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SEMILINEAR PARABOLIC PROBLEMS ON MANIFOLDS AND APPLICATIONS TO THE NON-COMPACT YAMABE PROBLEM

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Abstract. We show that the well-known non-compact Yamabe equation (of prescribing constant positive scalar curvature) on a manifold with non-negative Ricci curvature and positive scalar curvature behaving like $c/d(x)^2$ near infinity can not be solved if the volume of geodesic balls do not increase "fast enough". Even though both existence and nonexistence results have appeared in the case when the scalar curvature is negative somewhere([J], [AM]), or when the scalar curvature is positive ([Ki], [Zhan5]), the current paper seems to give the first nonexistence result in the case that the scalar curvature is positive and *Ricci* ≥ 0, which seems to be the fundamental part of the noncompact Yamabe problem. We also find some complete non-compact manifolds with positive scalar curvature which are conformal to complete manifolds with constant and with zero scalar curvature. This is a new phenomenon which does not happen in the compact case.

1. Introduction

We shall study the global existence and blow up of solutions to the homogeneous semilinear parabolic Cauchy problem

$$\begin{gathered} Hu \equiv H_0 u + u^p = \Delta u - Ru - \partial_t u + u^p = 0 \quad \text{in} \quad \mathbf{M}^n \times (0, \infty), \\ u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \mathbf{M}^n, \end{gathered} \tag{1.1}$$

where $\mathbf{M}^n$ with $n \geq 3$ is a non-compact complete Riemannian manifold, Δ is the Laplace-Beltrami operator and $R = R(x)$ is a bounded function.

Equation (1.1) with $R = 0$ contains the following important special cases. When the Riemannian metric is just the Euclidean metric, (1.1) becomes the semilinear parabolic equation which has been studied by many authors. In the paper [Fu], Fujita proved the following results:

(a) when $1 < p < 1 + \frac{2}{n}$ and $u_0 > 0$, problem (1.1) possesses no global positive solutions;

(b) when $p > 1 + \frac{2}{n}$ and u_0 is smaller than a small Gaussian, then (1.1) has global positive solutions. So $1 + \frac{2}{n}$ is the critical exponent.

For a reference to the rich literature of the subsequent development on the topic, we refer the reader to the survey paper [Le].

In recent years, many authors have undertaken research on semilinear elliptic operators on manifolds, including the well-known Yamabe problem (see [Sc] and

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[Yau]). The study of those elliptic problems and others such as Ricci flow lead naturally to semilinear or quasi-linear parabolic problems (see [H] and [Sh]). The first goal of the paper is to study when blow up of solutions occur and when global positive solutions exist for equation (1.1) on manifolds. Such an undertaking requires some new techniques. The prevailing methods of treating the above semi-linear problems i.e. variational and comparison method seem difficult to apply. The method we are using is based on new inequalities (see section 3 and Lemma 7.2) involving the heat kernels. We are able to find an explicit relation between the size of the critical exponent and geometric properties of the manifold such as the growth rate of geodesic balls (see Theorem B).

It is interesting to note that this technique also leads to a non-existence result of the well-known non-compact Yamabe problem of prescribing positive constant scalar curvatures; i.e. whether the following elliptic problem has "suitable" positive solutions on $\mathbf{M}^n$ with nonnegative scalar curvature (see Theorems C and D).

$$\Delta u - \frac{n-2}{4(n-1)} Ru + u^{(n+2)/(n-2)} = 0. \tag{1.1'}$$

This problem has been asked by J. Kazdan [K] and S. T. Yau [Yau]. The compact version of the problem was proposed by Yamabe [Yam], proved by Trudinger [Tr] and Aubin [Au1] in some cases and eventually proved by R. Schoen [Sc] completely.

In the non-compact case, Aviles and McOwen [AM] obtained some existence results for the problem of prescribing constant negative scalar curvature. Jin [Jin] gave a nonexistence result when R is negative somewhere. Some existence result when R is positive was obtained in [Ki]. Recently in [Zhan5] (Theorem A, B), we constructed complete noncompact manifolds with positive R, which do not have any conformal metric with positive constant scalar curvature. However those manifolds, obtained by deforming $\mathbf{R}^3 \times S^1$, do not have nonnegative Ricci curvature everywhere.

Since manifolds with nonnegative Ricci curvature are one of the most basic objects in geometry. It is the most natural to ask whether the Yamabe equation can be solved in this case. However, as far as we know, there has been no nonexistence result on the problem of prescribing constant positive scalar curvature when the Ricci curvature is nonnegative and R is positive.

When $\mathbf{M}^n$ is $\mathbf{R}^n$ with the Euclidean metric, then problem (1.1') becomes $\Delta u + u^{(n+2)/(n-2)} = 0$, which does have positive solutions (see [Ni]), that induce incomplete metric of constant positive scalar curvature.

$$u_\lambda(x) = [n(n-2)\lambda^2]^{(n-2)/4}/(\lambda^2 + |x|^2)^{(n-2)/2}, \quad \lambda > 0.$$

So it is reasonable to expect that (1.1') at least has a positive solution which gives rise to a incomplete metric of scalar curvature one. However Theorem D below asserts that unlike the compact Yamabe problem, equation (1.1') can not be solved in general, regardless of whether one requires the resulting metric is complete or not. In fact if the existing scalar curvature decays "too fast" and the volume of geodesic balls does not increase "fast enough", then (1.1') does not have any positive solution at all. This of course rules out the existence of complete or incomplete metric with constant positive scalar curvature.

Before stating the results precisely, let us list the basic assumptions and some notations to be used frequently in the paper. For theorems A, B and C in the paper we make the following assumptions (i), (ii) and (iii) unless stated otherwise. Instead, Theorem D is exclusively about manifolds with nonnegative Ricci curvature.

(i). There are positive constants b, C, K, q and Q such that

$$|B(x,r)| \le Cr^Q; \quad |B(x,2r)| \le C2^q|B(x,r)|,\ r>0; \quad \text{Ricci} \ge -K. \tag{1.2}$$

(ii). G, the fundamental solution of the linear operator $H_0 = \Delta - R - \partial_t$ in (1.1), has global Gaussian upper bound. i.e.

$$0 \le G(x,t;y,s) \le \frac{C}{|B(x,(t-s)^{1/2})|} e^{-b\frac{d(x,y)^2}{t-s}}, \tag{1.3}$$

for all $x, y \in \mathbf{M}^n$ and all $t > s$.

(iii). When $t - s \ge d(x,y)^2$, G satisfies

$$G(x,t;y,s) \ge \min\{\frac{1}{C|B(x,(t-s)^{1/2})|}, \frac{1}{C|B(y,(t-s)^{1/2})|}\}. \tag{1.4}$$

We mention that (1.4) actually is equivalent to G having global Gaussian lower bound by a well known argument in [FS].

At the first glance, the relation between conditions (1.3), (1.4) and the function R does not seem transparent. However we emphasize that in general (1.3) and (1.4) only require R to satisfy some decay conditions such as $R^-(x) \le \epsilon/[1+d^{2+\delta}(x,x_0)]$ for $\delta > 0$ and some $\epsilon > 0$ and $R^+(x) \le c/[1+d^{2+\delta}(x,x_0)]$ for $\delta > 0$ and an arbitrary $c > 0$ as indicated in Lemma 6.1 below and Theorem C in [Zhan5]. More specifically we have

Theorem *(Theorem C [Zhan5]). Suppose $\mathbf{M}$ is a complete noncompact manifold with nonnegative Ricci curvature outside a compact set and $R \ge 0$, then (1.4) and (1.3) hold if and only if $\sup_x \int \Gamma_0(x,y)R(y)dy < \infty$. Here Γ_0 is the fundamental solution of the free Laplacian Δ. In particular (1.4) and (1.3) hold if $0 \le R \le c/[1+d^{2+\delta}(x,x_0)]$ for $\delta > 0$ and an arbitrary $c > 0$*

In the context of the Yamabe equation, the relation is even more direct. If the Ricci curvature of $\mathbf{M}^n$ is non-negative and R is the scalar curvature, then the upper bound (1.3) automatically holds by [LY] and the maximum principle since $R^- = 0$.

Definition *1.1. A function $u = u(x,t)$ such that $u \in L^2_{loc}(\mathbf{M}^n \times (0,\infty))$ is called a solution of (1.1) if*

$$u(x,t) = \int_{\mathbf{M}^n} G(x,t;y,0)u_0(y)dy + \int_0^t \int_{\mathbf{M}^n} G(x,t;y,s)u^p(y,s)dyds$$

for all $(x,t) \in \mathbf{M}^n \times (0,\infty)$.

$G = G(x,t;y,s)$ will denote the fundamental solution of the linear operator H_0 in (1.1). For any $c > 0$, we write

$$G_c(x,t;y,s) = \begin{cases} \frac{1}{|B(x,(t-s)^{1/2})|} \exp(-c\frac{d(x,y)^2}{t-s}), & t > s, \\ 0, \quad t < s. \end{cases} \tag{1.5}$$

Let u_0 be a positive function in $L^\infty(\mathbf{M}^n)$ and $a>0$, we write

$$h_a(x,t)=\int_{\mathbf{M}^n} G_a(x,t;y,0)u_0(y)dy; \tag{1.6}$$

Given $V=V(x,t)$ and $c>0$, we introduce the notation

$$\begin{aligned} N_{c,\infty}(V) \equiv & \sup_{x\in\mathbf{M}^n,t>0}\int_{-\infty}^{\infty}\int_{\mathbf{M}^n}|V(y,s)|G_c(x,t;y,s)dyds \\ & +\sup_{y\in\mathbf{M}^n,s>0}\int_{\infty}^{\infty}\int_{\mathbf{M}^n}|V(x,t)|G_c(x,t;y,s)dxdt. \end{aligned} \tag{1.7}$$

We note that $N_{c,\infty}(V)$ may be infinite for some V. However, the fact that $N_{c,\infty}(V)$ is finite for some specific functions will play a key role in the proof of the theorems. When V is time independent, $N_{c,\infty}(V)/2=\sup_x\int_{\mathbf{M}^n}\Gamma_0(x,y)|V(y)|dy$ the quantity appeared in Theorem C of [Zhan5].

The main results of the paper are the next four theorems.

Theorem A. *Suppose $R=0$ and (1.2), (1.3) and (1.4) hold. Then the critical exponent of (1.1) is $p^*=1+\frac{1}{s^*}$, where*

$$\begin{aligned} s^*=\sup\{s \mid & \lim_{t\to\infty}\sup t^s\|W(.,t)\|_{L^\infty}<\infty, \\ & \text{for a non-negative and non-trivial } W \text{ such that } H_0W=0\}. \end{aligned}$$

Remark 1.1. Theorem A can be proved by following the proof of Theorem 1 in [Me]. However, this theorem does not provide any estimate of the size of the exponent. Our main concern is therefore to find an explicit relation between the critical exponent and geometrical properties of the manifold. This is done in

Theorem B. *Let $\mathbf{M}^n$ be any Riemannian manifold and $R=R(x)$ be any bounded function such that (1.2), (1.3) and (1.4) hold, then the following conclusions are true.*

(a). Suppose, for $t>s\geq 0$,

$$\sup_{x\in\mathbf{M}^n,t}\int_{r_0}^{\infty}\int_{\mathbf{M}^n}\frac{G_c(x,t;y,s)}{|B(y,s^{1/2})|^{p-1}}dyds+\sup_{y\in\mathbf{M}^n,s}\int_{r_0}^{\infty}\int_{\mathbf{M}^n}\frac{G_c(x,t;y,s)}{|B(x,t^{1/2})|^{p-1}}dxdt<\infty$$

for some $r_0>0$ and a suitable $c>0$, then (1.1) has global positive solutions for some $u_0\geq 0$.

(b). Suppose for some $x_0\in\mathbf{M}^n$,

$$\lim_{r\to\infty}\inf\frac{|B(x_0,r)|}{r^\alpha}<\infty$$

then (1.1) has no global positive solutions for any $p<1+\frac{2}{\alpha}$ and any $u_0\geq 0$.

Remark 1.2. In general the integral relation given in part (a) of Theorem B is necessary since, for fixed $s>0$, $|B(y,s^{1/2})|$ may tend to zero when y approaches ∞. The relation seems complicated at the beginning, however it is actually sharp as explained in the two corollaries below, where more specific assumptions are made on the manifold.

Corollary 1.1. *Under the same assumptions as in Theorem B, suppose*

$$\int_{r_0}^{\infty} \sup_{x\in \mathbf{M}^n} \frac{1}{|B(x,r^{1/2})|^{p-1}} dr < \infty$$

for some $r_0 > 0$*, then (1.1) has global positive solutions for some* $u_0 \geq 0$*. In particular if, for* $\alpha > 0$*,* $\inf_{x\in\mathbf{M}^n} |B(x,r)| \geq Cr^{\alpha}$ *when* r *is sufficiently large, then for* $p > 1 + \frac{2}{\alpha}$*, (1.1) has global positive solutions for some* u_0*.*

In the next corollary, we show that if $\mathbf{M}^n$ has bounded geometry in the sense of E. B. Davies (see p 172 in [D]), which means there exists a function $b(r)$ and $c > 0$ such that

$$c^{-1}b(r) \leq |B(x,r)| \leq cb(r) \tag{1.8}$$

for all $x \in \mathbf{M}^n$ and $r > 0$, then the critical exponent of (1.1) can be explicitly determined.

Corollary 1.2. *Suppose (1.2) , (1.3) , (1.4) and (1.8) hold, then* p^**, the critical exponent of (1.1), is given by* $p^* = 1 + \frac{2}{\alpha^*}$*, where*

$$\alpha^* = \inf\{\alpha > 0 \mid \lim_{r\to\infty} \inf \frac{|B(x,r)|}{r^{\alpha}} < \infty,\ x \in \mathbf{M}^n\}.$$

The above number α^* *is independent of the choice of* $x \in \mathbf{M}^n$*.*

Remark 1.3. Under the assumptions of part (a) of Theorem B, (1.1) has global positive solutions if $u_0 \geq 0$ satisfies $u_0 \in C^2(\mathbf{M}^n)$, $\lim_{d(x,0)\to\infty} u_0(x) = 0$, and $||u_0||_{L^\infty(\mathbf{M}^n)} + ||u_0||_{L^1(\mathbf{M}^n)} \leq b_0$where b_0 is a sufficiently small number and 0 is a point in $\mathbf{M}^n$. This result, to be proved in section 4, is new even in the Euclidean case.

Next we turn our attention to the non-compact Yamabe equation of prescribing constant positive scalar curvatures.

Theorem C. *Suppose (1.2), (1.3) and (1.4) hold and*

$$\lim_{r\to\infty} \inf \frac{|B(x_0,r)|}{r^{\alpha}} < \infty \tag{1.9}$$

for some x_0 *and* $\alpha < (n-2)/2$*, then the Yamabe equation (1.1') has no solutions.*

Examples of manifolds satisfying the conditions in Theorem C will be given in Corollary 1.3 below.

The next theorem gives a set of manifolds with nonnegative Ricci curvature, which are not conformal to manifolds with positive constant scalar curvature. The construction is more difficult since the scalar curvature does not have a rapid decay when *Ricci* ≥ 0 everywhere. This theorem marks one of the main differences between this paper and [Zhan5]. It is not known whether there exists a manifold with nonnegative Ricci curvature such that the scalar curvature satisfies (1.4). The scalar curvature G given below does *not* satisfy (1.4).

Theorem D. *Suppose* $\mathbf{M}$ *is a* $n(\geq 3)$ *dimensional complete noncompact manifold with nonnegative Ricci curvature and the scalar curvature* $R = R(x)$ *satisfies* $0 \leq R(x) \leq C_0/[1 + d^2(x, x_0)]$*, where* C_0 *is an arbitrary positive constant. There exists a positive integer* m *such that the manifold* $\mathbf{M} \times \Pi_1^m S^1$ *is not conformal to manifolds with positive constant scalar curvature. Here the metric is the product of the metric on* $\mathbf{M}$ *and the usual one on* S^1*.*

It is important to know what conditions should be imposed on the Ricci curvature and the scalar curvature R so that the conditions of Theorem C are met. We have

Corollary 1.3. *(a). Suppose the Ricci curvature of* $\mathbf{M}^n$ *is non-negative outside a compact set and for a suitable* $c > 0$*,* $N_{c,\infty}(R^+)$ *is finite and* $N_{c,\infty}(R^-)$ *is sufficiently small, then (1.2), (1.3) and (1.4) hold. Hence the Yamabe equation (1.1') has no solutions if (1.9) holds.*

(b). In particular suppose $\mathbf{M}^n$ *is a manifold with non-negative Ricci curvature outside a compact set and* $|B(x,r)| \sim r^\alpha$ *for* $2 < \alpha < (n-2)/2$ *and large* $r > 0$*. Then if*

$$0 \leq R(x) \leq c/(1 + d^{2+\delta}(x, 0))$$

for an arbitrary $c > 0$*, then the Yamabe problem (1.1') has no solution. Here* 0 *is a point in* $\mathbf{M}^n$*. An example is*

$$\mathbf{M}^9 = \mathbf{R}^3 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1.$$

Here the metric is the product of the usual ones on S^1 *and the Euclidean metric on* $\mathbf{R}^3$*. By Theorem B (a) in [Zhan5],* $\mathbf{M}^9$ *has a complete conformal metric with positive scalar curvature. But part (a) of Corollary 1.3 shows that* $\mathbf{M}^9$ *has no conformal metric with constant scalar curvature (*$\alpha = 4, n = 9$*).*

Remark 1.4. It seems hard if possible to construct the above $\mathbf{M}^9$ with $Ricci \geq 0$ and $R > 0$ everywhere. On the other hand manifolds required for Theorem D are well known to exist. Also by Theorem A in [Zhan5], the manifold $\mathbf{R}^3 \times S^1$ (and all its conformal deformations some of which have positive scalar curvature by Theorem B in [Zhan5]) also has no conformal metric with constant scalar curvature. The current more lengthy example is here to verify the claims in the announcement [Zhan4].

Remark 1.5. We would like to point out another fundamental difference between the non-compact Yamabe problem and the compact one. In the compact case, any manifold with positive scalar curvature is not conformal to a manifold with zero scalar curvature. This is reflected from the fact that the equation $\Delta u - Ru = 0$ with $R > 0$ has no positive solution on compact manifolds. However this is not the case for non-compact manifolds. The following is a complete non-compact manifold which is conformal to a complete manifold of constant positive scalar curvature, and to a complete manifold with zero scalar curvature. Let $\mathbf{M} = S^3 \times \mathbf{R}^1$ with the metric being the direct product of the usual ones on S^3 and $\mathbf{R}^1$. Then $R = 6$, $n = 4$ and hence equation (1.1') becomes $\Delta u - u + u^3 = 0$, which has a solution $u = 1$. At the same time the equation $\Delta u - u = 0$has a positive solution $u(x_1, x_2) = e^{x_2} + e^{-x_2} \geq 1$, where $x_1 \in S^3$ and $x_2 \in \mathbf{R}^1$. Clearly the above solution generates a complete metric of zero scalar curvature since it is bounded away from zero. Similar phenomenon is shown in Proposition 6.1 for $\mathbf{M} \times \mathbf{R}^k$ where $\mathbf{M}$ is any compact manifold with positive scalar curvatures.

Proposition 6.1. *Let $\mathbf{M}^n = \mathbf{M}_1 \times \mathbf{R}^k$, where $\mathbf{M}_1$ is a compact manifold with positive scalar curvature and the metric of $\mathbf{M}^n$ is the product of that of $\mathbf{M}_1$ and the Euclidean metric on $\mathbf{R}^k$, $k \geq 1$. Then $\mathbf{M}^n$ is conformal to a complete non-compact manifold with positive constant scalar curvature and to a complete manifold with zero scalar curvature.*

Let us briefly discuss the method we are going to adopt. We will use the Schauder fixed point theorem to achieve existence. This requires us to obtain some new estimates involving the heat kernel on $\mathbf{M}^n$. These estimates are presented in section three and seven. Theorem A will be proved in section 2. Theorem B part (a) and part (b) will be proved in sections 4 and 5, Theorem C and Corollary 1.3 in section 6 and Theorem D in section 7. The key idea is to obtain some global bounds for the fundamental solution of H_0 and to show that the parabolic problem (1.1) with $p = \frac{n+2}{n-2}$ has no global positive solutions under the assumptions of Theorem C. Since every positive solution of the Yamabe equation (1.1') is a global positive solution of the parabolic problem (1.1), the former can not exist either.

Recently we are able to treat (1.1) when R no longer satisfies (1.3) and (1.4) and hence belong to the slow decay case. For details please see [Zhan1].

2. Preliminaries

In this section we collect and obtain some preliminary results and prove Theorem A. Some results which are not new are here for completeness. We remark that C will always be absolute constants that may change from line to line. For simplicity we also assume that 0 is a reference point on $\mathbf{M}^n$.

Proposition 2.1. Given $0 < \alpha' < \alpha$, there is a positive constant C such that

$$\frac{1}{|B(x,(t-s)^{1/2})|} e^{-\alpha \frac{d(x,y)^2}{t-s}} \leq \frac{C}{|B(x,d(x,y))|} e^{-\alpha' \frac{d(x,y)^2}{t-s}}.$$

Proof. If $d^2(x,y) \leq t-s$, then the inequality is obvious. So we assume that $d^2(x,y) \geq t-s$. Let $r = (t-s)^{1/2}/d(x,y)$, then $r \in (0,1]$. By the doubling condition in (1.2) there exists a constant C such that

$$|B(x,(t-s)^{1/2})| = |B(x, rd(x,y))| \geq C r^q |B(x,d(x,y))|.$$

Then

$$\begin{aligned}
G_\alpha(x,t;y,s) &= \frac{1}{|B(x,(t-s)^{1/2})|} \exp(-\alpha \frac{d(x,y)^2}{t-s}) \\
&= \frac{C_1}{|B(x,(t-s)^{1/2})|} \exp(-(\alpha-\alpha')\frac{d(x,y)^2}{t-s}) \exp(-\alpha' \frac{d(x,y)^2}{t-s}) \\
&\leq \frac{C}{r^q} \exp(-\frac{\alpha-\alpha'}{r^2}) \frac{1}{|B(x,d(x,y))|} \exp(-\alpha' \frac{d(x,y)^2}{t-s}) \\
&\leq \frac{C}{|B(x,d(x,y))|} \exp(-\alpha' \frac{d(x,y)^2}{t-s}).
\end{aligned}$$

q.e.d.

Proposition 2.2. Given $0 < \alpha' < \alpha$, there is a positive constant C such that

$$\frac{1}{|B(x,(t-s)^{1/2})|} e^{-\alpha \frac{d(x,y)^2}{t-s}} \le \frac{C}{|B(y,(t-s)^{1/2})|} e^{-\alpha' \frac{d(x,y)^2}{t-s}}.$$

Proof. If $d^2(x,y) \le t-s$, then the inequality is obvious. This is because the doubling property implies that $|B(x,(t-s)^{1/2})|$ and $|B(y,(t-s)^{1/2})|$ are comparable. So we assume that $d^2(x,y) \ge t-s$. Proposition 2.1 gives

$$\frac{1}{|B(x,(t-s)^{1/2})|} e^{-\alpha \frac{d(x,y)^2}{t-s}} \le \frac{C}{|B(x,d(x,y))|} e^{-\alpha' \frac{d(x,y)^2}{t-s}}.$$

By the comparability of $|B(x,d(x,y))|$ and $|B(y,d(x,y))|$, we have

$$\frac{1}{|B(x,(t-s)^{1/2})|} e^{-\alpha \frac{d(x,y)^2}{t-s}} \le \frac{C}{|B(y,d(x,y))|} e^{-\alpha' \frac{d(x,y)^2}{t-s}}.$$

Recalling $d^2(x,y) \ge t-s$, we obtain

$$\frac{1}{|B(x,(t-s)^{1/2})|} e^{-\alpha \frac{d(x,y)^2}{t-s}} \le \frac{C}{|B(y,(t-s)^{1/2})|} e^{-\alpha' \frac{d(x,y)^2}{t-s}}.$$

q.e.d.

Proposition 2.3. Suppose $R = 0$ and $\mathbf{M}^n$ has non-negative Ricci curvature, then (1.2), (1.3) and (1.4) hold.

Proof. When the Ricci curvature is non-negative, (1.2) is standard. By the famous paper [LY], the fundamental solution G of H_0 satisfies the global Gaussian bounds

$$\frac{1}{C|B(x,(t-s)^{1/2})|} e^{-\frac{d(x,y)^2}{b(t-s)}} \le G(x,t;y,s) \le \frac{C}{|B(x,(t-s)^{1/2})|} e^{-b \frac{d(x,y)^2}{t-s}},$$

for some $b, C > 0$ and all $x, y \in \mathbf{M}^n$ and all $t > s$. Therefore (1.3) is clearly true. Using Proposition 2.2, we find, for a $b' < b$,

$$\frac{1}{C|B(y,(t-s)^{1/2})|} e^{-\frac{d(x,y)^2}{b'(t-s)}} \le G(x,t;y,s).$$

Hence (1.4) holds when $d(x,y)^2 \le t-s$. q.e.d.

Lemma 2.1. *Given $a > 0$, let*

$$h_a(x,t) = \int_{\mathbf{M}^n} G_a(x,t;y,0) u_0(y) dy, \tag{2.1}$$

where u_0 is a bounded non-negative function. Suppose $\lim_{d(x,0)\to\infty} u_0(x) = 0$, then $\lim_{d(x,0)\to\infty} h_a(x,t) = 0$ uniformly with respect to $t > 0$.

Proof. For any $\delta > 0$, let $R > 0$ be such that $u_0(y) < \delta/2$ when $d(y) \geq R$. When $d(x,0) \geq 2R$ we have

$$\begin{aligned} h_a(x,t) &= \int_{d(y)>R} \frac{e^{-a\ d(x,y)^2/t}}{|B(x,t^{1/2})|}\ u_0(y)dy + \int_{d(y)\leq R} \frac{e^{-a\ d(x,y)^2/t}}{|B(x,t^{1/2})|}\ u_0(y)dy \\ &\leq \frac{\delta}{2}\int_{d(y)>R} \frac{1}{|B(x,t^{1/2})|} e^{-a\ d(x,y)^2/t} dy \\ &\quad + \int_{d(y)\leq R} \frac{1}{|B(x,t^{1/2})|} e^{-a\ d(x,y)^2/(2t)} e^{-a\ d(x,y)^2/(2t)} u_0(y)dy. \end{aligned}$$

Note that

$$\frac{1}{|B(x,t^{1/2})|} e^{-a\ d(x,y)^2/(2t)} \leq C/|B(x,d(x,y))|$$

by Proposition 2.1 and $d(x,y) \geq d(x,0) - R$ when $d(y,0) \leq R$, we have

$$\begin{aligned} h_a(x,t) &\leq C\delta/2 + \frac{C}{|B(x,d(x,0)-R)|} \int_{d(y)\leq R} e^{-a\ d(x,y)^2/(2t)} u_0(y)dy \\ &\leq C\delta/2 + C\frac{|B(0,R)|}{|B(x,d(x,0)-R)|}||u_0||_{L^\infty} \\ &\leq C\delta, \end{aligned}$$

when $d(x,0)$ is sufficiently large. This proves (b). q.e.d.

Proof of Theorem A. Since the proof follows the lines of Theorem 1 in [Me], we will be sketchy.

(a). We first show that if $p > p^*$ then (1.1) has global positive solutions for some u_0. As in the case of [Me], it is enough to show that if there is a non-trivial positive solution W of $H_0 W = 0$ such that

$$\int_0^\infty ||W(.,t)||_{L^\infty}^{p-1} dt < \infty,$$

then (1.1) has global positive solutions for some u_0.

Let $a(t)$ be the solution of the initial value problem

$$a'(t) = ||W(.,t)||^{p-1} a(t)^p, \quad a(0) = a_0 > 0.$$

Then $u(x,t) \equiv a(t)W(x,t)$ is a super solution of (1.1) with $u_0(x) = a_0 W(x,0)$. Thus it is enough to show that choice of a_0 can ensure that $a(t)$ exists for all $t > 0$ and is uniformly bounded. As

$$a(t) = [a_0^{1-p} - (p-1)\int_0^t ||W(.,s)||^{p-1} ds]^{-1/(p-1)},$$

this follows from the condition. Since 0 is always a subsolution, the standard comparison theorem shows that (1.1) has a global positive solution.

(b). We want to show that if $p < p^*$ then all positive solutions of (1.1) blow up in finite time. Again by [Me], it is enough to show that if

$$\lim_{t\to\infty} \sup ||W(.,t)||_{L^\infty}^{p-1} t = \infty \tag{2.2}$$

for all non-trivial and non-negative solutions of $H_0 W = 0$, then every positive solution of (1.1) blows up in finite time.

Define $z(t; w)$ to be the solution of

$$\frac{dz}{dt} = z^p, \quad z(0, w) = w. \tag{2.3}$$

For arbitrary initial conditions let w be the solution of $H_0 w = 0$. Then it can easily be seen by using local coordinates and following [Me] that $u(x,t) \equiv z(t; w(x,t))$ is a subsolution for (1.1) that blows up in finite time. Indeed let $\Delta = a_{ij}(x) \frac{\partial^2}{\partial_{x_i} \partial_{x_j}} + b_i(x) \frac{\partial}{\partial_{x_i}}$ in a local coordinate system, then by direct computation one has, as in [Me],

$$Hu = p \frac{z^p}{w^{2p}} [z^{p-1} - w^{p-1}] a_{ij} w_{x_i} w_{x_j}.$$

Since $z \geq w$ by construction, we have $Hu \geq 0$ and hence u is a sub-solution. By (2.3),

$$t = \int_w^u \frac{ds}{s^p} = \frac{1}{(p-1)w^{p-1}} - \frac{1}{(p-1)u^{p-1}}.$$

We only need to prove that u blows up in finite time. Suppose the contrary is true, then

$$t - \frac{1}{(p-1)w^{p-1}} = -\frac{1}{(p-1)u^{p-1}} < 0,$$

for all $t > 0$. But this contradicts the assumption (2.2). q.e.d.

Remark 2.1. When the function $R = R(x)$ is not zero, we do not know whether Theorem A is still valid, except for the case when $\mathbf{M}^n$ has bounded geometry (see Corollary 1.2).

3. Two inequalities

In this section we present some inequalities for the heat kernel, which will play a key role in both the existence and blow up results. The proof, almost identical to that of Lemma 4.1 in [Zhan2], is given for completeness.

Lemma 3.1. *Suppose $0 < a < b$, there exist positive constants $C_{a,b}$ and c depending only on a and b such that, for $t > \tau > s$,*

(i). $G_a(x,t;z,\tau) G_b(z,\tau;y,s) \leq C[G_c(x,t;z,\tau) + G_c(z,\tau;y,s)] G_a(x,t;y,s)$,

(ii). $G_b(x,t;z,\tau) G_a(z,\tau;y,s) \leq C[G_c(x,t;z,\tau) + G_c(z,\tau;y,s)] G_a(x,t;y,s)$.

Remark 3.1. The condition $a < b$ is indispensable for Lemma 3.1.

Proof of the lemma. We will only give a proof of (i) since (ii) can be handled similarly.

Case 1. $\tau - s \leq \rho(t-s)$, where $\rho \in (0,1)$ is to be fixed later. Let us recall the following inequality that holds for all metric.

$$\frac{d(x,z)^2}{t-\tau} + \frac{d(z,y)^2}{\tau - s} \geq \frac{d(x,y)^2}{t-s}, \quad 0 < s < \tau < t. \tag{3.4}$$

Using (3.4) we have

$$\begin{aligned}
&G_a(x,t;z,\tau)G_b(z,\tau;y,s)\\
&=\frac{1}{|B(x,(t-\tau)^{1/2})|}\frac{1}{|B(z,(\tau-s)^{1/2})|}\exp(-a\frac{d(x,z)^2}{t-\tau})\exp(-b\frac{d(z,y)^2}{\tau-s})\\
&=\frac{1}{|B(x,(t-\tau)^{1/2})|}\exp(-a[\frac{d(x,z)^2}{t-\tau}+\frac{d(z,y)^2}{\tau-s}])\frac{\exp(-(b-a)\frac{d(z,y)^2}{\tau-s})}{|B(z,(\tau-s)^{1/2})|}\\
&\le\frac{1}{|B(x,(t-\tau)^{1/2})|}\exp(-a\frac{d(x,y)^2}{t-s})\frac{\exp(-(b-a)\frac{d(z,y)^2}{\tau-s})}{|B(z,(\tau-s)^{1/2})|}\\
&\le CG_{b-a}(z,\tau;y,s)G_a(x,t;y,s).
\end{aligned}$$

To reach the last step we used the doubling condition and the fact

$$t-\tau=t-s-(\tau-s)\ge(1-\rho)(t-s)$$

to get $1/|B(x,(t-\tau)^{1/2})|\le C/|B(x,(t-s)^{1/2})|$. Therefore (i) holds in this case.

Case 2. Picking two numbers a', b' such that

$$a<a'<b'<b$$

and using Proposition 2.2 on $G_b(z,\tau;y,s)$, we have

$$\frac{\exp(-b\frac{d(z,y)^2}{\tau-s})}{|B(z,(\tau-s)^{1/2})|}\le C\frac{\exp(-b'\frac{d(z,y)^2}{\tau-s})}{|B(y,(\tau-s)^{1/2})|}$$

When $d(z,y)\ge d(x,y)(a'/b')^{1/2}$ and $\tau-s\ge\rho(t-s)$, then

$$\frac{\exp(-b'\frac{d(z,y)^2}{\tau-s})}{|B(y,(\tau-s)^{1/2})|}\le\frac{\exp(-a'\frac{d(x,y)^2}{t-s})}{|B(y,[\rho(t-s)]^{1/2})|}.$$

Therefore

$$G_b(z,\tau,y,s)=\frac{\exp(-b\frac{d(z,y)^2}{\tau-s})}{|B(z,(\tau-s)^{1/2})|}\le C\frac{\exp(-a'\frac{d(x,y)^2}{t-s})}{|B(y,[\rho(t-s)]^{1/2})|}.$$

Using Proposition 2.2 we get

$$G_b(z,\tau,y,s)\le CG_a(x,t;y,s).$$

Case 3. Now we are left with only one case i.e. $d(z,y)\ge d(x,y)(a'/b')^{1/2}$ and $\tau-s\ge\rho(t-s)$. By Proposition 2.2

$$G_b(z,\tau;y,s)\le C/|B(y,(\tau-s)^{1/2})|\le C/|B(y,(t-s)^{1/2})|.$$

Hence

$$G_a(x,t;z,\tau)G_b(z,\tau;y,s)\le\frac{\exp(-a\frac{d(x,z)^2}{t-\tau})}{|B(x,(t-\tau)^{1/2})||B(y,(t-s)^{1/2})|}.$$

If $d(z,y) \le d(x,y)(a'/b')^{1/2}$, then

$$d(x,z) \ge d(x,y) - d(z,y) \ge d(x,y)\ (1-(a'/b')^{1/2}).$$

Hence

$$\begin{aligned}\exp(-a\frac{d(x,z)^2}{t-\tau}) &= \exp(-a\frac{d(x,z)^2}{2(t-\tau)})\exp(-a\frac{d(x,z)^2}{2(t-\tau)})\\ &\le \exp(-a\frac{d(x,z)^2}{2(t-\tau)})\exp(-a\frac{d(x,y)^2}{2(t-\tau)}(1-(a'/b')^{1/2})^2)\\ &\le \exp(-a\frac{d(x,z)^2}{2(t-\tau)})\exp(-a\frac{d(x,y)^2}{2(1-\rho)(t-s)}(1-(a'/b')^{1/2})^2).\end{aligned}$$

Here we have used the fact that $0 < t-\tau \le (1-\rho)(t-s)$. Now taking $\rho \in (0,1)$ so that

$$\frac{(1-(a'/b')^{1/2})^2}{4(1-\rho)} = 1, \tag{3.5}$$

we obtain,

$$\exp(-a\frac{d(x,z)^2}{t-\tau}) \le \exp(-a\frac{d(x,z)^2}{2(t-\tau)})\exp(-2a\frac{d(x,y)^2}{t-s}). \tag{3.6}$$

Therefore

$$\begin{aligned}&G_a(x,t;z,\tau)G_b(z,\tau;y,s) \le C\frac{\exp(-a\frac{d(x,z)^2}{t-\tau})}{|B(x,(t-\tau)^{1/2})||B(y,(t-s)^{1/2})|}\\ &\le \frac{\exp(-a\frac{d(x,z)^2}{2(t-\tau)})\exp(-2a\frac{d(x,y)^2}{t-s})}{|B(x,(t-\tau)^{1/2})||B(y,(t-s)^{1/2})|}.\end{aligned}$$

Using Proposition 2.2 again we have

$$\frac{\exp(-2a\frac{d(x,y)^2}{t-s})}{|B(y,(t-s)^{1/2})|} \le C\frac{\exp(-a\frac{d(x,y)^2}{t-s})}{|B(x,(t-s)^{1/2})|}$$

and hence

$$\begin{aligned}&G_a(x,t;z,\tau)G_b(z,\tau;y,s)\\ &\le C\frac{\exp(-a\frac{d(x,z)^2}{2(t-\tau)})}{|B(x,(t-\tau)^{1/2})|}G_a(x,t;y,s) = CG_{a/2}(x,t;z,\tau)G_a(x,t;y,s).\end{aligned}$$

Now we know the lemma holds for $c = \min\{b-a, a/2\}$. q.e.d.

Due to the notation $N_{c,\infty}(V)$ in (1.7), an immediate consequence of Lemma 3.1 is

Lemma 3.2. *Suppose $0 < a < b$, there exist positive constants $C_{a,b}$ and c depending only on a and b such that, for all $t > s \ge 0$,*

$$(i).\ \int_s^t \int_{\mathbf{M}^n} G_a(x,t;z,\tau)\ |V(z,\tau)|\ G_b(z,\tau;y,s)dzd\tau \le C_{a,b}\ N_{c,\infty}(V)G_a(x,t;y,s);$$

$$(ii).\int_s^t \int_{\mathbf{M}^n} G_b(x,t;z,\tau)\ |V(z,\tau)|\ G_a(z,\tau;y,s)dzd\tau \le C_{a,b}\ N_{c,\infty}(V)G_a(x,t;y,s).$$

Proof of Lemma 3.2. We will only give a proof of (i) since (ii) can be handled similarly. For simplicity we write

$$J(x,t;y,s)=\int_s^t \int_{\mathbf{M}^n} G_a(x,t;z,\tau)\ |V(z,\tau)|\ G_b(z,\tau;y,s)dzd\tau.$$

The desired estimate follows from the main inequality in Lemma 3.1 i.e.

$$G_a(x,t;z,\tau)G_b(z,\tau;y,s)\le C[G_c(x,t;z,\tau)+G_c(z,\tau;y,s)]G_a(x,t;y,s)$$

where c is a suitable positive constant. This is because the last inequality implies

$$J(x,t;y,s)\le C\int_s^t \int_{\mathbf{M}^n} [G_c(x,t;z,\tau)+G_c(z,\tau;y,s)]|V(z,\tau)|dzd\tau G_a(x,t;y,s).$$

Taking the maximum of the integral the right hand side, we finish the proof of the lemma. q.e.d.

4. Proof of Theorem B, part (a)

This section is divided into three parts. In the first part we list a number of notations and symbols, which include an integral operator and an appropriate function space. In the next part we will prove lemma 4.1 which states that the integral operator has a fixed point in the function space. Theorem B, part (a), will be proved in the end of the section.

First we recall and define a number of notations. Given a positive $u_0 \in L^\infty(\mathbf{M}^n)$, write

$$h(x,t)=\int_{\mathbf{M}^n} G(x,t;y,0)u_0(y)dy. \tag{4.1}$$

Here G is the fundamental solution of the operator H_0 in (1.1). By (1.3), there are positive constants C and b such that

$$G(x,t;y,s)\le \frac{C}{|B(x,(t-s)^{1/2})|}\exp(-b\frac{d(x,y)^2}{t-s})=CG_b(x,t;y,s),$$

for all $t>s$ and $x,y\in\mathbf{M}^n$.

For $u\in L^\infty(\mathbf{M}^n\times[0,\infty))$, we define T to be the integral operator:

$$Tu\ (x,t)=h(x,t)+\int_0^t\int_{\mathbf{M}^n} G(x,t;y,s)u^p(y,s)dyds. \tag{4.2}$$

For any constants $a>0$, $d>1$ and $M>1$, the space S_d is defined by

$$S_d=\{u(x,t)\in C(\mathbf{M}^n\times[0,d])\mid 0\le u(x,t)\le Mh_a(x,t)\}$$

where the function h_a is given by (2.1).

From this moment, we fix the number a to be a positive number strictly less than b, which is the constant in the Gaussian upper bound for G. This choice of a is crucial when we prove Lemma 3.1 below. Since $a < b$ we have

$$G(x,t;y,s) \le CG_b(x,t;y,s) \le CG_a(x,t;y,s),$$
$$h(x,t) \le Ch_b(x,t) \le Ch_a(x,t).$$

Next we present a Lemma which will lead to a proof of Theorem B. The idea is to show that the operator T has a fixed point in S_d.

Lemma 4.1. *Under the assumption of Theorem B, part (a), there exist constants $M > 1$ and $b_0 > 0$ independent of d such that the integral operator (4.2) has a fixed point in S_d, provided that*

$$u_0 \in C^2(\mathbf{M}^n), \quad \lim_{d(x,0)\to\infty} u_0(x) = 0, \quad \textit{and}, \quad ||u_0||_{L^\infty(\mathbf{M}^n)} + ||u_0||_{L^1(\mathbf{M}^n)} \le b_0.$$

Proof. *step* 1. We want to use the Schauder fixed point theorem. To this end we need to check the following conditions.
(i). S_d is nonempty, closed, bounded and convex.
(ii). $TS_d \subset S_d$.
(iii). TS_d is a compact subset of S_d in L^∞ norm.
(iv). T is continuous.

step 2. Condition (i) is obviously true. So let's verify (ii), which requires us to show that $0 \le Tu \le Mh_a$ when $0 \le u \le Mh_a$.
Since $0 \le u \le Mh_a$ we have

$$u^p(y,s) \le M^p h_a^{p-1}(y,s)h_a(y,s) = M^p h_a^{p-1}(y,s)\int_{\mathbf{M}^n} G_a(y,s;z,0)u_0(z)dz. \quad (4.3)$$

Recalling the definition of h_a and using the fact that $u_0 \in L^1(\mathbf{M}^n)$ we obtain

$$\begin{aligned} &h^{p-1}(y,s) = [\int_{\mathbf{M}^n} G_a(y,s;z,0)u_0(z)dz]^{p-1} \\ &= \frac{1}{|B(y,s^{1/2})|^{p-1}}[\int_{\mathbf{M}^n} u_0(z)dz]^{p-1} \le \frac{1}{|B(y,s^{1/2})|^{p-1}}||u_0||^{p-1}_{L^1(\mathbf{M}^n)}. \end{aligned} \quad (4.4)$$

Therefore

$$u^p(y,s) \le M^p||u_0||^{p-1}_{L^1(\mathbf{M}^n)}\frac{1}{|B(y,s^{1/2})|^{p-1}}.$$

Substituting (4.3) into (4.2) and using Fubini's theorem we obtain

$$\begin{aligned} &Tu(x,t) \le h(x,t) \\ &+ CM^p\int_{\mathbf{M}^n}\int_0^t\int_{\mathbf{M}^n} G(x,t;y,s)h_a^{p-1}(y,s)G_a(y,s;z,0)dyds\ u_0(z)dz, \end{aligned} \quad (4.5)$$

Remembering that

$$G(x,t;y,s) \le \frac{C}{|B(x,(t-s)^{1/2})|}\exp(-b\ d(x,y)^2/(t-s)) = CG_b(x,t;y,s),$$

we have

$$\int_0^t \int_{\mathbf{M}^n} G(x,t;y,s) h_a^{p-1}(y,s) G_a(y,s;z,0) dyds$$
$$\le C \int_0^t \int_{\mathbf{M}^n} G_b(x,t;y,s) h_a^{p-1}(y,s) G_a(y,s;z,0) dyds$$

At this stage we quote Lemma 3.2 which was first proved in [Zhan3] for the Euclidean case. Given $b > a$,

$$\int_0^t \int_{\mathbf{M}^n} G_b(x,t;y,s) h_a^{p-1}(y,s) G_a(y,s;z,0) dyds \le C C_{a,b} N_{c,\infty}(h_a^{p-1}) G_a(x,t;z,0), \tag{4.6}$$

for all $t > 0$ and some positive c and $C_{a,b}$. We claim that $N_{c,\infty}(h_a^{p-1})$, which is defined in (1.7), is a finite number. The proof is as follows. Since $s > 0$ and $h_a(x,t) = 0$ when $t < 0$, we have

$$N_{c,\infty}(h_a^{p-1}) \equiv \sup_{x,t>0} \int_0^t \int_{\mathbf{M}^n} |h_a^{p-1}(y,s)| G_c(x,t;y,s) dyds$$
$$+ \sup_{y,s\ge 0} \int_s^\infty \int_{\mathbf{M}^n} |h_a^{p-1}(x,t)| G_c(x,t;y,s) dxdt.$$

By (4.4) and the fact that $h_a(x,t) \le C||u_0||_{L^\infty}$ we have

$$\begin{aligned}
&\int_0^t \int_{\mathbf{M}^n} |h_a^{p-1}(y,s)| G_c(x,t;y,s) dyds \\
&= \int_{r_0}^t \int_{\mathbf{M}^n} |h_a^{p-1}(y,s)| G_c(x,t;y,s) dyds + \int_0^{r_0} \int_{\mathbf{M}^n} |h_a^{p-1}(y,s)| G_c(x,t;y,s) dyds \\
&\le ||u_0||_{L^1(\mathbf{M})}^{p-1} \int_{r_0}^t \int_{\mathbf{M}^n} \frac{1}{|B(y,s^{1/2})|^{p-1}} G_c(x,t;y,s) dyds + C||u_0||_{L^\infty}^{p-1} r_0.
\end{aligned}$$

Similarly

$$\int_s^\infty \int_{\mathbf{M}^n} |h_a^{p-1}(x,t)| G_c(x,t;y,s) dxdt$$
$$\le C||u_0||_{L^1(\mathbf{M})}^{p-1} \int_{r_0}^\infty \int_{\mathbf{M}^n} \frac{G_c(x,t;y,s)}{|B(x,t^{1/2})|^{p-1}} dxdt + C||u_0||_{L^\infty}^{p-1} r_0.$$

Using C_0 to denote

$$\sup_{x\in\mathbf{M}^n,t>0} \int_{r_0}^t \int_{\mathbf{M}^n} \frac{G_c(x,t;y,s)}{|B(y,s^{1/2})|^{p-1}} dyds + \sup_{y\in\mathbf{M}^n,s>0} \int_{r_0}^\infty \int_{\mathbf{M}^n} \frac{G_c(x,t;y,s)}{|B(x,t^{1/2})|^{p-1}} dxdt,$$

we have

$$N_{c,\infty}(h_a^{p-1}) \le C C_0 ||u_0||_{L^1(\mathbf{M})}^{p-1} + C||u_0||_{L^\infty}^{p-1} r_0 < \infty,$$

which proves the claim.

Combining (4.6) with (4.5), we reach

$$Tu(x,t) \le h(x,t) + CC_{a,b}M^p N_{c,\infty}(h_a^{p-1}) \int_{\mathbf{M}^n} G_a(x,t;z,0)u_0(z)dz,$$

which yields

$$Tu(x,t) \le (C + CC_{a,b}M^p N_{c,\infty}(h_a^{p-1}))h_a(x,t) \tag{4.7}$$

Since $p > 1$, by taking $M > 2C$ and $||u_0||_{L^1(M)} + ||u_0||_{L^\infty(M)}$ suitably small we find that

$$0 \le Tu(x,t) \le Mh_a(x,t). \tag{4.8}$$

Thus condition (ii) is satisfied.

step 3. Now we need to check condition (iii). We note that the local regularity theory for solutions of uniformly parabolic equations can be transplanted to the operator H_0 in (1.1). By our choice, functions u in S_d are uniformly bounded and therefore, Tu is equicontinuous and in fact Hölder continuous. This is because Tu actually satisfies, in the weak sense, $H_0(Tu) = -u^p$ in $\mathbf{M}^n \times (0,d)$ and $Tu(x,0) = u_0(x)$ and $u_0 \in C^2(\mathbf{M}^n)$. Taking into account that

$$0 \le \lim_{d(x,0)\to\infty} Tu(x,t) \le C \lim_{d(x,0)\to\infty} h_a(x,t) = 0$$

uniformly (by Lemma 2.1), we know that

$$\lim_{d(x,0)\to\infty} Tu(x,t) = 0$$

Hence TS_d is a relatively compact subset of S_d. This is an easy modification of the classical Ascoli-Arzela theorem (see [Zhao]). Hence we have verified (iii).

step 4. Finally we need to check condition (iv).

Given u_1 and u_2 in S_d, we have, by (3.2),

$$(Tu_1 - Tu_2)(x,t) = \int_0^t \int_{\mathbf{M}^n} G(x,t;y,s))\ [u_1^p(y,s) - u_2^p(y,s)]dyds. \tag{4.9}$$

Next we notice that

$$|u_1^p(y,s) - u_2^p(y,s)| \le p\max\{u_1^{p-1}(y,s), u_2^{p-1}(y,s)\}|u_1(y,s) - u_2(y,s)|.$$

Since u_1 and u_2 are bounded from above by Mh_a, we have

$$|u_1^p(y,s) - u_2^p(y,s)| \le CM^{p-1}h_a^{p-1}(y,s)|u_1(y,s) - u_2(y,s)|.$$

Substituting the last inequality to (4.9) we obtain

$$\begin{aligned} ||Tu_1 - Tu_2||_{L^\infty} &\le CM^{p-1}||u_1 - u_2||_{L^\infty} \int_0^t \int_{\mathbf{M}^n} G(x,t;y,s)h_a^{p-1}(y,s)dyds \\ &\le CM^{p-1}||u_1 - u_2||_{L^\infty} N_{c,\infty}(h_a^{p-1}). \end{aligned}$$

Here c is a suitable constant. Since $N_{c,\infty}(h_a^{p-1})$ is a finite constant we have proved the continuity of T and the lemma. q.e.d.

Now we are ready to give the

Proof of Theorem B, part (a).
For any $d > 1$, let u_d be a fixed point of T in the space S_d as given in Lemma 3.1. Define

$$U_d(x,t) = \begin{cases} u_d(x,t), & t \le d; \\ u_d(x,d), & t > d. \end{cases}$$

Then from the proof of Lemma 4.1 ((4.8) e.g.), we know that $\{U_d\}$ is uniformly bounded and equicontinuous. Hence there is a subsequence $\{U_{d_m} | m = 1, 2, ..\}$ which converges uniformly to a function u in any compact region of $\mathbf{M}^n \times [0, \infty)$. For any fixed $(x,t) \in \mathbf{M}^n \times [0,\infty)$ and m sufficiently large, we know that

$$U_{d_m}(x,t) = h(x,t) + \int_0^t \int_{\mathbf{M}^n} G(x,t;y,s) U_{d_m}^p(y,s) dy ds.$$

This is because U_{d_m} is a fixed point of T in S_{d_m}. Now by the dominated convergence theorem, u satisfies

$$u(x,t) = h(x,t) + \int_0^t \int_{\mathbf{M}^n} G(x,t;y,s) u^p(y,s) dy ds,$$

for all $(x,t) \in \mathbf{M}^n \times [0,\infty)$. Moreover, by (4.8) We know

$$0 < u(x,t) \le M h_a(x,t)$$

for all $(x,t) \in \mathbf{M}^n \times [0,\infty)$. Clearly u is a global positive solution of (1.1). This finishes the proof. q.e.d.

Next we give

Proof of Corollary 1.1. Note that

$$\int_{\mathbf{M}^n} G_c(x,t;y,s) dx, \quad \int_{\mathbf{M}^n} G_c(x,t;y,s) dy \le C,$$

for all $t > s$, we have

$$\begin{aligned} &\int_{r_0}^\infty \int_{\mathbf{M}^n} \frac{G_c(x,t;y,s)}{|B(y,s^{1/2})|^{p-1}} dy ds + \int_{r_0}^\infty \int_{\mathbf{M}^n} \frac{G_c(x,t;y,s)}{|B(x,t^{1/2})|^{p-1}} dx dt \\ &\le C \int_{r_0}^\infty \sup_{y \in \mathbf{M}} \frac{1}{|B(y,r^{1/2})|^{p-1}} dr < \infty. \end{aligned}$$

Now suppose $\inf_{x \in \mathbf{M}^n} |B(x,r)| \ge C r^\alpha$ when $r > r_0$, then, for $p > 1 + \frac{2}{\alpha}$,

$$\int_{r_0}^\infty \sup_{y \in \mathbf{M}} \frac{1}{|B(y,r^{1/2})|^{p-1}} dr \le C \int_{r_0}^\infty \frac{1}{r^{\alpha(p-1)/2}} < \infty,$$

since $\alpha(p-1)/2 > 1$. Therefore (1.1) has a global positive solution for some $u_0 > 0$ by Theorem B, part (a). This proves the corollary. q.e.d.

Proof of Corollary 1.2. Suppose $p < p^*$, then $p < 1 + \frac{2}{\alpha}$ for some α such that

$$\lim_{r\to\infty} \inf \frac{|B(x,r)|}{r^\alpha} < \infty$$

Hence clearly (1.1) has no global positive solution by Theorem B, part (b), which will be proved in the next section.

If $p > p^* = 1 + \frac{2}{\alpha^*}$, then there exists $\epsilon > 0$ such that $\frac{2}{p-1} + \epsilon < \alpha^*$ and hence

$$\lim_{r\to\infty} \inf \frac{|B(x,r)|}{r^{\frac{2}{p-1}+\epsilon}} = \infty.$$

Therefore, for some $r_0 > 0$ and all $r \geq r_0$,

$$1/|B(x, r^{1/2})|^{p-1} \leq C/r^{1+\epsilon(p-1)/2}.$$

By assumption (1.8), we have

$$\int_{r_0}^\infty \sup_{y\in\mathbf{M}} \frac{1}{|B(y, r^{1/2})|^{p-1}} dr \leq \int_{r_0}^\infty \frac{1}{r^{1+\epsilon(p-1)/2}} dr < \infty.$$

Theorem B, part (a) shows that (1.1) has global positive solutions for some $u_0 > 0$. q.e.d.

5. Proof of Theorem B, part (b)

Proof. Without loss of generality we take $x_0 = 0$.

Suppose u is a global positive solution of (1.1), by Definition 1.1, we know that u solves the integral equation

$$u(x,t) = \int_{\mathbf{M}^n} G(x,t;y,0)u_0(y)dy + \int_0^t \int_{\mathbf{M}^n} G(x,t;y,s)u^p(y,s)dyds, \tag{5.1}$$

for all $(x,t) \in \mathbf{M}^n \times [0,\infty)$.

Given $t > 0$, choosing $T > t$, multiplying $G(x,T;0,t)$ on both sides of (5.1) and integrating with respect to x, we obtain

$$\begin{aligned}\int_{\mathbf{M}^n} G(x,T;0,t)u(x,t)dx \geq C \int_{\mathbf{M}^n} \int_{\mathbf{M}^n} G(x,T;0,t)\ G(x,t;y,0)dx\ u_0(y)dy+ \\ + C\int_0^t \int_{\mathbf{M}^n} \int_{\mathbf{M}^n} G(x,T;0,t)\ G(x,t;y,s)dx\ u^p(y,s)dyds.\end{aligned} \tag{5.2}$$

Even though H_0 is an operator with variable coefficients, the fundamental solution G still enjoys the symmetry

$$G(x,T;y,t) = G(y,T;x,t)$$

for all $x, y \in \mathbf{M}^n$ and $T > t$ (see [D]). Therefore by the reproducing property of the heat kernel, we reach

$$\int_{\mathbf{M}^n} G(x,T;0,t)\ G(x,t;y,0)dx = CG(0,T;y,0) = CG(y,T;0,0),$$

$$\int_{\mathbf{M}^n} G(x,T;0,t)\ G(x,t;y,s)dx = CG(0,T;y,s) = CG(y,T;0,s).$$

Substituting the last two equalities into (5.2), we see that

$$\begin{aligned}\int_{\mathbf{M}^n} & G(x,T;0,t)u(x,t)dx \\ & \ge C\int_{\mathbf{M}^n} G(y,T;0,0)u_0(y)dy + C\int_0^t\int_{\mathbf{M}^n} G(y,T;0,s)u^p(y,s)dyds.\end{aligned} \tag{5.3}$$

By (1.3), $\int_{\mathbf{M}^n} G(y,T;0,s)dy \le C$. Using Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbf{M}^n} G(y,T;0,s)u(y,s)dy &= \int_{\mathbf{M}^n} G^{1/q}(y,T;0,s)G^{1/p}(y,T;0,s)u(y,s)dy \\ &\le [\int_{\mathbf{M}^n} G(y,T;0,s)dy]^{1/q}[\int_{\mathbf{M}^n} G(y,T;0,s)u^p(y,s)dy]^{1/p} \\ &\le C[\int_{\mathbf{M}^n} G(y,T;0,s)u^p(y,s)dy]^{1/p},\end{aligned}$$

where $1/p + 1/q = 1$. Inequality (5.3) then implies

$$\begin{aligned}\int_{\mathbf{M}^n} & G(x,T;0,t)u(x,t)dx \\ & \ge C\int_{\mathbf{M}^n} G(y,T;0,0)u_0(y)dy + C\int_0^t [\int_{\mathbf{M}^n} G(y,T;0,s)u(y,s)dy]^p ds.\end{aligned} \tag{5.4}$$

Without loss of generality we assume that u_0 is strictly positive in a neighborhood of 0. Using the lower bound in (1.4) for G, we can then find a constant $C > 0$ so that, for $T > 1$,

$$\int_{\mathbf{M}^n} G(y,T;0,0)u_0(y)dy \ge \int_{d(y,0)^2\le 1} \frac{C}{|B(0,T^{1/2})|}u_0(y)dy \ge \frac{C}{|B(0,T^{1/2})|}. \tag{5.5}$$

Going back to (5.4) and writing $J(t) \equiv \int_{\mathbf{M}^n} G(x,T;0,t)u(x,t)dx$, we have

$$J(t) \ge C/|B(0,T^{1/2})| + C\int_0^t J^p(s)ds, \quad T > t, T > 1. \tag{5.6}$$

Using the notation $g(t) \equiv \int_0^t J^p(s)ds$, we obtain,

$$g'(t)/(|B(0,T^{1/2})|^{-1} + g(t))^p \ge C. \tag{5.7}$$

Integrating (5.7) from 0 to T and noticing $g(0) = 0$, we have

$$-\frac{1}{(|B(0,T^{1/2})|^{-1} + g(t))^{p-1}}\Big|_0^T \ge (p-1)CT \tag{5.8}$$

and therefore

$$|B(0,T^{1/2})|^{p-1} \ge (p-1)CT,$$

for all $T > 1$. This is possible only when

$$|B(0,T)| \geq CT^{2/(p-1)} \tag{5.9}$$

when T is large. Under the assumption of part (b) of Theorem B,

$$\lim_{r\to\infty} \inf \frac{|B(x,r)|}{r^\alpha} < \infty$$

and $p < 1 + \frac{2}{\alpha}$. Therefore

$$\lim_{r\to\infty} \inf \frac{|B(x,r)|}{r^{2/(p-1)}} = 0.$$

This contradicts (5.9). Hence no global positive solutions exist for such p. q.e.d.

6. Proof of Theorem C: non-existence result for the non-compact Yamabe problem

Proof of Theorem C. Since the properties (1.2), (1.3) and (1.4) hold, we can apply part (b) of Theorem B, which says, for any $p < 1 + \frac{2}{\alpha}$, equation (1.1) has no global positive solutions. Since

$$\alpha < (n-2)/2, \tag{6.1}$$

we know that equation (1.1) has no global positive solutions for $p = \frac{n+2}{n-2}$. Therefore the Yamabe equation can not have solutions. This is so because any solution $u = u(x)$ of the Yamabe equation is a stationary and hence global positive solution of (1.1). Following the arguments in Lemma 1 of [P], which can be easily generalized to manifolds with Ricci bounded from below, we know that

$$u(x) \geq \int_{\mathbf{M}^n} G(x,t;y,0)u(y)dy + \int_0^t \int_{\mathbf{M}^n} G(x,t;y,s)u^p(y)dyds.$$

By Theorem B, part (b), such a u does not exist in this case. q.e.d.

To prove Corollary 1.3 we need to show that the Green's function G of the operator $\Delta - R - \partial_t$ have global bounds (1.3) and (1.4). Due to the presence of R, we need much more effort. This is done in the next

Lemma 6.1. *Let G_0 be the fundamental solution of the unperturbed operator $\Delta - \partial_t$. Suppose there are constants $b, C > 0$ such that*

$$\frac{1}{C|B(x,(t-s)^{1/2})|} e^{-\frac{d(x,y)^2}{b(t-s)}} \leq G_0(x,t;y,s) \leq \frac{C}{|B(x,(t-s)^{1/2})|} e^{-b\frac{d(x,y)^2}{t-s}}, \tag{6.2}$$

for all $x, y \in \mathbf{M}^n$ and all $t > s$. Let G be the fundamental solution of the operator $\Delta - V - \partial_t$ where $V = V(x)$ is a bounded function. Suppose $N_{c,\infty}(V^-)$ is sufficiently small and $N_{c,\infty}(V^+)$ is finite then properties (1.3) and (1.4) hold for G.

Proof. This essentially follows from combining Theorem C in [Zhan5] with the proof of Theorem A part (b) in [Zhan2]. For the sake of completeness we include the proof here.

First we show that (1.3) holds. Without any loss of generality we assume that $V^+ = 0$ otherwise the maximum principle gives the desired result.

We pick a number a such that $0 < a < b$, where b is given in (6.2). Let B be the smallest positive number such that

$$G(x, t; y, s) \le BG_a(x, t; y, s), \tag{6.3}$$

for all $x, y \in \mathbf{M}^n$ and $s < t$. We claim that such a B does exist by our extra assumption that $V(x, t) = 0$ if $t > T$. The claim can be checked easily by using the reproducing formula

$$G(x, t; y, 0) = \int_{\mathbf{M}^n} G(x, t; z, T)G(z, T; y, 0)dz, \quad t > T,$$

and the fact that $G(x, t; z, T) = G_0(x, t; z, T)$ for $t > T$. The main task is to show that B depends on V only in the form of $N_{c,\infty}(V)$.

From the Duhamel's principle and (6.3) we have

$$G(x, t; y, s) \le G_0(x, t; y, s) + BC_0 \int_s^t \int_{\mathbf{M}^n} G_a(x, t; z, \tau)\ |V(z, \tau)|\ G_b(z, \tau; y, s)dzd\tau,$$

$$G(x, t; y, s) \ge G_0(x, t; y, s) - BC_0 \int_s^t \int_{\mathbf{M}^n} G_a(x, t; z, \tau)\ |V(z, \tau)|\ G_b(z, \tau; y, s)dzd\tau,$$

where $x, y \in \mathbf{M}^n$ and $s < t$. Lemma 3.2 then implies

$$G(x, t; y, s) \le C_0G_b(x, t; y, s) + BC_0C_{a,b}N_{c,\infty}(V)\ G_a(x, t; y, s), \tag{6.4}$$

$$G(x, t; y, s) \ge C_0G_{1/b}(x, t; y, s) - BC_0C_{a,b}N_{c,\infty}(V)\ G_a(x, t; y, s), \tag{6.4'}$$

for all $x, y \in \mathbf{M}^n$ and $s < t$. Since $a < b$ we know that

$$G(x, t; y, s) \le [C_0 + BC_0C_{a,b}N_{c,\infty}(V)]\ G_a(x, t; y, s)$$

for all $x, y \in \mathbf{M}^n$ and $s < t$. Hence, by the definition of B, we obtain,

$$B \le C_0 + BC_0C_{a,b}N_{c,\infty}(V). \tag{6.5}$$

When $C_0C_{a,b}N_{c,\infty}(V) < 1/2$ we have

$$B \le 2C_0.$$

This together with (6.3) proves that the upper bound (1.3) holds.

The lower bound (1.4) is an immediate consequence of the maximum principle and Theorem C in [Zhan5]. Here we note that Theorem C in that paper was stated for manifolds with nonnegative Ricci curvature. However it is still valid under the current assumptions. The proof is identical. □

Remark 6.1. In fact Theorem C in [Zhan5] states: if $V^+ = 0$, then the lower bound (1.4) in Lemma 6.1 holds if and only if $N_{c,\infty}(V)$ is finite. In contrast for the upper bound (1.3) to hold some smallness of V^+ is necessary.

Proof of Corollary 1.3. (a). Since $\mathbf{M}^n$ has non-negative Ricci curvature outside a compact set, by [CS-C], (6.2) is true. By choosing $V = -R(x)$ in Lemma 6.1, we know that (1.2), (1.3) and (1.4) are satisfied.

(b). In this case, by [LY], the Green's function Γ of Δ exists and

$$\Gamma(x,y) \sim \int_{d(x,y)^2}^{\infty} \frac{1}{|B(x,t^{1/2})|} dt \sim \frac{1}{d(x,y)^{\alpha-2}}, \quad d(x,y) \geq c > 0.$$

Moreover, since R is independent of time,

$$\begin{aligned} N_{c,\infty}(R) &= \sup_{x,t} \int_0^{\infty} \int_{\mathbf{M}^n} G_c(x,t;y,s) R(y) dy ds \leq C \sup_{x\in\mathbf{M}^n} \int_{\mathbf{M}^n} \Gamma(x,y)|R(y)|dy \\ &\leq \sup_{x\in\mathbf{M}^n} \int_{\mathbf{M}^n - B(x,c)} \Gamma(x,y)|R(y)|dy + \sup_{x\in\mathbf{M}^n} \int_{B(x,c)} \Gamma(x,y)|R(y)|dy. \end{aligned}$$

Therefore

$$\begin{aligned} &N_{c,\infty}(R) \leq \sup_{x\in\mathbf{M}^n} \int_{\mathbf{M}^n} \frac{C}{1+d(y,0)^{2+\delta}} \frac{\epsilon}{d(x,y)^{\alpha-2}} dy + C\epsilon \\ &= \sup_{x\in\mathbf{M}^n} \int_{d(x,y)\geq d(y,0)} \frac{C}{1+d(y,0)^{2+\delta}} \frac{\epsilon}{d(x,y)^{\alpha-2}} dy \\ &\quad + \sup_{x\in\mathbf{M}^n} \int_{d(x,y)\leq d(y,0)} \frac{C}{1+d(y,0)^{2+\delta}} \frac{\epsilon}{d(x,y)^{\alpha-2}} dy + C\epsilon \\ &\leq \sup_{x\in\mathbf{M}^n} \int_{d(x,y)\geq d(y,0)} \frac{C}{1+d(y,0)^{2+\delta}} \frac{\epsilon}{d(y,0)^{\alpha-2}} dy + \\ &\quad \sup_{x\in\mathbf{M}^n} \int_{d(x,y)\leq d(y,0)} \frac{C}{1+d(x,y)^{2+\delta}} \frac{\epsilon}{d(x,y)^{\alpha-2}} dy + C\epsilon \\ &\leq \int_{\mathbf{M}^n} \frac{C}{1+d(y,0)^{2+\delta}} \frac{\epsilon}{d(y,0)^{\alpha-2}} dy + \sup_{x\in\mathbf{M}^n} \int_{\mathbf{M}^n} \frac{C}{1+d(x,y)^{2+\delta}} \frac{\epsilon}{d(x,y)^{\alpha-2}} dy + C\epsilon \\ &\leq C\epsilon \int_0^{\infty} \frac{r}{1+r^{2+\delta}} dr + C\epsilon. \end{aligned}$$

Since $\delta > 0$, by taking ϵ sufficiently small, we have $N_{c,\infty}(V)$ is sufficiently small. It is clear that the size of ϵ can be chosen to depend only on n since the Ricci curvature is nonnegative. Now we can use part (a) of the Corollary to conclude that the non-compact Yamabe problem (1.1') has no solution. This proves (b).

Finally, for $\mathbf{M}^9 = \mathbf{R}^3 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1$ with the metric tensor being the direct sum of the usual ones on $\mathbf{R}^3$ and S^1, we know $Ricci = 0$, $R = 0$, $n = 9$ and (1.9) holds for $\alpha = 3$. Therefore (1.1') has no solution. By Theorem B (a) in [Zhan5], $\mathbf{M}^9$ has a complete conformal metric with positive scalar curvature. But part (a) of Corollary 1.3 shows that $\mathbf{M}^9$ has no conformal metric with constant scalar curvature ($\alpha = 4, n = 9$)

As a comparison, on another Ricci flat manifold $\mathbf{R}^n$ with Euclidean metric, problem (1.1') has infinitely many solutions (see [Ni])

$$u_\lambda(x) = \frac{[n(n-2)\lambda^2]^{(n-2)/4}}{(\lambda^2 + |x|^2)^{(n-2)/2}}, \quad \lambda > 0.$$

q.e.d.

In the next Proposition, we shall give examples of a complete non-compact manifold with positive scalar curvature, which is not only conformal to a complete non-compact manifold with positive constant scalar curvature but also to a complete manifold with zero scalar curvature. It is well known that this situation does not exist in the compact case.

Proposition 6.1. *Let* $\mathbf{M}^n = \mathbf{M}_1 \times \mathbf{R}^k$, *where* $\mathbf{M}_1$ *is a compact manifold with positive scalar curvature and the metric of* $\mathbf{M}^n$ *is the product of that of* $\mathbf{M}_1$ *and the Euclidean metric on* $\mathbf{R}^k$, $k \geq 1$. *Then* $\mathbf{M}^n$ *is conformal to a complete non-compact manifold with positive constant scalar curvature and to a complete manifold with zero scalar curvature.*

Proof. (i). $\mathbf{M}^n$ is conformal to a complete non-compact manifold with positive constant scalar curvature.

Let $(x_1, x_2) \in \mathbf{M}_1 \times \mathbf{R}^k$. It is clear that the scalar curvature of $\mathbf{M}^n$ is only a function of x_1. Let Δ, Δ_1 and Δ_2 be the Laplace-Beltrami operators on $\mathbf{M}^n$, $\mathbf{M}_1$ and $\mathbf{R}^k$ respectively, then $\Delta = \Delta_1 + \Delta_2$. Moreover $0 < R(x) = R_1(x_1)$ where R and R_1 are the scalar curvature of $\mathbf{M}^n$ and $\mathbf{M}_1$ respectively. Since the exponent $(n+2)/(n-2)$ is subcritical for the lower dimensional manifold $\mathbf{M}_1$, we know, by the standard variational technique (see [Au2] Theorem 5.5), that the following equation has a positive solution $u_1 = u_1(x_1)$ on $\mathbf{M}_1$.

$$\Delta_1 u_1 - \frac{n-2}{4(n-1)} R_1 u_1 + u_1^{(n+2)/(n-2)} = 0.$$

Define $u(x) = u(x_1, x_2) = u_1(x_1)$, then clearly u is a positive solution of equation (1.1'), i.e.

$$\Delta u - \frac{n-2}{4(n-1)} R u + u^{(n+2)/(n-2)} = 0.$$

Since u is bounded away from zero, we know that the metric it generates is complete. Therefore $\mathbf{M}^n$ is conformal to a complete non-compact manifold with positive constant scalar curvature.

(ii). $\mathbf{M}^n$ is conformal to a complete non-compact manifold with zero constant scalar curvature.

Without any loss of generality we take $k = 1$. It is enough to show that following linear equation has a positive solution that is bounded away from zero.

$$\Delta u - Ru = \Delta_1 u + \Delta_2 u - R_1 u = 0. \tag{6.6}$$

Suppose $-\lambda < 0$ is the first eigenvalue of the Schrödinger operator $\Delta_1 - R_1$ on $\mathbf{M}_1$. At this moment we notice that by the manifold version of the well-known node theorem (see [GJ], Corollary 3.3.4, which can be easily generalized to compact manifolds), the eigenfunctions corresponding to $-\lambda$ do not change sign. Therefore we can find $u_1 = u_1(x_1) \geq c > 0$ on the compact $\mathbf{M}_1$ such that

$$\Delta_1 u_1 - R_1 u_1 = -\lambda u_1.$$

Now let

$$u_2 = u_2(x_2) = e^{\lambda^{1/2} x_2} + e^{-\lambda^{1/2} x_2},$$

then $\Delta_2 u_2 = \lambda u_2$ and $u_2 \geq 1$. We claim that $u \equiv u_1(x_1)u_2(x_2)$ is a solution of (6.6), which is also bounded away from zero. This can be seen from the following calculation.

$$\Delta u - Ru = u_2\Delta_1 u_1 + u_1\Delta_2 u_2 - R_1(x_1)u_1u_2 = u_2(\Delta_1 u_1 + \lambda u_1 - R_1(x_1)u_1) = 0.$$

It is obvious that the metric u generates has zero scalar curvature. Since $u \geq c > 0$, we know that the new metric is also complete. q.e.d.

7. Proof of Theorem D: a non-existence result when $Ricci \geq 0$

In order to prove Theorem D we need to obtain some lower bound on the fundamental solution G. Since the decay of the scalar curvature is quadratic, more efforts are needed. However this task is well worth doing for two reasons. One, it is very easy to construct nonparablic manifolds with nonnegative Ricci curvature and with scalar curvature decaying like $1/d^2(x, 0)$. Second, the lower bound estimates seems interesting in its own right as quadratic decay is the borderline between long range and short range potentials.

The key estimate of the section is Lemma 7.2 below where we show that the heat kernel of $\Delta + V$ has at most polynomial (on diagonal) decay provided V has quadratic decay.

Lemma 7.1. *Assume that Ricci ≥ 0 and $|V(x,t)| \leq C_0/[1 + d^2(x, x_0)]$ for an arbitrary positive constant C_0. Suppose that u is a nonnegative solution of*

$$\Delta u - Vu - u_t = 0, \quad t > 0, \tag{7.1}$$

and $d(x, x_0) = t^{1/2}$, then exist positive constants α and C_1 depending on V such that

$$u(x_0, 2t) \geq C_1 t^{-\alpha} u(x, t), \quad t \geq 1.$$

Moreover α is a linear function of C_0.

Proof. Let γ be a shortest geodesic connecting x_0 and x, which is parameterized by length. For $i = 0, 1, ..., k$, we write $y_i = \gamma(2^i)$, where k is the greatest integer smaller than or equal to $\log_2 d(x, x_0)$. Clearly $y_i, y_{i+1} \in B(y_{i+1}, 2^i) \subset B(y_{i+1}, 2^i 11/10)$. For any $y \in B(y_{i+1}, 2^i 11/10)$, we have

$$d(y, x_0) \geq d(x_0, y_{i+1}) - d(y_{i+1}, y) \geq 2^{i+1} - 2^i 11/10 = 2^i 9/10.$$

Therefore, there is $C > 0$ such that

$$\beta_i \equiv \sup_{B(y_{i+1}, 2^i 11/10) \times (0,\infty)} |V(x,t)| \leq CC_0/2^{2i}.$$

By the Harnack inequality stated in Corollary 5.3 of [Sa1], we have for $y, y' \in B(y_{i+1}, 2^i)$ and $s > s'$,

$$\ln[u(y', s')/u(y, s)] \leq C[\frac{d^2(y, y')}{s - s'} + (\beta_i + \frac{1}{s'})(s - s')].$$

It follows, for a $C_2 = e^{c(1+C_0)} > 0$,

$$u(y_{i+1}, 2t - 2^{2(i+1)}) \le C_2 u(y_i, 2t - 2^{2i}),$$
$$u(y_0, 2t-1) \le C_2 u(x_0, 2t).$$

Hence

$$u(x,t) \le C_2^{k+1} u(x_0, 2t) \le C_2^{2+\log_2 t^{1/2}} u(x_0, 2t) = C_2^2 2^{\log_2 C_2 \log_2 t^{1/2}} u(x_0, 2t).$$

Taking $\alpha = (\log_2 C_2)/2$ we have

$$u(x_0, 2t) \ge C t^{-\alpha} u(x,t). \tag{7.1'}$$

Noting that $C_2 \sim e^{c(1+C_0)}$ we know that α is a linear function of C_0. $\square$

Lemma 7.2. *Assume that Ricci ≥ 0 and $|V(x,t)| \le C_0/[1 + d^2(x, x_0)]$ for an arbitrary positive constant C_0. Suppose that G is the fundamental solution of (7.1), then exist positive constants α_1 and C_3 depending on V such that*

$$G(x_0, t; y, 0) \ge \frac{C_3}{t^{\alpha_1} |B(x_0, t^{1/2})|}, \quad \textit{for} \quad t \ge 1, \quad d(x_0, y) \le 1.$$

Moreover α_1 is a linear function of C_0.

Proof. Since $d(x_0, y) \le 1$ and $t \ge 1$, by Harnack inequality and the doubling condition of the balls, it is enough to prove, for $t \ge 1$, that

$$G(x_0, t; x_0, 0) \ge \frac{C_3}{t^{\alpha_1} |B(x_0, t^{1/2})|}. \tag{7.2}$$

To this end, we pick a point x_1 such that $d(x_0, x_1) = t^{1/2}$.

Let $\phi \in C_0^\infty(B(x_1, t^{1/2}/2))$ be such that $\phi(x) = 1$ when $x \in B(x_1, t^{1/2}/4)$ and $0 \le \phi \le 1$ everywhere. Consider the function

$$u(x,t) = \int_{\mathbf{M}} G(x, t; y, 0) \phi(y) dy. \tag{7.3}$$

As in [Sa1], we extend u by assigning $u(x,t) = 1$ when $t < 0$ and $x \in B(x_1, t^{1/2}/4)$, then u is a positive solution of (7.1) in $B(x_1, t^{1/2}/4) \times (-\infty, \infty)$. Here we take $V(x,t) = 0$ when $t < 0$ and we note that no continuity of V is needed. For any $y \in B(x_1, t^{1/2}/2)$, we have

$$d(y, x_0) \ge d(x_0, x_1) - d(x_1, y) \ge t^{1/2} - t^{1/2}/2 = t^{1/2}/2.$$

Hence, by the decay condition on V, there is a constant $C > 0$ such that

$$\beta \equiv \sup_{B(x, t^{1/2}/2)} |V(y,s)| \le C/t.$$

Using twice the Harnack inequality as stated in Theorem 5.2 in [Sa1], we obtain

$$u(x_1, 0) \le C e^{\beta t} u(x_1, t/4) \le C u(x_1, t/4),$$

$$G(y,t/4;x_1,0) \le Ce^{\beta t} G(x_1,t;x_1,0) \le CG(x_1,t;x_1,0),$$

for $y \in B(x_1, t^{1/2}/2)$. Hence

$$\begin{aligned}
1 = u(x_1,0) &\le Cu(x_1,t/4) = C\int_{B(x_1,t^{1/2}/2)} G(x_1,t/4;y,0)\phi(y)dy \\
&= C\int_{B(x_1,t^{1/2}/2)} G(y,t/4;x_1,0)\phi(y)dy \le CG(x_1,t;x_1,0)\int_{B(x_1,t^{1/2}/2)} \phi(y)dy \\
&\le CG(x_1,t;x_1,0)|B(x_1,t^{1/2})|.
\end{aligned}$$

The doubling property implies that $|B(x_0,t^{1/2})|$ and $|B(x_1,t^{1/2})|$ are comparable since $d(x_0,x_1) = t^{1/2}$ by choice. Therefore

$$G(x_1,t;x_1,0) \ge \frac{C}{|B(x_0,t^{1/2})|}. \tag{7.4}$$

Using Lemma 7.1, we have for some $\alpha > 0$,

$$G(x_0,2t;x_1,0) \ge Ct^{-\alpha} G(x_1,t;x_1,0),$$

which is, by (7.4),

$$G(x_1,2t;x_0,0) \ge Ct^{-\alpha} G(x_1,t;x_1,0) \ge \frac{C}{t^{\alpha}|B(x_0,t^{1/2})|},$$

By the doubling condition

$$G(x_1,t;x_0,0) \ge \frac{C}{t^{\alpha}|B(x_0,t^{1/2})|}.$$

Using Lemma 7.1 again

$$G(x_0,2t;x_0,0) \ge Ct^{-\alpha} G(x_1,t;x_0,0) \ge \frac{C}{t^{2\alpha}|B(x_0,t^{1/2})|}.$$

The lemma is proved by the doubling condition again. q.e.d.

Lemma 7.3. *Suppose*
(a). $\mathbf{M}_1$ is a n dimensional complete noncompact manifold with nonnegative Ricci curvature and the function $V = V(x)$ satisfies $0 \le R(x) \le C_0/[1 + d^2(x,x_0)]$, where C_0 is an arbitrary positive constant and $x \in \mathbf{M}_1$;
(b). $\mathbf{M}_2 \equiv \mathbf{M}_1 \times \mathbf{M}_0$ is equipped with the product metric, where $\mathbf{M}_0$ is a m dimensional compact manifold;
(c). G_2 is the fundamental solution of

$$\Delta_2 u - Vu - u_t = 0, \tag{7.5}$$

on $\mathbf{M}_2 \times (0,\infty)$, where Δ_2 is the Laplace-Beltrami operator on $\mathbf{M}_2$.

Then for some $\delta > 0$, there exist a constant α independent of $\mathbf{M}_0$ and C_δ such that, when $t \ge 1$, $d_1(x_0,x) \le \delta$ and $d_0(\xi_0,\xi) \le \delta$,

$$G_2(x_0,\xi_0,t;x,\xi,0) \ge C_\delta / t^{\alpha+(n/2)}.$$

Here $x, x_0 \in \mathbf{M}_1$ and $\xi, \xi_0 \in \mathbf{M}_0$; d_1, d_0 are the distances on $\mathbf{M}_1$ and $\mathbf{M}_0$ respectively.

Proof. Let Δ_1 and Δ_0 be the Laplace-Beltrami operator on $\mathbf{M}_1$ and $\mathbf{M}_0$ respectively. Let G_1 be the fundamental solution of

$$\Delta_1 u - Vu - u_t = 0$$

on $\mathbf{M}_1 \times (0, \infty)$ and G_0 be the fundamental solution of

$$\Delta_0 u - u_t = 0$$

on $\mathbf{M}_0 \times (0, \infty)$. It is easy to see that

$$G_2(x, \xi, t; x_0, \xi_0, 0) = G_1(x, t; x_0, 0) G_0(\xi, t; \xi_0, 0). \tag{7.6}$$

By Lemma 7.2, there exist α and C_3, which are independent of $\mathbf{M}_0$ such that

$$G_1(x, t; x_0, 0) \geq \frac{C_3}{t^\alpha |B(x_0, t^{1/2})|}. \tag{7.7}$$

Since G_0 is the free heat kernel of $\mathbf{M}_0$, we have

$$G_0(\xi_0, t; \xi, 0) = \int_{\mathbf{M}_0} G_0(\xi_0, t; \eta, 1) G(\eta, 1; \xi, 0) d\eta.$$

As $\mathbf{M}_0$ is a compact manifold, we know that $\min_{\eta, \xi \in \mathbf{M}_0} G(\eta, 1; \xi, 0) \geq C_4 > 0$, which shows, for $t \geq 1$,

$$G_0(\xi_0, t; \xi, 0) \geq C_4 \int_{\mathbf{M}_0} G_0(\xi_0, t; \eta, 1) d\eta = C_4. \tag{7.8}$$

Combining (7.6)-(7.8) and using the fact that $|B(x_0, t^{1/2})| \leq C t^{n/2}$, we obtain

$$G_2(x_0, \xi_0, t; x, \xi, 0) \geq C_\delta / t^{\alpha + (n/2)}. \text{ q.e.d.}$$

Now we are ready to give the

Proof of Theorem D. For a positive integer m, let $\mathbf{M}_2 = \mathbf{M} \times S^1 \times ... \times S^1$ equipped with the metric prescribed in the theorem. Here there are m S^1 in the product. Then the dimension of $\mathbf{M}_2$ is $n + m$ and the exponent in the nonlinear term of the Yamabe equation is $\frac{n+m+2}{n+m-2}$. Let us consider the following semilinear parabolic equation on $\mathbf{M}_2 \times (0, \infty)$.

$$\Delta_2 u - \frac{n+m-2}{4(n+m-1)} R u + u^p - u_t = 0. \tag{7.9}$$

Note that R is also the scalar curvature of $\mathbf{M}_2$ since $S^1 \times ... \times S^1$ is Ricci flat. By our assumption on the scalar curvature, we have, for $x \in \mathbf{M}$,

$$0 \leq \frac{n+m-2}{4(n+m-1)} R(x) \leq \frac{1}{4} R(x) \leq \frac{C_0}{1 + d^2(x, x_0)}.$$

By Lemma 7.3 (taking $V = \frac{n+m-2}{4(n+m-1)} R(x)$) , we can find a constant α, which is independent of m and another constant C_δ, such that when $t \geq 1$, $d(x_0, x) \leq \delta$ and $d_0(\xi_0, \xi) \leq \delta$,

$$G_2(x_0, \xi_0, t; x, \xi, 0) \geq C_\delta / t^{\alpha+(n/2)}. \tag{7.10}$$

Here $x, x_0 \in \mathbf{M}$ and $\xi, \xi_0 \in S^1 \times ... \times S^1$. We emphasize that the independence of α with respect to m is crucial since we will select a large m in the end. Obviously G_2 has a Gaussian upper bound since $R \geq 0$. Now we can follow the proof of Theorem B part (b) to show the following: if $p < 1 + \frac{2}{2\alpha+n}$ then (7.9) does not have global positive solutions. The only change in the proof is to replace $|B(0, T^{1/2})|$ in (5.5)-(5.8) by $T^{\alpha+(n/2)}$. Indeed, if $u = u(x, \xi, t)$ is a solution of (7.9), then

$$\begin{aligned} & u(x,t) \\ & \geq \int_{\mathbf{M}_2} G_2(x, \xi, t; y, \eta, 0) u_0(y, \eta) dy d\eta + \int_0^t \int_{\mathbf{M}_2} G_2(x, \xi, t; y, \eta, s) u^p(y, \eta, s) dy d\eta ds, \end{aligned}$$

for all $(x, \xi, t) \in \mathbf{M}_2 \times [0, \infty)$.

Since $R \geq 0$ we know that , $\int_{\mathbf{M}_2} G_2(y, \eta, T; x_0, \xi_0, s) dy d\eta \leq C$. As in section 5, using the above inequality, the symmetry of the heat kernel and Hölder's inequality, we obtain, for $T > 1$ and $T > t$,

$$\begin{aligned} & \int_{\mathbf{M}_2} G_2(x, \xi, T; x_0, \xi_0, t) u(x, \xi, t) dx d\xi \geq C \int_{\mathbf{M}_2} G_2(y, \eta, T; x_0, \xi_0, 0) u_0(y, \eta) dy d\eta \\ & + C \int_0^t [\int_{\mathbf{M}_2} G_2(y, \eta, T; x_0, \xi_0, s) u(y, \eta, s) dy d\eta]^p ds. \end{aligned} \tag{7.11}$$

Without loss of generality we assume that u_0 is strictly positive in a neighborhood of (x_0, ξ_0). Using the lower bound in (7.10) for G_2, we can then find a constant $C > 0$ so that, for $T > 1$,

$$\int_{\mathbf{M}_2} G(y, \eta, T; x_0, \xi_0, 0) u_0(y, \eta) dy d\eta \geq \frac{C}{T^{\alpha+(n/2)}}.$$

Going back to (7.11) and writing $J(t) \equiv \int_{\mathbf{M}_2} G(x, \xi, T; x_0, \xi_0, t) u(x, \xi, t) dx d\xi$, we have $J(t) \geq C/T^{\alpha+(n/2)} + C \int_0^t J^p(s) ds, \quad T > t, T > 1$. Using the notation $g(t) \equiv \int_0^t J^p(s) ds$, we obtain, as in section 5,

$$g'(t)/(T^{-\alpha-(n/2)} + g(t))^p \geq C. \tag{7.12}$$

Integrating (7.12) from 0 to T and noticing $g(0) = 0$, we have $|T^{\alpha+(n/2)}|^{p-1} \geq (p-1)CT$, for all $T > 1$. Therefore global positive solutions can not exist if $p < 1 + \frac{2}{2\alpha+n}$.

Since α is independent of m we can choose m sufficiently large so that

$$\frac{n+m+2}{n+m-2} < 1 + \frac{2}{2\alpha+n}.$$

Hence the Yamabe equation

$$\Delta_2 u - \frac{n+m-2}{4(n+m-1)} R u + u^{\frac{n+m+2}{n+m-2}} = 0$$

can not have global positive solution. q.e.d.

We conclude by pointing out another impact of the nonexistence result on the noncompact Yamabe problem. Since we have shown that there are noncompact manifolds with positive scalar curvature, which are not conformal to manifolds with positive constant scalar curvature, the problem of prescribing zero scalar curvature should now be considered seriously. Some progress has been made in [ZZ].

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