Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 102, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NON-TRIVIAL SOLUTIONS OF FRACTIONAL SCHRÖDINGER-POISSON SYSTEMS WITH SUM OF PERIODIC AND VANISHING POTENTIALS

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Abstract. We consider the fractional Schrödinger-Poisson system

$$(-\Delta)^{\alpha} u + V(x)u + K(x)\Phi(x)u = f(x,u) - \Gamma(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^3,$$
$$(-\Delta)^{\beta}\Phi = K(x)u^2 \quad \text{in } \mathbb{R}^3,$$

where $\alpha, \beta \in (0,1]$, $4\alpha + 2\beta > 3$, $4 \le q < 2^*_{\alpha}$, $K(x)$, $\Gamma(x)$ and $f(x,u)$ are periodic in x, V is coercive or $V = V_{\rm per} + V_{\rm loc}$ is a sum of a periodic potential $V_{\rm per}$ and a localized potential $V_{\rm loc}$. If f has the subcritical growth, but higher than $\Gamma(x)|u|^{q-2}u$, we establish the existence and nonexistence of ground state solutions are dependent on the sign of $V_{\rm loc}$. Moreover, we prove that such a problem admits infinitely many pairs of geometrically distinct solutions provided that V is periodic and f is odd in u. Finally, we investigate the existence of ground state solutions in the case of coercive potential V.

1. Introduction

This article concerns the nonlinear fractional Schöding-Poisson system

$$\begin{gathered}(-\Delta)^{\alpha} u + V(x)u + K(x)\phi(x)u = f(x,u) - \Gamma(x)|u|^{q-2}u, \quad \text{in } \mathbb{R}^3,\\ (-\Delta)^{\beta}\phi = K(x)u^2, \quad \text{in } \mathbb{R}^3,\end{gathered} \tag{1.1}$$

where $\alpha, \beta \in (0,1]$, $4\alpha + 2\beta > 3$, the fractional Laplacian $(-\Delta)^{\alpha}(\alpha \in (0,1))$ can be defined by the Fourier transform $(-\Delta)^{\alpha} = \mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}u)$, $\mathcal{F}u$ being the usual Fourier transform in $\mathbb{R}^3$. System (1.1) stems from not only an expansion of the Feynman path integral from Brownian-like to Levy-like quantum mechanical paths [17, 18], but also a system of identically charged particles interacting with each other in the case when magnetic effects can be neglected [3].

When $\alpha = \beta = 1$, system (1.1) reduces to the Schrödinger-Poisson system

$$\begin{gathered}-\Delta u + V(x)u + K(x)\phi(x)u = f(x,u) - \Gamma(x)|u|^{q-2}u, \quad \text{in } \mathbb{R}^3\\ -\Delta\phi = K(x)u^2, \quad \text{in } \mathbb{R}^3\end{gathered} \tag{1.2}$$

which has important physical meanings because it appears in quantum mechanical models (see [19]) and in semiconductor theory [4, 20]. During the past few years, there has been increasing attention to problems like (1.2) on the existence of ground

2010 *Mathematics Subject Classification.* 35B38, 35G99.
Key words and phrases. Fractional Schrödinger-Poisson system; coercive potential; periodic and localized potential; Nehari manifold; variational method.

Submitted March 22, 2018. Published September 4, 2019.

state solutions, positive solutions, multiple solutions and so on. For more details in studying on these kinds of problems, we refer the readers to [1, 2, 7, 15, 22, 27, 32, 33, 39] and the references therein.

When $\phi \equiv 0$, system (1.1) reduces to the fractional Schrödinger problem

$$(-\Delta)^{\alpha} u + V(x)u = f(x,u) - \Gamma(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

which appears in different areas of mathematical physics. For fractional Schrödinger problems like (1.3), pioneered from [12] and [14] via variational methods, there have been many works in the existence and multiplicity of solutions, such as [6, 30, 36]. For other related problems, we can refer to [9, 10, 23, 31, 24, 28, 36, 11]. At least in our knowledge, however, the only paper considering problem (1.3) is Bieganowski [6], where the authors assumed that

- (A1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable, $\mathbb{Z}^N-$ periodic in $x \in \mathbb{R}^N$ and continuous in $u \in \mathbb{R}^N$ a.e. $x \in \mathbb{R}^N$, and there are $c > 0$, $2 \le q < p < 2_{\alpha}^* = \frac{2N}{N-2\alpha}$ such that
$$|f(x,u)| \le c(1 + |u|^{p-1}) \quad \forall u \in \mathbb{R},\ x \in \mathbb{R}^N;$$
- (A2) $f(x,u) = o(|u|)$ uniformly in x as $|u| \to 0$;
- (A3) $F(x,u)/|u|^q \to \infty$ uniformly in x as $|u| \to \infty$, where $F(x,u) = \int_0^u f(x,s)ds$ is the primitive of f with respect to u;
- (A4) $u \to f(x,u)/|u|^{q-1}$ is strictly increasing on $(-\infty, 0)$ and $(0, +\infty)$;
- (A5) $\Gamma \in L^{\infty}(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic in $x \in \mathbb{R}^N$, $\Gamma(x) \ge 0$ for a.e. $x \in \mathbb{R}^N$;
- (A6) $V_0 := \operatorname{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0$ for $0 < \alpha < 1$, $\inf \sigma(-\Delta + V(x)) > 0$ for $\alpha = 1$;
- (A7) $V \in C(\mathbb{R}^N, \mathbb{R})$ is such that $\lim_{|x|\to\infty} V(x) = \infty$ and $V_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$;
- (A8) $V = V_{\rm per} + V_{\rm loc}$, $V_{\rm per} \in L^{\infty}(\mathbb{R}^3)$ is $\mathbb{Z}^3$-periodic, $V_{\rm loc} \in L^{\infty}(\mathbb{R}^3)$ and $V_{\rm loc}(x) \to 0$ as $|x| \to \infty$.

To obtain the ground state solutions of (1.3), the authors first adopted the techniques in [34] to show that the corresponding Nehari manifold is a topological manifold homeomorphic with the unit sphere in the work space, where a minimizing sequence can be found. Then they use the abstract setting introduced [5] which extends some results from [24, 25] to indicate that such a minimizing sequence is bounded. Finally, they decomposed the bounded (PS) sequences including the above minimizing sequence to achieve their aim. The similar reduction techniques are successful used by other authors to study semilinear elliptic systems in [9] and Schrödinger-Poisson system in [37]. Bieganowski also applied the methods in [34] to obtain infinitely many pairs of geometrically distinct solutions of (1.3).

Presently, more attention has been paid to the study of problems concerning the fractional Schöding-Poisson system (1.1) without $\Gamma(x)|u|^{q-2}u$. And there has been a few results about these kinds of problems, we can refer to [8, 13, 16, 21, 35, 38]. Among them, Ji [16] considered the fractional Schrödinger-Poisson system

$$\begin{gathered} (-\Delta)^s u + V(x)u + \lambda\phi u = f(x,u), \quad \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.4}$$

where $\lambda \in \mathbb{R}^+$ is a parameter, $s, t \in (0,1)$ and $4s + 2t > 3$, $V(x)$ satisfies $(V_{\alpha}2)$ and f fulfills subcritical growth, 4-suplinear at infinity and monotonicity assumption. By constraint variational method and quantitative deformation lemma, Ji proved that the system (1.4) admits a least an energy sign-changing solution.

Motivated by the results mentioned previously, we study the non-trivial solutions of (1.1). In what follows, we assume that $N = 3$ in conditions (A5)–(A8) for convenience. To state our main results, we make the following assumptions:

(A9) $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is measurable, $\mathbb{Z}^3$-periodic in $x \in \mathbb{R}^3$ and continuous in $u \in \mathbb{R}^3$ a.e. $x \in \mathbb{R}^3$, and there are $c > 0$, $4 \le q < p < 2^*_\alpha = \frac{6}{3-2\alpha}$ such that
$$|f(x,u)| \le c(1 + |u|^{p-1}) \quad \forall u \in \mathbb{R},\ x \in \mathbb{R}^3;$$

(A10) $K \in L^\infty(\mathbb{R}^3)$ is $\mathbb{Z}^3$-periodic, $0 \le K(x) \le K_\infty$, for all $x \in \mathbb{R}^3$ and $K(x) \not\equiv 0$;

Now we state our main results.

Theorem 1.1. *Let $\alpha \in (0,1]$. Suppose that* (A2)–(A6), (A8)–(A10) *are satisfied and either $V_{\rm loc} \equiv 0$ or $V_{\rm loc}(x) < 0$ a.e. $x \in \mathbb{R}^3$. Then* (1.1) *has a ground state of Nehari type.*

Theorem 1.2. *Let $\alpha \in (0,1]$. Suppose that* (A2)–(A6), (A8)–(A10) *are satisfied and $\inf \sigma(-\Delta + V_{\rm per}(x)) > 0$. If $V_{\rm loc} > 0$ a.e. $x \in \mathbb{R}^3$, then* (1.1) *has no ground state solutions.*

Theorem 1.3. *Let $\alpha \in (0,1]$. Suppose that* (A2)–(A6), (A8)–(A10)*are satisfied, $V_{\rm loc} \equiv 0$ and f is odd in u. Then* (1.1) *admits infinitely many pairs $\pm u$ of geometrically distinct solutions.*

Theorem 1.4. *Let $\alpha \in (0,1]$. Suppose that* (A2)–(A5), (A7), (A9), (A10) *are satisfied. Then* (1.1) *has a ground state solution of Nehari type.*

The remainder of this articleis organized as follows. In Section 2, some preliminary results are presented. In Section $3-6$, we give the proof of our main results.

2. Variational setting and preliminaries

In this section, we introduce some necessary information to be used in this paper. We will denote by $\mathcal{F}u$ the usual Fourier transform of u.

Fractional Sobolev spaces are the convenient setting for our equations. A very detailed introduction about it can be found in [26], we offer a review below.

We see that the fractional Sobolev space $W^{\alpha,p}(\mathbb{R}^3)$ is defined for any $p \in [1,+\infty)$ and $\alpha \in (0,1)$ as
$$W^{\alpha,p}(\mathbb{R}^3) = \Big\{u \in L^p(\mathbb{R}^3) : \int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p+3}}dxdy < \infty\Big\}.$$
This space is endowed with the Gagliardo norm
$$\|u\|_{W^{\alpha,p}} = \Big(\int_{\mathbb{R}^3} |u|^p dx + \int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p+3}}dxdy\Big)^{1/p}.$$
When $p = 2$, these spaces are also denoted by $H^\alpha(\mathbb{R}^3)$.

If p =2, an equivalent definition of fractional Sobolev spaces is possible, based on Fourier analysis. Indeed, it turns out that
$$H^\alpha(\mathbb{R}^3) = \Big\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2\alpha}|\hat{u}|^2 + |u|^2)d\xi < \infty\Big\},$$
and the norm can be equivalently written as
$$\|u\|_{H^\alpha} = \Big(\|u\|_2^2 + \int_{\mathbb{R}^3} |\xi|^{2\alpha}|\hat{u}|^2 d\xi\Big)^{1/2}.$$

Furthermore, this norm is equivalent to the norm

$$\|u\|_\alpha = \Big(\int_{\mathbb{R}^3} (|(-\Delta)^{\alpha/2} u|^2 + u^2) dx\Big)^{1/2}.$$

The spaces $D^{\alpha,2}(\mathbb{R}^3)$ is defined as the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{D^{\alpha,2}} = \Big(\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 dx\Big)^{1/2} = \Big(\int_{\mathbb{R}^3} |\xi|^{2\alpha} |\hat{u}(\xi)|^2 d\xi\Big)^{1/2}.$$

In this article, in view of the potential $V(x)$, we consider its subspace

$$E^\alpha = \big\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2\alpha} |\hat{u}(\xi)|^2 + V(x) u^2) dx < \infty\big\} \tag{2.1}$$

endowed with the norm

$$\|u\| = \Big(\int_{\mathbb{R}^3} (|\xi|^{2\alpha} |\hat{u}(\xi)|^2 + V(x) u^2) dx\Big)^{1/2}, \tag{2.2}$$

and the scalar product

$$(u, v) = \int_{\mathbb{R}^3} (|\xi|^{2\alpha} \hat{u}(\xi)\hat{v} + V(x) u(x) v(x)) dx.$$

Furthermore, we know that $\|\cdot\|_{E^\alpha}$ is equivalent to the norm

$$\|u\| = \Big(\int_{\mathbb{R}^3} (|(-\Delta)^{\alpha/2} u|^2 + V(x) u^2) dx\Big)^{1/2}.$$

The corresponding inner product is

$$(u, v) = \int_{\mathbb{R}^3} ((-\Delta)^{\alpha/2} u (-\Delta)^{\alpha/2} v + V(x) uv) dx.$$

First of all, let us study the variational setting of problem (1.1). Under (A10), for any $u \in E^\alpha(\mathbb{R}^3)$, the linear operator $T : D^{\beta,2}(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$T(v) = \int_{\mathbb{R}^3} K(x) u^2 v \, dx.$$

By Hölder's inequality and fractional Sobolev inequality, one has

$$\begin{aligned} T(v) &= \int_{\mathbb{R}^3} K(x) u^2 v \, dx \\ &\le K_\infty \Big(\int_{\mathbb{R}^3} |u(x)|^{\frac{12}{3+2\beta}} dx\Big)^{\frac{3+2\beta}{6}} \Big(\int_{\mathbb{R}^3} |v(x)|^{2_\beta^*} dx\Big)^{1/2_\beta^*} \\ &\le C\|u\|^2 \|v\|_{D^{\beta,2}} \end{aligned} \tag{2.3}$$

is well defined on $D^{\beta,2}(\mathbb{R}^3)$ and is continuous, where $2 \le \frac{12}{3+2\beta} < 2_\alpha^*$ because of $4\alpha + 2\beta > 3$. Then, by the Lax-Milgram theorem, there exists $\phi_u^\beta \in D^{\beta,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} (-\Delta)^{\beta/2} \phi_u^\beta (-\Delta)^{\beta/2} v \, dx = \int_{\mathbb{R}^3} K(x) u^2 v \, dx, \quad \forall v \in D^{\beta,2}(\mathbb{R}^3). \tag{2.4}$$

Therefore, $(-\Delta)^\beta \phi_u^\beta = K(x) u^2$ in a weak sense, and the representation formula holds

$$\phi_u^\beta(x) = c_\beta \int_{\mathbb{R}^3} \frac{K(y) u^2(y)}{|x-y|^{3-2\beta}} dy, \forall x \in \mathbb{R}^3 \quad \text{where } c_\beta = \pi^{-\frac{3}{2}} 2^{-2\beta} \frac{\Gamma(3-2\beta)}{\Gamma(\beta)}.$$

This formula is called t-Riesz potential. Substituting ϕ_u^β into (1.1), then (1.1) can be reduced to the fractional Schrödinger equation

$$(-\Delta)^\alpha u + V(x)u + K(x)\phi_u^\beta(x)u = f(x,u) - \Gamma(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^3, \tag{2.5}$$

whose solutions are the critical points of the function $J(u) : E^\alpha(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\beta u^2 dx - I(u), \tag{2.6}$$

where

$$I(u) = \int_{\mathbb{R}^3} (F(x,u(x)) - \frac{1}{q}\Gamma(x)|u|^q)dx.$$

Obviously, the assumptions of our theorems imply that J is a well-defined of class C^1 functional and that

$$\begin{aligned}(J'(u),v) = &\int_{\mathbb{R}^3} ((-\Delta)^{\alpha/2}u(-\Delta)^{\alpha/2}v + V(x)uv)dx + \int_{\mathbb{R}^3} K(x)\phi_u^\beta(x)uv\,dx \\ &- \int_{\mathbb{R}^3} f(x,u)v - \Gamma(x)|u|^{q-1}u\,dx.\end{aligned} \tag{2.7}$$

Hence, if $u \in E^\alpha(\mathbb{R}^3)$ is a critical point of J, then the pair (u, ϕ_u^β) is a solution of (1.1).

Our goal is to find a critical point being a minimizer of J on the Nehari manifold $\mathcal{N} := \{u \in E^\alpha(\mathbb{R}^3) \setminus \{0\} : J'(u)(u) = 0\}$. Obviously, $\mathcal{N}$ contains all nontrivial critical points, hence a ground state is the least energy solution.

Lemma 2.1 ([35])**.** *If $\alpha, \beta \in (0,1)$ and $4\beta + 2\alpha \ge 3$, then for any $u \in H^\alpha(\mathbb{R}^3)$, we have*

- (i) *$\phi_u^t : H^\alpha(\mathbb{R}^3) \to D^{\beta,2}(\mathbb{R}^3)$ is continuous, and maps bounded sets into bounded sets;*
- (ii) *$\phi_{\tau u}^\beta(x) = \tau^2\phi_u^\beta(x)$ for all $\tau \in \mathbb{R}$, $\phi_{u(\cdot+y)}^\beta = \phi_u^\beta(x+y)$;*
- (iii) *if $u_n \rightharpoonup u$ in $H^\alpha(\mathbb{R}^3)$, then $\phi_{u_n}^\beta \rightharpoonup \phi_u^\beta$ in $D^{\beta,2}(\mathbb{R}^3)$;*
- (iv) *if $u_n \rightharpoonup u$ in $H^\alpha(\mathbb{R}^3)$, then $\int_{\mathbb{R}^3} \phi_{u_n}^\beta u_n^2 = \int_{\mathbb{R}^3} \phi_{(u_n-u)}^\beta (u_n-u)^2 + \int_{\mathbb{R}^3} \phi_u^\beta u^2 + o(1)$.*

3. Proof of Theorem 1.1

First, we consider the norm in the working space $H^{\alpha/2}(\mathbb{R}^3)$.

Lemma 3.1 ([6])**.** *Let $\alpha \in (0,2)$ and denote by*

$$\|u\|_{H^{\alpha/2}}^2 = \int_{\mathbb{R}^3} |\xi|^\alpha |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^3} |u(x)|^2 dx$$

the classical norm on the fractional Sobolev space $H^{\alpha/2}(\mathbb{R}^3)$. Suppose that (A6) *and* (A8) *hold. Then the norm*

$$\|u\|^2 = \int_{\mathbb{R}^3} |\xi|^\alpha |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^3} V(x)|u(x)|^2 dx$$

is equivalent to $\|\cdot\|_{H^{\alpha/2}}$.

Remark 3.2. Obviously, the norm equivalence is also true when (A7) or (A8) holds. And the above lemma implies that $E^\alpha(\mathbb{R}^3)$ coincides with $H^\alpha(\mathbb{R}^3)$. Thus our functional $J : H^\alpha(\mathbb{R}^3) \to \mathbb{R}$ has the form

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\beta(x)u^2 - I(u).$$

The Nehari manifold has the form

$$\mathcal{N} = \big\{u \in H^\alpha(\mathbb{R}^3) \setminus \{0\} : \|u\|^2 = I'(u)(u) - \int_{\mathbb{R}^3} K(x)\phi_u^\beta(x)u^2\big\}.$$

The following lemmas are crucial to the proof of the Theorem 1.1.

Lemma 3.3. *Suppose that the following conditions hold:*

- (A11) *there exists $r \in (0,1)$ such that $a := \inf_{\|u\|=r} J(u) > J(0) = 0$;*
- (A12) *for any $t_n \to \infty$ and $u_n \to u \neq 0$ as $n \to \infty$, $I(t_n u_n)/t_n^q \to \infty$;*
- (A13) *for $t \in (0,\infty) \setminus \{1\}$ and $u \in \mathcal{N}$, $\frac{t^4-1}{4} I'(u)(u) - I(tu) + I(u) < 0$;*
- (A14) *J is coercive on $\mathcal{N}$.*

Then $\inf_{\mathcal{N}} J > 0$ and there exists a bounded minimizing sequence for J on $\mathcal{N}$, i.e. there is a sequence $\{u_n\} \subset \mathcal{N}$ such that $J(u_n) \to \inf_{\mathcal{N}} J$ and $J'(u_n) \to 0$.

Since the proof of Lemma 3.3 is similar to that of [37, Theorem 3.2], we omit it here.

Lemma 3.4. *Suppose that* (A2)–(A10) *are satisfied. Then* (A11)–(A14) *hold.*

Proof. To prove (A11) we fix $\epsilon > 0$. Observe that (A2) and (A9) imply that $F(x,u) \leq \epsilon|u|^2 + C_\epsilon|u|^p$ for some $C_\epsilon > 0$. Therefore,

$$\int_{\mathbb{R}^3}(F(x,u(x)) - \frac{1}{q}\Gamma(x)|u|^q)dx \leq \int_{\mathbb{R}^3} F(x,u(x)dx \leq C(\epsilon\|u\|^2 + C_\epsilon\|u\|^p)$$

for a positive constant C provided by the Sobolev embedding theorem. Thus there is $r > 0$ such that

$$\int_{\mathbb{R}^3}(F(x,u(x)) - \frac{1}{q}\Gamma(x)|u|^q)dx \leq \frac{1}{4}\|u\|^2$$

for $\|u\| \leq r$. Therefore

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\beta(x)u^2 - I(u) \geq \frac{1}{4}\|u\|^2 = \frac{1}{4}r^2 > 0$$

for all $u \in H^\alpha(\mathbb{R}^3)$, $p > 4$. So there exists $r \in (0,1)$ such that $\|u\| = r$ and $J(u) > 0$, then $a = \inf_{\|u\|=r} J(u) > 0$.

To prove (A12), by (A4) and Fatou's lemma we obtain

$$I(t_n u_n)/t_n^q = \int_{\mathbb{R}^3} \frac{F(x,t_n u_n)}{t_n^q}dx - \frac{1}{q}\int_{\mathbb{R}^3}\Gamma(x)|u_n|^q dx \to \infty.$$

To prove (A13) we fix $u \in \mathcal{N}$ and consider

$$\psi(t) = \frac{t^4-1}{2} I'(u)(u) - I(tu) + I(u) \quad \forall t \geq 0.$$

Then $\psi(1) = 0$ and

$$\frac{d\psi(t)}{dt} = 2t^3 I'(u)(u) - u\frac{dI(tu)}{dt}$$

$$
\begin{aligned}
&= 2t^3 \int_{\mathbb{R}^3} f(x,u)u - \Gamma(x)|u|^q dx - \int_{\mathbb{R}^3} f(x,tu)u + \Gamma(x)t^{q-1}|u|^{q-1} dx \\
&= \int_{\mathbb{R}^3} 2t^3 f(x,u)u - f(x,tu)u\, dx - (2t^3 - t^{q-1}) \int_{\mathbb{R}^3} \Gamma(x)|u|^q dx.
\end{aligned}
$$

Since $u \in \mathcal{N}$,

$$
I'(u)(u) = \int_{\mathbb{R}^3} f(x,u)u - \Gamma(x)|u|^q dx = \|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u^\beta u^2 > 0.
$$

Therefore, for $t > 1$, we have $\frac{d\psi(t)}{dt} < 0$ by (A4). Similarly, $\frac{d\psi(t)}{dt} > 0$, for $t < 1$. Therefore $\psi(t) < \psi(1) = 0$ for $t = 1$, that is

$$
\frac{t^4 - 1}{2} I'(u)(u) - I(tu) + I(u) < 0.
$$

To prove (A14), for $u \in \mathcal{N}$, (A4) implies that $f(x,u)u \geq qF(x,u)$. Therefore

$$
\begin{aligned}
J(u) &= J(u) - \frac{1}{q} J'(u)(u) \\
&= (\frac{1}{2} - \frac{1}{q})\|u\|^2 + (\frac{1}{4} - \frac{1}{q}) \int_{\mathbb{R}^3} K(x)\phi_u^\beta(x) u^2 dx + \int_{\mathbb{R}^3} (\frac{1}{q} f(x,u)u - F(x,u)) dx \\
&\geq (\frac{1}{2} - \frac{1}{q})\|u\|^2
\end{aligned}
$$

which implies that J is coercive on $\mathcal{N}$. □

Now, we will introduce a decomposition result of bounded (PS) sequences which is important to the proof of Theorem 1.1. The following lemma generalizes the decomposition result from [5]. We will denote

$$
J_{\text{per}} = J(u) - \frac{1}{2} \int_{\mathbb{R}^3} V_{\text{loc}}(x) u^2 dx.
$$

Lemma 3.5. *Suppose that the assumptions of Theorem 1.1 are satisfied. Let (u_n) be a bounded Palais–sequence for J. Then passing to a subsequence of (u_n), there exist an integer $l \geq 0$ and sequences $(y_n^k) \subset \mathbb{Z}^3$, $w_k \in H^\alpha(\mathbb{R}^3)$, $k = 1, \dots, l$ such that*

(a) $u_n \rightharpoonup u_0$ *and* $J'(u_0) = 0$;
(b) $|y_n^k| \to \infty$ *and* $|y_n^k - y_n^{k'}| \to \infty$ *for* $k \neq k'$;
(c) $w^k \neq 0$ *and* $J'_{\text{per}}(w^k) = 0$ *for each* $1 \leq k \leq l$;
(d) $u_n - u_0 - \sum_{k=1}^l w^k(\cdot - y_n^k) \to 0$ *in* $H^\alpha(\mathbb{R}^3)$ *as* $n \to \infty$;
(e) $J(u_n) \to J(u_0) + \sum_{k=1}^l J_{\text{per}}(w^k)$.

The proof of Lemma 3.5 resembles the proof of [5, Theorem 4.1]. So we omit it.

Proof of Theorem 1.1. By Lemmas 3.3 and 3.4, there exists a bounded minimizing sequence $(u_n) \in \mathcal{N}$ for J, that is $J'(u_n) \to 0$ and $J(u_n) \to c := \inf_{\mathcal{N}} J > 0$. By Lemma 3.5 we have that

$$
J(u_n) \to J(u_0) + \sum_{k=1}^l J_{\text{per}}(w^k),
$$

where w^k satisfies

$$
w^k \neq 0 \quad \text{and} \quad J'_{\text{per}}(w_k) = 0 \quad \text{for } 1 \leq k \leq l,
$$

that is w^k are critical points of the periodic part of the functional J. Suppose that $V_{\rm loc} \equiv 0$, so $J = J_{\rm per}$. If $u_0 = 0$, we have

$$c + o(1) = J(u_n) \to \sum_{k=1}^{l} J_{\rm per}(w^k) \geq lc.$$

Then $l = 1$ and $w^1 \neq 0$ is a ground state solution. If $u_0 \neq 0$, we have

$$c + o(1) = J(u_n) \to J(u_0) + \sum_{k=1}^{l} J_{\rm per}(w^k) \geq (l+1)c.$$

Therefore, $l = 0$ and $J(u_n) \to J(u_0) = c$, so u_0 is a ground state solution. Suppose that $V_{\rm loc} < 0$ for a.e. $x \in \mathbb{R}^3$. We denote $c_{\rm per} := \inf_{\mathcal{N}_{\rm per}} J_{\rm per}$, where $\mathcal{N}_{\rm per} = \big\{u \neq 0 : J'_{\rm per}(u)(u) = 0\big\}$. Because $V_{\rm loc} < 0$, we have $J(u) < J_{\rm per}(u)$. From the above arguments, there exists a ground state solution $u_{\rm per}$ of $J_{\rm per}$ such that $J_{\rm per}(u_{\rm per}) = c_{\rm per}$. Let $t > 0$ be such that $tu_{\rm per} \in \mathcal{N}$. Then

$$c \leq J(tu_{\rm per}) < J_{\rm per}(tu_{\rm per}) \leq J_{\rm per}(u_{\rm per}) = c_{\rm per}.$$

If $u_0 = 0$, we have

$$c + o(1) = J(u_n) = \sum_{k=1}^{l} J_{\rm per}(w^k) \geq lc_{\rm per} > lc,$$

which implies $l = 0$ and $J(u_n) \to c = 0$. Obviously, it is a contraction. Therefore, $u_0 \neq 0$ and note that

$$c + o(1) = J(u_n) \to J(u_0) + \sum_{k=1}^{l} J_{\rm per}(w^k) \geq c + lc_{\rm per}.$$

So $l = 0$ which means that $J(u_n) \to J(u_0) = c$ and u_0 is a ground solution. □

4. Proof of Theorem 1.2

Observe that the condition $\inf \sigma(-\Delta + V_{\rm per}) > 0$ implies that the norm $\|u\|_0^2 = \int_{\mathbb{R}^3} (|\xi|^{2\alpha} |\hat{u}(\xi)|^2 + V_{\rm per}(x) u^2) dx$ is equivalent to $\|\cdot\|$ in $H^\alpha(\mathbb{R}^3)$.

Proof of Theorem 1.2. By contradiction, suppose that there exists a ground state solution $u_0 \in \mathcal{N}$ of J. Similar to the proof of [37, Theorem 3.2], there exists $t_p > 0$ such that $t_p u_0 \in \mathcal{N}_{\rm per}$. Since $V_{\rm loc} > 0$ for a.e. $x \in \mathbb{R}^3$, one has $\int_{\mathbb{R}^3} V_{\rm loc}(x) u_0^2 > 0$. Then by Lemma 3.4, we obtain that (A13) holds; that is, for $t \in (0, \infty) \setminus \{1\}$ and $u \in \mathcal{N}$,

$$\frac{t^4 - 1}{4} I'(u)(u) - I(tu) + I(u) < 0.$$

Since $u_0 \in \mathcal{N}$, one has

$$\begin{aligned}
&J(u_0) - J(t_p u_0) \\
&= \frac{1 - t_p^2}{2} \|u_0\|^2 - \frac{t_p^4 - 1}{4} \int_{\mathbb{R}^3} K(x) \phi_u^\beta u_0^2 - I(u_0) + I(t_p u_0) \\
&= \Big(\frac{1 - t_p^2}{2}\Big)^2 \|u_0\|^2 - \frac{t_p^4 - 1}{4} J'(u_0) u_0 - \frac{t_p^4 - 1}{4} I'(u_0)(u_0) + I(t_p u_0) - I(u_0) \\
&= \frac{(1 - t_p^2)^2}{4} \|u_0\|^2 - \frac{t_p^4 - 1}{4} I'(u_0)(u_0) + I(t_p u_0) - I(u_0)
\end{aligned}$$

$$> \frac{(t_p^2-1)^2}{4}\|u_0\|^2$$

Then $J(u_0) > J(t_p u_0)$. Therefore,

$$c_{\rm per} := \inf_{\mathcal{N}_{\rm per}} J_{\rm per} \le J_{\rm per}(t_p u_0) < J(t_p u_0) < J(u_0) = c. \tag{4.1}$$

On the other hand, let $u \in \mathcal{N}_{\rm per}$ and $u_y(\cdot) = u(\cdot - y)$ for $y \in \mathbb{Z}^3$. Then there exist $t_u > 0$ such that $t_u u_y \in \mathcal{N}$. Observe that

$$\begin{aligned} J_{\rm per}(u) &= J_{\rm per}(u_y) \ge J_{\rm per}(t_u u_y) \\ &= J(t_u u_y) - \frac{1}{2}\int_{\mathbb{R}^3} V_{\rm loc}(x)(t_u u_y)^2 \\ &\ge c - \int_{\mathbb{R}^3} V_{\rm loc}(x)(t_u u_y)^2. \end{aligned} \tag{4.2}$$

We are going to show that

$$\int_{\mathbb{R}^3} V_{\rm loc}(x)(t_u u_y)^2 \to 0 \tag{4.3}$$

Indeed,

$$\int_{\mathbb{R}^3} V_{\rm loc}(x)(t_u u_y)^2 = t_u^2 \int_{\mathbb{R}^3} V_{\rm loc}(x+y)u^2. \tag{4.4}$$

By (A8), one has

$$\int_{\mathbb{R}^3} V_{\rm loc}(x+y)u^2 \to 0 \quad \text{as } |y| \to \infty. \tag{4.5}$$

If $t_u \to \infty$, then by (A4) one has

$$\begin{aligned} o(1) = \frac{c}{t_u^q} &\le \frac{J(_u u_y)}{t_u^q} \\ &= \frac{1}{2}\|u_y\|^2 t_u^{2-q} + \frac{1}{4} t_u^{4-q}\int_{\mathbb{R}^3} K(x)\phi_{u_y}^\beta u_y^2 \\ &\quad - \int_{\mathbb{R}^3} \frac{F(x,t_u u_y)}{|t_u u_y|^q} u_y^q dx + \int_{\mathbb{R}^3} \frac{1}{q}\Gamma(x)|u_y|^q dx \to -\infty. \end{aligned}$$

Thus $t_u < \infty$, which, together with (4.5), implies that (4.3) holds. By (4.2), we have

$$J_{\rm per}(u) \ge c - \int_{\mathbb{R}^3} V_{\rm loc}(x)(t_u u_y)^2 \to c.$$

Taking infimum over all $u \in \mathcal{N}_{\rm per}$ we have $c_{\rm per} \ge c$, which contradicts with (4.1). The proof is complete. $\square$

5. Proof of Theorem 1.3

Let $\beta := \inf_{\mathcal{N}} \|u\| > 0$, where $\|\cdot\|$ is the norm defined by (2.2). Theorem 1.1 provides that c is attained at some function in $\mathcal{N}$. By τ_u we denote the $\mathbb{Z}^3$-action on $H^\alpha(\mathbb{R}^3)$, i.e. $\tau_k u = u(\cdot - k)$. Obviously, $\tau_k\tau_{-k}u = u$. For given $k \in \mathbb{Z}^3$, let us consider τ_k as an operator $\tau_k : H^\alpha(\mathbb{R}^3) \to H^\alpha(\mathbb{R}^3)$. It is easy to prove that τ_k is linear.

Lemma 5.1 ([6]). . *For every $u, v \in H^\alpha(\mathbb{R}^N)$ and $k \in \mathbb{Z}^N$, we have*

$$(\tau_k u, v) = (u, \tau_{-k} v)\,.$$

Remark 5.2. When $N = 3$, the conclusion in Lemma 5.1 still holds. Obviously, by the definition of τ_k, we have $\|\tau_k\| = \|u\|$. Thus τ_k is a bounded operator and $\|\tau_k\| = 1$. Then we may consider an adjoint operator $\tau_k^* : H^\alpha(\mathbb{R}^3) \to H^\alpha(\mathbb{R}^3)$. From Lemma 5.1 we have $\tau_k^* = \tau_{-k}$. Moreover, τ_k is an isomorphism and $\tau_k^{-1} = \tau_{-k} = \tau_k^*$. So τ_k is an orthogonal operator.

The proof of Lemmas 5.3, 5.5, 5.6 are similar to those in [6, Lemmas 5.3, 5.4 5.6, 5.7]. So we omit them here.

Lemma 5.3. *Let $\alpha \in (0, 1]$.*
(1) *The functional J is $\mathbb{Z}^3$-invariant;*
(2) *$\mathcal{N}$ is $\mathbb{Z}^3$-invariant.*

Remark 5.4. Lemma 5.1 implies that the unit sphere S^1 is $\mathbb{Z}^3$-invariant.

From [5] we know for each $u \in H^\alpha(\mathbb{R}^3)$ there is a unique number $t(u) > 0$ such that $t(u)u \in \mathcal{N}$ and the function $m : S^1 \to \mathcal{N}$ given by $m(u) = t(u)u$ is a homeomorphism. The inverse $m^{-1} : \mathcal{N} \to S^1$ is given by $m^{-1}(u) = \frac{u}{\|u\|}$. Assume that $A, B \subset H^\alpha(\mathbb{R}^3)$ are $\mathbb{Z}^3$-invariant subsets. We say that the function $h : A \to B$ is $\mathbb{Z}^3$-equivariant if

$$h(\tau_k u) = \tau_k h(u)$$

for any $u \in A$ and $k \in \mathbb{Z}^3$.

Lemma 5.5. *The following four functions are $\mathbb{Z}^3$-equivariant:*
(1) *$m : S^1 \to \mathcal{N}$,*
(2) *$m^{-1} : \mathcal{N} \to S^1$,*
(3) *$\nabla J : H^\alpha(\mathbb{R}^3) \to H^\alpha(\mathbb{R}^3)$,*
(4) *$\nabla(J \circ m) : S^1 \to H^\alpha(\mathbb{R}^3)$*

Lemma 5.6. *The function $m^{-1} : \mathcal{N} \to S^1$ is Lipschitz continuous.*

We denote $\mathcal{O}(u) = \{u(\cdot - k) : k \in \mathbb{Z}^3\}$. If u is a solution of (1.1) and $k \in \mathbb{Z}^3$, then $u(\cdot - k)$ is also a solution, provided that $V_{\rm loc} \equiv 0$. Therefore, all elements of the orbit $\mathcal{O}(u)$ of u under the $\mathbb{Z}^3$-action are solutions. We define that u_1 and u_2 are geometrically distinct if their orbits satisfy $\mathcal{O}(u_1) \cap \mathcal{O}(u_2) = \emptyset$.

To prove Theorem 1.3 we use the method introduced by Szulkin and Weth [34]. Let $\mathcal{L} = \{u \in S^1 : (J \circ m)'(u) = 0\}$. Take a set $F \subset \mathcal{L}$ such that $F = -F$ and for each orbit $\mathcal{O}(w)$ there is a unique representative $v \in F$. Observe the [24, Theorem 3.1(b)], we know that to prove Theorem 1.3 it is sufficient to show $J \circ m$ has infinitely many geometrically distinct critical points. Therefore, we only need to show that F is infinite.

As in [34], we put

$$\kappa = \inf_{v,w \in \mathcal{L}, v \neq w} \|v - w\| > 0.$$

Next, we show a lemma which is crucial to the proof of the Theorem 1.3.

Lemma 5.7. *Let $d \geq c$. If $(v_n^1), (v_n^2) \subset S^1$ are two (PS) sequences for $J \circ m$ such that $J(m(v_n^i)) \leq d$, $i = 1, 2$, then*

$$\|v_n^1 - v_n^2\| \to 0$$

or

$$\liminf_{n \to \infty} \|v_n^1 - v_n^2\| \geq \rho(d) > 0,$$

where $\rho(d)$ depends only on d, but not on the particular choice of the sequences.

Proof. Note that $u_n^i = m(v_n^i)$, $i=1,2$ are (PS) sequences for J. Moreover, they are bounded in $H^\alpha(\mathbb{R}^3)$, since J is coercive on $\mathcal{N}$. Therefore (u_n^1) and (u_n^2) are bounded in $L^2(\mathbb{R}^3)$, say $|u_n^1|_2 + |u_n^2|_2 \le M$ for some $M>0$.

Case 1: Assume that $|u_n^1 - u_n^2|_p \to 0$. Fix $\epsilon > 0$, by (A2) and (A9) we have

$$
\begin{aligned}
&\|u_n^1 - u_n^2\|^2\\
&= J'(u_n^1)(u_n^1-u_n^2) - J'(u_n^2)(u_n^1-u_n^2) + \int_{\mathbb{R}^3}[f(x,u_n^1)-f(x,u_n^2)](u_n^1-u_n^2)dx\\
&\quad - \int_{\mathbb{R}^3}\Gamma(x)(|u_n^1|^{q-2}u_n^1 - |u_n^2|^{q-2}u_n^2)(u_n^1-u_n^2)dx\\
&\quad + \int_{\mathbb{R}^3}K(x)(\phi^\beta_{u_n^1}u_n^1 - \phi^\beta_{u_n^2}u_n^2)(u_n^1-u_n^2)\\
&\le \epsilon\|u_n^1-u_n^2\| + \int_{\mathbb{R}^3}(\epsilon(|u_n^1|+|u_n^2|) + C_\epsilon(|u_n^1|^{p-1}+|u_n^2|^{p-1}))|u_n^1-u_n^2|dx\\
&\quad - \int_{\mathbb{R}^3}\Gamma(x)(|u_n^1|^{q-2}u_n^1 - |u_n^2|^{q-2}u_n^2)(u_n^1-u_n^2)dx\\
&\quad + \int_{\mathbb{R}^3}K(x)\phi^\beta_{u_n^1}u_n^1(u_n^1-u_n^2)dx + \int_{\mathbb{R}^3}K(x)\phi^\beta_{u_n^2}u_n^2(u_n^1-u_n^2)dx\\
&\le (1+C_0)\epsilon\|u_n^1-u_n^2\| + D_\epsilon|u_n^1-u_n^2|_p + C_1|\Gamma|_\infty|u_n^1-u_n^2|_q^q\\
&\quad + \int_{\mathbb{R}^3}K(x)\phi^\beta_{u_n^1}u_n^1(u_n^1-u_n^2)dx + \int_{\mathbb{R}^3}K(x)\phi^\beta_{u_n^2}u_n^2(u_n^1-u_n^2)dx
\end{aligned}
$$

for each $n \ge n_\epsilon$ and some constants $C_0, C_1, D_\epsilon > 0$. By assumption we have that $D_\epsilon|u_n^1-u_n^2|_p \to 0$. Observe that

$$
C_1|\Gamma|_\infty|u_n^1-u_n^2|_q^q \le C_1|\Gamma|_\infty|u_n^1-u_n^2|_2^{\theta q}|u_n^1-u_n^2|_p^{(1-\theta)q},
$$

where $\theta\in(0,1)$ is such that $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{p}$. Thus

$$
C_1|\Gamma|_\infty|u_n^1-u_n^2|_q^q \le C_1|\Gamma|_\infty M^{\theta q}|u_n^1-u_n^2|_p^{(1-\theta)q} \to 0.
$$

And by Hölder inequality,

$$
\begin{aligned}
&\int_{\mathbb{R}^N}K(x)\phi^\beta_{u_n^1}u_n^1(u_n^1-u_n^2)dx\\
&\le |K|_\infty\Big(\int_{\mathbb{R}^3}|\phi^\beta_{u_n^1}|^{2^*_\beta}dx\Big)^{1/2^*_\beta}\Big(\int_{\mathbb{R}^3}|u_n^1(u_n^1-u_n^2)|^{\frac{6}{3+2\beta}}dx\Big)^{\frac{3+2\beta}{6}}\\
&\le C_2\|\phi^\beta_{u_n^1}\|_{D^{\beta,2}}\Big(\int_{\mathbb{R}^3}|u_n^1|^{\frac{6}{3+2\beta}}|u_n^1-u_n^2|^{\frac{6}{3+2\beta}}dx\Big)^{\frac{3+2\beta}{6}}\\
&\le C_3\|\phi^\beta_{u_n^1}\|_{D^{\beta,2}}|u_n^1|_{\frac{12}{3+2\beta}}|u_n^1-u_n^2|_{\frac{12}{3+2\beta}}\\
&\le C_4|u_n^1-u_n^2|_{\frac{12}{3+2\beta}},
\end{aligned}
\tag{5.1}
$$

where $C_2, C_3, C_4 > 0$ are constants. By $\beta\in(0,1]$, we have $\frac{12}{3+2\beta} < 4 < p$. Thus there exists $\theta'\in(0,1)$ such that

$$
\frac{1}{\frac{12}{3+2\beta}} = \frac{\theta'}{2} + \frac{1-\theta'}{p}
$$

and

$$
|u_n^1-u_n^2|_{\frac{12}{3+2\beta}} \le |u_n^1-u_n^2|_2^{\theta'}|u_n^1-u_n^2|_p^{1-\theta'} \le M^{\theta'}|u_n^1-u_n^2|_p^{1-\theta'}. \tag{5.2}
$$

Then

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n^1}^{\beta} u_n^1(u_n^1 - u_n^2)dx \le C_4 M^{\theta'} |u_n^1 - u_n^2|_p^{1-\theta'} \to 0. \tag{5.3}$$

Similarly we have

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n^2}^{\beta} u_n^2(u_n^1 - u_n^2)dx \le C_5 M^{\theta''} |u_n^1 - u_n^2|_p^{1-\theta''} \to 0, \tag{5.4}$$

where $C_5 > 0$ is some constant and $\theta'' \in (0,1)$. Finally

$$\begin{aligned}
\limsup_{n\to\infty} \|u_n^1 - u_n^2\|^2 &\le \limsup_{n\to\infty} (1+C_0)\epsilon \|u_n^1 - u_n^2\| \\
&\quad + \limsup_{n\to\infty} D_\epsilon |u_n^1 - u_n^2|_p + \limsup_{n\to\infty} C_1 |\Gamma|_\infty |u_n^1 - u_n^2|_q^q \\
&\quad + \limsup_{n\to\infty} C_4 M^{\theta'} |u_n^1 - u_n^2|_p^{1-\theta'} + \limsup_{n\to\infty} C_5^{\theta''} |u_n^1 - u_n^2|_p^{1-\theta''} \\
&= (1+C_0)\epsilon \limsup_{n\to\infty} \|u_n^1 - u_n^2\|
\end{aligned} \tag{5.5}$$

for every $\epsilon > 0$. Thus $\lim_{n\to\infty} \|u_n^1 - u_n^2\| = 0$. Finally

$$\|v_n^1 - v_n^2\| = \|m^{-1}(u_n^1) - m^{-1}(u_n^2)\| \le L\|u_n^1 - u_n^2\| \to 0,$$

where $L > 0$ is a Lipschitz constant for m^{-1}.

Case 2: Assume that $|u_n^1 - u_n^2|_p \nrightarrow 0$. By the Lions lemma, there are $y_n \in \mathbb{R}^3$ such that

$$\int_{B(y_n,1)} |u_n^1 - u_n^2|^2 dx = \sup_{y\in\mathbb{R}^3} \int_{B(y,1)} |u_n^1 - u_n^2|^2 dx \ge \epsilon$$

for some $\epsilon > 0$. In view of Lemma 5.5 we can assume that (y_n) is bounded. Therefore, up to a subsequence we have

$$u_n^1 \rightharpoonup u^1,\ u_n^2 \rightharpoonup u^2,$$

where $u^1 \ne u^2$ and $J'(u^1) = J'(u^2) = 0$ and

$$\|u_n^1\| \to \alpha^1, \|u_n^2 \to \alpha^2,$$

where $\beta \le \alpha^i \le \nu(d) = \sup\{\|u\| : u \in \mathcal{N}, J(u) \le d\}$, $i = 1,2$. Suppose that $u^1 \ne 0$ and $u^2 \ne 0$. Then $u^i \in \mathcal{N}$ for $i = 1,2$. Moreover

$$v^i = m^{-1}(u^i) \in S^1, \quad i = 1,2 \text{ and } v^1 \ne v^2.$$

Therefore,

$$\liminf_{n\to\infty} \|v_n^1 - v_n^2\| = \liminf_{n\to\infty} \|\frac{u_n^1}{\|u_n^1\|} - \frac{u_n^1}{\|u_n^2\|}\| \ge \|\frac{u^1}{\alpha^1} - \frac{u^2}{\alpha^2}\| = \|\beta_1 v^1 - \beta_2 v^2\|,$$

where $\beta_i = \frac{\|u^i\|}{\alpha^i} \ge \frac{\beta}{\nu(d)}$, $i = 1,2$. Of course $\|v^1\| = \|v^2\| = 1$. So

$$\liminf_{n\to\infty} \|v_n^1 - v_n^2\| \ge \|\beta_1 v^1 - \beta_2 v^2\| \ge \min_{i=1,2}\{\beta_i\} \|v^1 - v^2\| \ge \frac{\beta\kappa}{\nu(d)}.$$

If $u^2 = 0$, then $u^1 \ne u^2 = 0$. Therefore,

$$\liminf_{n\to\infty} \|v_n^1 - v_n^2\| = \liminf_{n\to\infty} \|\frac{u_n^1}{\|u_n^1\|} - \frac{u_n^1}{\|u_n^2\|}\| \ge \|\frac{u^1}{\alpha^1} - \frac{u^2}{\alpha^2}\| = \|\frac{u^1}{\alpha^1}\| \ge \frac{\beta}{\nu(d)}.$$

The case $u^1 = 0$ is similar, the proof is complete. $\square$

Proof of Theorem 1.3. Suppose by contraction that F is finite. Because the unit sphere $S^1 \subset H^\alpha(\mathbb{R}^3)$ is a Finsler $C^{1,1}-$manifold and by [29, Lemma II.3.9], we obtain $J \circ m : S^1 \to \mathbb{R}$ has a pseudo-gradient vector field. Because of the obtained discreteness of (PS) sequences in Lemma 5.7, we can apply the methods in [34, Lemma 2.15, Lemma 2.16 and Theorem 1.2] in our case. In fact, for every $k \in \mathbb{N}$ there is $u \in S^1$ such that

$$(J \circ m)'(u) = 0 \quad \text{and} \quad J(m(u)) = c_k,$$

where

$$c_k = \inf \big\{ d \in \mathbb{R} : \gamma(\{v \in S^1 : J(m(v)) \le d\}) \ge k \big\}$$

is the Lusternik-Schnirelmann value and γ denotes the Krasnoselskii genus (see [29]). Moreover $c_k \le c_{k+1}$, thus we get contradiction (for the detailed arguments see [34]). The proof is complete. $\square$

6. Proof of Theorem 1.4

Before showing the proof, We state the following form of the Sobolev-Gagliardo-Nirenberg inequality.

Lemma 6.1 ([28, Proposition II.3])**.** *Let $r > 1$. Then there is a positive constant $C > 0$ such that for every function $u \in H^\alpha(\mathbb{R}^N)$ there holds*

$$|u|_{r+1}^{r+1} \le C\|u\|_{H^\alpha}^{\frac{(r-1)N}{\alpha}} |u|_2^{r+1-\frac{(r-1)N}{\alpha}}.$$

In [11], it was proven that E^α is compactly embedded into $L^r(\mathbb{R}^N)$ for $N > \alpha$ and $2 \le r < 2_\alpha^*$. We will show that the method introduced by Secchi in [28] may be applied in this case.

Proof of Theorem 1.4. In view of Lemma 3.3, we obtain a bounded minimizing sequence $(u_n) \subset \mathcal{N}$, i.e.

$$J(u_n) \to \inf_{\mathcal{N}} J =: c, \quad J'(u_n) \to 0.$$

Then we assume that $u_n \rightharpoonup u_0$ in $E^\alpha(\mathbb{R}^3)$ and $u_n \to u_0$ in $L^r_{\rm loc}(\mathbb{R}^3)$ for $1 \le r < 2_\alpha^*$. It is a easy to proof that u_0 is a weak solution to our problem. Then we just need to check whether $u_0 \ne 0$. Observe that for $n \ge n_0$

$$\begin{aligned}
\frac{c}{2} &\le J(u_n) = J(u_n) - \frac{1}{2}J'(u_n)(u_n) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} f(x,u_n)u_n - 2F(x,u_n)dx - \big(\frac{1}{2} - \frac{1}{q}\big)\int_{\mathbb{R}^3} \Gamma(x)|u|^q dx \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\Phi_u^\beta(x)u^2 dx \\
&\le \frac{1}{2}\int_{\mathbb{R}^3} f(x,u_n)u_n - 2F(x,u_n)dx \\
&\le \frac{1}{2}\int_{\mathbb{R}^3} f(x,u_n)u_n dx \\
&\le \frac{1}{2}\int_{\mathbb{R}^3} (\epsilon|u_n|^2 + C_\epsilon|u_n|^p)dx,
\end{aligned}$$

where $n_0 \ge 1$ is large enough. Then

$$\frac{c}{2} \le \frac{\epsilon}{2}|u_n|_2^2 + \frac{C_\epsilon}{2}|u_n|_p^p.$$

By Lemma 6.1, we have

$$\frac{c}{2} \le \frac{\epsilon}{2}|u_n|_2^2 + \frac{CC_\epsilon}{2}\|u_n\|_{H^\alpha(\mathbb{R}^3)}^{\frac{3(p-2)}{\alpha}}|u_n|_2^{p-\frac{3(p-2)}{\alpha}}.$$

Observe that the boundedness of (u_n) in H^α with respect to the norm $\|\cdot\|$ implies the boundedness of (u_n) in E^α with respect to the classical norm $\|\cdot\|_{H^\alpha}(\mathbb{R}^3)$. Therefore $\|u_n\|_{H^\alpha}(\mathbb{R}^3) \le D$ for some $D > 0$. Thus

$$\frac{c}{2} \le \frac{\epsilon}{2}|u_n|_2^2 + \frac{CC_\epsilon}{2}D^{\frac{3(p-2)}{\alpha}}|u_n|_2^{p-\frac{3(p-2)}{\alpha}}.$$

Denote $\hat{C}_\epsilon = \frac{C\cdot C_\epsilon}{2}D^{\frac{3(p-2)}{\alpha}}$. Then

$$\frac{c}{2} \le \frac{\epsilon}{2}|u_n|_2^2 + \hat{C}_\epsilon|u_n|_2^{p-\frac{3(p-2)}{\alpha}}.$$

Take any $\epsilon \le \frac{c}{2(\sup_n\|u_n\|)^2}$. Then

$$\frac{c}{2} \le \frac{c|u_n|_2^2}{4(\sup_n\|u_n\|)^2} + \hat{C}_\epsilon|u_n|_2^{p-\frac{3(p-2)}{\alpha}}.$$

Obviously,

$$\frac{|u_n|_2^2}{(\sup_n\|u_n\|)^2} \le 1,$$

and therefore

$$\frac{c}{2} \le \frac{c}{4} + \hat{C}_\epsilon|u_n|_2^{p-\frac{3(p-2)}{\alpha}}.$$

Finally

$$\frac{c}{4} \le \hat{C}_\epsilon|u_n|_2^{p-\frac{3(p-2)}{\alpha}}.$$

So

$$\ln\frac{c}{4} \le \Big(p - \frac{3(p-2)}{\alpha}\Big)\ln(C_1|u_n|_2),$$

where $C_1 = \hat{C}_\epsilon^{p-\frac{3(p-2)}{\alpha}}$. Thus

$$\frac{\alpha}{\alpha p - 3(p-2)}\ln\frac{c}{4} \le \ln(C_1|u_n|_2).$$

And finally

$$|u_n|_2^2 \ge \Big(\frac{1}{C_1}\exp\Big(\frac{\alpha}{\alpha p - 3(p-2)})\ln\frac{c}{4}\Big)\Big)^2 =: \hat{c} > 0.$$

Take any $R > 0$ and observe that

$$|u_n|_2^2 = \int_{B(0,R)}|u_n|^2dx + \int_{\mathbb{R}^3\setminus B(0,R)}|u_n|^2dx.$$

Assume by a contraction that $u_0 = 0$, then $u_n \to 0$ in $L^2_{\rm loc}(\mathbb{R}^3)$. Then for every $R > 0$ there is n_0 such that for $n \ge n_0$ we have

$$\int_{B(0,R)}|u_n|^2dx \le \frac{\hat{c}}{2}.$$

Thus

$$\int_{\mathbb{R}^3\setminus B(0,R)}|u_n|^2dx \ge \frac{\hat{c}}{2}.$$

On the other hand

$$\frac{\hat{c}}{2} \le \int_{\mathbb{R}^3\setminus B(0,R)}|u_n|^2dx = \int_{\mathbb{R}^3\setminus B(0,R)}\frac{V(x)|u_n|^2}{V(x)}dx$$

$$
\begin{aligned}
&\le \frac{1}{\inf_{|x|\ge R} V(x)} \int_{\mathbb{R}^3\setminus B(0,R)} V(x)|u_n|^2 dx \\
&\le \frac{\|u_n\|^2}{\inf_{|x|\ge R} V(x)} \le \frac{\sup_n \|u_n\|^2}{\inf_{|x|\ge R} V(x)}.
\end{aligned}
$$

Taking $R > 0$ big enough we obtain a contradiction, since $V(x) \to \infty$ as $|x| \to \infty$. Therefore, $u_0 \neq 0$. The proof is complete. $\square$

Acknowledgments. This work was supported by Natural Science Foundation of China 11671403.

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