ \cdot dg < 0} |f(w_q)| \cdot dg_q \right| < e$

(D) $\displaystyle \sum_{D\pm} \left| \sum_{{}_iM^- \cdot dg \geq 0} |f(w_q)| \cdot dg_q \right| < e$

(E) $\displaystyle \sum_{D\pm} \left| \sum_{{}_iM^- \cdot dg < 0} |f(w_q)| \cdot dg_q \right| < e$

(F) $\displaystyle \sum_{D\pm} \left| \sum_{{}_iD_1^+ \cup {}_iD_1^-} |f(z_p)| \cdot dg_p \right| < e$

Proof:

(A) Let $e > 0$. By Theorem 2.2, since $\int_a^b f dg$ exists and $e > 0$ then there is a subdivision $D = (x_i)_{i=0}^n$ of (a,b) such that if $D_1 = (x'_p)_{p=0}^m$ is a refinement of D and $(z_p)_{p=1}^m$ and $(t_i)_{i=1}^n$ are interpolated sequences for D_1 and D, respectively, then

$$\sum_{D^+ \cup D^-} \left| \, |f(t_i)| \, dg_i - \sum_{{}_iD_1} |f(z_p)| \, dg_p \right| < e$$

Let $D_1 = (x'_p)_{p=0}^m$ be a refinement of D. For each x_i in D, let ${}_iM$ be defined as in hypothesis of theorem and let $M = \bigcup_{i=1}^n {}_iM$. Thus, M is a refinement of D and D_1 is a refinement of M. Also, for each i, let $(w_q)_{q=1}^{l_i}$ be an interpolating sequence for ${}_iM$. Hence,

$$\begin{aligned} e &> \sum_{M^+ \cup M^-} \left| \, |f(w_q)| \, dg_q - \sum_{{}_qD_1} |f(z_p)| \, dg_p \right| \\ &= \sum_{D^+ \cup D^-} \left| \quad \cdot \quad \right| + \sum_{D^\pm} \sum_{{}_iM^+ \cup {}_iM^-} \left| \quad \cdot \quad \right| \\ &\geq \sum_{D^\pm} \left| \sum_{{}_iM^+ \cup {}_iM^-} |f(w_q)| \, dg_q - \sum_{{}_iM^+ \cup {}_iM^-} \sum_{{}_qD_1} |f(z_p)| \, dg_p \right| \\ &\geq -\sum_{D^\pm} \left| \sum_{{}_iM^+ \cup {}_iM^-} |f(w_q)| \, dg_q \right| + \sum_{D^\pm} \left| \sum_{{}_iM^+ \cup {}_iM^-} \sum_{{}_qD_1} |f(z_p)| \, dg_p \right| . \end{aligned}$$

Therefore,

$$e + \sum_{D^\pm} \left| \sum_{{}_iM^+ \cup {}_iM^-} |f(w_q)| \, dg_q \right| > \sum_{D^\pm} \left| \sum_{{}_iM^+ \cup {}_iM^-} \sum_{{}_qD_1} |f(z_p)| \, dg_p \right|$$

(B) Let $e > 0$. Since $\int_a^b f dg$ exists and $\frac{e}{3} > 0$ then, by Theorem 1.7, there is a subdivision $D_2 = (x_i)_{i=0}^d$ such that if $A_1 = (x'_p)_{p=0}^s$ is a refinement of D_2 and $A_2 = (w_q)_{q=0}^r$ is a refinement of A_1 and $(z_p)_{p=1}^s$ and $(t_q)_{q=1}^r$ are interpolating sequences for A_1 and A_2, respectively, then

$$\sum_{A_1} \left| f(z_p)dg_p - \sum_{p^{A_2}} f(t_q)dg_q \right| < \frac{e}{3}.$$

Since $\int_a^b fdg$ exists and $\frac{e}{3} > 0$ then there is a subdivision D_3 of (a,b) such that if $D_1 = (x'_p)_{p=0}^m$ is a refinement of D_3 and $(z_p)_{p=1}^m$ is an interpolating sequence for D_1 then

$$\sum_{D_1^{\pm}} \left| f(z_p)dg_p \right| < \frac{e}{3}.$$

Let $D = D_2 \cup D_3 = (x_i)_{i=0}^n$. Let $D_1 = (x'_p)_{p=0}^m$ be a refinement of D and for each x_i in D, let ${}_iM$ be defined as in hypothesis and $M = \bigcup_{i=1}^n {}_iM$. Let M_1 be the refinement of D such that x belongs to M_1 if and only if x is in D or there is an x_i in D such that x is in ${}_iM^{\pm}$, ${}_iM^-$ or ${}_iM^+ \cdot dg < o$. For each i, let ${}_iM_1 = (y_j)_{j=0}^{k_i}$. Notice that M is a refinement of M_1. For each y_j in ${}_iM_1$, let z'_j be in (y_{j-1}, y_j). Thus, $\sum_{M_1^{\pm}} \left| f(z'_j)dg_j \right| < \frac{e}{3}$ and $\sum_{M^{\pm}} \left| f(w_q)dg_q \right| < \frac{e}{3}$.

Therefore,

$$\sum_{D^{\pm}} \left| \sum_{{}_iM^+ \cdot dg \geq o} f(w_q)[g(y_q) - g(y_{q-1})] \right|$$

$$= \sum_{D^{\pm}} \left| \sum_{{}_iM^+ \cdot dg \geq o} f(w_q)dg_q \right|$$

$$= \sum_{D^{\pm}} \sum_{{}_iM^{\mp} \cdot dg \geq o} f(w_q)dg_q + \sum_{D^{\pm}} \sum_{{}_iM^{\pm}} f(w_q)dg_q - \sum_{D^{\pm}} \sum_{{}_iM^{\pm}} f(w_q)dg_q$$

$$\leq \sum_{D^{\pm}} \sum_{{}_iM^{\pm} \cup {}_iM^{+} \cdot dg_q \geq o} f(w_q)dg_q + \sum_{M^{\pm}} \left| f(w_q)dg_q \right|$$

$$< \sum_{D^{\pm}} \sum_{{}_iM^{+} \cdot dg_q \geq o \cup {}_iM^{\pm}} f(w_q)dg_q + \sum_{D^{\pm}} \sum_{{}_iM_1^{\pm}} f(z'_j)dg_j - \sum_{D^{\pm}} \sum_{{}_iM_1^{\pm}} f(z'_j)dg_j + \frac{e}{3}$$

$$\leq \sum_{D^{\pm}} \left| \sum_{{}_iM^{\pm} U_i M^{+} \cdot dg \geq 0} f(w_q)dg_q - \sum_{{}_iM_1^{\pm}} f(z_j')dg_j \right| + \sum_{M_1^{\pm}} \left| f(z_j')dg_j \right| + \frac{e}{3}$$

$$< \sum_{D^{\pm}} \left| \sum_{{}_jM_1} f(w_q)dg_q - f(z_j')dg_j \right| + \frac{e}{3} + \frac{e}{3}$$

$$\leq \sum_{M_1} \left| \quad \cdot \quad \right| + \frac{2}{3} e$$

$$< \frac{e}{3} + \frac{2}{3} e$$

$$= e .$$

Thus,
$$\sum_{D^{\pm}} \left| \sum_{{}_iM^{+} \cdot dg \geq 0} \left| f(w_q) \right| \left[g(y_q) - g(y_{q-1}) \right] \right| < e$$

By similar argument, parts C, D and E are also true. Using these results, the following establishes part F as the main conclusion of the theorem.

(F) For each of the previous parts, A, B, C, D and E, let the arbitrary positive number be $\frac{e}{5}$. Since $\int_a^b fdg$ exists and $\frac{e}{5} > 0$ then there is a subdivision $D = (x_i)_{i=0}^{n}$ such that if $D_1 = (x_p')_{p=0}^{m}$ is a refinement of D and $M = \bigcup_{i=1}^{n} ({}_iM)$, as defined in hypothesis, is a refinement of D then parts A, B, C, D and E are true.

Let $D_1 = (x_p')_{p=0}^{m}$ be a refinement of D and $(z_p)_{p=1}^{m}$ be an interpolating sequence for D_1. For each ${}_iM$, let w_q be in (y_{q-1}, y_q) for each y_q in ${}_iM$. Hence,

$$\sum_{D^{\pm}} \left| \sum_{{}_iD_1^{+} U_i D_1^{-}} \left| f(z_p) \right| dg_p \right|$$

$$= \sum_{D^{\pm}} \left| \sum_{{}_iM^{+} U_i M^{-}} \sum_{{}_qD_1} \left| f(z_p) \right| dg_p \right|$$

$$< \frac{e}{5} + \sum_{D^{\pm}} \left| \sum_{{}_iM^{+} U {}_iM^{-}} \left| f(w_q) \right| dg_q \right|$$

$$\leq \frac{e}{5} + \sum_{D^\pm} \left| \sum_{i^{M+} \cdot dg \geq 0} |f(w_q)|\, dg_q \right| + \sum_{D^\pm} \left| \sum_{i^{M+} \cdot dg < 0} |f(w_q)|\, dg_q \right|$$

$$\sum_{D^\pm} \left| \sum_{i^{M-} \cdot dg \geq 0} |f(w_q)|\, dg_q \right| + \sum_{D^\pm} \left| \sum_{i^{M-} \cdot dg \leq 0} |f(w_q)|\, dg_q \right|$$

$$< \frac{e}{5} + \frac{e}{5} + \frac{e}{5} + \frac{e}{5} + \frac{e}{5}$$

$$= e.$$

Thus,
$$\sum_{D^\pm} \left| \sum_{i^{D_1^+} \cup i^{D_1^-}} |f(z_p)|\, dg_p \right| < e$$

Finally, with the preceding theorems we can establish the following result.

THEOREM 2.4: If $\int_a^b f dg$ exists then $\int_a^b |f|\, dg$ exists.

Proof:

Let $e > 0$. Since $\int_a^b f dg$ exists and $\frac{e}{4} > 0$ then, by Theorem 2.1, there is a subdivision $D_2 = (x_i)_{i=0}^{k}$ of (a,b) such that if $D_1 = (x_p)_{p=0}^{m}$ is a refinement of D_2 and $(t_p')_{p=1}^{m}$ is an interpolating sequence for D_1 then
$$\sum_{D_1 \pm} \left| f(t_p')\ dg_p \right| < \frac{e}{4}$$

Since $\int_a^b f dg$ exists and $\frac{e}{4} > 0$ then, by Theorem 2.2, there is a subdivision $D_3 = (x_i)_{i=0}^{l}$ of (a,b) such that if $D_1 = (x_p')_{p=0}^{m}$ is a refinement of D_3 and $(t_i)_{i=1}^{l}$ and $(t_p')_{p=1}^{m}$ are interpolating sequences for D_3 and D_1, respectively, then
$$\sum_{D_3^+ \cup D_3^-} \left| |f(t_i)|\, dg_i - \sum_{i^{D_1}} |f(t_p')|\, dg_p \right| < \frac{e}{4}$$

Since $\int_a^b f dg$ exists and $\frac{e}{4} > 0$ then, by Theorem 2.3, there exists a subdivision $D_4 = (x_i)_{i=0}^{j}$ of (a,b) such that if $D_1 = (x_p')_{p=0}^{m}$ is a refinement of D_4 and $(t'_p)_{p=1}^m$ is an interpolating sequence for D_1 then

$$\sum_{D_4^{\pm}} \left| \sum_{{}_iD_1^+ \cup {}_iD_1^-} |f(t'_p)|\, dg_p \right| < \frac{e}{4}$$

Let $D = D_2 \cup D_3 \cup D_4 = (x_i)_{i=0}^n$. Let $D_1 = (x'_p)_{p=0}^m$ be a refinement of D and $(t_i)_{i=1}^n$ and $(t'_p)_{p=1}^m$ be interpolating sequences for D and D_1, respectively. Thus,

$$\left| \sum_D |f(t_i)|\, dg_i - \sum_{D_1} |f(t'_p)|\, dg_p \right|$$

$$\leq \sum_D \left| |f(t_i)|\, dg_i - \sum_{{}_iD_1} |f(t'_p)|\, dg_p \right|$$

$$\leq \sum_{D^+ \cup D^-} \left| |f(t_i)|\, dg_i - \sum_{{}_iD_1} |f(t'_p)|\, dg_p \right| + \sum_{D^{\pm}} \left| f(t_i) dg_i \right|$$

$$+ \sum_{D^{\pm}} \left| \sum_{{}_iD_1} |f(t'_p)|\, dg_p \right|$$

$$< \frac{e}{4} + \frac{e}{4} + \sum_{D^{\pm}} \left| \sum_{{}_iD_1^{\pm}} |f(t'_p)|\, dg_p \right| + \sum_{D^{\pm}} \left| \sum_{{}_iD_1^+ \cup {}_iD_1^-} |f(t'_p)|\, dg_p \right|$$

$$< \frac{e}{2} + \sum_{D_1^{\pm}} \left| f(t'_p) dg_p \right| + \frac{e}{4}$$

$$< \frac{3}{4} e + \frac{e}{4}$$

$$= e .$$

Since for each $e > 0$ there is a subdivision $D = (x_i)_{i=0}^n$ of (a,b) such that if $D_1 = (x'_p)_{p=0}^m$ is a refinement of D and $(t_i)_{i=1}^n$ and $(t'_p)_{p=1}^m$ are interpolating sequences for D and D_1, respectively, then

$$\left| \sum_D |f(t_i)|\, dg_i - \sum_{D_1} |f(t'_p)|\, dg_p \right| < e ,$$

therefore, $\int_a^b |f|\, dg$ exists [2, p. 28].

Using this theorem, another relationship can be established between $\int_a^b f\,dg$ and $\int_a^b f\,d|g|$.

THEOREM 2.5: If $\int_a^b f\,dg$ exists then $\int_a^b f\,d|g|$.

Proof:

Since $\int_a^b f\,dg$ exists then $\int_a^b g\,df$ exists [2, p. 53] and is

$$f(b)g(b) - f(a)g(a) - \int_a^b f\,dg.$$

Since $\int_a^b g\,df$ exists then, by Theorem 2.4, $\int_a^b |g|\,df$ exists. Since $\int_a^b |g|\,df$ exists then $\int_a^b f\,d|g|$ exists.

CHAPTER III

RELATIONSHIPS BETWEEN $\int_a^b f dg$ AND $\int_a^b f|dg|$

The next relationship to be shown is between the integrals $\int_a^b f dg$ and $\int_a^b f|dg|$. It has been found that if g is of bounded variation on (a,b), then equivalent statements can be made regarding these integrals. The following theorem allows us to prove an equivalent statement as the next theorem.

THEOREM 3.1: If g is of bounded variation on (a,b) and $e > 0$ then there is a subdivision $D = (x_i)_{i=0}^{n}$ of (a,b) such that if $D_1 = (x_p')_{p=0}^{m}$ is a refinement of D then

$$\sum_{+D} \sum_{-D_1} |dg_p| + \sum_{-D} \sum_{+D_1} |dg_p| < e .$$

Proof:

Let $e > 0$. Since g is of bounded variation on (a,b) and $\frac{e}{2} > 0$ then there is a subdivision $D = (x_i)_{i=0}^{n}$ of (a,b) such that if $D' = (x_p')_{p=0}^{m}$ is a refinement of D then

$$\sum_{D'} |dg_p| \geq \sum_{D} |dg_i| \geq V_a^b g - \frac{e}{2}.$$

Let $D' = (x_p')_{p=0}^{m}$ be a refinement of D. Since $V_a^b g$ is the least upper bound of such summations on (a,b) then

$$V_a^b g - \sum_{D'} |dg_p| \geq 0 \quad \text{and} \quad V_a^b g - \sum_{D} |dg_i| \geq 0 ;$$

also, $$\left| V_a^b g - \sum_{D'} |dg_p| \right| < \frac{e}{2} \quad \text{and} \quad \left| V_a^b g - \sum_{D} |dg_i| \right| < \frac{e}{2}.$$

Thus, $\sum_{D'} |dg_p| - \sum_{D} |dg_i|$

$$= \left| \sum_{D'} |dg_p| - \sum_{D} |dg_i| \right|$$

$$= \left| \sum_{D'} |dg_p| - V_a^b g + V_a^b g - \sum_{D} |dg_i| \right|$$

$$\leq \left| \sum_{D'} |dg_p| - V_a^b g \right| + \left| V_a^b g - \sum_{D} |dg_i| \right|$$

$$< \frac{e}{2} + \frac{e}{2}$$

$$= e.$$

Therefore, $\sum_{D'} |dg_p| - \sum_{D} |dg_i| < e.$

Also notice that

$$\sum_{+D} dg_i = \sum_{+D} \sum_{+D'} dg_p + \sum_{+D} \sum_{-D'} dg_p .$$

Hence, $\sum_{+D} |dg_i| = \sum_{+D} \sum_{+D'} |dg_p| + \sum_{+D} \sum_{-D'} -|dg_p|$

$$\leq \sum_{+D} \sum_{+D'} |dg_p|$$

and $\sum_{+D} \sum_{+D'} |dg_p| - \sum_{+D} |dg_i| \geq 0.$

Similarly, $\sum_{-D} \sum_{-D'} |dg_p| \geq \sum_{-D} |dg_i| .$

Therefore,

$$\sum_{+D} \sum_{-D'} |dg_p| + \sum_{-D} \sum_{+D'} |dg_p|$$

$$\leq \sum_{+D} \sum_{-D'} |dg_p| + \sum_{-D} \sum_{+D'} |dg_p| + [\sum_{+D} \sum_{+D'} |dg_p| - \sum_{+D} |dg_i|]$$

$$+ [\sum_{-D} \sum_{-D'} |dg_p| - \sum_{-D} |dg_i|]$$

$$= \sum_{D'} |dg_p| - \sum_{D} |dg_i|$$

$$< e .$$

Hence, $\sum_{+D} \sum_{-D'} |dg_p| + \sum_{-D} \sum_{+D'} |dg_p| < e.$

THEOREM 3.2: If g is of bounded variation on (a,b) then the following two statements are equivalent:

(1) $\int_a^b fdg$ exists.

(2) $\int_a^b f|dg|$ exists.

Proof:

If either integral exists then there is a subdivision $(y_r)_{r=0}^p$ of (a,b) such that for each r, either f is bounded on (y_{r-1}, y_r) or g is constant on (y_{r-1}, y_r) [2, p. 51]. Thus, $\int_{y_{r-1}}^{y_r} fdg = 0$ or $\int_{y_{r-1}}^{y_r} f|dg| = 0$ for each (y_{r-1}, y_r) on which f is not bounded. Hence, in the following proof we shall consider the case where f is bounded on (a,b).

(2) implies (1)

Let $e > 0$. Since $\int_a^b f|dg|$ exists then f is bounded by some number $M > 1$ on each subinterval of (a,b) on which g is not constant. Since $\int_a^b f|dg|$ exists and $\frac{e}{2} > 0$ then, by Theorem 1.7, there is a subdivision $D_1 = (x_i)_{i=0}^j$ of (a,b) such that if $D' = (x'_p)_{p=0}^m$ is a refinement of D_1 and $(t_i)_{i=1}^j$ and $(t'_p)_{p=1}^m$ are interpolating sequences for D_1 and D', respectively, then

$$\left| \sum_{D_1} f(t_i)|dg_i| - \sum_{D'} f(t'_p)|dg_p| \right| < \frac{e}{2}.$$

Since g is of bounded variation on (a,b) and $\frac{e}{4M} > 0$ then there is a subdivision $D_2 = (x_i)_{i=0}^k$ of (a,b) such that if $D' = (x'_p)_{p=0}^m$ is a refinement of D_2 then

$$\sum_{+D} \sum_{-D'} |dg_p| + \sum_{-D} \sum_{+D'} |dg_p| < \frac{e}{4M} .$$

Let $D = D_1 \cup D_2 = (x_i)_{i=0}^{n}$. Let $D' = (x'_p)_{p=0}^{m}$ be a refinement of D and $(t_i)_{i=1}^{n}$ and $(t'_p)_{p=1}^{m}$ be interpolating sequences for D and D', respectively. Hence,

$$\left| \sum_{D} f(t_i) dg_i - \sum_{D'} f(t'_p) dg_p \right|$$

$$= \Big| \sum_{-D} f_i dg_i + \sum_{+D} f_i dg_i - \sum_{-D} \sum_{-D'} f_p dg_p - \sum_{+D} \sum_{+D'} f_p dg_p$$

$$- \sum_{+D} \sum_{-D'} f_p dg_p - \sum_{-D} \sum_{+D'} f_p dg_p \Big|$$

$$\leq \left| \sum_{-D} f_i (-|dg_i|) - \sum_{-D} \sum_{-D'} f_p (-|dg_p|) \right|$$

$$+ \left| \sum_{+D} f_i |dg_i| - \sum_{+D} \sum_{+D'} f_p |dg_p| \right| + \left| \sum_{+D} \sum_{-D'} f_p (-|dg_p|) \right|$$

$$+ \left| \sum_{-D} \sum_{+D'} f_p |dg_p| \right|$$

$$\leq \sum_{-D} \left| f_i |dg_i| - \sum_{-D'_i} f_p |dg_p| \right| + \sum_{+D} \left| f_i |dg_i| - \sum_{+D'_i} f_p |dg_p| \right|$$

$$+ \sum_{+D} \sum_{-D'} |f_p| \, |dg_p| + \sum_{-D} \sum_{+D'} |f_p| \, |dg_p|$$

$$\leq \sum_{-D} \left| \quad \cdot \quad \right| + \sum_{+D} \left| \quad \cdot \quad \right|$$

$$+ \sum_{+D} \sum_{-D'} M |dg_p| + \sum_{-D} \sum_{+D'} M |dg_p|$$

$$= \sum_{-D} \left| \quad \cdot \quad \right| + \sum_{+D} \left| \quad \cdot \quad \right|$$

$$+ M \left(\sum_{+D} \sum_{-D'} |dg_p| + \sum_{-D} \sum_{+D'} |dg_p| \right)$$

$$< \sum_{-D} \left| \quad \cdot \quad \right| + \sum_{+D} \left| \quad \cdot \quad \right| + M \left(\frac{e}{4M} \right)$$

$$= \sum_{-D} \left| \quad \cdot \quad \right| + \sum_{+D} \left| \quad \cdot \quad \right| + \frac{e}{4}$$

$$= \sum_{-D} \left| f_i |dg_i| - \sum_{i^{-D'}} f_p |dg_p| - \sum_{i^{+D'}} f_p |dg_p| + \sum_{i^{+D'}} f_p |dg_p| \right|$$

$$+ \sum_{+D} \left| f_i |dg_i| - \sum_{i^{+D'}} f_p |dg_p| - \sum_{i^{-D'}} f_p |dg_p| + \sum_{i^{-D'}} f_p |dg_p| \right|$$

$$+ \frac{e}{4}$$

$$\leq \sum_{-D} \left| f_i |dg_i| - \sum_{i^{-D'}} f_p |dg_p| \right| + \sum_{+D} \left| f_i |dg_i| - \sum_{i^{+D'}} f_p |dg_p| \right|$$

$$+ \sum_{-D} \sum_{+D'} |f_p| |dg_p| + \sum_{+D} \sum_{-D'} |f_p| |dg_p| + \frac{e}{4}$$

$$\leq \sum_{D} \left| f_i |dg_i| - \sum_{i^{D'}} \sum f_p |dg_p| \right| + \sum_{-D} \sum_{+D'} M |dg_p|$$

$$+ \sum_{+D} \sum_{-D'} M |dg_p| + \frac{e}{4}$$

$$< \frac{e}{2} + M \left(\sum_{-D} \sum_{+D'} |dg_p| + \sum_{+D} \sum_{-D'} |dg_p| \right) + \frac{e}{4}$$

$$< \frac{e}{2} + M \left(\frac{e}{4M} \right) + \frac{e}{4}$$

$$= e .$$

Since for each $e > 0$ there is a subdivision $D = (x_i)_{i=0}^{n}$ of (a,b) such that if $D' = (x'_p)_{p=0}^{m}$ is a refinement of D and $(t_i)_{i=1}^{n}$ and $(t'_p)_{p=1}^{m}$ are interpolating sequences for D and D', respectively, then

$$\left| \sum_{D} f(t_i)dg_i - \sum_{D'} f(t'_p)dg_p \right| < e ,$$

therefore, $\int_a^b f dg$ exists [2, p. 28].

(1) implies (2)

Let $e > 0$. Since $\int_a^b f dg$ exists and g is of bounded variation on (a,b) then $\int_a^b f dV_g$ exists, where $V_g(x) = V_a^x g$ for each x in (a,b) [2, p. 66]. Since f is bounded on (a,b) then there is an $M > 1$ such that $M > |f(x)|$ for each x in (a,b). Since $\int_a^b f dV_g$ exists and

$\frac{e}{2} > 0$ then there is a subdivision D_1 of (a,b) such that if $D' = (x_i)_{i=0}^{n}$ is a refinement of D_1 and $(t_i)_{i=1}^{n}$ is an interpolating sequence for D' then
$$\left| \sum_{D'} f(t_i) dVg_i - \int_a^b f dVg \right| < \frac{e}{2}.$$
Since g is of bounded variation on (a,b) and $\frac{e}{2M} > 0$ then there is a subdivision D_2 of (a,b) such that if $D' = (x_i)_{i=0}^{n}$ is a refinement of D_2 then
$$\sum_{D'} \left| V_{x_{i-1}}^{x_i} g - |dg_i| \right| < \frac{e}{2M}.$$

Let $D = D_1 \cup D_2$. Let $D' = (x_i)_{i=0}^{n}$ be a refinement of D and $(t_i)_{i=1}^{n}$ be an interpolating sequence for D'. For each i, let $V_{x_{i-1}}^{x_i} g$ be denoted by Vg_i. Hence,

$$\left| \sum_{D'} f(t_i) |dg_i| - \int_a^b f dVg \right|$$
$$\leq \left| \sum_{D'} f(t_i) |dg_i| - \sum_{D'} f(t_i) Vg_i + \sum_{D'} f(t_i) Vg_i - \int_a^b f dVg \right|$$
$$\leq \left| \sum_{D'} f(t_i) |dg_i| - \sum_{D'} f(t_i) Vg_i \right| + \left| \sum_{D'} f(t_i) Vg_i - \int_a^b f dVg \right|$$
$$< \sum_{D'} \left| f(t_i) |dg_i| - f(t_i) Vg_i \right| + \frac{e}{2}$$
$$= \sum_{D'} |f(t_i)| \left| Vg_i - |dg_i| \right| + \frac{e}{2}$$
$$\leq \sum_{D'} M \left| Vg_i - |dg_i| \right| + \frac{e}{2}$$
$$= M \sum_{D'} \left| Vg_i - |dg_i| \right| + \frac{e}{2}$$
$$< M \left(\frac{e}{2M} \right) + \frac{e}{2}$$
$$= e.$$

Since $\int_a^b f dVg$ is a number such that if $e > 0$ then there is a subdivision D of (a,b) such that if $D' = (x_i)_{i=0}^{n}$ is a refinement of

D and $(t_i)_{i=1}^{n}$ is an interpolating sequence for D' then

$$\left| \sum_{D'} f(t_i)|dg_i| - \int_a^b f dV g \right| < e,$$

therefore, $\int_a^b f|dg|$ exists, by Definition 1.3.

With further investigation, it has been found that given the existence of $\int_a^b f|dg|$, the proof of the existence of $\int_a^b fdg$ does not require the condition of bounded variation for g on (a,b). The following are two preliminary theorems in preparation for the desired result.

THEOREM 3.3: If $\int_a^b f|dg|$ exists then $\int_a^b |fdg|$ exists.

Proof:

By Theorem 2.4, since $\int_a^b f|dg|$ exists then $\int_a^b |f||dg|$ exists and $\int_a^b |fdg|$ exists.

THEOREM 3.4: If $\int_a^b f|dg|$ exists and g is not of bounded variation on (a,b) then for each $e > 0$ there is a subinterval (c,d) of (a,b) such that $|f(x)| < e$ for each x in (c,d).

Proof:

Assume the conclusion is false. Therefore, there is an $e > 0$ such that if (c,d) is any subinterval of (a,b) then there is an x in (c,d) such that $|f(x)| \geq e$. Since $\int_a^b |fdg|$ exists and $e > 0$ then there is a subdivision D of (a,b) such that if $D' = (x_i)_{i=0}^{n}$ is a refinement of D and $(t_i)_{i=1}^{n}$ is an interpolating sequence for D' then, by Definition 1.3,

$$\left| \sum_{D'} | f(t_i)dg_i| - \int_a^b |fdg| \right| < e.$$

Since g is not of bounded variation on (a,b) and $1 + \frac{1}{e}\int_a^b |fdg| > 0$ then there is a refinement $D' = (x_i)_{i=0}^{n}$ of D such that

$$\sum_{D'} |dg_i| > 1 + \frac{1}{e} \int_a^b |fdg|.$$

From our assumption, there exists an interpolating sequence for D', $(t_i)_{i=1}^n$, such that for each (x_{i-1}, x_i), $|f(t_i)| \geq e$. Hence,

$$\left| \sum_{D'} |f_i dg_i| - \int_a^b |fdg| \right| < e$$

and

$$\sum_{D'} |f_i dg_i| < e + \int_a^b |fdg| .$$

Thus,

$$\begin{aligned} e + \int_a^b |fdg| &> \sum_{D'} |f_i dg_i| \\ &\geq \sum_{D'} e \, |dg_i| \\ &= e \sum_{D'} |dg_i| \\ &\geq e \left(1 + \frac{1}{e} \int_a^b |fdg| \right) \\ &= e + \int_a^b |fdg| . \end{aligned}$$

Therefore,

$$e + \int_a^b |fdg| > e + \int_a^b |fdg| .$$

This is a contradiction. Thus, the assumption is false and the theorem is true.

THEOREM 3.5: If $\int_a^b f|dg|$ exists then $\int_a^b fdg$ exists.

Proof:

Let $e > 0$. Since $\int_a^b f\,|dg|$ exists then, by Theorem 3.3, $\int_a^b |fdg|$ exists. Since $\int_a^b |fdg|$ exists and $\frac{e}{6} > 0$ then, by Theorem 1.7, there is a subdivision $D_1 = (z_l)_{l=0}^n$ of (a,b) such that if $D_2 = (x_i)_{i=0}^m$ is a refinement of D_1, $D_3 = (x'_p)_{p=0}^k$ is a refinement of D_2 and

$(t_i)_{i=1}^{m}$ and $(t'_p)_{p=1}^{k}$ are interpolating sequences for D_2 and D_3, respectively, then

$$\sum_{D_2} \left| |f(t_i)dg_i| - \sum_{{}_iD_3} |f(t_p)dg_p| \right| < \frac{e}{6}.$$

Let A be the set such that z_l belongs to A if and only if z_l is in D_1 and g is of bounded variation on (z_{l-1}, z_l). Since for each z_l in A, $\int_{z_{l-1}}^{z_l} f|dg|$ exists and g is of bounded variation on (z_{l-1}, z_l) then, by Theorem 3.2, $\int_{z_{l-1}}^{z_l}$ fdg exists. Since for each z_l in A, $\int_{z_{l-1}}^{z_l}$ fdg exists and $\frac{e}{6n} > 0$ then, by Theorem 1.7, there is a subdivision $A_l = (c_r)_{r=0}^{k_l}$ of (z_{l-1}, z_l) such that if $A'_l = (c'_p)_{p=0}^{j_l}$ is a refinement of A_l and $(t_p)_{p=1}^{j_l}$ is an interpolating sequence for A'_l then

$$\sum_{A'_l} \left| f(t_p)dg_p - \int_{c'_{p-1}}^{c'_p} fdg \right| < \frac{e}{6n}$$

and

$$\sum_{A_l} \left| \int_{c_{r-1}}^{c_r} fdg - \sum_{{}_rA'_l} f(t_p)dg_p \right| < \frac{e}{6n}.$$

For each z_l in D_1 which is not A, let A_l be the set such that x belongs to A_l if and only if $x = z_l$.

Let $D = D_1 \cup \left(\bigcup_{l=1}^{n} A_l \right) = (x_i)_{i=0}^{\alpha}$ and $D' = (x'_p)_{p=0}^{m}$ be a refinement of D. Thus, D and D' are refinements of D_1 such that D' is a refinement of D. Let $(t_i)_{i=1}^{\alpha}$ and $(t'_p)_{p=1}^{m}$ be interpolating sequences for D and D', respectively.

Let C be the set such that x belongs to C if and only if x is in D and there is a z_l in A such that $z_{l-1} < x \leq z_l$. Let C' be the set such that x belongs to C' if and only if x is in D' and there is an x_i in D such that x_i is in C and $x_{i-1} < x \leq x_i$. Let B be the set D − C and B' be the set D' − C'. Therefore, for each x_i in B, g is not of bounded variation on (x_{i-1}, x_i). For each x_i in B, since g is not of bounded variation on (x_{i-1}, x_i), $\int_{x_{i-1}}^{x_i} f|dg|$ exists and $\frac{e}{6n(|dg_i| + 1)} > 0$ then, by Theorem 3.4, there is a subinterval $(c,d)_i$ of (x_{i-1}, x_i) such that for each x in $(c,d)_i$, $|f(x)| < \frac{e}{6n(|dg_i| + 1)}$.

For each x_i in B, let q_i be in $(c,d)_i$. Hence,

$$\left|\sum_D f(t_i)dg_i - \sum_{D'} f(t'_p)dg_p\right|$$

$$\leq \left|\sum_C f(t_i)dg_i - \sum_{C'} f(t'_p)dg_p\right| + \sum_B \left|f(t_i)dg_i\right| + \sum_{B'} \left|f(t'_p)dg_p\right|$$

$$\leq \left|\sum_C f_i dg_i - \sum_C \int_{x_{i-1}}^{x_i} fdg\right| + \left|\sum_C \int_{x_{i-1}}^{x_i} fdg - \sum_{C'} f_p dg_p\right|$$

$$+ \sum_B \left| \quad \cdot \quad \right| + \sum_{B'} \left| \quad \cdot \quad \right|$$

$$\leq \sum_C \left| f_i dg_i - \int_{x_{i-1}}^{x_i} fdg \right| + \sum_C \left| \int_{x_{i-1}}^{x_i} fdg - \sum_{{}_iC'} f_p dg_p \right|$$

$$+ \sum_B \left| \quad \cdot \quad \right| + \sum_{B'} \left| \quad \cdot \quad \right|$$

$$= \sum_{\substack{l=1 \\ D_l}}^{n} \sum_{A_l} \left| f_i dg_i - \int_{x_{i-1}}^{x_i} fdg \right| + \sum_{\substack{l=1 \\ D_l}}^{n} \sum_{A_l} \left| \int_{x_{i-1}}^{x_i} fdg - \sum_{{}_iC'} f_p dg_p \right|$$

$$+ \sum_B \left| \quad \cdot \quad \right| + \sum_{B'} \left| \quad \cdot \quad \right|$$

$$< \sum_{l=1}^{n} \frac{e}{6n} + \sum_{l=1}^{n} \frac{e}{6n} + \sum_B \left| \quad \cdot \quad \right| + \sum_{B'} \left| \quad \cdot \quad \right|$$

$$= \frac{e}{6} + \frac{e}{6} + \sum_B \left| f_i dg_i \right| + \sum_{B'} \left| f_p dg_p \right|$$

$$= \frac{e}{3} + \sum_B \left| f(t_i) dg_i \right| + \sum_{B'} \left| f(t_p) dg_p \right|$$

$$= \frac{e}{3} + \sum_B \left| f(t_i) dg_i \right| - \sum_B \left| f(q_i) dg_i \right|$$

$$+ \sum_{B'} \left| f(t_p) dg_p \right| - \sum_B \left| f(q_i) dg_i \right| + 2 \sum_B \left| f(q_i) dg_i \right|$$

$$\leq \frac{e}{3} + \sum_B \left| \left| f(t_i) dg_i \right| - \left| f(q_i) dg_i \right| \right|$$

$$+ \sum_B \left| \left| f(q_i) dg_i \right| - \sum_{iB'} \left| f(t'_p) dg_p \right| \right| + 2 \sum_B \left| f(q_i) \right| \left| dg_i \right|$$

$$< \frac{e}{3} + \frac{e}{6} + \frac{e}{6} + 2 \sum_B \frac{e}{6n(\left| dg_i \right| + 1)} \left| dg_i \right|$$

$$\leq \frac{2}{3} e + 2 \left(\frac{e}{6} \right) \sum_B \frac{1}{n}$$

$$\leq \frac{2}{3} e + \frac{e}{3} (1)$$

$$= e .$$

Since for each $e > 0$ there is a subdivision $D = (x_i)_{i=0}^{n}$ of (a,b) such that if $D' = (x'_p)_{p=0}^{m}$ is a refinement of D and $(t_i)_{i=1}^{n}$ and $(t'_p)_{p=1}^{m}$ are interpolating sequences for D and D', respectively, then

$$\left| \sum_D f(t_i) dg_i - \sum_{D'} f(t'_p) dg_p \right| < e ,$$

therefore, $\int_a^b f dg$ exists [2, p. 28].

THEOREM 3.6: If $\int_a^b f dg$ and $\int_a^b |f dg|$ both exist then $\int_a^b f |dg|$ exists.

Proof:

Let $e > 0$. Since $\int_a^b |f dg|$ exists and $\frac{e}{4} > 0$ then, by Theorem 1.7, there is a subdivision $D_2 = (x_i)_{i=0}^{k}$ of (a,b) such that if $D_1 = (x'_p)_{p=0}^{m}$

is a refinement of D_2 and $(t_i)_{i=1}^{k}$ and $(t'_p)_{p=1}^{m}$ are interpolating sequences for D_2 and D_1, respectively, then

$$\sum_{D_2} \left| |f(t_i)dg_i| - \sum_{i^{D_1}} |f(t'_p)dg_p| \right| < \frac{e}{4}.$$

Since $\int_a^b fdg$ exists and $\frac{e}{4} > 0$ then, by Theorem 2.1, there is a subdivision D_3 of (a,b) such that if $D_1 = (x'_p)_{p=0}^{m}$ is a refinement of D_3 and $(t'_p)_{p=1}^{m}$ is an interpolating sequence for D_1 then

$$\sum_{D_1^{\pm}} |f(t'_p)dg_p| < \frac{e}{4}.$$

Since $\int_a^b fdg$ exists and $\frac{e}{4} > 0$ then, by Theorem 2.3, there is a subdivision D_4 of (a,b) such that if $D_1 = (x'_p)_{p=0}^{m}$ is a refinement of D_4 and $(t'_p)_{p=1}^{m}$ is an interpolating sequence for D_1 then

$$\sum_{D_4^{\pm}} \left| \sum_{D_1^{+} \cup D_1^{-}} |f(t'_p)dg_p| \right| < \frac{e}{4}.$$

Let $D = D_2 \cup D_3 \cup D_4 = (x_i)_{i=0}^{n}$. Let $D_1 = (x'_p)_{p=0}^{m}$ be a refinement of D and $(t_i)_{i=1}^{n}$ and $(t'_p)_{p=1}^{m}$ be interpolating sequences for D and D_1, respectively. Hence,

$$\left| \sum_{D} f(t_i)|dg_i| - \sum_{D_1} f(t'_p)|dg_p| \right|$$

$$\leq \sum_{D^+} \left| |f(t_i)|\,|dg_i| - \sum_{i^{D_1}} |f(t'_p)|\,|dg_p| \right|$$

$$+ \sum_{D^-} \left| -|f(t_i)|\,|dg_i| - \sum_{i^{D_1}} -|f(t'_p)|\,|dg_p| \right| + \sum_{D^{\pm}} |f(t_i)dg_i|$$

$$+ \sum_{D^{\pm}} \sum_{i^{D_1}} |f(t'_p)dg_p|$$

$$< \sum_{D^+ \cup D^-} \left| |f_i dg_i| - \sum_{{}_iD_1} |f_p dg_p| \right| + \frac{e}{4} + \sum_{D^\pm} \sum_{{}_iD_1} |f_p dg_p|$$

$$\leq \sum_{D^+ \cup D^- \cup D^\pm} \left| \qquad \cdot \qquad \right| + \frac{e}{4} + \sum_{D^\pm} \sum_{{}_iD_1^+ \cup {}_iD_1^-} |f_p dg_p|$$

$$+ \sum_{D^\pm} \sum_{{}_iD_1^\pm} |f_p dg_p|$$

$$< \sum_{D} \left| \qquad \cdot \qquad \right| + \frac{e}{4} + \frac{e}{4} + \sum_{D_1^\pm} |f_p dg_p|$$

$$< \frac{e}{4} + \frac{e}{2} + \frac{e}{4}$$

$$= e.$$

Since for each $e > 0$ there is a subdivision $D = (x_i)_{i=0}^n$ of (a,b) such that if $D_1 = (x'_p)_{p=0}^m$ is a refinement of D and $(t'_p)_{p=1}^m$ and $(t_i)_{i=1}^n$ are interpolating sequences for D_1 and D, respectively, then

$$\left| \sum_D f(t_i)|dg_i| - \sum_{D_1} f(t'_p)|dg_p| \right| < e,$$

therefore, $\int_a^b f|dg|$ exists [2, p. 28].

The questions of reciprocity of the relationships between several of the integrals arise. If $\int_a^b |f| dg$ exists then $\int_a^b f dg$ does not necessarily exist. For example, if f is the function defined as follows:

$f(x) = 1$, if x is a rational number

$f(x) = -1$, if x is an irrational number

and $g(x) = x$, for each x in (a,b), then $\int_a^b |f| dg$ exists but $\int_a^b f dg$ does not exist; $\int_a^b g d|f|$ exists but $\int_a^b g df$ does not; and $\int_a^b |f dg|$ exists but $\int_a^b f|dg|$ does not.

If $\int_a^b f dg$ exists then $\int_a^b f|dg|$ does not necessarily exist. For example, if f and g are functions such that $f(x) = 1$ for each number x and $g(x) = x \sin \frac{1}{x}$ for each number $x \neq 0$ and $g(0) = 0$, then $\int_0^2 f dg$ exists but $\int_0^2 f|dg|$ does not.

BIBLIOGRAPHY

1. Helton, Burrell W. "Integral Equations and Product Integrals." *Pacific Journal of Mathematics*, Vol. XVI, 1966.

2. Hildebrandt, T. H. *Theory of Integration.* New York: Academic Press, 1963.