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Solutions to nonlinear elliptic equations with a nonlocal boundary condition *

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Abstract

We study an elliptic equation and its evolution problem on a bounded domain with nonlocal boundary conditions. Eigenvalue problems, existence, and dynamic behavior of solutions for linear and semilinear equations are investigated. We use the comparison principle and a semigroup approach.

1 Introduction

In this paper we consider the following nonlinear equation with nonlocal boundary conditions

$$
\begin{gathered}
Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) = f(x,u), \text{ in } \Omega \\
u|_{\partial\Omega} = \int_\Omega K(x,y)u(y)\,dy
\end{gathered}
\tag{1.1}
$$

and its corresponding evolution problem. Firstly, we consider the eigenproblem for the special case $u|_{\partial\Omega} = k\int_\Omega u(y)\,dy$ with k a constant. As we know from the literature [3, 12, 13, 16], the comparison principle may not apply, unless $K(x,y) \geq 0$ and $\int_\Omega K(x,y)\,dy < 1$. However, using special techniques one can obtain the behavior of solutions when $K(x,y)$ alternates signs [3, 12, 13]. But we wondered how the boundary kernel $K(x,y)$ influences results such as those on the eigenvalues and on the decay of solutions for evolution equations. Because these questions are not easy, we expect to have only a partial answer by considering a simple case. We will find that there are no negative eigenvalues unless $k > 1/|\Omega|$. Also we will obtain some estimates on the eigenvalues. In section 3, we prove the existence of solutions for linear problem. In section 4, the method of quasilinearization is used to prove that monotonic iterative sequences converge quadratically to the solution of the nonlinear problem. Lastly, we discuss the long time behavior of solution in Sobolev-Slobodeckii spaces.

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Throughout this paper we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^{2+\mu}$-boundary $\partial\Omega$, $a_{ij} \in C^{1+\mu}$ $(i,j = 1,2,\cdots,n)$ with $\mu \in (0,1)$ and that there exists a positive number α such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2, \quad \forall\,(x,\xi_1,\cdots,\xi_n) \in \overline{\Omega} \times \mathbb{R}^n. \tag{1.2}$$

2 Eigenvalue Problems

Let us consider a special eigenvalue problem for (1.1) with $K(x,y) = k$ a constant.

$$\begin{gathered} L\varphi(x) \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial \varphi(x)}{\partial x_j}) = \lambda\varphi(x), \quad \text{in } \Omega \\ \varphi|_{\partial\Omega} = k\int_\Omega \varphi(y)\,dy. \end{gathered} \tag{2.1}$$

We expect to obtain some information about the relation between the eigenvalue λ and the constant k. First integrate over Ω on the first equation of (2.1):

$$-\int_{\partial\Omega}\sum_{i,j=1}^{n} a_{ij}\frac{\partial\varphi}{\partial x_j}\cos(\nu,x_i)\,dS = \lambda\int_\Omega \varphi(x)\,dx. \tag{2.2}$$

Then multiplying by $\varphi(x)$ and integrate again

$$\begin{aligned} \int_\Omega \varphi L\varphi\,dx &= -\int_{\partial\Omega}\sum_{i,j=1}^{n} a_{ij}\frac{\partial\varphi}{\partial x_j}\cos(\nu,x_i)\,dS\cdot\gamma(\varphi) + \int_\Omega\sum_{i,j=1}^{n} a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\,dx \\ &= \lambda\int_\Omega \varphi^2(x)\,dx, \end{aligned} \tag{2.3}$$

where γ is the trace operator $\gamma(\varphi) = \varphi|_{\partial\Omega}$. Combining the above equations with the boundary condition in (2.1), we have

$$\lambda\Big\{\int_\Omega \varphi^2\,dx - k(\int_\Omega \varphi\,dx)^2\Big\} = \int_\Omega\sum_{i,j=1}^{n} a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\,dx \geq \alpha\int_\Omega |\nabla\varphi|^2\,dx. \tag{2.4}$$

It follows directly from Jensen's inequality and (2.4) that if there exists an eigenvalue $\lambda < 0$, then $k > 1/|\Omega|$. Moreover, for f_1 and $f_2 \in C(\overline{\Omega})$, Cauchy's inequality

$$\Big(\int_\Omega f_1(x)f_1(x)\,dx\Big)^2 \leq \int_\Omega f_1^2(x)\,dx\int_\Omega f_2^2(x)\,dx \tag{2.5}$$

becomes equality if and only if $f_1(x) = lf_2(x)$, in $\overline{\Omega}$. Therefore, if $\lambda_0 = 0$ is an eigenvalue, then its corresponding eigenfunction is $\varphi_0 = 1$. This implies that $k = 1/|\Omega|$. On the other hand, if $k = 1/|\Omega|$ then 0 is an eigenvalue of (2.1). Hence, all eigenvalues of (2.1) are positive when $k < 1/|\Omega|$. Thus, we have

Proposition 2.1 *For the linear eigenproblem (2.1) the following holds:*

i) 0 is an eigenvalue (with eigenfunction 1) if and only if $k = 1/|\Omega|$

ii) If there exists one eigenvalue $\lambda < 0$*, then* $k > 1/|\Omega|$

iii) If $k < 1/|\Omega|$ *then all eigenvalues of (2.1) are positive.*

Proposition 2.2 *The linear eigenproblem (2.1) has at most one negative eigenvalue.*

Proof. First, we claim that the eigenfunction $\varphi(x)$ corresponding to one negative eigenvalue λ does not alternate its sign on $\overline{\Omega}$.

Actually, the positive maximum $\varphi(x_M)$ can not be attained at $x_M \in \Omega$, otherwise

$$0 \le L\varphi(x_M) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial \varphi(x_M)}{\partial x_j}) = \lambda\varphi(x_M) < 0, \tag{2.6}$$

this is impossible. So, $\varphi(x_M) > 0$ can be attained only on the boundary $\partial\Omega$. Also $\varphi(x)$ can not have a negative minimum, $\varphi(x_m) < 0$, in Ω: It is easy to get contradiction as the one above. Hence, if $\varphi(x)$ is an eigenfunction with positive maximum on $\partial\Omega$ for a negative eigenvalue λ, then $\varphi(x) \ge 0$ for all $x \in \overline{\Omega}$.

Similarly, if $\varphi(x)$ is an eigenfunction with negative minimum on $\partial\Omega$ for a negative eigenvalue λ, then $\varphi(x) \le 0$ for all $x \in \overline{\Omega}$.

For $k > 1/|\Omega|$, we suppose that there exist two eigenvalues $\lambda_1 < \lambda_2 < 0$ and that $\varphi_1(x)$ and $\varphi_2(x)$ are the corresponding eigenfunctions, with $\varphi_1(x) \ge 0$, $\varphi_2(x) \ge 0$, satisfying $\varphi_1|_{\partial\Omega} = \varphi_2|_{\partial\Omega}$. Then the positive maxima for $\varphi_1(x)$ and $\varphi_2(x)$ can be attained only on $\partial\Omega$. We claim that $\varphi_1(x) \le \varphi_2(x)$ on $\overline{\Omega}$. If it is not true, there is $x^* \in \Omega$ such that $\varphi_1(x^*) > \varphi_2(x^*)$, with x^* a positive maximum point for $\varphi_1 - \varphi_2$, then

$$0 \le L(\varphi_1 - \varphi_2)|_{x^*} = \lambda_1\varphi_1(x^*) - \lambda_2\varphi_2(x^*).$$

From $\lambda_1 < \lambda_2 < 0$, it follows that $\varphi_1(x^*) \le |\frac{\lambda_2}{\lambda_1}|\varphi_2(x^*) < \varphi_1(x^*)$, which is a contradiction.

The inequality $\lambda_1 < \lambda_2$ implies $\varphi_1(x) \le \varphi_2(x)$, but

$$0 = \varphi_1|_{\partial\Omega} - \varphi_2|_{\partial\Omega} = k\int_{\Omega}(\varphi_1(y) - \varphi_2(y))\,dy \le 0\,.$$

There exists only one possibility: $\varphi_1(x) = \varphi_2(x)$ on $\overline{\Omega}$. Therefore, $\lambda_1 = \lambda_2$. □

Naturally, the next step is to estimate the minimal eigenvalue for (2.1). As mentioned, if $k = 1/|\Omega|$ then the minimal eigenvalue $\lambda = 0$. Now we consider the issue for $k < 1/|\Omega|$.

Proposition 2.3 *Let* d *be the diameter of* Ω*. Then* *i)* $\lambda \geq \frac{2\alpha}{nd^2}\big[1 + \frac{|\Omega|}{1-k|\Omega|}(k - \frac{1}{d^n})\big]$ *for* $k \leq 0$

ii) $\lambda \geq \frac{2\alpha}{nd^2}\big[1 - \frac{|\Omega|}{1-k|\Omega|}(k + \frac{1}{d^n})\big]$ *for* $0 < k < 1/|\Omega|$.

Proof. Let $\varphi(x)$ be an eigenfunction for the minimal eigenvalue λ. Let D be the cube in $\mathbb{R}^n$ with edges of length d containing Ω. Extend φ into D with $\gamma(\varphi) = k\int_\Omega \varphi(y)\,dy$, denote the extension by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & \text{in } \Omega \\ k\int_\Omega \varphi\,dx, & \text{in } D-\Omega. \end{cases} \tag{2.7}$$

Define $\Phi = \int_\Omega \varphi\,dx$, obviously,

$$\int_D \tilde{\varphi}^2\,dx = \int_\Omega \varphi^2\,dx + k^2\Phi^2|D-\Omega|.$$

Applying Poincaré's inequality in the cube D, we have

$$\int_\Omega \varphi^2\,dx + k^2\Phi^2|D-\Omega| \leq \frac{1}{d^n}(k\Phi|D-\Omega| + \Phi)^2 + \frac{nd^2}{2}\int_\Omega |\nabla\varphi|^2\,dx.$$

From the elliptic hypothesis (1.2) and (2.4),

$$\lambda\frac{nd^2}{2\alpha}\big[\int_\Omega \varphi^2\,dx - k\Phi^2\big] \geq \int_\Omega \varphi^2\,dx + k^2\Phi^2|D-\Omega| - \frac{\Phi^2(k|D-\Omega|+1)^2}{d^n}.$$

Let $\int_\Omega \varphi^2\,dx = 1$, take note of $\Phi^2 = (\int_\Omega \varphi\,dx)^2 < |\Omega|$, Since $\varphi(x)$ is not constant for $k \neq 1/|\Omega|$ (see Proposition 2.1), then $\Phi^2 \in [0, |\Omega|)$. By the assumption $k < 1/|\Omega|$,

$$\begin{aligned} \lambda &\geq \frac{2\alpha}{nd^2(1-k\Phi^2)}\big[1 + \Phi^2(k^2|D-\Omega| - \frac{(k|D-\Omega|+1)^2}{d^n})\big] \\ &= \frac{2\alpha}{nd^2} + \frac{2\alpha\Phi^2}{nd^2(1-k\Phi^2)}(k + k^2|D-\Omega| - \frac{k^2|D-\Omega|^2 + 2k|D-\Omega| + 1}{d^n}) \\ &\geq \frac{2\alpha}{nd^2} + \frac{2\alpha\Phi^2}{nd^2(1-k\Phi^2)}(k - \frac{2k|D-\Omega|+1}{d^n}), \end{aligned} \tag{2.8}$$

in the last inequality above, the relation $|D-\Omega| \leq |D| = d^n$ is used. It is not difficult to get that the nonnegative function $h(t) = \frac{t}{1-kt}$ reach its maximum $\frac{|\Omega|}{1-k|\Omega|}$ at $t = |\Omega|$ (for $t \in [0, |\Omega|]$).

Hence, if $0 \leq k < 1/|\Omega|$, then

$$\lambda \geq \frac{2\alpha}{nd^2} - \frac{2\alpha}{nd^2}\frac{|\Omega|}{1-k|\Omega|}(k + \frac{1}{d^n}). \tag{2.9}$$

If $k < 0$, then

$$\lambda \geq \frac{2\alpha}{nd^2} + \frac{2\alpha}{nd^2}\frac{|\Omega|}{1-k|\Omega|}(k - \frac{1}{d^n}). \tag{2.10}$$

The assertion are proved. □

Because the domain is smooth, $|\Omega|/d^n < 1$. Certainly, (2.10) deduces $\lambda > 0$ for $k < 0$. On the other hand, the estimate (2.8) is more accurate than (2.9), one can obtain easily from (2.8) that $\lambda > 0$ provided with $k < 1/(2|\Omega|)$. Now we see a special example in one-dimension:

$$-\phi'' = \rho\phi, \quad x \in (-\pi, \pi); \quad \phi(-\pi) = \phi(\pi) = k \int_{-\pi}^{\pi} \phi(x)\, dx. \tag{2.11}$$

For this problem, the relationship between k and ρ is as follows: If $\rho < 0$, then $k = \frac{1}{2}\sqrt{-\rho}\coth(\pi\sqrt{-\rho})$; if $\rho = 0$, then $k = 1/(2\pi)$; and if $\rho > 0$ and not the square of an integer, then $k = \frac{1}{2}\sqrt{\rho}\cot(\pi\sqrt{\rho})$. See Figure 1, where the eigenvalues correspond to the values k for which the graph crosses the horizontal axis.

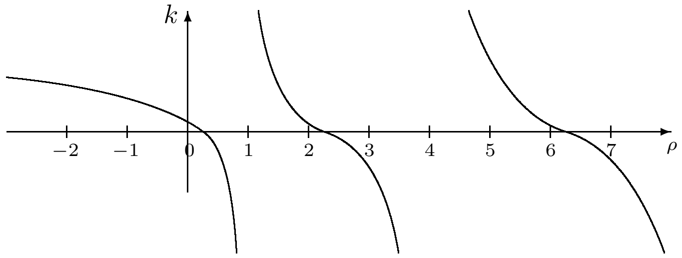

Figure 1: k as a function of ρ for problem (2.11)

3 Linear Problems

We investigate the linear problem before using the monotonic iteration method for nonlinear equations. Throughout this sections we assume that $k < 1/|\Omega|$. To get the existence of solutions for the linear problem

$$\begin{gathered}(L + c)u \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial u}{\partial x_j}\big) + c(x)u = F(x), \quad \text{in } \Omega \\ u|_{\partial\Omega} = k\int_{\Omega} u(y)\, dy,\end{gathered} \tag{3.1}$$

we discuss the Dirichlet problem

$$\begin{gathered}(L + c)U + \frac{k\, c(x)}{1 - k|\Omega|}\int_{\Omega} U(x)\, dx = F(x), \quad \text{in } \Omega \\ U|_{\partial\Omega} = 0.\end{gathered} \tag{3.2}$$

Lemma 3.1 *For $F(x), c(x) \in C^{\mu}(\overline{\Omega})$ and $c(x) \geq 0$, the linear problem (3.2) admits a unique solution $u \in C^{2+\mu}$.*

Proof. From the theory on elliptic equations [5], we know that (3.2) has a unique solution when $k=0$, *i.e.* the operator $L+c$ has a compact inverse operator $(L+c)^{-1}$. According to Riesz-Schauder theory [17], if 0 is not an eigenvalues for the eigenproblem

$$\begin{gathered}(L+c)\varphi+\frac{k\,c(x)}{1-k|\Omega|}\int_\Omega \varphi(x)\,dx=\lambda\varphi, \quad \text{in } \Omega\\ \varphi|_{\partial\Omega}=0,\end{gathered} \tag{3.3}$$

then (3.2) has a unique solution. Now we show that 0 is an eigenvalue of (3.3). Otherwise, the problem

$$\begin{gathered}(L+c)\varphi+\frac{k\,c(x)}{1-k|\Omega|}\int_\Omega \varphi(x)\,dx=0, \quad \text{in } \Omega\\ \varphi|_{\partial\Omega}=0\end{gathered} \tag{3.4}$$

has a solution $\varphi(x)\not\equiv 0$ ($l\varphi$ is also a solution for all $l\in\mathbb{R}$). From the maximum principle, $\int_\Omega \varphi(x)\,dx\neq 0$. Denote $\varphi_0=\varphi/\int_\Omega\varphi(x)\,dx$. Then the Dirichlet problem

$$(L+c)\varphi_0(x)=-\frac{k\,c(x)}{1-k|\Omega|}, \quad \varphi_0|_{\partial\Omega}=0$$

has a unique solution $\varphi_0(x)$ for any $c(x)\geq 0$ and k. The maximal principle for nonhomogeneous equations [5, chpater 3] shows that there is a constant C, independent of the nonhomogeneous term $-\frac{k\,c(x)}{1-k|\Omega|}$, such that

$$\sup_\Omega \varphi_0(x)\leq \sup_{\partial\Omega}\varphi_0(x)+\frac{C}{\alpha}\sup_\Omega\Big\|\frac{-k\,c(x)}{1-k|\Omega|}\Big\|=\Big\|\frac{Ck}{\alpha(1-k|\Omega|)}\Big\|\sup_\Omega c(x)\overset{k\to 0}{\longrightarrow}0.$$

But $\int_\Omega\varphi_0(x)\,dx=1$ for all $k<1/|\Omega|$ and $c\geq 0$, this contradicts the above inequality. Therefore, there is no eigenfunction $\varphi\not\equiv 0$ and 0 is not the eigenvalue of (3.3). It follows that (3.2) has a unique solution $U\in C^{2+\mu}$. $\square$

In the above proof, we observe that the mapping $\tilde{k}(\varphi)\equiv\frac{k\,c(x)}{1-k|\Omega|}\int_\Omega\varphi(x)\,dx$ with the domain and the range $C^\mu(\overline{\Omega})$, is linear and bounded.

The proof consists of finding an $H_0^1(\Omega)$-solution, then to strengthening the regularity by estimates and Sobolev inequalities.

Take $u=U+\frac{k}{1-k|\Omega|}\int_\Omega U\,dx$ with U being the solution (3.2), then

$$(L+c)u=(L+c)\big(U+\frac{k}{1-k|\Omega|}\int_\Omega U\,dx\big)=F(x),$$

and

$$\begin{aligned}u|_{\partial\Omega}&=\frac{k}{1-k|\Omega|}\int_\Omega U\,dx=\frac{k(1-k|\Omega|)+k^2|\Omega|}{1-k|\Omega|}\int_\Omega U\,dx\\ &=k\int_\Omega Udx+\frac{k^2|\Omega|}{1-k|\Omega|}\int_\Omega U\,dx\\ &=k\int_\Omega\big(U+\frac{k}{1-k|\Omega|}\int_\Omega U\,dx\big)dx=k\int_\Omega udx.\end{aligned}$$

Theorem 3.2 *For $c > 0$ and $F(x) \in C^\mu$, the linear nonlocal boundary problem (3.1) admits a unique solution $u \in C^{2+\mu}$.*

Proof. We prove only the uniqueness. If there are two solutions, then the problem

$$Lu + cu = 0, \quad \text{for } \ x \in \Omega; \quad u|_{\partial\Omega} = k \int_\Omega u(x)dx$$

has nonzero solution. This is not possible for $c(x) > 0$ and $u(x)$ being constant on $\partial\Omega$. □

From the above discussion, one can see that when $k \to 0$, the solution of (3.1) approaches U_0, the solution of

$$\begin{gathered}(L+c)u \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + c(x)u = F(x), \quad \text{in } \Omega \\ u|_{\partial\Omega} = 0.\end{gathered} \tag{3.5}$$

More generally, if $k = K(x,y)$ is smooth enough on $\overline{\Omega} \times \overline{\Omega}$, then the solution of the corresponding linear problem with boundary condition $u|_{\partial\Omega} = \int_\Omega K(x,y)u(y)dx$ approaches the solution of (3.5) when $\epsilon \equiv \max_{\overline{\Omega}\times\overline{\Omega}} |K(x,y)| \to 0$. We assume that $K \in C^{1+\mu}(\overline{\Omega}) \times C(\overline{\Omega})$ satisfies

$$K(x,y) \geq 0, \quad \int_\Omega K(x,y)\,dy < 1, \quad \text{for } x \in \partial\Omega, y \in \overline{\Omega}. \tag{3.6}$$

Now we give a comparison and an existence result.

Lemma 3.3 *Let $K(x,y)$ satisfy (3.6), and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy*

$$Lu + cu \leq 0, \quad \textit{for } \ x \in \Omega; \quad u|_{\partial\Omega} \leq \int_\Omega K(x,y)u(y)dx,$$

with $c(x) \geq 0$. Then $u(x) \leq 0$ for all $x \in \overline{\Omega}$.

Lemma 3.4 *Let $K(x,y)$ satisfy (3.6), $C \in C^\mu(\overline{\Omega})$, and $c(x) \geq 0$, then the linear problem*

$$Lu + cu = F(x), \quad \textit{for } x \in \Omega; \quad u|_{\partial\Omega} \leq \int_\Omega K(x,y)u(y)dx$$

has a unique solution $u \in C^{2+\mu}(\overline{\Omega})$ for all $F \in C^\mu(\overline{\Omega})$.

Proof. The assertion in Lemma 3.3 can be proved using a method similar to the one in [12, Lemma 3.1]. The existence is deduced from [12, Theorem 3.3]. □

4 Nonlinear Problems

We use the method of upper and lower solutions to discuss the existence of solutions for nonlinear problem (1.1). In this section, we assume that $K(x,y)$ satisfies (3.6).

A pair of a lower solution $\underline{u}(x)$ and an upper solution $\overline{u}(x)$ in $C^2(\Omega)\cap C(\overline{\Omega})$ of (1.1) is defined as

$$L\underline{u} \le f(x,\underline{u}), \quad \underline{u}|_{\partial\Omega} \le \int_\Omega K(x,y)\underline{u}(y)\,dy; \tag{4.1}$$

$$L\overline{u} \ge f(x,\overline{u}), \quad \overline{u}|_{\partial\Omega} \ge \int_\Omega K(x,y)\overline{u}(y)\,dy. \tag{4.2}$$

We construct two iteration sequences $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ starting with $\underline{u} = \underline{u}_0$ and $\overline{u} = \overline{u}_0$ as follows

$$L\underline{u}_n + c\underline{u}_n = c\underline{u}_{n-1} + f(x,\underline{u}_{n-1}), \quad \underline{u}_n|_{\partial\Omega} = \int_\Omega K(x,y)\underline{u}_n(y)\,dy; \tag{4.3}$$

$$L\overline{u}_n + c\overline{u}_n = c\overline{u}_{n-1} + f(x,\overline{u}_{n-1}), \quad \overline{u}_n|_{\partial\Omega} = \int_\Omega K(x,y)\overline{u}_n(y)\,dy. \tag{4.4}$$

Though the construction of iteration sequences are not the same as that in [12], the convergence can be proved by an analogous argument.

Theorem 4.1 *If there exists one ordered pair of a lower and an upper solution $\underline{u}$ and $\overline{u}$, $\underline{u} \le \overline{u}$, and there is a constant $c > 0$ such that*

$$f(x,u) - f(x,v) \ge -c(u-v), \quad \text{for } u \ge v, \quad \text{and} \quad u,v \in [\underline{u},\overline{u}],$$

where $u \in [\underline{u},\overline{u}]$ means $\underline{u}(x) \le u(x) \le \overline{u}(x)$, for all $x \in \overline{\Omega}$. Then the problem (1.1) has solutions $\underline{u}_s$ and $\overline{u}_s$ satisfying $\underline{u}(x) \le \underline{u}_s \le \overline{u}_s \le \overline{u}(x)$.

Proof. According to the definition of iteration sequences $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ in (4.3) and (4.4), we get

$$L(\underline{u}_1 - \underline{u}_0) + c(\underline{u}_1 - \underline{u}_0) \ge 0, \quad (\underline{u}_1 - \underline{u}_0)|_{\partial\Omega} \ge \int_\Omega K(x,y)(\underline{u}_1 - \underline{u}_0)(y)\,dy.$$

From Lemma 3.3, it follows that $\underline{u}_1 \ge \underline{u}_0$. Similarly

$$L(\underline{u}_2 - \underline{u}_1) + c(\underline{u}_2 - \underline{u}_1) = c(\underline{u}_1 - \underline{u}_0) + f(x,\underline{u}_1) - f(x,\underline{u}_0) \ge 0,$$
$$(\underline{u}_2 - \underline{u}_1)|_{\partial\Omega} \ge \int_\Omega K(x,y)(\underline{u}_2 - \underline{u}_1)(y)\,dy.$$

As in the discussion above, one can prove that the sequence $\{\underline{u}_n\}$ is monotone nondecreasing, the sequence $\{\overline{u}_n\}$ is monotone non-increasing. and

$$L(\overline{u}_1 - \underline{u}_1) + c(\overline{u}_1 - \underline{u}_1) = c(\overline{u}_1 - \underline{u}_0) + f(x,\overline{u}_0) - f(x,\underline{u}_0) \ge 0,$$
$$(\overline{u}_1 - \underline{u}_1)|_{\partial\Omega} \ge \int_\Omega K(x,y)(\overline{u}_1 - \underline{u}_1)(y)\,dy.$$

Then, $\overline{u}_1 \geq \underline{u}_1$, generally,

$$\underline{u} = \underline{u}_0 \leq \underline{u}_1 \leq \cdots \leq \underline{u}_n \leq \overline{u}_n \leq \cdots \leq \overline{u}_1 = \overline{u}_0,$$

it follows that $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ converge, respectively, to some limits $\underline{u}_s$ and $\overline{u}_s$, and satisfy the relation $\underline{u}_s \leq \overline{u}_s$. A regularity argument shows that $\underline{u}_s$ and $\overline{u}_s$ are solutions of (1.1) [11], the details are omitted here. □

In fact, $\underline{u}_s$ and $\overline{u}_s$ are the minimal and the maximal solution in $[\underline{u}, \overline{u}]$, it is easy to obtain that $\underline{u}_s \leq u \leq \overline{u}_s$ if (1.1) has another solution $u \in [\underline{u}, \overline{u}]$. Certainly, $\underline{u}_s$ and $\overline{u}_s$ may be equal, for example, when $f(x,u)$ is monotone non-increasing on u, $\underline{u}_s = \overline{u}_s$ [12].

Furthermore, assume that

- H1: $f(x,u) = F(x,u) + G(x,u)$ and that F_u, G_u, F_{uu}, G_{uu} exist, are continuous, and $F_{uu} \geq 0$, $G_{uu} \leq 0$ on $\overline{\Omega} \times \mathrm{R}$.

Employing the quasilinearization idea in [7], we have

Theorem 4.2 *Under assumption H1, if there exist one pair of ordered lower and upper solutions $\underline{u}$ and $\overline{u}$ for the problem (1.1), and there is a positive constant c such that*

$$F_u(x, \overline{u}) + G_u(x, \underline{u}) \leq -c < 0.$$

Then there exist monotone sequences $\{\underline{u}_n\}$, $\{\overline{u}_n\} \in C^{2+\mu}(\overline{\Omega})$ such that $\underline{u}_n \to u \leftarrow \overline{u}_n$, u is the unique solution of (1.1) satisfying $\underline{u} \leq u \leq \overline{u}$, and the convergence is quadratic.

Proof. The hypotheses $F_{uu} \geq 0$ and $G_{uu} \leq 0$, yield inequalities

$$\begin{aligned} F(x,u) - F(x,v) &\geq F_u(x,v)(u-v), \\ G(x,u) - G(x,v) &\geq G_u(x,u)(u-v), \end{aligned} \quad \text{for } u \geq v. \tag{4.5}$$

We construct new iterative sequences $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$, starting with $\underline{u}_0 = \underline{u}$ and $\overline{u}_0 = \overline{u}$, by linear equations

$$\begin{gathered} L\underline{u}_n = F(x, \underline{u}_{n-1}) + G(x, \underline{u}_{n-1}) + (F_u(x, \underline{u}_{n-1}) + G_u(x, \overline{u}_{n-1}))(\underline{u}_n - \underline{u}_{n-1}), \\ L\overline{u}_n = F(x, \overline{u}_{n-1}) + G(x, \overline{u}_{n-1}) + (F_u(x, \underline{u}_{n-1}) + G_u(x, \overline{u}_{n-1}))(\overline{u}_n - \overline{u}_{n-1}); \\ \underline{u}_n|_{\partial\Omega} = \int_\Omega K(x,y)\underline{u}_n\,dx, \quad \overline{u}_n|_{\partial\Omega} = \int_\Omega K(x,y)\overline{u}_n\,dx. \end{gathered} \tag{4.6}$$

It is obvious that

$$F_u(x, \overline{u}_n) + G_u(x, \underline{u}_n) \leq -c < 0 \quad \text{for } \ \underline{u} \leq \underline{u}_{n-1},\ \overline{u}_{n-1} \leq \overline{u}; \tag{4.7}$$

$(n = 1, 2, \cdots,)$. As we know, for $\eta \in C^2(\overline{\Omega})$ with $\underline{u} \leq \eta \leq \overline{u}$, the function $h(x) = F(x,\eta) + G(x,\eta) - F_u(x,\eta)\eta - G_u(x,\eta)\eta$ belongs to $C^\mu(\overline{\Omega})$ [7]. Hence the linear problems (4.6) have unique solutions $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ in $C^{2+\mu}(\overline{\Omega})$. Also,

$$\begin{gathered}L(\underline{u}_1-\underline{u}_0)\geq (F_u(x,\underline{u}_0)+G_u(x,\underline{u}_0))(\underline{u}_1-\underline{u}_0),\\ (\underline{u}_1-\underline{u}_0)|_{\partial\Omega}\geq k\int_\Omega(\underline{u}_1-\underline{u}_0)\,dx.\end{gathered}$$

Taking notice of (4.7), Lemma 3.3 yields $\underline{u}_0\leq\underline{u}_1$.

A similar argument gives $\overline{u}_1\leq\overline{u}_0$. We show next that $\underline{u}_1\leq\overline{u}_0$ on $\overline{\Omega}$. Using the inequalities in (4.5), we get

$$\begin{aligned}&L(\overline{u}_0-\underline{u}_1)\\ &\geq F(x,\overline{u}_0)+G(x,\overline{u}_0)-F(x,\underline{u}_0)-G(x,\underline{u}_0)\\ &\quad -(F_u(x,\underline{u}_0)+G_u(x,\underline{u}_0))(\underline{u}_1-\underline{u}_0)\\ &\geq (F_u(x,\underline{u}_0)+G_u(x,\overline{u}_0))(\overline{u}_0-\underline{u}_0)-(F_u(x,\underline{u}_0)+G_u(x,\underline{u}_0))(\underline{u}_1-\underline{u}_0)\\ &\geq (F_u(x,\overline{u}_0)+G_u(x,\overline{u}_0))(\overline{u}_0-\underline{u}_1)+(G_u(x,\overline{u}_0)-G_u(x,\underline{u}_0)(\underline{u}_1-\underline{u}_0)\\ &\geq (F_u(x,\overline{u}_0)+G_u(x,\overline{u}_0))(\overline{u}_0-\underline{u}_1).\end{aligned}$$

The condition $G_{uu}\leq 0$ is used for the last inequality. Lemma 3.3 implies $\underline{u}_1\leq\overline{u}_0$. Similarly one can get that $\underline{u}_0\leq\overline{u}_1$. Also, since that $F_u(x,u)$ and $G_u(x,u)$ are nondecreasing and non-increasing in u respectively, from (4.5) we arrive at

$$\begin{aligned}&L(\underline{u}_2-\underline{u}_1)\\ &= F(x,\underline{u}_1)+G(x,\underline{u}_1)-F(x,\underline{u}_0)+G(x,\underline{u}_0)\\ &\quad +(F_u(x,\underline{u}_1)+G_u(x,\overline{u}_1))(\underline{u}_2-\underline{u}_1)-(F_u(x,\underline{u}_0)+G_u(x,\overline{u}_0))(\underline{u}_1-\underline{u}_0)\\ &\geq (F_u(x,\underline{u}_1)+G_u(x,\underline{u}_1))(\underline{u}_1-\underline{u}_0)-(F_u(x,\underline{u}_0)+G_u(x,\overline{u}_0))(\underline{u}_1-\underline{u}_0)\\ &\quad +(F_u(x,\underline{u}_1)+G_u(x,\overline{u}_1))(\underline{u}_2-\underline{u}_1)\\ &\geq (F_u(x,\underline{u}_1)+G_u(x,\overline{u}_1))(\underline{u}_2-\underline{u}_1).\end{aligned}$$

It then follows by Lemma 3.3 that $\underline{u}_1\leq\underline{u}_2$ on $\overline{\Omega}$. And $\overline{u}_2\leq\overline{u}_1$ can be obtained similarly. In the same way, we get

$$\begin{aligned}L(\overline{u}_1-\underline{u}_1) &= F(x,\overline{u}_0)+G(x,\overline{u}_0)+(F_u(x,\underline{u}_0)+G_u(x,\overline{u}_0))(\overline{u}_1-\overline{u}_0)\\ &\quad -F(x,\underline{u}_0)-G(x,\underline{u}_0)-(F_u(x,\underline{u}_0)+G_u(x,\overline{u}_0))(\underline{u}_1-\underline{u}_0)\\ &\geq (F_u(x,\underline{u}_0)+G_u(x,\overline{u}_0))(\overline{u}_0-\underline{u}_0)\\ &\quad +(F_u(x,\underline{u}_0)+G_u(x,\overline{u}_0))(\overline{u}_1-\overline{u}_0-\underline{u}_1+\underline{u}_0)\\ &\geq (F_u(x,\underline{u}_0)+G_u(x,\overline{u}_0))(\overline{u}_1-\underline{u}_1).\end{aligned}$$

Hence, $\underline{u}_1\leq\overline{u}_1$. From a similar argument, we can show $\underline{u}_2\leq\overline{u}_2$. By the above process, step by step, we have

$$\underline{u}_0\leq\underline{u}_1\leq\underline{u}_2\leq\cdots\leq\underline{u}_n\leq\overline{u}_n\leq\cdots\leq\overline{u}_2\leq\overline{u}_1\leq\overline{u}_0.$$

The convergence for $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$, and regularity for the limits can be proved by a similar process to [7] or [11], we omit the details. The uniqueness of the solution follows from the assumption (4.2). Hence, we obtain that $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ converge, nondecreasing and nonincreasing respectively, to the unique solution $u \in C^{2+\mu}(\overline{\Omega})$ between $\underline{u}$ and $\overline{u}$.

To prove the quadratic convergence of $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$, we define $P_n = u - \underline{u}_n$, and $Q_n = \overline{u}_n - u$, then

$$\begin{aligned}
LP_n &= F(x,u) + G(x,u) - [F(x,\underline{u}_{n-1}) + G(x,\underline{u}_{n-1}) \\
&\quad + (F_u(x,\underline{u}_n) + G_u(x,\overline{u}_n)(\underline{u}_n - \underline{u}_{n-1})] \\
&\le [F_u(x,u) - F_u(x,\underline{u}_{n-1})]P_{n-1} + [G_u(x,\underline{u}_{n-1}) - G_u(x,\overline{u}_{n+1})]P_{n-1} \\
&\quad + [F_u(x,\underline{u}_{n-1}) + G_u(x,\overline{u}_{n-1})]P_n \\
&= F_{uu}(x,\xi)P_{n-1}^2 + G_{uu}(x,\zeta)(\underline{u}_{n-1} - \overline{u}_{n-1})P_{n-1} \\
&\quad + [F_u(x,\underline{u}_{n-1}) + G_u(x,\overline{u}_{n-1})]P_n,
\end{aligned}$$

where $\underline{u}_{n-1} \le \xi \le u$, $\underline{u}_{n-1} \le \zeta \le \overline{u}_{n-1}$. Because

$$\begin{aligned}
&F_{uu}(x,\xi)P_{n-1}^2 + G_{uu}(x,\zeta)(\underline{u}_{n-1} - \overline{u}_{n-1})P_{n-1} \\
&\le F_{uu}(x,\xi)P_{n-1}^2 - G_{uu}(x,\zeta)(P_{n-1} + Q_{n-1})P_{n-1} \\
&\le \delta_1(P_{n-1}^2 + P_{n-1}Q_{n-1}) \le \frac{3\delta_1}{2}(P_{n-1}^2 + Q_{n-1}^2)
\end{aligned}$$

where $\delta_1 = \max\{|G_{uu}(x,u)| : x \in \overline{\Omega}, \underline{u} \le u \le \overline{u}\}$. Take $\delta = 3\delta_1/2$, then

$$LP_n - [F_u(x,\underline{u}_{n-1}) + G_u(x,\overline{u}_{n-1})]P_n \le \delta(P_{n-1}^2 + Q_{n-1}^2).$$

Hence

$$LP_n + cP_n \le \delta(P_{n-1}^2 + Q_{n-1}^2).$$

On the other hand, $\phi(x) \equiv \delta[\max_{\overline{\Omega}} P_{n-1}^2 + \max_{\overline{\Omega}} Q_{n-1}^2]/c$ satisfies

$$\begin{gathered}
L(\phi - P_n) + c(\phi - P_n) \ge c\phi(x) - \delta(P_{n-1}^2 + Q_{n-1}^2) \ge 0 \\
(\phi(x) - P_n(x))|_{\partial\Omega} \ge \int_\Omega K(x,y)(\phi(y) - P_n(y))\,dy.
\end{gathered}$$

By Lemma 3.3, we have $\phi(x) \ge P_n(x)$, that is

$$0 \le u - \underline{u}_n = P_n \le \frac{\delta}{c}[\max_{\overline{\Omega}} P_{n-1}^2 + \max_{\overline{\Omega}} Q_{n-1}^2]. \tag{4.8}$$

A similar estimate for Q_n can be obtained. Therefore, the assertion is proved. $\square$

5 Parabolic Equations

In this section we study the large time behavior of solutions for the evolution equation

$$\begin{gathered} u_t+(L+c)\,u\equiv u_t-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j})+c(x)u=f(x,u),\quad \text{in } \Omega\times(0,T],\\ u|_{\partial\Omega}=\int_\Omega K(x,y)u(y,t)\,dy.\\ u(x,0)=u_0(x),\quad \text{on}\overline{\Omega}. \end{gathered} \tag{5.1}$$

The authors of [3, 4, 12, 13] have obtained some results. Here, we use semigroup methods to discuss the decay of solutions. We should also mention the work of Triggiani [15], Lasiecka [8, 9], and Amann [2].

Let $W_p^s(\Omega)$ be the standard Sobolev-Slobodeckii spaces for $s\in\mathbb{R}^+$, $p>1$, $1/p+1/p'=1$, and

$$W^{2\beta}_{p,\gamma}\equiv\begin{cases} W_p^{2\beta}, & \text{for } 2\beta\in[0,1/p),\\ (W_{p'}^{-2\beta})', & \text{for } 2\beta\in[-2,0]\setminus\{-2+1/p,-1+1/p\}\end{cases}$$

where X' is the duality space of X with respect to the duality pairing which is obtained naturally from $\int_\Omega v(x)u(x)\,dx$, $v\in L_{p'}$, $u\in L_p$. Hence $W^{2\beta}_{p,\gamma}$ is a closed linear subspace of $W_p^{2\beta}$. And the boundary space is defined as

$$\partial W_p^{2\beta}\equiv W_p^{2\beta-1/p}(\partial\Omega),\quad \text{for } 2\beta\in[0,1/p).$$

Denote $\mathbf{K}(u)=\int_\Omega K(x,y)u(y,t)dy$ and $\mathcal{F}(u)=f(x,u)$, a $W_p^{2\beta}$-*weak solution* on J of (5.1) is defined as one function $u\in C(J,W_p^{2\beta})$ satisfying the initial data $u(x,0)=u_0$, where J is one perfect subinterval of $\mathbb{R}^+$ containing 0, such that

$$\begin{gathered}\int_0^t\int_\Omega\Big\{-\dot\phi u+\sum_{i,j=1}^{n}a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial\phi}{\partial x_i}+cu\phi dx+\int_{\partial\Omega}\sum_{i,j=1}^{n}a_{ij}\frac{\partial\phi}{\partial x_i}\mathbf{K}(u)\cos(\nu,x_i)dS\Big\}dt\\ =\int_0^t\int_\Omega\phi\mathcal{F}(u)dx\,dt+\int_\Omega\phi(0)u_0dx\end{gathered}$$

for every $t\in J\setminus\{0\}$ and every $\phi\in C([0,t],W^{2(1-\beta)}_{p',\gamma})\cap C^1([0,t],W^{-2\beta}_{p',\gamma})$ satisfying $\phi(t)=0$.

The above definition of solution of (5.1) is meaningful. By using Green's formula, if $u\in C(J,W_p^2)\cap C^1(J,L_p)$ satisfies (5.1) (pointwise in t) then u is a solution of (5.1) on J. Let $\mathcal{H}(X)$ be the infinitesimal generator of a strongly continuous analytic semigroup $\{e^{-tA};\,t\geq0\}$ on a Banach space X. Let $\sigma(A)$ be the spectrum of A.

Lemma 5.1 ([2, Lemma 4.1]) *Put* $A_0=(L+c)|W^2_{p,\gamma}\equiv\{u\in W_p^2;\gamma u=0\}$. *Then*

1) $A_0 \in \mathcal{L}(W^2_{p,\gamma}, L_p)$ *with compact resolvent belongs to* $\mathcal{H}(L_p)$, *where* $\mathcal{L}(X,Y)$ *is defined as the bounded linear operators from Banach spaces* X *to* Y.

2) There exists a unique $A_{\beta-1} \in \mathcal{H}((W^{2(1-\beta)}_{p'})') \cap \mathcal{L}(W^{2\beta}_{p,\gamma}, (W^{2(1-\beta)}_{p'})')$ *with compact resolvent, so that* $A_{\alpha-1}$ *is the* $(W^{2(1-\alpha)}_{p'})'$*-realization of* $A_{\beta-1}$, *i.e.* $D(A_{\alpha-1}) \equiv \{y \in (W^{2\alpha}_{p'})' \cap D(A_{\beta-1});\ A_{\alpha-1}y \in (W^{2\alpha}_{p'})'\}$, *for* $\alpha \geq \beta$.

3) There exist $\sigma \in \mathbb{R}$ *and* $\mathcal{R}_\beta \in \mathcal{L}(\partial W^{2\beta}_p, W^{2\beta}_{p,\gamma})$, $\beta \in [0, 1/p)$, *so that* $\mathcal{R}_\alpha = \mathcal{R}_\beta | \partial W^{2\alpha}_p$ *for* $\alpha \geq \beta$ *and*

$$\int_\Omega v(\sigma + A_{\beta-1})\mathcal{R}_\beta u dx = \int_{\partial\Omega} \sum_{i,j=1}^n a_{ij} \frac{\partial \phi}{\partial x_i} \gamma u \cos(\nu, x_i) dS,$$

for $(v, u) \in W^{2(1-\beta)}_{p',\gamma} \times W^{2\beta}_{p,\gamma}$.

4) Let U_β *be a nonempty open subset of* $W^{2\beta}_{p,\gamma}$, $\mathcal{F} \in C(U_\beta, (W^{2(1-\beta)}_{p'})')$ *and* $u_0 \in U_\beta$. *Then* u *is a solution of (5.1) on* J *if and only if* u *is a solution on* J *of the evolution equation*

$$\dot{u} + (A_{\beta-1} - (\sigma + A_{\beta-1})\mathcal{R}_\beta \mathbf{K})u = \mathcal{F}(u),\ t \in J,\ u(0) = u_0. \tag{5.2}$$

5) $E_{\alpha-1}$ *is imbedded densely into* $(E_{\beta-1}, E_\beta)_\theta$ *for* $\alpha - \beta > \theta > 0$.

6) $\sigma(A_{\beta-1}) = \sigma(A_0)$ *for* $\beta \in [0, 1/p]$ *and the geometric eigenspaces, ker*$(\lambda + A_{\beta-1})$, *and the algebraic eigenspaces,* $\cup_{k\geq 1}$ *ker*$(\lambda + A_{\beta-1})^k$, *are independent of* β *for* $\lambda \in \sigma(A_{\beta-1}) = \sigma(A_0)$.

In fact, for the linear elliptic problem

$$A_{\beta-1}u = f_1, \quad \gamma u = \psi, \tag{5.3}$$

there exists one $\sigma \in \mathbb{R}$ and $\mathcal{R}_\beta$ such that the problem (5.3) has solution if and only if the equation

$$A_{\beta-1}u = f_1 + (\sigma + A_{\beta-1})\mathcal{R}_\beta \psi \tag{5.4}$$

has solution [1]. So, we can treat the linear nonlocal problem

$$A_{\beta-1}u = f_1, \quad \gamma u = \mathbf{K}u, \tag{5.5}$$

as

$$(A_{\beta-1} - (\sigma + A_{\beta-1})\mathcal{R}_\beta \mathbf{K})u = f_1. \tag{5.6}$$

i.e. by a solution u of (5.5) we mean a $W^{2\beta}_{p,\gamma}$-solution of (5.6). The existence of a solution for (5.5) is changed into the existence for a new operator equation (5.6). Of course, this is a generalization of the discussion in section 3. Particularly, when $\sigma = 0$, the problem (5.6) becomes

$$(A_{\beta-1} - A_{\beta-1}\mathcal{R}_\beta \mathbf{K})u = A_{\beta-1}(I - \mathcal{R}_\beta \mathbf{K})u = f_1, \tag{5.7}$$

in which I is the identity operator. We make an observation on (5.7), if 1 does not belong to the eigenvalue set of $\mathcal{R}_\beta \mathbf{K}$ and $A_{\beta-1}(I-\mathcal{R}_\beta \mathbf{K})$ is invertible, then (5.7) has a unique solution in $W^{2\beta}_{p,\gamma}$. This is consistent with the discussion in the section 2 when $\mathbf{K}u = k\int_\Omega u dx$.

The asymptotic behavior of a solution to the evolution equation (5.2) can be investigated through the study of properties of $A_{\beta-1}$, $\mathbf{K}$ and $\mathcal{F}$. This idea appeared in [6, 14, 9]. Let $u_s \in U_\beta$ satisfy

$$(A_{\beta-1} - (\sigma + A_{\beta-1}\mathcal{R}_\beta \mathbf{K})u_s = \mathcal{F}(u_s),$$

i.e. u_s is an equilibrium point, and suppose that:

(H2) $\mathcal{F}(u) = f(x,u)$ is locally Lipschitzian in u on U_β and

$$f(x,u) = f(x,u_0) + B(u-u_0) + g(x,u-u_0) \tag{5.8}$$

where B is a bounded linear map from $W^{2\beta}_{p,\gamma}$ to L_p and $\|g(x,v)\|_{L_p} = o(\|v\|_{W^{2\beta}_{p,\gamma}})$ as $\|v\|_{W^{2\beta}_{p,\gamma}} \to 0$, uniformly in $x \in \overline{\Omega}$.

Theorem 5.2 *Let $\mathcal{F}$ be as in (H2) and u_s be an equilibrium point. If $A_{\beta-1} - (\sigma + A_{\beta-1}\mathcal{R}_\beta \mathbf{K}) \in \mathcal{H}((W^{2(1-\beta)}_{p'})')$ and the spectrum $\sigma(A_0 - (\sigma + A_0\mathcal{R}_0\mathbf{K}) - B) \subset \{\operatorname{Re}\lambda > \lambda_0\}$ for some $\lambda_0 > 0$, then there exist $\rho > 0$, $M > 1$ such that if $\|u_0 - u_s\|_{W^{2\beta}_{p,\gamma}} \le \rho/2M$ then a unique solution of (5.2) exists and satisfies*

$$\|u(x,t) - u_s(x)\|_{W^{2\beta}_{p,\gamma}} \le 2Me^{-\lambda_0 t}\|u_0 - u_s\|_{W^{2\beta}_{p,\gamma}}, \quad \textit{for } t \ge 0. \tag{5.9}$$

Proof. Denote $\bar{A}_{\beta-1} = A_{\beta-1} - (\sigma + A_{\beta-1}\mathcal{R}_\beta\mathbf{K}) - B$. By using semigroup theories and Lemma 5.1, there exists one semigroup $\{e^{-\bar{A}_{\beta-1}t}\}$ such that

$$u(x,t) = e^{-\bar{A}_{\beta-1}t}u_0 + \int_0^t e^{-\bar{A}_{\beta-1}(t-\tau)}\mathcal{F}(u(x,\tau))\,d\tau; \tag{5.10}$$

$$u(x,t) - u_s = e^{-\bar{A}_{\beta-1}t}(u_0 - u_s) + \int_0^t e^{-\bar{A}_{\beta-1}(t-\tau)}g(x,u(x,\tau))\,d\tau \tag{5.11}$$

and there exists $\lambda \in (\lambda_0, \operatorname{Re}\sigma(\bar{A}_{\beta-1}))$, $M \ge 1$ such that for $t > 0$ and $v \in W^{2\beta}_{p,\gamma}$,

$$\|e^{-\bar{A}_{\beta-1}t}\| \le Me^{-\lambda t}\|v\|_{W^{2\beta}_{p,\gamma}}, \quad \|e^{-\bar{A}_{\beta-1}t}v\|_{W^{2\beta}_{p,\gamma}} \le Mt^{-2\beta}e^{-\lambda t}\|v\|_{L_p},$$

One can choose $\delta > 0$ and $\rho > 0$ small so that

$$M\delta\int_0^{+\infty}\tau^{-2\beta}e^{-(\lambda-\lambda_0)\tau}\,d\tau < \frac{1}{2} \tag{5.12}$$

and

$$\|g(x,v)\|_{L_p} \le \delta\|v\|_{W^{2\beta}_{p,\gamma}} \quad \text{for } \|v\|_{W^{2\beta}_{p,\gamma}} \le \rho\,. \tag{5.13}$$

Take u_0 with $\|u_0 - u_s\|_{W^{2\beta}_{p,\gamma}} \le \rho/2M$, then the local solution for (5.10) exists and satisfies $\|u(x,t) - u_s\|_{W^{2\beta}_{p,\gamma}} \le \rho$ for $t \in J$. On the other hand, from (5.11),

$$\begin{aligned}
&\|u - u_s\|_{W^{2\beta}_{p,\gamma}} \\
&\le Me^{-\lambda t}\|u_0 - u_s\|_{W^{2\beta}_{p,\gamma}} + \int_0^t \|e^{-\bar{A}_{\beta-1}(t-\tau)} g(x, u(x,\tau))\|_{W^{2\beta}_{p,\gamma}}\, d\tau \\
&\le Me^{-\lambda t}\|u_0 - u_s\|_{W^{2\beta}_{p,\gamma}} + \delta M \int_0^t (t-\tau)^{-2\beta} e^{-\lambda(t-\tau)} \|u - u_s\|_{W^{2\beta}_{p,\gamma}}\, d\tau \\
&\le \rho\left(\frac{1}{2} + \delta M \int_0^t (t-\tau)^{-2\beta} e^{-\lambda(t-\tau)}\, d\tau\right) < \rho\,.
\end{aligned}$$

In the last inequality above, (5.10) and (5.11) are used. Moreover,

$$\begin{aligned}
&e^{\lambda_0 t}\|u(x,t) - u_s(x)\|_{W^{2\beta}_{p,\gamma}} \\
&\le M\|u_0 - u_s\|_{W^{2\beta}_{p,\gamma}} + \delta M \int_0^t (t-\tau)^{-2\beta} e^{-(\lambda-\lambda_0)(t-\tau)} e^{\lambda_0 \tau} \|u - u_s\|_{W^{2\beta}_{p,\gamma}}\, d\tau \\
&\le M\|u_0 - u_s\|_{W^{2\beta}_{p,\gamma}} + \frac{1}{2} \sup_{0 \le \tau \le t} \left\{ e^{\lambda_0 \tau} \|u(x,\tau) - u_s(x)\|_{W^{2\beta}_{p,\gamma}} \right\}.
\end{aligned}$$

Hence, $\|u(x,\tau) - u_s(x)\|_{W^{2\beta}_{p,\gamma}} \le 2Me^{-\lambda_0 t}\|u_0 - u_s\|_{W^{2\beta}_{p,\gamma}}$. ☐

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