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SOLUTION TO NONLINEAR GRADIENT DEPENDENT SYSTEMS WITH A BALANCE LAW

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Abstract. In this paper, we are concerned with the initial boundary value problem (IBVP) and with the Cauchy problem to the reaction-diffusion system

$$\begin{aligned} u_t - \Delta u &= -u^n |\nabla v|^p, \\ v_t - d\Delta v &= u^n |\nabla v|^p, \end{aligned}$$

where $1 \leq p \leq 2$, d and n are positive real numbers. Results on the existence and large-time behavior of the solutions are presented.

1. Introduction

In the first part of this article, we are interested in the existence of global classical nonnegative solutions to the reaction-diffusion equations

$$\begin{aligned} u_t - \Delta u &= -u^n |\nabla v|^p =: -f(u,v), \\ v_t - d\Delta v &= u^n |\nabla v|^p, \end{aligned} \tag{1.1}$$

posed on $\mathbb{R}^+ \times \Omega$ with initial data

$$u(0;x) = u_0(x), \quad v(0;x) = v_0(x) \quad \text{in } \Omega \tag{1.2}$$

and boundary conditions (in the case Ω is a bounded domain in $\mathbb{R}^n$)

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \quad \text{on } \mathbb{R}^+ \times \partial\Omega. \tag{1.3}$$

Here Δ is the Laplacian operator, u_0 and v_0 are given bounded nonnegative functions, $\Omega \subset \mathbb{R}^n$ is a regular domain, η is the outward normal to $\partial\Omega$. The diffusive coefficient d is a positive real. One of the basic questions for (1.1)-(1.2) or (1.1)-(1.3) is the existence of global solutions. Motivated by extending known results on reaction-diffusion systems with conservation of the total mass but with non linearities depending only for the unknowns, Boudiba, Mouley and Pierre succeeded in obtaining L^1 solutions only for the case $u^n|\nabla v|^p$ with $p < 2$. In this article, we are interested essentially in classical solutions in the case where $p = 2$ (Ω bounded or $\Omega = \mathbb{R}^n$; in the latter case, there are no boundary conditions).

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2. Results

The existence of a unique classical solution over the whole time interval $[0, T_{\max}[$ can be obtained by a known procedure: a local solution is continued globally by using a priori estimates on $\|u\|_\infty, \|v\|_\infty, \||\nabla u|\|_\infty$, and $\||\nabla v|\|_\infty$.

2.1. **The Cauchy problem.**

Uniform bounds for u and v. First, we consider the auxiliary problem

$$\begin{gathered} L_\lambda \omega := \omega_t - \lambda \Delta \omega = b \nabla \omega, \quad t > 0, \ x \in \mathbb{R}^N \\ \omega(0, x) = \omega_0(x) \in L^\infty, \end{gathered} \tag{2.1}$$

where $b = (b_1(t,x), \dots, b_N(t,x)), b_i(t,x)$ are continuous on $[0,\infty) \times \mathbb{R}^N$, ω is a classical solution of (2.1).

Lemma 2.1. *Assume that $\omega_t, \nabla\omega, \omega_{x_i x_i}$, $i = 1, \dots, N$ are continuous,*

$$L_\lambda \omega \le 0, \quad (\ge) \quad (0,\infty) \times \mathbb{R}^N \tag{2.2}$$

and $\omega(t,x)$ satisfies $(2.1)_2$. Then

$$\begin{gathered} \omega(t,x) \le C := \sup_{x \in \mathbb{R}^N} \omega_0(x), \quad (0,\infty) \times \mathbb{R}^N. \\ \omega(t,x) \ge C := \inf_{x \in \mathbb{R}^N} \omega_0(x), \quad (0,\infty) \times \mathbb{R}^N. \end{gathered}$$

The proof of the above lemma is elementary and hence is omitted. Now, we consider the problem (1.1)-(1.2). It follows by the maximun principle that

$$u, v \ge 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N.$$

Uniform bounds of u. We have

$$u \le C_1 := \sup_{\mathbb{R}^N} u_0(x),$$

thanks to the maximum principle.

Uniform bounds of v. Next, we derive an upper estimate for v. Assume that $1 \le p < 2$. We transform $(1.1)_2$ by the substitution $\omega = e^{\lambda v} - 1$ into

$$\omega_t - \lambda \Delta \omega = \lambda e^{\lambda v}(v_t - d\Delta v - d\lambda \, |\nabla v|^2) = \lambda e^{\lambda v}(u^n |\nabla v|^p - d\lambda \, |\nabla v|^2).$$

Let

$$\phi(x) \equiv C x^p - d\lambda x^2; \quad C > 0, \ x \ge 0.$$

By elementary computations,

$$\phi(x) \ge 0 \quad \text{when } x \le \Big(\frac{C}{\lambda d}\Big)^{1/(2-p)}.$$

But in this case

$$|\nabla v| \le \Big(\frac{c}{\lambda d}\Big)^{1/(2-p)}.$$

In the case $x \ge (\frac{c}{\lambda d})^{1/(2-p)}$,

$$\phi(x) \le 0 \tag{2.3}$$

and hence $\omega \le M$ where

$$M = C\Big(\frac{pC}{2d\lambda}\Big)^{p/2-p}(\frac{2-p}{2}). \tag{2.4}$$

Then we have $v \le C_2$.

2.1.1. *Uniform bounds for $|\nabla u|$ and $|\nabla v|$.* At first, we present the uniform bounds for $|\nabla v|$. We write $(1.1)_2$ in the form

$$L_d v + kv = kv + u^n|\nabla v|^p \tag{2.5}$$

and transform it by the substitutions $\omega = e^{kt}v$ to obtain

$$\begin{gathered} L_d\omega = e^{kt}(L_d v + kv) = e^{kt}(kv + u^n|\nabla v|^p), \quad t > 0,\ x \in \mathbb{R}^N \\ \omega(0,x) = v_0(x). \end{gathered}$$

Now let

$$G_\lambda = G_\lambda(t-\tau; x-\xi) = \frac{1}{[4\pi\lambda(t-\tau)]^{\frac{N}{2}}} \exp\Big(\frac{|x-\xi|^2}{4\lambda(t-\tau)}\Big)$$

be the fundamental solution related to the operator L_λ. Then, with $Q_t = (0,t)\times\mathbb{R}^N$, we have

$$\omega = e^{kt}v = v^0(t,x) + \int_{Q_t} G_d(t-\tau; x-\xi)e^{k\tau}(kv + u^n|\nabla v|^p)d\xi d\tau$$

or

$$v = e^{-kt}v^0 + \int_{Q_t} e^{-k(t-\tau)}G_d(t-\tau; x-\xi)(kv + u^n|\nabla v|^p)d\xi d\tau, \tag{2.6}$$

where $v^0(t,x)$ is the solution of the homogeneous problem

$$L_d v^0 = 0, \quad v^0(0,x) = v_0(x).$$

From (2.6) we have

$$\nabla v = e^{-kt}\nabla v^0 + \int_{Q_t} e^{-k(t-\tau)}\nabla_x G_d(t-\tau; x-\xi)(kv + u^n|\nabla v|^p)d\xi d\tau. \tag{2.7}$$

Now we set $\nu_1 = \sup|\nabla v|$ and $\nu_1^0 = \sup|\nabla v^0|$, in Q_t. From (2.6), and using $v \le C_2$, we have

$$\nu_1 = \nu_1^0 + (kC_2 + C_1^n\nu_1^p)\int_0^t e^{-k(t-\tau)}\Big(\int_{\mathbb{R}^N}|\nabla_x G_d(t-\tau; x-\xi)|d\xi\Big)d\tau.$$

We also have

$$\int_{\mathbb{R}^N}|\nabla_x G_d(t-\tau; x-\xi)|d\xi = \int_{\mathbb{R}^N}\frac{|x-\xi|}{2d(t-\tau)}|G_d(t-\tau,; x-\xi)|d\xi$$

which is transformed by the substitution $\rho = 2\sqrt{d(t-\tau)\nu}$ into

$$\int_{\mathbb{R}^N}|\nabla_x G_d|d\rho = \frac{w_N}{\pi^{N/2}}\int_0^\infty e^{-\nu^2}d\nu = \frac{\chi}{\sqrt{d(t-\tau)}}$$

where $\chi = \frac{w_N}{2\pi^{N/2}}\Gamma(\frac{N+1}{2}) = \frac{\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N}{2})}$. It follows that

$$\nu_1 = \nu_1^0 + (kC_2 + C_1^n\nu_1^p)\frac{\chi}{\sqrt{d}}\int_0^t e^{-k(t-\tau)}\frac{d\tau}{\sqrt{t-\tau}}. \tag{2.8}$$

Recall that

$$\int_0^t e^{-k(t-\tau)}\frac{d\tau}{\sqrt{t-\tau}} = \frac{2}{\sqrt{k}}\int_0^t e^{-z^2}dz < \sqrt{\frac{\pi}{k}}.$$

If we set $s = \sqrt{k}$ in (2.8) then we have

$$\nu_1 \le \nu_1^0 + \Big(sC_2 + \frac{C_1^n}{s}\nu_1^p\Big)\chi\sqrt{\frac{\pi}{d}}. \tag{2.9}$$

Now we minimize the right hand side of (2.9) with respect to s to obtain

$$\nu_1 \leq \nu_1^0 + \frac{2\chi\sqrt{\pi}}{d}\Big(C_2 C_1^n \nu_1^p\Big)^{1/p}. \tag{2.10}$$

Note that $\nu_1^0 = C_2$.

We have two cases: Case (i) $1 \leq p < 2$. In this case (2.10) implies

$$|\nabla v| \leq \nu_1 \leq \overline{\nu}(p) = D, \quad \text{in } Q_t, \tag{2.11}$$

where D is a positive constant.

Case (ii) $p = 2$. In this case (2.10) holds under the additional condition

$$C_2 C_1^n \leq \frac{d}{4\pi\chi}. \tag{2.12}$$

Similarly we obtain from $(1.1)_1$,

$$U_1 := \sup_{Q_T} |\nabla u| \leq C_1 + C_1 \frac{2\sqrt{\pi}\chi}{\sqrt{d}} \nu_1^{p/2} \leq Constant. \tag{2.13}$$

The estimates (2.10) and (2.13) are independent of t, hence $T_{\max} = +\infty$.

Finally, we have the main result.

Theorem 2.2. *Let $p = 2$ and (u_0, v_0) be bounded such that (2.12) holds, then system (1.1)-(1.2) admits a global solution.*

2.2. **The Neumann Problem.** In this section, we are concerned with the Neumann problem

$$\begin{aligned} u_t - \Delta u &= -u^n |\nabla v|^2 \\ v_t - d\Delta v &= u^n |\nabla v|^2 \end{aligned} \tag{2.14}$$

where Ω be a bounded domain in $\mathbb{R}^N$, with the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \mathbb{R}^+ \times \partial\Omega \tag{2.15}$$

subject to the initial conditions

$$u(0;x) = u_0(x); \quad v(0;x) = v_0(x) \quad \text{in } \Omega. \tag{2.16}$$

The initial nonnegative functions u_0, v_0 are assumed to belong to the Holder space $C^{2,\alpha}(\Omega)$.

Uniform bounds for u and v. In this section a priori estimates on $\|u\|_\infty$ and $\|v\|_\infty$ are presented.

Lemma 2.3. *For each $0 < t < T_{\max}$ we have*

$$0 \leq u(t,x) \leq M, \quad 0 \leq v(t,x) \leq M, \tag{2.17}$$

for any $x \in \Omega$.

Proof. Since $u_0(x) \geq 0$ and $f(0,v) = 0$, we first obtain $u \geq 0$ and then $v \geq 0$ as $v_0(x) \geq 0$. Using the maximum principle, we conclude that

$$0 \leq u(t,x) \leq M, \quad \text{on } Q_T$$

where

$$M \geq M_1 := \max_{x\in\Omega} u_0(x).$$

Using $\omega = e^{\lambda v} - 1$, with $d\lambda \geq M_1^n$, from (2.14), we obtain

$$\omega_t - d\Delta\omega = \lambda|\nabla v|^2(u^n - d\lambda)e^{\lambda v}, \quad \text{on } Q_T$$
$$\frac{\partial u}{\partial v} = 0 \quad \text{on } \partial S_T.$$

Consequently as $d\lambda > \max_\Omega u^n$, we deduce from the maximum principle that

$$0 \leq \omega(t,x) \leq \exp(\lambda|v_0|_\infty) - 1.$$

Hence

$$v(x,t) \leq \frac{1}{\lambda}\ln(|\omega|_\infty + 1) \leq Constant < \infty.$$

□

Uniform bounds for $|\nabla v|$ *and* $|\nabla u|$. To obtain uniform a priori estimates for $|\nabla v|$, we make use of some techniques already used by Tomi [8] and von Wahl [9]

Lemma 2.4. *Let* (u,v) *be a solution to* (2.10) -(2.12) *in its maximal interval of existence* $[0,T_{\max}[$. *Then there exist a constant* C *such that*

$$\|u\|_{L^\infty([0,T[,W^{2,q}(\Omega))} \leq C \quad \text{and} \quad \|v\|_{L^\infty([0,T[,W^{2,q}(\Omega))} \leq C.$$

Proof. Let us introduce the function

$$f_{\sigma,\epsilon}(t,x,u,\nabla v) = \sigma u^n(t,x)\frac{\epsilon + |\nabla v|^2}{1 + \epsilon|\nabla v|^2}.$$

It is clear that $|f_{\sigma,\epsilon}(t,x,u,\nabla v)| \leq C(1 + |\nabla v|^2)$ and a global solution $v_{\sigma,\epsilon}$ differentiable in σ for the equation

$$v_t - d\Delta v = f_{\sigma,\epsilon}(t,x,u,\nabla v)$$

exists. Moreover, $v_{\sigma,\epsilon} \to v$ as $\sigma \to 1$ and $\epsilon \to 0$, uniformly on every compact of $[0,T_{\max}[$.

The function $\omega_\sigma := \frac{\partial v_{\sigma,\epsilon}}{\partial \sigma}$ satisfies

$$\partial_t\omega_\sigma - d\Delta\omega_\sigma = u^n(t,x)\frac{\epsilon + |\nabla v_\sigma|^2}{1 + \epsilon|\nabla v\sigma|^2} - 2\sigma u^n\frac{(\epsilon^2 - 1)\nabla v_\sigma.\nabla\omega_\sigma}{(1 + \epsilon|\nabla v_\sigma|^2)^2}. \tag{2.18}$$

Hereafter, we derive uniform estimates in σ and ϵ. Using Solonnikov's estimates for parabolic equation [5] we have

$$\|\omega_\sigma\|_{L^\infty([0,T(u_0,v_0)[,W^{2,p}(\Omega))} \leq C[\|\nabla v_\sigma\|^2_{L^p(\Omega)} + \|\nabla v_\sigma.\nabla\omega_\sigma\|^2_{L^p(\Omega)}].$$

The Gagliardo-Nirenberg inequality [5] in the in the form

$$\|u\|_{W^{1,2p}(\Omega)} \leq C\|u\|^{1/2}_{L^\infty(\Omega)}C\|u\|^{1/2}_{W^{2,p}(\Omega)}$$

and the δ-Young inequality (where $\delta > 0$)

$$\alpha\beta \leq \frac{1}{2}(\delta\alpha^2 + \frac{\beta^2}{\delta}),$$

allows one to obtain the estimate

$$\|\omega_\sigma\|_{L^\infty([0,T(u_0,v_0)[,W^{2,p}(\Omega))} \leq C(1 + \|\omega_\sigma\|_{W^{2,p}(\Omega)}).$$

But $\omega_\sigma = \frac{\partial v_\sigma}{\partial \sigma}$, hence by Gronwall's inequality we have

$$\|v_\sigma\|_{L^\infty([0,T[,W^{2,p}(\Omega))} \leq Ce^{C\sigma}.$$

Letting $\sigma \to 1$ and $\epsilon \to 0$, we obtain

$$\|v\|_{L^\infty([0,T[,W^{2,p}(\Omega))} \leq C.$$

On the other hand, the Sobolev injection theorem allows to assert that $u \in C^{1,\alpha}(\Omega)$. Hence in particular $|\nabla u| \in C^{0,\alpha}(\Omega)$. Since $|\nabla v|$ is uniformly bounded, it is easy then to bound $|\nabla u|$ in $L^\infty(\Omega)$. As a consequence, one can affirm that the solution (u, v) to problem (2.14) -(2.16) is global; that is $T_{\max} = \infty$. □

2.3. **Large-time behavior.** In this section, the large time behavior of the global solutions to (2.14)-(2.16) is briefly presented.

Theorem 2.5. *Let $(u_0, v_0) \in C^{2,\epsilon}(\Omega) \times C^{2,\epsilon}(\Omega)$ for some $0 < \epsilon < 1$. The system (2.14)-(2.16) has a global classical solution. Moreover, as $t \to \infty$, $u \to k_1$ and $v \to k_2$ uniformly in x, and*

$$k_1 + k_2 = \frac{1}{|\Omega|} \int_\Omega [u_0(x) + v_0(x)]dx.$$

Proof. The proof of the first part of the Theorem is presented above. Concerning the large time behavior, observe first that for any $t \geq 0$,

$$\int_\Omega [u(t,x) + v(t,x)]dx = \int_\Omega [u_0(x) + v_0(x)]dx.$$

Then, the function $t \to \int_\Omega u(x)dx$ is bounded; as it is decreasing, we have

$$\int_\Omega u(x)dx \to k_1 \quad \text{as } t \to \infty;$$

the function $t \to \int_\Omega v(x)dx$ is increasing and bounded, hence admits a finite limit k_2 as $t \to \infty$. As $\bigcup_{t\geq 0}\{(u(t), v(t))\}$ is relatively compact in $C(\overline{\Omega}) \times C(\overline{\Omega})$,

$$u(\tau_n) \to \widetilde{u}, \quad v(\tau_n) \to \widetilde{v} \quad \text{in } C(\overline{\Omega}),$$

through a sequence $\tau_n \to \infty$. It is not difficult to show that in fact $(\widetilde{u}, \widetilde{v})$ is the stationary solution to (2.14)-(2.16) (see [3]).

As the stationary solution (u_s, v_s) to (2.14)-(2.16) satisfies

$$-\Delta u_s = -u_s^n |\nabla v_s|^2, \quad \text{in } \Omega,$$

$$-d\Delta v_s = u_s^n |\nabla v_s|^2, \quad \text{in } \Omega, \frac{\partial u_s}{\partial \nu} = \frac{\partial v_s}{\partial \nu} = 0, \quad \text{on } \partial\Omega,$$

we have

$$-\int_\Omega \Delta u_s . u_s dx = -\int_\Omega u_s^{n+1} |\nabla v_s|^2 dx$$

which in the light of the Green formula can be written

$$\int_\Omega |\nabla u_s|^2 dx = -\int_\Omega u_s^{n+1} |\nabla v_s|^2 dx$$

hence $|\nabla u_s| = |\nabla v_s| = 0$ implies $u_s = k_1$ and $v_s = k_2$. □

Remarks. (1) It is very interesting to address the question of existence global solutions of the system (2.14)-(2.16) with a genuine nonlinearity of the form $u^n |\nabla v|^p$ with $p \geq 2$.

(2) It is possible to extend the results presented here for systems with nonlinear boundary conditions satisfying reasonable growth restrictions.

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