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ON PERIODIC BOUNDARY VALUE PROBLEMS OF FIRST-ORDER PERTURBED IMPULSIVE DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we present an existence result for a first order impulsive differential inclusion with periodic boundary conditions and impulses at the fixed times under the convex condition of multi-functions.

1. Introduction

In this paper, we study the existence of solutions to a periodic nonlinear boundary value problems for first order Carathéodory impulsive ordinary differential inclusions with convex multi-functions. Given a closed and bounded interval $J := [0, T]$ in $\mathbb{R}$, the set of real numbers, and given the impulsive moments $t_1, t_2, \dots, t_p$ with $0 = t_0 < t_1 < t_2 < \dots < t_p < t_{p+1} = T$, $J' = J \setminus \{t_1, t_2, \dots, t_p\}$, $J_j = (t_j, t_{j+1})$, consider the following periodic boundary-value problem for impulsive differential inclusions (in short IDI):

$$x'(t) \in F(t, x(t)) + G(t, x(t)) \text{ a.e. } t \in J', \tag{1.1}$$

$$x(t_j^+) = x(t_j^-) + I_j(x(t_j^-)), \tag{1.2}$$

$$x(0) = x(T), \tag{1.3}$$

where $F, G : J \times \mathbb{R} \to P_f(\mathbb{R})$ are impulsive multi-functions, $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \dots, p$ are the impulse functions and $x(t_j^+)$ and $x(t_j^-)$ are respectively the right and the left limit of x at $t = t_j$.

Let $C(J, \mathbb{R})$ and $L^1(J, \mathbb{R})$ denote the space of continuous and Lebesgue integrable real-valued functions on J. Consider the Banach space

$$X := \{x : J \to \mathbb{R} : x \in C(J', \mathbb{R}), x(t_j^+), x(t_j^-) \text{ exist}, \ x(t_j^-) = x(t_j), \ j = 1, 2, \dots, p\}$$

equipped with the norm $\|x\| = \max\{|x(t)| : t \in J\}$, and the space

$$Y := \{x \in X : x \text{ is differentiable a.e. on } (0, T), x' \in L^1(J, \mathbb{R})\}.$$

By a solution of (1.1)–(1.3), we mean a function x in $Y_T := \{v \in Y : v(0) = v(T)\}$ that satisfies the differential inclusion (1.1), and the impulsive conditions (1.2).

Several papers have been devoted to the study of initial and boundary value problems for impulsive differential inclusions (see for example [2, 3]). Some basic

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results in the theory of periodic boundary value problems for first order impulsive differential equations may be found in [12, 13, 14] and the references therein. Also, for a general theory on impulsive differential equations we refer the interested reader to [15] and the monographs [10] and [16]. Our aim is to provide sufficient conditions on the multifunctions F, G and the impulsive functions I_j, that insure the existence of solutions of problem IDI (1.1)–(1.3).

2. Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space and let $P_f(E)$ denote the class of all non-empty subsets of E with the property f. Thus $P_{cl}(E), P_{bd}(E), P_{cv}(E)$ and $P_{cp}(E)$ denote respectively the classes of all closed, bounded, convex and compact subsets of E. Similarly $P_{cl,cv,bd}(E)$ and $P_{cp,cv}(E)$ denote the classes of all closed, convex and bounded and compact and convex subsets of E. For $x \in E$ and $Y, Z \in P_{bd,cl}(E)$ we denote by $D(x, Y) = \inf\{\|x - y\| : y \in Y\}$, and $\rho(Y, Z) = \sup_{a \in Y} D(a, Z)$.

Define a function $H : P_{bd,cl}(E) \times P_{bd,cl}(E) \to \mathbb{R}^+$ by

$$H(A, B) = \max\{\rho(A, B, \rho(B, A)\}.$$

The function H is called a Hausdorff metric on E. Note that $\|Y\| = H(Y, \{0\})$.

A map $F : E \to P(E)$ is called a *multi-valued mapping* on E. A point $u \in E$ is called a *fixed point* of the multi-valued operator $F : E \to P(E)$ if $u \in F(u)$. The fixed points set of F will be denoted by $\mathrm{Fix}(F)$.

A multivalued map $F : [a, b] \subset \mathbb{R} \to P_{cl,bd}(E)$ is said to be *measurable* if for each $x \in X$, the distance between x and $F(t)$ is a measurable function on $[a, b]$. A function $f : [a, b] \to E$ is called *measurable selector* of the multi-function F if f is measurable and $f(t) \in F(t)$ for almost everywhere $t \in [a, b]$.

Definition 2.1. Let $F : E \to P_{bd,cl}(E)$ be a multi-valued operator. Then F is called a multi-valued contraction if there exists a constant $\alpha \in (0, 1)$ such that for each $x, y \in E$ we have

$$H(F(x), F(y)) \le \alpha\|x - y\|.$$

The constant α is called a contraction constant of F.

A multifunction F is called *upper semi-continuous (u.s.c.)* if for each $x_0 \in E$, the set $F(x_0)$ is a nonempty and closed subset of E, and for each open set $N \subset E$ containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M) \subset N$. If F is nonempty and compact-valued, then F is u.s.c. if and only if F has a closed graph, i.e., given sequences $\{x_n\}_{n=1}^{\infty} \to x_0, \{y_n\}_{n=1}^{\infty} \to y_0\,, y_n \in F(x_n)$ for every $n = 1, 2, \dots$ imply $y_0 \in F(x_0)$.

F is *bounded on bounded sets* if $\bigcup F(S)$ is bounded in E for every bounded set $S \subset E$, i.e., $\sup_{x \in S}\{\sup\{|y| : y \in F(x)\}\} < +\infty$. Again the operator F is called *compact* if $\overline{\bigcup F(E)}$ is a compact subset of E. F is said to be *completely continuous* if it is u.s.c. and $\bigcup F(S)$ is relatively compact set in E for every bounded subset S of E. Finally a multi-valued operator F is called *convex (resp. compact) valued* if $F(x)$ is a convex (resp. compact) set in E for each $x \in E$.

The following form of a fixed point theorem of Dhage [6] will be used while proving our main existence result.

Theorem 2.1 (Dhage [6]). *Let $B(0,r)$ and $B[0,r]$ denote respectively the open and closed balls in a Banach space E centered at origin and of radius r and let $A : E \to P_{cl,cv,bd}(E)$ and $B : B[0,r] \to P_{cp,cv}(E)$ be two multi-valued operators satisfying*

(i) *A is multi-valued contraction , and*
(ii) *B is completely continuous.*

Then either

(a) *the operator inclusion $x \in Ax + Bx$ has a solution in $B[0,r]$, or*
(b) *there exists an $u \in E$ with $\|u\| = r$ such that $\lambda u \in Au + Bu$ for some $\lambda > 1$.*

In the following section we prove the main existence results of this paper.

3. Main Results

Consider the following linear periodic problem with some given impulses $\theta_j \in \mathbb{R}$, $j = 1, 2, \dots, p$:

$$x'(t) + kx(t) = \sigma(t), \text{a.e. } t \in J', \tag{3.1}$$

$$x(t_j^+) - x(t_j^-) = \theta_j, j = 1, 2, \dots, p, \tag{3.2}$$

$$x(0) = x(T), \tag{3.3}$$

where $k > 0$, and $\sigma \in L^1(J)$. The solution of (3.1)–(3.3) is given by (see [12, Lemma 2.1])

$$x(t) = \int_0^T g_k(t,s)\sigma(s)\,ds + \sum_{j=1}^{p} g_k(t,t_j)\theta_j, \tag{3.4}$$

where

$$g_k(t,s) = \begin{cases} \dfrac{e^{-k(t-s)}}{1-e^{-kT}}, & 0 \le s \le t \le T \\ \dfrac{e^{-k(T+t-s)}}{1-e^{-kT}}, & 0 \le t < s \le T\,. \end{cases}$$

Clearly the function $g_k(t,s)$ is discontinuous and nonnegative on $J \times J$ and has a jump at $t = s$.

Let

$$M_k := \max\{|g_k(t,s)| : t, s \in [0,T]\} = \frac{1}{1-e^{-kT}}.$$

Now $x \in Y_T$ is a solution of(1.1)–(1.3) if and only if

$$x(t) \in B_k^1 x(t) + B_k^2 x(t), \quad t \in J \tag{3.5}$$

where the multi-valued operators B_k^1 and B_k^2 are defined by

$$\mathcal{B}_k^1 x(t) = \int_0^T g_k(t,s)F(s,x(s))\,ds, \tag{3.6}$$

$$\mathcal{B}_k^2 x(t) = \int_0^T g_k(t,s)[kx(s) + G(s,x(s))]\,ds + \sum_{j=1}^{p} g(t,t_j)I_j(x(t_j^-)). \tag{3.7}$$

Definition 3.1. A multi-function $\beta : J \times \mathbb{R} \to P_f(\mathbb{R})$ is called an impulsive Carathéodory if

(i) $\beta(\cdot,x)$ is measurable for every $x \in \mathbb{R}$ and
(ii) $\beta(t,\cdot)$ is upper semi-continuous a.e. on J.

Further the impulsive Carathéodory multifunction β is called impulsive L^1-Carathéodory if

(iii) for every $r > 0$ there exists a function $h_r \in L^1(J)$ such that
$$\|\beta(t,x)\| = \sup\{|u| : u \in \beta(t,x)\} \le h_r(t) \text{a.e. } t \in J$$
for all $x \in \mathbb{R}$ with $|x| \le r$.

Denote
$$S^1_\beta(x) = \{v \in L^1(J,\mathbb{R}) : v(t) \in \beta(t,x) \text{ a.e. } t \in J\}.$$

Lemma 3.1 (Lasota and Opial [11]). *Let E be a Banach space. Further if* $\dim(E) < \infty$ *and $\beta : J \times E \to P_{bd,cl}(E)$ is L^1-Carathéodory, then $S^1_\beta(x) \neq \emptyset$ for each $x \in E$.*

Definition 3.2. A measurable multi-valued function $F : J \to P_{cp}(\mathbb{R})$ is said to be integrably bounded if there exists a function $h \in L^1(J,\mathbb{R})$ such that $|v| \le h(t)$ a.e. $t \in J$ for all $v \in F(t)$.

Remark 3.1. It is known that if $F : J \to \mathbb{R}$ is an integrably bounded multi-function, then the set S^1_F of all Lebesgue integrable selections of F is closed and non-empty. See Covitz and Nadler [4].

We now introduce the following assumptions:

(H1) The functions $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1,2,\dots,p$ are continuous, and there exist $c_j \in \mathbb{R}$, $j = 1,2,\dots,p$ such that $|I_j(x)| \le c_j$, $j = 1,2,\dots,p$ for every $x \in \mathbb{R}$.

(H2) $G : J \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ is an impulsive Carathéodory multi-function.

(H3) There exist a real number $k > 0$ and a Carathéodory function $\omega : J \times \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing with respect to its second argument such that
$$\|G(t,x) + kx\| = \sup\{|v| : v \in G(t,x) + kx\} \le \omega(t,|x|)$$
a.e. $t \in J', x \in \mathbb{R}$.

(H4) The multi-function $t \mapsto F(t,x)$ is measurable and integrally bounded for each $x \in \mathbb{R}$.

(H5) The multi-function $F(t,x)$ is $F : J \times \mathbb{R} \to P_{cl,cv,bd}(\mathbb{R})$ and there exists a function $\ell \in L^1(J,\mathbb{R})$ such that
$$H(F(t,x),F(t,y)) \le \ell(t)|x-y| \quad \text{a.e. } t \in J$$
for all $x, y \in \mathbb{R}$.

Note that the hypotheses (H1)–(H5) are not new, they have been used extensively in the literature on differential inclusions. Also (H3) in the special case $\omega(t,r) = \phi(t)\psi(r)$ has been used by several authors. See Dhage [6] and the references therein.

Lemma 3.2. *Assume that (H2)–(H3) hold. Then the operator $S^1_{k+G} : Y_T \to P_f(L^1(J,\mathbb{R}))$ defined by*
$$S^1_{k+G}(x) := \big\{v \in L^1(J,\mathbb{R}) : v(t) \in kx(t) + G(t,x(t)) \text{ a.e. } t \in J\big\} \tag{3.8}$$
is well defined, u.s.c., closed and convex valued, and sends bounded subsets of Y_T into bounded subsets of $L^1(J,\mathbb{R})$.

Proof. Since (H2) holds, by Lemma 3.1 $S^1_{k+G}(x) \neq \emptyset$ for each $x \in Y_T$. Below we show that S^1_{k+G} has the desired properties on Y_T.

Step I: First we show that S^1_{k+G} has closed values on Y_T. Let $x \in Y_T$ be arbitrary and let $\{\omega_n\}$ be a sequence in $S^1_{k+G}(x) \subset L^1(J,\mathbb{R})$ such that $\omega_n \to \omega$. Then $\omega_n \to \omega$

in measure. So there exists a subset S of positive integers such that $\omega_n \to \omega$ a.e. $n \to \infty$ through S. Since the hypothesis (H2) holds, we have $\omega \in S^1_{k+G}(x)$. Therefore, $S^1_{k+G}(x)$ is a closed set in $L^1(J,\mathbb{R})$. Thus for each $x \in Y_T$, $S^1_{k+G}(x)$ is a non-empty, closed subset of $L^1(J,\mathbb{R})$ and consequently S^1_{k+G} has non-empty and closed values on Y_T.

Step II: Next we show that $S^1_{k+G}(x)$ is convex subset of $L^1(J,\mathbb{R})$ for each $x \in Y_T$. Let $v_1, v_2 \in S^1_{k+G}(x)$ and let $\lambda \in [0,1]$. Then there exist functions $f_1, f_2 \in S^1_{k+G}(x)$ such that

$$v_1(t) = kx(t) + f_1(t) \quad \text{and} \quad v_2(t) = kx(t) + f_2(t)$$

for $t \in J$. Therefore we have

$$\begin{aligned}
\lambda v_1(t) + (1-\lambda)v_2(t) &= \lambda\big[kx(t) + f_1(t)\big] + (1-\lambda)\big[kx(t) + f_2(t)\big] \\
&= \lambda kx(t) + (1-\lambda)kx(t) + \lambda f_1(t) + (1-\lambda)f_2(t) \\
&= kx(t) + f_3(t)
\end{aligned}$$

where $f_3(t) = \lambda f_1(t) + (1-\lambda)f_2(t)$ for all $t \in J$. Since $G(t,x)$ is convex for each $x \in \mathbb{R}$, one has $f_3(t) \in G(t,x(t))$ for all $t \in J$. Therefore,

$$\lambda v_1(t) + (1-\lambda)v_2(t) \in kx(t) + G(t,x(t))$$

for all $t \in J$ and consequently $\lambda v_1 + (1-\lambda)v_2 \in S^1_{k+G}(x)$. As a result $S^1_{k+G}(x)$ is a convex subset of $L^1(J,\mathbb{R})$.

Step III: Next we show that S^1_{k+G} is an u.s.c. multi-valued operator on Y_T. Let $\{x_n\}$ be a sequence in Y_T such that $x_n \to x_*$ and let $\{y_n\}$ be a sequence such that $y_n \in S^1_{k+G}(x_n)$ and $y_n \to y_*$. To finish, it suffices to show that $y_* \in S^1_{k+G}(x_*)$. Since $y_n \in S^1_{k+G}(x_n)$, there is a function $f_n \in S^1_{k+G}(x_n)$ such that $y_n(t) = kx_n(t) + f_n(t)$ for all $t \in J$ and that $y_*(t) = kx_*(t) + f_*(t)$, where $f_n \to f_*$ as $n \to \infty$. Now the multi-function $G(t,x)$ is an upper semi-continuous in x for all $t \in J$, one has $f_*(t) \in G(t,x_*(t))$ for all $t \in J$. Hence it follows that $y_* \in S^1_{k+G}(x_*)$.

Step IV: Finally we show that S^1_{k+G} maps bounded sets of Y_T into bounded sets of $L^1(J,\mathbb{R})$. Let M be a bounded subset of Y_T. Then there is a real number $r > 0$ such that $\|x\| \le r$ for all $x \in M$. Let $y \in S^1_{k+G}(S)$ be arbitrary. Then there is an $x \in M$ such that $y \in S^1_{k+G}(x)$ and therefore $y(t) \in kx(t) + G(t,x(t))$ a.e. $t \in J$. Now by (H3),

$$\begin{aligned}
\|y\|_{L^1} &= \int_0^T |y(t)|\,dt \\
&\le \int_0^T \|kx(t) + G(t,x(t))\|\,dt \\
&\le \int_0^T \omega(t,|x(t|)\,dt \\
&\le \int_0^T \omega(t,r)\,dt.
\end{aligned}$$

Hence $S^1_{k+G}(S)$ is a bounded set in $L^1(J,\mathbb{R})$.

Thus the multi-valued operator S^1_{k+G} is an upper semi-continuous and has closed, convex values on Y_T. The proof is complete. □

Lemma 3.3. *Assume $(H_1)-(H_3)$. The multivalued operator $\mathcal{B}^2_k$ defined by (3.7) is completely continuous and has convex, compact values on Y_T.*

Proof. Since S^1_{k+G} is as upper semi-continuous and has closed and convex values and since (H1) holds, $\mathcal{B}^2_k$ is u.s.c. and has closed-convex values on Y_T. To show $\mathcal{B}^2_k$ is relatively compact, we use the Arzelá-Ascoli theorem. Let $M \subset B[0,r]$ be any set. Then $\|x\| \le r$ for all $x \in M$. First we show that $\mathcal{B}^2_k(M)$ is uniformly bounded. Now for any $x \in M$ and for any $y \in \mathcal{B}^2_k(x)$ one has

$$\begin{aligned}
|y(t)| &\le \int_0^T |g_k(t,s)|\|[kx(s)+G(s,x(s))]\|\,ds + \sum_{j=1}^p |g_k(t,t_j)||I_j(x(t_j^-))| \\
&\le \int_0^T M_k\omega(s,|x(s)|)\,ds + M_k\sum_{j=1}^p c_j \\
&\le M_k\int_0^T \omega(s,r)\,ds + M_k\sum_{j=1}^p c_j,
\end{aligned}$$

where M_k is the bound of g_k on $[0,T]\times[0,T]$. Taking the supremum over t,

$$\|\mathcal{B}^2_k x\| \le M_k\Big[\int_0^T \omega(s,r)\,ds + \sum_{j=1}^p c_j\Big]$$

for all $x \in M$. Hence $\mathcal{B}^2_k(M)$ is a uniformly bounded set in Y_T. Next we prove the equi-continuity of the set $\mathcal{B}^2_k(M)$ in Y_T. Let $y \in B^2_k(M)$ be arbitrary. Then there is a $v \in S_{k+G}(x)$ such that

$$y(t) = \int_0^T g_k(t,s)v(s)\,ds + \sum_{j=1}^p g_k(t,t_j)I_j(x(t_j^-)), \quad t \in J,$$

for some $x \in M$.

To finish, it is sufficient to show that y' is bounded on $[0,T]$. Now for any $t \in [0,T]$,

$$\begin{aligned}
|y'(t)| &\le \Big|\int_0^T \frac{\partial}{\partial t}g_k(t,s)v(s)\,ds + \sum_{j=1}^p \frac{\partial}{\partial t}g_k(t,t_k)I_j(y_j(t_j^-))\Big| \\
&= \Big|\int_0^T (-k)g_k(t,s)v(s)\,ds + \sum_{j=1}^p (-k)g_k(t,t_k)I_j(y_j(t_j^-))\Big| \\
&\le kM_k\int_0^T \omega(s,r)\,ds + kM_k\sum_{j=1}^p c_j = c.
\end{aligned}$$

Hence for any $t,\tau \in [0,T]$ and for all $y \in B^2_k(M)$ one has

$$|y(t)-y(\tau)| \le c|t-\tau| \to 0 \quad \text{as} \quad t \to \tau.$$

This shows that $\mathcal{B}^2_k(M)$ is a equi-continuous set and consequently relatively compact in view of Arzelá-Ascoli theorem. Obviously $\mathcal{B}^2_k(x) \subset \mathcal{B}^2_k(B[0,r])$ for each $x \in B[0,r]$. Since $\mathcal{B}^2_k(B[0,r])$ is relatively compact, $\mathcal{B}^2_k(x)$ is relatively compact and which is compact in view of hypothesis (H2). Hence $\mathcal{B}^2_k$ is a completely continuous multi-valued operator on Y_T. The proof of the lemma is complete. □

Lemma 3.4. *Assume that the hypotheses (H4)–(H5) hold. Then the operator* B_k^1 *defined by* (3.6) *is a multi-valued contraction operator on* Y_T*, provided* $M_k\|\ell\|_{L^1} < 1$.

Proof. Define a mapping $\mathcal{B}_k^1 : Y_T \to Y_T$ by (3.6). We show that $\mathcal{B}_k^1$ is a multi-valued contraction on Y_T. Let $x, y \in Y_T$ be arbitrary and let $u_1 \in \mathcal{B}_k^1(x)$. Then $u_1 \in Y_T$ and

$$u_1(t) = \int_0^T g_k(t,s)v_1(s)\,ds$$

for some $v_1 \in S_F^1(x)$. Since $H(F(t,x(t)), F(t,y(t)) \leq \ell(t)|x(t)-y(t)|$, one obtains that there exists a $w \in F(t,y(t))$ such that

$$|v_1(t) - w| \leq \ell(t)|x(t) - y(t)|.$$

Thus the multi-valued operator U defined by $U(t) = S_F^1(y)(t) \cap K(t)$,where

$$K(t) = \{w \mid |v_1(t) - w| \leq \ell(t)|x(t) - y(t)|\}$$

has nonempty values and is measurable. Let v_2 be a measurable selection for U (which exists by Kuratowski-Ryll-Nardzewski's selection theorem. See [3]). Then $v_2 \in F(t, y(t))$ and

$$|v_1(t) - v_2(t)| \leq \ell(t)|x(t) - y(t)| \quad \text{a.e. } t \in J.$$

Define

$$u_2(t) = \int_0^T g_k(t,s)v_2(s)\,ds.$$

It follows that $u_2 \in \mathcal{B}_k^1(y)$ and

$$\begin{aligned}
|u_1(t) - u_2(t)| &\leq \Big|\int_0^T g_k(t,s)v_1(s)\,ds - \int_0^T g_k(t,s)v_2(s)\,ds\Big| \\
&\leq \int_0^T M_k|v_1(s) - v_2(s)|\,ds \\
&\leq \int_0^T M_k\ell(s)|x(s) - y(s)|\,ds \\
&\leq M_k\|\ell\|_{L^1}\|x - y\|.
\end{aligned}$$

Taking the supremum over t, we obtain

$$\|u_1 - u_2\| \leq M_k\|\ell\|_{L^1}\|x - y\|.$$

From this and the analogous inequality obtained by interchanging the roles of x and y we get that

$$H(\mathcal{B}_k^1(x), \mathcal{B}_k^1(y)) \leq \mu\|x - y\|,$$

for all $x, y \in Y_T$. This shows that $\mathcal{B}_k^1$ is a multi-valued contraction, since $\mu = M_k\|\ell\|_{L^1} < 1$. □

Theorem 3.1. *Assume (H1)–(H5) are satisfied. Further if there exists a real number* $r > 0$ *such that*

$$r > \frac{M_k \int_0^T \omega(s,r)\,ds + M_k F_0 + M_k \sum_{j=1}^p c_j}{1 - M_k\|\ell\|_{L^1}} \tag{3.9}$$

where $M_k\|\ell\|_{L^1} < 1$ *and* $F_0 = \int_0^T \|F(s,0)\|\,ds$*, then the problem IDI (1.1)–(1.3) has at least one solution on* J.

Proof. Define an open ball $B(0,r)$ in Y_T, where the real number r satisfies the inequality given in condition (3.9). Define the multi-valued operators $\mathcal{B}_k^1$ and $\mathcal{B}_k^2$ on Y_T by (3.6) and (3.7). We shall show that the operators $\mathcal{B}_k^1$ and $\mathcal{B}_k^2$ satisfy all the conditions of Theorem 2.1.

Step I: The assumptions (H2)–(H3) imply by Lemma 3.3 that $\mathcal{B}_k^2$ is completely continuous multi-valued operator on $B[0,r]$. Again since (H4)–(H5) hold, by Lemma 3.4, $\mathcal{B}_k^1$ is a multi-valued contraction on Y_T with a contraction constant $\mu = M_k\|\ell\|_{L^1}$. Now an application of Theorem 2.1 yields that either the operator inclusion $x \in \mathcal{B}_k^1 x + \mathcal{B}_k^2 x$ has a solution in $B[0,r]$, or, there exists an $u \in Y_T$ with $\|u\| = r$ satisfying $\lambda u \in B_k^1 u + B_k^2 u$ for some $\lambda > 1$.

Step II: Now we show that the second assertion of Theorem 2.1 is not true. Let $u \in Y_T$ be a possible solution of $\lambda u \in B_k^1 u + B_k^2 u$ for some real number $\lambda > 1$ with $\|u\| = r$. Then we have,

$$\begin{aligned} u(t) \in \lambda^{-1} \int_0^T g_k(t,s)F(s,u(s))\,ds + \lambda^{-1} \int_0^T g_k(t,s)[ku(s) + G(s,u(s))]\,ds \\ + \lambda^{-1} \sum_{j=1}^p g_k(t,t_j) I_j(u(t_j^-))\,. \end{aligned}$$

Hence by (H3)-(H5),

$$\begin{aligned} |u(t)| &\le \int_0^T |g_k(t,s)|\omega(s,|u(s)|)\,ds + \int_0^T |g_k(t,s)||\ell(s)||u(s)|\,ds \\ &\quad + \int_0^T |g_k(t,s)|\|F(s,0)\|\,ds + \sum_{j=1}^p |g_k(t,s)||I_j(u(t_j^-)| \\ &\le M_k \int_0^T \omega(s,\|u\|)\,ds + M_k \int_0^T |\ell(s)|\|u\|\,ds + M_k F_0 + M_k \sum_{j=1}^p c_j \\ &\le M_k \int_0^T \omega(s,\|u\|)\,ds + M_k\|\ell\|_{L^1}\|u\| + M_k F_0 + M_k \sum_{j=1}^p c_j. \end{aligned}$$

Taking the supremum over t we get

$$\|u\| \le M_k \int_0^T \omega(s,\|u\|)\,ds + M_k\|\ell\|_{L^1}\|u\| + M_k F_0 + M_k \sum_{j=1}^p c_j.$$

Substituting $\|u\| = r$ in the above inequality yields

$$r \le \frac{M_k \int_0^T \omega(s,r)\,ds + M_k F_0 + M_k \sum_{j=1}^p c_j}{1 - M_k\|\ell\|_{L^1}}$$

which is a contradiction to (3.9). Hence the operator inclusion $x \in \mathcal{B}_k^1 x + \mathcal{B}_k^2 x$ has a solution in $B[0,r]$. This further implies that the IDI (1.1)–(1.3) has a solution on J. The proof is complete. □

Remark 3.2. On taking $F(t,x) \equiv 0$ on $J' \times \mathbb{R}$ in Theorem 3.1 we obtain as a special case the existence result in [7] for the impulsive differential inclusion (1.1)–(1.3) with $F(t,x) \equiv 0$.

Remark 3.3. *In this paper we have dealt with the perturbed impulsive differential inclusions involving convex multi-functions. Note that the continuity of the multi-function is important here, however in a forthcoming paper we will relax the continuity condition of one of the multi-functions and discuss the existence results for mild discontinuous perturbed impulsive differential inclusions.*

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