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k-HESSIAN CURVATURE TYPE EQUATIONS IN SPACE FORMS

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Abstract. In this article, we study closed star-shaped (η, k)-convex hypersurfaces in space forms satisfying a class of k-Hessian curvature type equations. Firstly, using the maximum principle, we obtain a priori estimates for the class of Hessian curvature type equations. Secondly, we obtain an existence result by using standard degree theory based on a priori estimates.

1. Introduction

Suppose that M is an immersed hypersurface in Euclidean space $\mathbb{R}^{n+1}$. Define a $(0,2)$-tensor η on M by

$$\eta_{ij} = Hg_{ij} - h_{ij},$$

where g_{ij}, h_{ij} and H are the first, second fundamental forms and mean curvature of M respectively. In fact, η is the first Newton transformation of h with respect to g, see [18]. Let $\kappa = (\kappa_1, \dots, \kappa_n)$ be the vector whose components κ_i are the principal curvatures of M. Using $\lambda(\eta)$ to denote the vector whose components are the eigenvalues of η, we have that

$$\lambda(\eta) = (H - \kappa_1, \dots, H - \kappa_n).$$

Then k-Hessian equation of $\lambda(\eta)$ can be written as

$$\sigma_k(\lambda(\eta)) = f(X, \nu(X)), \quad 1 \le k \le n,\ X \in M, \tag{1.1}$$

where ν is the normal vector field along M and σ_k is the k-th elementary symmetric function

$$\sigma_k(\lambda) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \dots \lambda_{i_k}.$$

If $\lambda(\eta)$ is replaced by the principal curvature vector κ of the hypersurface, Equation (1.1) becomes the classical prescribed curvature equation

$$\sigma_k(\kappa) = f(X, \nu),\ for \quad X \in M \subset \mathbb{R}^n, \tag{1.2}$$

which has been widely studied in [2, 3, 6, 9, 10, 11]. In fact, curvature estimates are the key to the existence of star-shaped k-convex hypersurface satisfying Equation (1.2). In the case $k = 2$, Guan, Ren, and Wang [12] obtained a global C^2 estimate for strictly star-shaped 2-convex hypersurfaces. Spruck and Xiao [23] extended the estimate for 2-convex hypersurfaces to space forms. Further more, Li, Ren, and Wang [17] showed that the convex hypersurface in [12] can be substituted by

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$(k+1)$-convex hypersurface. Ren and Wang [19, 20] solved the case $k=n-1$ and $k=n-2$. For $3\le k\le n-3$, the existence of star-shaped k-convex hypersurface satisfying (1.2) is still open.

Equation (1.1) is motivated by some geometric problems. To ensure the ellipticity of (1.1), so called (η,k)-convex hypersurface is introduced in [5]. Namely

$$\lambda(\eta)\in\Gamma_k=\{\lambda\in\mathbb{R}^n:\sigma_i(\lambda)>0,\forall 1\le i\le k\}.$$

For example, when $k=n$, it becomes

$$\det(\eta(X))=f(X,\nu),\quad \text{for } X\in M. \tag{1.3}$$

The (η,n)-convex hypersurface has been studied intensively by Sha [21, 22], Wu [24], Harvey and Lawson [14]. (η,n)-convexity is called $(n-1)$-convexity in [14, 21, 22]. In complex geometry, when $k=n$, Equation (1.1) is called the $(n-1)$ Monge-Ampère equation, which is related to the Gauduchon conjecture (see [8]). Compared to (1.2), it is interesting that the curvature estimate of (1.1) can be established for $1\le k\le n$. Chu and Jiao [5] established curvature estimates for (η,k)-convex hypersurface and proved the existence for (1.1). Chen, Tu and Xiang [4] extended it to a class of Hessian quotient equations.

In this article, we give a simpler proof of the result of Chu and Jiao [5], and extend it to space forms. Let $N^{n+1}(K)$ be a space form of sectional curvature $K=-1,0$, or 1. It is known that the space forms can be viewed as Euclidean space $\mathbb{R}^{n+1}$ equipped with a metric tensor g^N, that is,

$$N^{n+1}(K)=(\mathbb{R}^{n+1},g^N),\ g^N=d\rho^2+\phi^2(\rho)dz^2,$$

where

$$\phi(\rho)=\begin{cases}\sin(\rho),\ \rho\in[0,\frac{\pi}{2}), & \text{if } K=1,\\ \rho,\ \rho\in[0,+\infty), & \text{if } K=0,\\ \sinh(\rho),\ \rho\in[0,+\infty), & \text{if } K=-1,\end{cases}$$

where dz^2 denotes the standard metric on $\mathbb{S}^n$ induced from $\mathbb{R}^{n+1}$. We define the vector field $V=\phi(\rho)\frac{\partial}{\partial\rho}$. In fact, V is a conformal Killing field in $N^{n+1}(K)$ and V is just the position vector field in $\mathbb{R}^{n+1}$. We consider the k-Hessian equation of $\lambda(\eta)$ in $N^{n+1}(K)$,

$$\sigma_k(\lambda(\eta))=f(V,\nu),\ 2\le k\le n, \tag{1.4}$$

and obtain the main result as follows.

Theorem 1.1. *Let $f(V,\nu)\in C^2(\Gamma)$ be a positive function and Γ be an open neighborhood of the unit normal bundle of M in $N^{n+1}\times\mathbb{S}^n$. Assume that there exist two positive constants r_1,r_2 and $r_1<1<r_2$, such that*

$$f(V,\frac{V}{|V|})\le C_n^k(n-1)^k\big(\frac{\phi'(r_2)}{\phi(r_2)}\big)^k,\quad \textit{for } \rho=r_2, \tag{1.5}$$

$$f(V,\frac{V}{|V|})\ge C_n^k(n-1)^k\big(\frac{\phi'(r_1)}{\phi(r_1)}\big)^k,\quad \textit{for } \rho=r_1, \tag{1.6}$$

$$\frac{\partial}{\partial\rho}\big[\phi^kf(V,\nu)\big]\le 0,\quad \textit{for } r_1\le\rho\le r_2. \tag{1.7}$$

Then there exists a $C^{4,\delta}$ closed star-shaped (η,k)-convex hypersurface satisfying (1.4) for any $\delta\in(0,1)$.

The rest of this article is organized as follows. In Section 2, we give some definitions and important formulas. In Section 3, we prove C^0, C^1 and C^2 estimates of (1.4). In Section 4, we give the proof for the existence, that is Theorem 1.1.

2. Preliminaries

In this section, we recall some geometric objects and related formulas on hypersurfaces in space forms. Let M be an immersed star-shaped hypersurface in $N^{n+1}(K)$, which is expressed as

$$M = \{(z, \rho(z)) : z \in \mathbb{S}^n\}.$$

Let ∇' and ∇ denote the covariant derivatives with respect to the standard spherical metric and the covariant derivatives with respect to the induced metric on M, respectively. Following the notations in [1], the induced metric, its inverse, unit normal vector and second fundamental form on M are respectively by

$$g_{ij} = \phi^2 e_{ij} + \nabla'_i\rho\nabla'_j\rho, \quad g^{ij} = \frac{1}{\phi^2}\Big(e^{ij} - \frac{\rho^i\rho^j}{\phi^2 + |\nabla'\rho|^2}\Big), \tag{2.1}$$

$$\nu = \frac{-\nabla'\rho + \phi^2\frac{\partial}{\partial\rho}}{\sqrt{\phi^4 + \phi^2|\nabla'\rho|^2}}\,, \tag{2.2}$$

$$h_{ij} = \frac{\phi}{\sqrt{\phi^2 + |\nabla'\rho|^2}}\Big(- \nabla'_{ij}\rho + \frac{2\phi'}{\phi}\nabla'_i\rho\nabla'_j\rho + \phi\phi' e_{ij}\Big). \tag{2.3}$$

where e_{ij} is the standard spherical metric and e^{ij} is inverse of it. We define $\Phi(\rho) = \int_0^\rho \phi(r)dr$ and $u = \langle V, \nu\rangle$. Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame on M. By direct calculations, we have the following formulas (see [13, 23]):

$$\nabla_i\Phi = \langle V, e_i\rangle, \quad \nabla_{ij}\Phi = \phi' g_{ij} - u h_{ij}, \tag{2.4}$$

$$\nabla_i u = g^{kl}h_{ik}\nabla_l\Phi, \tag{2.5}$$

$$\nabla_{ij}u = g^{kl}\nabla_k h_{ij}\nabla_l\Phi + \phi' h_{ij} - u g^{kl}h_{ik}h_{jl}, \tag{2.6}$$

$$\nabla_i\nu = g^{kl}h_{ik}e_l, \tag{2.7}$$

$$\begin{aligned} \nabla_{ij}h_{kl} = \nabla_{kl}h_{ij} - h_{ml}(h_{im}h_{kj} - h_{ij}h_{mk}) - h_{mj}(h_{mi}h_{kl} - h_{il}h_{mk}) \\ + K h_{ml}(\delta_{ij}\delta_{km} - \delta_{im}\delta_{kj}) + K h_{mj}(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}). \end{aligned} \tag{2.8}$$

For simplicity, we denote

$$G(\eta) := \sigma_k^{1/k}(\lambda(\eta)), \quad G^{ij}(\eta) := \frac{\partial G}{\partial\eta_{ij}}, \quad G^{ij,rs}(\eta) := \frac{\partial^2 G}{\partial\eta_{ij}\eta_{rs}}, \quad F^{ii} = \sum_{k\neq i} G^{kk}.$$

If (h_{ij}) is diagonal and $h_{11} \geq \dots \geq h_{nn}$, then

$$\eta_{11} \leq \dots \leq \eta_{nn}, \quad G^{11} \geq \dots \geq G^{nn}, \quad F^{11} \leq \dots \leq F^{nn}.$$

3. A priori estimates

In this section, we obtain C^0, C^1 and C^2 estimates for (1.4). Let us consider a family of functions, for $t \in [0, 1]$,

$$f^t(V, \nu) = tf(V, \nu) + (1-t)C_n^k(n-1)^k\Big[\Big(\frac{\phi'(\rho)}{\phi(\rho)}\Big)^k + \varepsilon\Big(\Big(\frac{\phi'(\rho)}{\phi(\rho)}\Big)^k - \Big(\frac{\phi'(1)}{\phi(1)}\Big)^k\Big)\Big], \tag{3.1}$$

where the constant ε is small sufficiently such that

$$\min_{r_1\le\rho\le r_2}\Big[\big(\frac{\phi'(\rho)}{\phi(\rho)}\big)^k+\varepsilon\Big(\big(\frac{\phi'(\rho)}{\phi(\rho)}\big)^k-\big(\frac{\phi'(1)}{\phi(1)}\big)^k\Big)\Big]\ge c_0>0.$$

It is easy to see that $f^t(V,\nu)$ satisfies (1.5), (1.6) and (1.7) with strict inequalities for $0<t<1$. To prove Theorem 1.1, we consider the family of equations

$$\sigma_k(\lambda(\eta))=f^t(V,\nu),\quad 0\le t\le 1. \tag{3.2}$$

3.1. C^0 **estimates.** Now, we prove the following proposition which asserts that the solutions of (3.2) have uniform C^0 bounds.

Proposition 3.1. *Let $f^t(V,\nu)\in C^2(N^{n+1}\times\mathbb{S}^n)$ is a positive function. Under assumptions* (1.5) *and* (1.6)*, if $M_t=\{(z,\rho(z)):z\in\mathbb{S}^n\}\subset N^{n+1(K)}$ is a star-shaped (η,k)-convex hypersurface satisfying Equation* (3.2) *for $0<t<1$, then $r_1<\rho_t<r_2$.*

Proof. Suppose that $\rho_t(z)$ attains its maximum at $z_0\in\mathbb{S}^n$ and $\rho_t(z_0)\ge r_2$. Then $\nabla'\rho=0$, at z_0. Therefore, from (2.1) and (2.3) we obtain

$$g^{ij}=\phi^{-2}e^{ij},\quad h_{ij}=-\nabla'_{ij}\rho+\phi\phi' e_{ij},$$

which implies that

$$h^i_j=g^{ik}h_{kj}=-\frac{e^{ik}\nabla'_{kj}\rho}{\phi^2}+\frac{\phi'}{\phi}\delta^i_j\ge\frac{\phi'}{\phi}\delta^i_j.$$

It follows that

$$\eta^i_j=H\delta^i_j-h^i_j\ge(n-1)\frac{\phi'}{\phi}\delta^i_j.$$

Noticing that σ_k is elliptic in Γ_k, we have

$$\sigma_k(\lambda(\eta))\ge C^k_n(n-1)^k\big(\frac{\phi'}{\phi}\big)^k. \tag{3.3}$$

On the other hand, the unit outer normal vector $\nu=\frac{V}{|V|}$ at z_0 and $f^t(V,\nu)$ satisfies (1.5) with strict inequality for $0<t<1$. If $\rho_t(z_0)=r_2$, then

$$C^k_n(n-1)^k\Big(\frac{\phi'(r_2)}{\phi(r_2)}\Big)^k>f^t(V,\frac{V}{|V|})=f^t(V,\nu)=\sigma_k(\lambda(\eta)). \tag{3.4}$$

This contradicts (3.3), and shows that . $\sup_{M_t}\rho_t<r_2$. Similarly, we prove $\inf_{M_t}\rho_t>r_1$. □

Now, we prove the following uniqueness result.

Proposition 3.2. *For $t=0$, there exists unique (η,k)-convex solution of Equation* (3.2)*, namely, M_0 is an unit sphere in $N^k(K)$.*

Proof. Let M_0 be a solution of (3.2) for $t=0$. Assume the height function $\rho(z)$ of M_0 achieves its maximum $\rho_{\max}$ at $z_0\in\mathbb{S}^n$, then

$$\begin{aligned}&C^k_n(n-1)^k\Big[\big(\frac{\phi'(\rho_{\max})}{\phi(\rho_{\max})}\big)^k+\varepsilon\Big(\big(\frac{\phi'(\rho_{\max})}{\phi(\rho_{\max})}\big)^k-\big(\frac{\phi'(1)}{\phi(1)}\big)^k\Big)\Big]\\&=\sigma_k(\lambda(\eta))\\&\ge C^k_n(n-1)^k\big(\frac{\phi'(\rho_{\max})}{\phi(\rho_{\max})}\big)^k,\end{aligned}$$

which implies

$$\frac{\phi'(\rho_{\max})}{\phi(\rho_{\max})} \geq \frac{\phi'(1)}{\phi(1)}. \tag{3.5}$$

Noting that

$$\frac{\phi'(\rho)}{\phi(\rho)} = \begin{cases} \cot(\rho), & \text{if } K = 1, \\ \frac{1}{\rho}, & \text{if } K = 0, \\ \coth(\rho), & \text{if } K = -1, \end{cases}$$

we obtain $\rho_{\max} \leq 1$. Similarly, $\rho_{min} \geq 1$. Thus, $\rho = 1$ is the unique solution of (3.2) for $t = 0$. □

3.2. C^1 **estimates.** In this section, we follow the ideas in [3] and [10] to obtain C^1 estimates for the height function ρ.

Proposition 3.3. *Let M be a closed star-shaped (η, k)-convex hypersurface in $N^k(K)$ satisfying* (3.2)*. Under assumption* (1.7)*, if ρ has positive upper and lower bounds, there exists a constant C depending on $inf_M \rho$, $\sup_M \rho$, and $\|f\|_{C^1(M)}$ such that $|\nabla\rho| \leq C$.*

Proof. Since

$$u = \langle V, \nu\rangle = \frac{\phi^2}{\phi^2 + |\nabla'\rho|^2},$$

it is sufficient to obtain a positive lower bound of u. We consider a test function

$$\varphi = -\log u + \gamma(\Phi(\rho)),$$

where $\gamma(t)$ is a function which will be chosen later. Assume that φ achieves its maximum value at $z_0 \in \mathbb{S}^n$, we will show that $u(z_0) = |V(z_0)|$, that is, $V(z_0) = \phi(\rho(z_0))\nu(z_0)$, which implies a uniform lower bound for u on M. If not, we may choose a local orthonormal frame $\{e_1, \dots, e_n\}$ around $(z_0, \rho(z_0)) \in M$ such that $\langle V, e_1\rangle \neq 0$ and $\langle V, e_i\rangle = 0$, $i \geq 2$. Using (2.5), we have at $(z_0, \rho(z_0)) \in M$,

$$0 = \nabla_i\varphi = -\frac{\nabla_i u}{u} + \gamma'\nabla_i\Phi = -\frac{h_{i1}\langle V, e_1\rangle}{u} + \gamma'\langle V, e_i\rangle. \tag{3.6}$$

It follows from (3.6) that

$$h_{11} = u\gamma', \quad h_{i1} = 0, \quad i \geq 2. \tag{3.7}$$

Rotate $\{e_2, \dots, e_n\}$ around $(z_0, \rho(z_0)) \in M$ such that h_{ij} is diagonal. Covariantly differentiating φ twice yields

$$\begin{aligned} 0 &\geq F^{ii}\nabla_{ii}\varphi \\ &= -F^{ii}\frac{\nabla_{ii}u}{u} + F^{ii}\frac{|\nabla_i u|^2}{u^2} + \gamma'' F^{ii}|\nabla_i\Phi|^2 + \gamma' F^{ii}\nabla_{ii}\Phi \\ &= -\frac{1}{u}F^{ii}\big(h_{ii1}\nabla_1\Phi + \phi' h_{ii} - u h_{ii}^2\big) + \big((\gamma')^2 + \gamma''\big)F^{ii}|\nabla_i\Phi|^2 \\ &\quad + \gamma' F^{ii}\big(\phi'\delta_{ii} - u h_{ii}\big), \end{aligned} \tag{3.8}$$

where the second equality is given by using (2.4), (2.5) and (2.6). Then

$$\eta_{ii} = \sum_{j\neq i} h_{jj}$$

implies

$$\sum_i \eta_{ii} = (n-1)\sum_i h_{ii}, \quad h_{ii} = \frac{1}{n-1}\sum_k \eta_{kk} - \eta_{ii},$$

which results in

$$\begin{aligned} \sum_i F^{ii}h_{ii} &= \sum_i \big(\sum_k G^{kk} - G^{ii}\big)\big(\frac{1}{n-1}\sum_k \eta_{kk} - \eta_{ii}\big) \\ &= \sum_i G^{ii}\eta_{ii} = f^{1/k}(V,\nu), \end{aligned} \tag{3.9}$$

$$\sum_i F^{ii}h_{iij} = \sum_i G^{ii}\eta_{iij}. \tag{3.10}$$

Notice that (1.4) can be written as

$$G(\eta) = f^{1/k}(V,\nu) = \widetilde{f}(V,\nu). \tag{3.11}$$

By (2.7) and covariantly differentiating (3.11) with respect to e_1, we have

$$G^{ii}\eta_{ii1} = d_V\widetilde{f}(\nabla_{e_1}V) + h_{11}d_\nu\widetilde{f}(e_1). \tag{3.12}$$

Taking (2.4), (3.9), (3.10) and (3.12) in (3.8) yields

$$\begin{aligned} 0 &\geq -\frac{1}{u}\Big(d_V\widetilde{f}(\nabla_{e_1}V)\langle V,e_1\rangle + \phi'\widetilde{f} + h_{11}d_\nu\widetilde{f}(e_1)\langle V,e_1\rangle\Big) \\ &\quad + \big((\gamma')^2+\gamma''\big)F^{11}\langle V,e_1\rangle^2 + \gamma'\phi'\sum_i F^{ii} - \gamma' u\widetilde{f} \\ &\geq -\frac{1}{u}\Big(d_V\widetilde{f}(\nabla_{e_1}V)\langle V,e_1\rangle + \phi'\widetilde{f}\Big) - \gamma' d_\nu\widetilde{f}(e_1)\langle V,e_1\rangle \\ &\quad + \big((\gamma')^2+\gamma''\big)F^{11}\langle V,e_1\rangle^2 + \gamma'\phi'\sum_i F^{ii} - \gamma' u\widetilde{f}, \end{aligned} \tag{3.13}$$

where the second inequality is obtained by (3.7). Since $V = \langle V,e_1\rangle e_1 + \langle V,\nu\rangle\nu$ at z_0,

$$d_V\widetilde{f}(V) = \langle V,e_1\rangle\big(d_V\widetilde{f}\big)(\nabla_{e_1}V) + u\big(d_V\widetilde{f}\big)(\nabla_\nu V). \tag{3.14}$$

From this and the assumption (1.7), we see that

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial\rho}\big(\phi^k f(V,\nu)\big) = k\big(\phi\widetilde{f}\big)^{k-1}\big(\phi'\widetilde{f} + d_V\widetilde{f}(V)\big) \\ &= k\big(\phi\widetilde{f}\big)^{k-1}\Big(\phi'\widetilde{f} + \langle V,e_1\rangle\big(d_V\widetilde{f}\big)(\nabla_{e_1}V) + u\big(d_V\widetilde{f}\big)(\nabla_\nu V)\Big). \end{aligned} \tag{3.15}$$

Combining this with (3.13) gives

$$\begin{aligned} 0 &\geq d_V\widetilde{f}(\nabla_\nu V) + \big((\gamma')^2+\gamma''\big)F^{11}\langle V,e_1\rangle^2 + \gamma'\phi'\sum_i F^{ii} \\ &\quad - \gamma' u\widetilde{f} - \gamma' d_\nu\widetilde{f}(e_1)\langle V,e_1\rangle. \end{aligned} \tag{3.16}$$

Now we choose

$$\gamma(t) = \frac{\alpha}{t}, \tag{3.17}$$

where α is sufficiently large. Recalling that $h_{11} = \gamma' u$ at $(z_0,\rho(z_0))$, we have $h_{11} < 0$. Since $H > 0$, there exists k_0 with $2 \leq k_0 \leq n$ such that $h_{k_0k_0} > h_{11}$. Combining this with the definitions of η_{ii} and G^{ii} yields

$$\eta_{K_0k_0} < \eta_{11}, \quad G^{k_0k_0} \geq G^{11}.$$

Thus,

$$F^{11} = \sum_{j\neq 1} G^{jj} \geq \frac{1}{2}\sum_i G^{ii} = \frac{1}{2(n-1)}\sum_i F^{ii} \geq \frac{1}{2}\big(C_n^k\big)^{1/k}. \tag{3.18}$$

Putting (3.17) and (3.18) in (3.16), we obtain

$$\begin{aligned} 0 \geq & \frac{\langle V, e_1\rangle^2}{2(n-1)}\big(\alpha^2\Phi^{-4} + 4\alpha^2\Phi^{-6}\big)\sum_i F^{ii} - \alpha\Phi^{-2}\phi'\sum_i F^{ii} \\ & - \alpha\Phi^{-2}\big|V\big|\big|d_\nu \widetilde{f}(e_1)\big| - \big|d_V\widetilde{f}(\nabla_\nu V)\big|, \end{aligned} \tag{3.19}$$

which leads to a contradiction when α is large. Therefore $u(z_0) = |V(z_0)|$. $\square$

3.3. C^2 **estimates.** To obtain C^2 estimates for (3.2), we prove that the principal curvatures have uniform bounds.

Proposition 3.4. *Let $M = \{(z,\rho(z)) : z \in \mathbb{S}^n\}$ be a closed star-shaped (η,k)-convex hypersurface in $N^k(K)$ satisfying (3.2), where $f(V,\nu) \in C^2(\Gamma)$ is a positive function and Γ is an open neighborhood of the unit normal bundle of M in $N^{n+1} \times \mathbb{S}^n$. If $0 < r_1 \leq \rho(z) \leq r_2$, $\|\rho\|_{C^1} \leq r_3$, then there exists a constant C depending on n, k, r_1, r_2, r_3, $\|f\|_{C^2(M)}$ and $\inf_M f$ such that*

$$\max_{\mathbb{S}^n} |\kappa_i| \leq C, \quad \textit{for} \ \ 1 \leq i \leq n,$$

where $(\kappa_1,\dots,\kappa_n)$ is the principal curvatures vector of M.

Proof. Since $H > 0$, it suffices to prove that the largest curvature $\kappa_{\max}$ is uniformly bounded from above. From Propositions 3.1 and 3.3, we know that

$$\frac{1}{C} \leq \inf_M u \leq u \leq \sup_M u \leq C,$$

where the positive constant C depends on $\inf_M \rho$ and $\|\rho\|_{C^1}$. Taking the auxiliary function

$$Q = \frac{e^{\beta\Phi}\kappa_{\max}}{u-a}, \tag{3.20}$$

where $a = \frac{1}{2}\inf_M u$ and β is a large constant to be determined later. Assume that $(z_0,\rho(z_0))$ is the maximum point of the function Q, we can choose a local orthonormal frame $\{e_1,\dots,e_n\}$ around $(z_0,\rho(z_0))$ such that h_{ij} is diagonal and $h_{11} \geq \dots \geq h_{nn}$ at $(z_0,\rho(z_0))$. In the rest of proof, all computations will be carried out at $(z_0,\rho(z_0))$. Since $h_{11} = \kappa_{\max}$, the function

$$\log Q = \log h_{11} - \log(u-a) + \beta\Phi$$

has a local maximum at $(z_0,\rho(z_0))$. Therefore,

$$0 = \frac{\nabla_i h_{11}}{h_{11}} - \frac{\nabla_i u}{u-a} + \beta\nabla_i\Phi, \tag{3.21}$$

$$0 \geq \frac{F^{ii}\nabla_{ii}h_{11}}{h_{11}} - \frac{F^{ii}\big(\nabla_i h_{11}\big)^2}{h_{11}^2} - \frac{F^{ii}\nabla_{ii}u}{u-a} + \frac{F^{ii}\big(\nabla_i u\big)^2}{(u-a)^2} + \beta F^{ii}\nabla_{ii}\Phi. \tag{3.22}$$

By (2.4) and (3.9), we have

$$\beta F^{ii}\nabla_{ii}\Phi = \beta\phi'\sum_i F^{ii} - \beta u\widetilde{f}. \tag{3.23}$$

It follows from (2.6) and (3.12) that

$$\begin{aligned} -\frac{F^{ii}\nabla_{ii}u}{u-a} &= -\frac{F^{ii}h_{iij}\nabla_j\Phi}{u-a} - \frac{\phi'\widetilde{f}}{u-a} + \frac{uF^{ii}h_{ii}^2}{u-a} \\ &\geq -\frac{d_V\widetilde{f}(\nabla_{e_i}V)\nabla_i\Phi}{u-a} - \frac{h_{ii}d_\nu\widetilde{f}(e_i)\nabla_i\Phi}{u-a} - \frac{\phi'\widetilde{f}}{u-a} + \frac{uF^{ii}h_{ii}^2}{u-a}. \end{aligned} \tag{3.24}$$

Applying (2.8) and (3.9), we obtain

$$\begin{aligned} F^{ii}\nabla_{ii}h_{11} &= F^{ii}\nabla_{11}h_{ii} - h_{11}F^{ii}h_{ii}^2 + F^{ii}h_{ii}h_{11}^2 \\ &\quad - KF^{ii}(h_{11}\delta_{1i}^2 - h_{11}\delta_{ii} + h_{ii} - h_{i1}\delta_{i1}) \\ &= F^{ii}\nabla_{11}h_{ii} - h_{11}F^{ii}h_{ii}^2 + \widetilde{f}h_{11}^2 + Kh_{11}\sum_i F^{ii} - \widetilde{f}K. \end{aligned} \tag{3.25}$$

Covariantly differentiating (3.11) twice yields

$$F^{ii}\nabla_{11}h_{ii} = G^{ii}\nabla_{11}\eta_{ii} \geq -G^{ij,rs}\nabla_1\eta_{ij}\nabla_1\eta_{rs} + \sum_i h_{11i}d_\nu\widetilde{f}(e_i) - C_1(1+h_{11}^2), \tag{3.26}$$

where the positive constant C_1 depends on $\|f\|_{C^2}$. The concavity of G and Codazzi formula give

$$G^{ij,rs}\nabla_1\eta_{ij}\nabla_1\eta_{rs} \geq -2\sum_{i\geq 2} G^{1i,i1}|\nabla_1\eta_{1i}|^2 = -2\sum_{i\geq 2} G^{1i,i1}|\nabla_i h_{11}|^2. \tag{3.27}$$

Combining (3.25) and (3.26) with (3.27), we obtain

$$\begin{aligned} \frac{F^{ii}\nabla_{ii}h_{11}}{h_{11}} &\geq -\frac{2}{h_{11}}\sum_{i\geq 2} G^{1i,i1}|\nabla_i h_{11}|^2 - F^{ii}h_{ii}^2 + \frac{h_{11i}d_\nu\widetilde{f}(e_i)}{h_{11}} \\ &\quad + K\sum_i F^{ii} + h_{11}\widetilde{f} - \frac{K\widetilde{f}}{h_{11}} - C_1(\frac{1}{h_{11}} + h_{11}). \end{aligned} \tag{3.28}$$

Putting (3.23), (3.24) and (3.28) in (3.22),

$$\begin{aligned} 0 &\geq -\frac{2}{h_{11}}\sum_{i\geq 2} G^{1i,i1}|\nabla_i h_{11}|^2 - \frac{F^{ii}|\nabla_i h_{11}|^2}{h_{11}^2} + \frac{a}{u-a}F^{ii}h_{ii}^2 + \frac{F^{ii}|\nabla_i u|^2}{(u-a)^2} \\ &\quad + \sum_i\Big(\frac{\nabla_i h_{11}}{h_{11}} - \frac{h_{ii}\nabla_i\Phi}{u-a}\Big)d_\nu\widetilde{f}(e_i) + (K+\beta\phi')\sum_i F^{ii} - C_2(1+h_{11}) \\ &\geq -\frac{2}{h_{11}}\sum_{i\geq 2} G^{1i,i1}|\nabla_i h_{11}|^2 - \frac{F^{ii}|\nabla_i h_{11}|^2}{h_{11}^2} + \frac{a}{u-a}F^{ii}h_{ii}^2 + \frac{F^{ii}|\nabla_i u|^2}{(u-a)^2} \\ &\quad + (K+\beta\phi')\sum_i F^{ii} - C_2(\beta + h_{11}), \end{aligned} \tag{3.29}$$

where C_2 depends r_1, r_2, r_3, and $\|f\|_{C^2}$. The second inequality is obtained by (3.21).

We divide the rest of proof into three steps.

Step 1. We prove that

$$\frac{a}{2(u-a)}F^{ii}h_{ii}^2 + \frac{1}{2}(K+\beta\phi')\sum_i F^{ii} \geq C_2 h_{11}. \tag{3.30}$$

The proof of step 1 is split into two cases.

Case 1. $|h_{ii}| \le \delta h_{11}$ for all $2 \le i \le n$, δ is a small constant to be chosen. We obtain

$$|\eta_{11}| \le (n-1)\delta h_{11}, \quad \big(1-(n-2)\delta\big)h_{11} \le \eta_{22} \le \dots \le \eta_{nn} \le \big(1+(n-2)\delta\big)h_{11}. \tag{3.31}$$

This shows that

$$\begin{aligned} \sigma_{k-1}(\eta) &= \sigma_{k-1}(\eta|1) + \eta_{11}\sigma_{k-2}(\eta|1) \\ &\ge C_{n-1}^{k-1}\big(1-(n-2)\delta\big)^{k-1} h_{11}^{k-1} \\ &\quad - C_{n-1}^{k-2}\big(1+(n-2)\delta\big)\big(1-(n-2)\delta\big)^{k-2} h_{11}^{k-1}. \end{aligned} \tag{3.32}$$

Choosing δ sufficiently small and using $k \ge 2$, we have

$$\sigma_{k-1}(\eta) \ge \frac{1}{2} h_{11}^{k-1} \ge \frac{1}{2} h_{11}. \tag{3.33}$$

It follows from (3.33) and the definitions of G^{ii} and F^{ii} that

$$\begin{aligned} \sum_i F^{ii} = (n-1)\sum_i G^{ii} &= \frac{(n-1)(n-k+1)}{k}\sigma_k^{\frac{1}{k}-1}(\eta)\sigma_{k-1}(\eta) \\ &\ge \frac{(n-1)(n-k+1)}{2k\inf_M f^{1-\frac{1}{k}}} h_{11}. \end{aligned} \tag{3.34}$$

Choosing β sufficiently large gives

$$\frac{1}{2}(K+\beta\phi')\sum_i F^{ii} \ge C_2 h_{11}. \tag{3.35}$$

Case 2. $h_{22} > \delta h_{11}$ or $h_{nn} < -\delta h_{11}$. We obtain

$$\begin{aligned} \frac{a}{2(u-a)} F^{ii} h_{ii}^2 &\ge \frac{a}{2(\sup_M u - a)}\big(F^{22}h_{22}^2 + F^{nn}h_{nn}^2\big) \\ &\ge \frac{a\delta^2}{2(\sup_M u - a)} F^{22} h_{11}^2. \end{aligned} \tag{3.36}$$

Applying Maclaurin's inequality, we have

$$F^{22} = \sum_{i\ne 2} G^{ii} \ge \frac{1}{2}\sum_i G^{ii} \ge \frac{1}{2}(C_n^k)^{1/k}. \tag{3.37}$$

Inserting into (3.36) yields

$$\frac{a}{2(u-a)} F^{ii} h_{ii}^2 \ge \frac{a\delta^2}{4(\sup_M u - a)}(C_n^k)^{1/k} h_{11}^2 \ge C_2 h_{11}, \tag{3.38}$$

where the second inequality is obtained from

$$h_{11} \ge \frac{4(\sup_M u - a)}{a\delta^2}(C_n^k)^{-\frac{1}{k}} C_2,$$

otherwise, the proof is complete.

Step 2. We prove that

$$|h_{ii}| \le \beta C_3, \quad \text{for } 2 \le i \le n,$$

where C_3 depends r_1, r_2, r_3, and $\|f\|_{C^2}$. Combining step 1 and (3.29) gives

$$\begin{aligned} 0 \geq & -\frac{2}{h_{11}} \sum_{i \geq 2} G^{1i,i1} |\nabla_i h_{11}|^2 - \frac{F^{ii}|\nabla_i h_{11}|^2}{h_{11}^2} + \frac{a}{2(u-a)} F^{ii} h_{ii}^2 \\ & + \frac{F^{ii}|\nabla_i u|^2}{(u-a)^2} + \frac{1}{2}(K + \beta\phi') \sum_i F^{ii} - C_2 \beta. \end{aligned} \tag{3.39}$$

From (3.21) and Cauchy-Schwarz inequality, we have

$$-\frac{F^{ii}|\nabla_i h_{11}|^2}{h_{11}^2} \geq -\frac{1+\varepsilon}{(u-a)^2} F^{ii} |\nabla_i u|^2 - (1 + \frac{1}{\varepsilon}) \beta^2 F^{ii} |\nabla_i \Phi|^2. \tag{3.40}$$

Note that

$$-\frac{2}{h_{11}} \sum_{i \geq 2} G^{1i,i1} |\nabla_i h_{11}|^2 \geq 0. \tag{3.41}$$

Using (3.40) and (3.41) in (3.39) yields

$$\begin{aligned} 0 \geq & \Big(\frac{a}{2(u-a)} - \frac{\varepsilon |\nabla \Phi|^2}{(u-a)^2} \Big) F^{ii} h_{ii}^2 - C_2 \beta \\ & + \Big(\frac{1}{2}(K + \beta\phi') - (1 + \frac{1}{\varepsilon}) \beta^2 |\nabla \Phi|^2 \Big) \sum_i F^{ii}, \end{aligned} \tag{3.42}$$

where $\nabla_i u = h_{ii} \nabla_i \Phi$. Recalling that

$$F^{ii} \geq F^{22} \geq \frac{1}{2(n-1)} \sum_i F^{ii} \geq \frac{1}{2} (C_n^k)^{1/k}$$

and choosing ε sufficiently small such that

$$\frac{a}{2(u-a)} - \frac{\varepsilon |\nabla \Phi|^2}{(u-a)^2} \geq c_0 > 0,$$

we deduce that

$$0 \geq \frac{c_0}{2(n-1)} \sum_{j \geq 2} h_{jj}^2 + \Big(\frac{1}{2}(K + \beta\phi') - (1 + \frac{1}{\varepsilon}) \beta^2 |\nabla \Phi|^2 \Big) - \frac{C_2 \beta}{\sum_i F^{ii}}. \tag{3.43}$$

Therefore, $\sum_{i \geq 2} h_{ii}^2 \leq \beta^2 C_3^2$.

Step 3. We show that there exists a constant C depending r_1, r_2, r_3, $\|f\|_{C^2}$, and $\inf_M f$, such that $h_{11} \leq C$.

From (3.21) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & -\frac{F^{ii}|\nabla_i h_{11}|^2}{h_{11}^2} \\ & \geq -\frac{1+\varepsilon}{(u-a)^2} F^{11} |\nabla_1 u|^2 - (1 + \frac{1}{\varepsilon}) \beta^2 F^{11} |\nabla_1 \Phi|^2 - \sum_{i \geq 2} \frac{F^{ii}|\nabla_i h_{11}|^2}{h_{11}^2}. \end{aligned} \tag{3.44}$$

Choosing ε sufficiently small, we obtain

$$-\frac{\varepsilon}{(u-a)^2} F^{11} |\nabla_1 u|^2 = -\frac{\varepsilon |\nabla_1 \Phi|^2}{(u-a)^2} F^{11} h_{11}^2 \geq -\frac{a}{16(u-a)} F^{ii} h_{ii}^2. \tag{3.45}$$

Without loss of generality, we assume that

$$h_{11}^2 \geq \max \Big\{ \frac{32(\sup_M u - a)\beta^2}{a\varepsilon} |\nabla \Phi|^2, \frac{\beta^2 C_3^2}{\alpha^2} \Big\},$$

where α will be determined later ($\alpha < 1$). This gives

$$-(1+\frac{1}{\varepsilon})\beta^2 F^{11}|\nabla_1\Phi|^2 \geq -\frac{2}{\varepsilon}\beta^2 F^{11}|\nabla\Phi|^2 \geq -\frac{a}{16(u-a)}F^{ii}h_{ii}^2. \tag{3.46}$$

By step 2,

$$|h_{ii}| \leq \alpha h_{11}, \quad \text{for } i \geq 2, \tag{3.47}$$

which implies that

$$\frac{1}{h_{11}} \leq \frac{1+\alpha}{h_{11}-h_{ii}}. \tag{3.48}$$

Noting that

$$-G^{1i,i1} = \frac{G^{11}-G^{ii}}{\eta_{ii}-\eta_{11}} = \frac{F^{ii}-F^{11}}{h_{11}-h_{ii}},$$

we have

$$\begin{aligned}
-\sum_{i\geq 2}\frac{F^{ii}|\nabla_i h_{11}|^2}{h_{11}^2} &\geq -\sum_{i\geq 2}\frac{F^{ii}-F^{11}}{h_{11}^2}|\nabla_i h_{11}|^2 - \sum_{i\geq 2}\frac{F^{11}|\nabla_i h_{11}|^2}{h_{11}^2} \\
&\geq -\frac{1+\alpha}{h_{11}}\sum_{i\geq 2}\frac{F^{ii}-F^{11}}{h_{11}-h_{ii}}|\nabla_i h_{11}|^2 - \sum_{i\geq 2}\frac{F^{11}|\nabla_i h_{11}|^2}{h_{11}^2} \\
&= \frac{1+\alpha}{h_{11}}\sum_{i\geq 2}G^{i1,1i}|\nabla_i h_{11}|^2 - \sum_{i\geq 2}\frac{F^{11}|\nabla_i h_{11}|^2}{h_{11}^2}.
\end{aligned} \tag{3.49}$$

Using (3.21), (3.47), and Cauchy-Schwarz inequality we have

$$\begin{aligned}
&-\sum_{i\geq 2}\frac{F^{11}|\nabla_i h_{11}|^2}{h_{11}^2} \\
&\geq -2\sum_{i\geq 2}\frac{F^{11}|\nabla_i u|^2}{(u-a)^2} - 2\beta^2\sum_{i\geq 2}F^{11}|\nabla_i\Phi|^2 \\
&\geq -\frac{2(n-1)\alpha^2|\nabla\Phi|^2}{a^2}\frac{aF^{11}h_{11}^2}{u-a} - \frac{\varepsilon(u-a)}{16(\sup_M u - a)}\frac{aF^{11}h_{11}^2}{u-a}.
\end{aligned} \tag{3.50}$$

Choosing α sufficiently small gives

$$-\sum_{i\geq 2}\frac{F^{11}|\nabla_i h_{11}|^2}{h_{11}^2} \geq -\frac{aF^{11}h_{11}^2}{8(u-a)} \geq -\frac{aF^{ii}h_{ii}^2}{8(u-a)}. \tag{3.51}$$

Putting (3.44), (3.45), (3.46), (3.49), and (3.51) in (3.39) yields

$$0 \geq \frac{F^{ii}|\nabla_i u|^2}{4(u-a)^2} + \frac{1}{2}(K+\beta\phi')\sum_i F^{ii} - C_2\beta \geq \frac{C_2}{2}h_{11} - C_2\beta. \tag{3.52}$$

Thus $h_{11} \leq 2\beta$. □

4. Existence

In this section, we use the degree theory for nonlinear elliptic equation developed in [16] to prove Theorem 1.1. After establishing the a priori estimates in Propositions 3.1, 3.3 and 3.4, we know that (3.2) is uniformly elliptic. From Evans-Krylov estimates [7, 15], and Schauder estimates, we obtain

$$\|\rho\|_{C^{4,\delta}} \leq C \tag{4.1}$$

for any (η,k)-convex solution $M=\{(z,\rho(z)):z\in\mathbb{S}^n\}$ to (1.4). We consider a family of the mappings for $t\in[0,1]$, $F(\cdot;t):C_0^{4,\delta}(\mathbb{S}^n)\to C^{2,\delta}(\mathbb{S}^n)$, defined by

$$F(z,\rho(z);t)=\sigma_k(\lambda(\eta))-f^t(V,\nu),$$

where

$$f^t(V,\nu)=tf(V,\nu)+(1-t)C_n^k(n-1)^k\Big[\big(\frac{\phi'(\rho)}{\phi(\rho)}\big)^k+\varepsilon\Big(\big(\frac{\phi'(\rho)}{\phi(\rho)}\big)^k-\big(\frac{\phi'(1)}{\phi(1)}\big)^k\Big)\Big],$$

where the constant ε is sufficiently small such that

$$\min_{r_1\le\rho\le r_2}\Big[\big(\frac{\phi'(\rho)}{\phi(\rho)}\big)^k+\varepsilon\Big(\big(\frac{\phi'(\rho)}{\phi(\rho)}\big)^k-\big(\frac{\phi'(1)}{\phi(1)}\big)^k\Big)\Big]\ge c_0>0,$$

for some positive constant c_0. We set

$$\mathcal{O}_R=\{\rho\in C_0^{4,\delta}(\mathbb{S}^n):\|\rho\|_{C^{4,\delta}(\mathbb{S}^n)}<R\},$$

which is an open set of $C_0^{4,\delta}(\mathbb{S}^n)$. If R is sufficiently large, $F(z,\rho(z);t)=0$ has no solution on $\partial\mathcal{O}_R$ by the a priori estimates in (4.1). Therefore, the degree of $\deg(F(\cdot;t),\mathcal{O}_R,0)$ is well-defined. Using the homotopic invariance of the degree, we have

$$\deg(F(\cdot;1),\mathcal{O}_R,0)=\deg(F(.;0),\mathcal{O}_R,0).$$

At $t=0$, by Proposition 3.2, $\rho_0=1$ is the unique solution of (3.2) in $\mathcal{O}_R$. Direct calculations yields

$$F(z,\rho;0)=-\varepsilon C_n^k(n-1)^k\Big(\big(\frac{\phi'(\rho)}{\phi(\rho)}\big)^k-\big(\frac{\phi'(1)}{\phi(1)}\big)^k\Big).$$

By the definition of $\phi(\rho)$, we obtain

$$\begin{aligned}\delta_{\rho_0}F(z,\rho_0;0)&=\frac{d}{ds}|_{s=1}F(z,s\rho_0;0)\\&=-\varepsilon kC_n^k(n-1)^k\big(\frac{\phi'(1)}{\phi(1)}\big)^{k-1}\frac{\phi''(1)\phi(1)-\phi'(1)\phi'(1)}{\big(\phi(1)\big)^2}>0,\end{aligned}$$

where $\delta F(z,\rho_0;0)$ is the linearized operator of F at ρ_0. Then $\delta F(z,\rho_0;0)$ takes the form

$$\delta_\varphi F(z,\rho_0;0)=-a^{ij}\nabla'_{ij}\varphi+b^i\nabla'_i\varphi-\varepsilon kC_n^k(n-1)^k\big(\frac{\phi'(1)}{\phi(1)}\big)^{k-1}\frac{\phi''(1)\phi(1)-\phi'(1)\phi'(1)}{\big(\phi(1)\big)^2},$$

where (a^{ij}) is a positive definite matrix. Clearly, $\delta_{\rho_0}F(z,\rho_0;0)$ is an invertible operator. Therefore,

$$\deg(F(.;1),\mathcal{O}_R,0)=\deg(F(.;0),\mathcal{O}_R,0)\neq 0.$$

It implies that there is a solution of Equation (3.2) at $t=1$. This completes the proof of Theorem 1.1.

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