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WELL-POSEDNESS AND ENERGY DECAY OF A TRANSMISSION PROBLEM OF KIRCHHOFF TYPE WAVE EQUATIONS WITH DAMPING AND DELAY TERMS

ZHIQING LIU, CUNCHEN GAO, ZHONG BO FANG

Abstract. We consider a transmission problem of Kirchhoff type wave equations with delay and damping terms, subject to a memory condition on one part of the boundary. Under appropriate hypotheses on the relaxation function and the relationship between weights of damping and delay terms, we establish well-posedness of the problem. Using the Faedo-Galerkin approximation technique, and introducing suitable energy and Lyapunov functionals, we obtain estimates for exponential, polynomial, and logarithmic decay.

1. Introduction

We consider a transmission problem of Kirchhoff type wave equations with damping and delay terms,

$$u_{tt} - (1 + \|\nabla u\|^2_{\Omega_1})\Delta u + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0, \quad (x,t) \in S_1, \tag{1.1}$$

$$v_{tt} - (1 + \|\nabla v\|^2_{\Omega_2})\Delta v = 0, \quad (x,t) \in S_2, \tag{1.2}$$

subject to boundary and transmission conditions

$$v = 0, \quad (x,t) \in \partial S_0, \tag{1.3}$$

$$(1 + \|\nabla u\|^2_{\Omega_1})\frac{\partial u}{\partial \nu} = (1 + \|\nabla v\|^2_{\Omega_2})\frac{\partial v}{\partial \nu}, \quad u = v, \quad (x,t) \in \partial S_1, \tag{1.4}$$

$$u + \int_0^t g(t-s)(1 + \|\nabla u(s)\|^2_{\Omega_1})\frac{\partial u(s)}{\partial \nu}\mathrm{d}s = 0, \quad (x,t) \in \partial S_2, \tag{1.5}$$

and initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega_1, \tag{1.6}$$

$$u_t(x,t-\tau) = f_0(x,t-\tau), \quad (x,t) \in \Omega_1 \times (0,\tau), \tag{1.7}$$

$$v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega_2. \tag{1.8}$$

Here, $S_i := \Omega_i \times (0,+\infty)$ and $\partial S_j := \Gamma_j \times (0,+\infty)$ with $i = 1,2$ and $j = 0,1,2$, where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_2$, $\overline{\Gamma_0} \cap \overline{\Gamma_2} = \emptyset$. Γ_0 is the boundary of small ball $B(x_0)$ containing x_0 in Ω, $\Omega_2 \subset \Omega$ is a subdomain with smooth boundary $\Gamma_0 \cup \Gamma_1$ in the outside of $B(x_0)$, and $\Omega_1 =$

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$\Omega \setminus (\overline{\Omega_2} \cup B(x_0))$ is a subdomain with smooth boundary $\Gamma_1 \cup \Gamma_2$. ν denotes the unit outer normal vector pointing towards the exterior of Ω_1 and there exists $\delta > 0$, such that $m \cdot \nu \geq \delta > 0$ on Γ_2, where $m := m(x) = x - x_0$ (see Figure 1 for an example). Moreover, μ_1 and μ_2 are positive constants, $\tau > 0$ is the delay, g is a positive function, and f_0 is the given history belonging to suitable spaces.

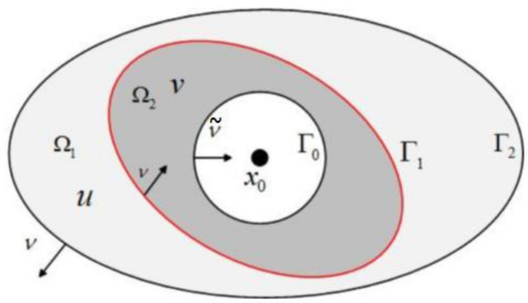

FIGURE 1. Domain Ω.

Our transmission model (1.1)-(1.8) arises in several applications in physics and biology, such as models of the transverse vibrations of a membrane composed by two different materials in Ω_1 and Ω_2.

In the past decades, there many authors investigated wave equations and systems with damping terms and showed that the dissipation produced by internal or boundary damping can lead to the decay of solutions, see [8, 9, 16, 17, 18, 19, 24, 33] and the references therein. For examples, Cavalcanti et al. [16] studied the mixed initial boundary value problem of linear degenerate wave equations with nonlinear boundary damping and boundary memory sources

$$\begin{gathered}
\rho_1(x,t)u_{tt} + \rho_2(x,t)u_t - \Delta u = 0,\ (x,t) \in \Omega \times (0,+\infty), \\
\frac{\partial u}{\partial \nu} + u + u_t + g(t)|u_t|^\rho u_t = g * |u|^\gamma u, \quad (x,t) \in \partial S_0, \\
u = 0, \quad (x,t) \in \partial S_1, \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega,
\end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ of C^2, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Meantime, Γ_0 and Γ_1 possess positive measures with

$$\begin{gathered}
\Gamma_0 := \{x \in \partial\Omega :\ \nu \cdot (x - x_0) \leq 0\}, \\
\Gamma_1 := \{x \in \partial\Omega :\ \nu \cdot (x - x_0) > 0\}.
\end{gathered}$$

They established the existence and exponential decay estimates of the global solutions. Later, Park and Bae [8] considered a Kirchhoff type wave equation

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u - \Delta u_t = 0, \quad (x,t) \in \Omega \times (0,+\infty),$$

and obtained the same conclusion with [16] under similar conditions. Santos et al. [17] investigated the Kirchhoff type wave equation

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u - \Delta u_t + f(u) = 0, \quad (x,t) \in \Omega \times (0,+\infty),$$

with boundary conditions

$$u = 0, \quad (x,t) \in \partial S_0,$$
$$u + \int_0^t g(t-s)(M(\|\nabla u\|_2^2)\frac{\partial u}{\partial \nu}(s) + \frac{\partial u_s}{\partial \nu}(s))\mathrm{d}s = 0, \quad (x,t) \in \partial S_1.$$

They proved that the energy decays with the same rate to the relaxation function, that is, the energy decays exponentially or polynomially provided the relaxation function decays exponentially or polynomially, respectively. Bae [9] considered the coupled wave equation of Kirchhoff type

$$u_{tt} - (1 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2)\Delta u + |u|^\alpha u = 0, \quad (x,t) \in \Omega \times (0,+\infty),$$
$$v_{tt} - (1 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2)\Delta v + |v|^\beta v = 0, \quad (x,t) \in \Omega \times (0,+\infty),$$

subject to mixed boundary conditions, and obtained the similar conclusion with [17]. We refer to [18, 19, 24] on the decay estimates of degenerate wave equations with localized damping and viscoelastic damping and linear systems with boundary memory dissipation.

Most recently, for the research advances on the ground state solutions for quasilinear equations of Kirchhoff type and multiple positive solutions to the fractional Kirchhoff problem, one can see [13, 14].

It is well known that delay effects, which arise in many practical problems, may be the sources of instability. Hence, the control of PDEs with delay effects has become an active area of research in recent years. For examples, it was proved in [6, 21, 22, 25, 26] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay, unless additional conditions or control terms were imposed. A boundary stabilization problem for the wave equation with interior delay was studied in [15]. The authors proved an exponential stability result under some Lions geometric conditions. Kirane and Said-Houari [20] considered the viscoelastic wave equation with delay

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0, \quad (x,t) \in \Omega \times (0,+\infty),$$

where μ_1 and μ_2 are positive constants. Under the hypothesis of $0 \le \mu_1 \le \mu_2$, they established general decay estimate of the energy. Later, Liu [31] improved this result by considering the equation with a time-varying delay term, with coefficient μ_2 not necessarily positive.

For the transmission problems, we can see [1, 2, 4, 10, 11, 23, 32] for the studies of existence, regularity, controllability and decay estimates of solutions for the transmission problems with Laplacian operators. For example, Marzocchi [1] proved that the solution for a semilinear transmission problem between an elastic and thermoelastic material in one-dimensional space decays exponentially. This result was extended to the case of N-dimensional space by Marzocchi and Naso [2]. Bastos and Raposo [32] investigated the transmission problem with frictional damping and showed the well-posedness and exponential stability of the total energy. Recently, There are many new results on transmission problems with operators of Kirchhoff type, see [3, 5, 7, 12, 27, 28, 29, 30]. Bae [12] concerned the transmission problem for the wave equations given by

$$u_{tt} - \|\nabla u\|_{\Omega_1}^2 \Delta u + |u|^\alpha u = 0, \quad (x,t) \in S_1,$$
$$v_{tt} - \|\nabla v\|_{\Omega_2}^2 \Delta v + |v|^\beta v = 0, \quad (x,t) \in S_2,$$

subject to boundary and transmission conditions

$$
\begin{gathered}
v=0, \quad (x,t)\in \partial S_0\times(0,+\infty),\\
u=v, \quad \|\nabla u\|_{\Omega_1}^2\frac{\partial u}{\partial \nu}=\|\nabla v\|_{\Omega_2}^2\frac{\partial v}{\partial \nu}, \quad (x,t)\in\partial S_1,\\
u+\int_0^t g(t-s)\|\nabla u(s)\|_{\Omega_1}^2\frac{\partial u(s)}{\partial \nu}\mathrm{d}s=0, \quad (x,t)\in\partial S_2,
\end{gathered}
$$

and initial conditions

$$
\begin{gathered}
u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega_1,\\
v(x,0)=v_0(x),\ v_t(x,0)=v_1(x), \quad x\in\Omega_2.
\end{gathered}
$$

He studied the global existence of solutions and showed that if the relaxation function decays exponentially or polynomially, the solutions decays with the same rates. Later, Park [27, 28] considered the transmission problem of the Kirchhoff type wave equations

$$
\begin{gathered}
u_{tt}-(1+\|\nabla u\|_{\Omega_1}^2)\Delta u=0, \quad (x,t)\in S_1,\\
v_{tt}-(1+\|\nabla v\|_{\Omega_2}^2)\Delta v=0, \quad (x,t)\in S_2,
\end{gathered}
$$

subject to the same boundary and transmission conditions with [12]. He established general decay results depending on the behavior of the relaxation function.

On the other hand, for the transmission problems with delay terms, Benseghir [3] investigated the linear transmission problem with a delay term in one-dimensional space

$$
\begin{gathered}
u_{tt}-au_{xx}+\mu_1u_t+\mu_2u_t(t-\tau)=0, \quad (x,t)\in\Omega\times(0,+\infty),\\
v_{tt}-bv_{xx}=0, \quad (x,t)\in(L_1,L_2)\times(0,+\infty),
\end{gathered}
$$

subject to boundary and transmission conditions

$$
\begin{gathered}
u(0,t)=v(L_3,t)=0,\\
u(L_i,t)=v(L_i,t),\ au_x(L_i,t)=bv_x(L_i,t), \quad i=1,2,
\end{gathered}
$$

and initial conditions

$$
\begin{gathered}
u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), \quad x\in\Omega,\\
u_t(x,t-\tau)=f_0(x,t-\tau), \quad (x,t)\in\Omega\times(0,\tau),\\
v(x,0)=v_0(x),\ v_t(x,0)=v_1(x), \quad x\in(L_1,L_2),
\end{gathered}
$$

where $0<L_1<L_2<L_3$, $\Omega=(0,L_1)\cup(L_2,L_3)$, a and b are positive constants. Under the assumption $\mu_2<\mu_1$, he showed the exponential stability of the solution by introducing a suitable Lyaponov functional. Li et al. [7] studied the linear transmission system with long time memory and delay terms

$$
\begin{gathered}
u_{tt}-au_{xx}+\int_0^{+\infty}g(t-s)u_{xx}(s)\mathrm{d}s+\mu_1u_t+\mu_2u_t(t-\tau)=0,\ (x,t)\in\Omega\times(0,+\infty),\\
v_{tt}-bv_{xx}=0, \quad (x,t)\in(L_1,L_2)\times(0,+\infty),
\end{gathered}
$$

with the same boundary, transmission and initial conditions with [3]. Under the assumption $\mu_2\le\mu_1$, they proved the well-posedness result by means of semigroup theory and Hille-Yosida theorem. Furthermore, they established a general decay result, of which the exponential and polynomial decays are only special cases. Moreover, we refer to [5] for the similar transmission problem with short time memory term.

In view of the works mentioned above, on can see that the studies on transmission problem for a Kirchhoff type wave system (1.1)-(1.8) with damping and delay terms has not been started. The main difficulty encountered arises from the simultaneous appearance of the Kirchhoff type operators, delay and damping terms and memory damping on one part of the boundary. Our first goal is to establish the well-posedness of problem (1.1)-(1.8) by means of Faedo-Galerkin approximation together with priori energy estimates. As for the asymptotic behavior, we establish a general decay result under a wider class of relaxation functions and some conditions on the boundary, by introducing suitable energy and Lyapunov functionals.

The remaining of this paper is organized as follows: In Sect.2, we present some preliminaries and state the main results. In Sect.3, we establish well-possedness of problem (1.1)-(1.8) and the general decay estimate of energy is derived in Sect.4.

2. Preliminaries and main results

In this section, we present some materials needed in the proof and state the main results. Throughout this paper, we define

$$H^1_\Gamma(\Omega_2) := \{v \in H^1(\Omega_2) : v = 0 \text{ on } \Gamma_0\},$$

$$V := \{(u,v) \in H^1(\Omega_1) \times H^1_\Gamma(\Omega_2) : u = v,\ (1 + \|\nabla u\|^2_{\Omega_1})\frac{\partial u}{\partial \nu} = (1 + \|\nabla v\|^2_{\Omega_2})\frac{\partial v}{\partial \nu}\},$$

$$(u,v)_{\Omega_i} := \int_{\Omega_i} u(x)v(x)\mathrm{d}x,\ i = 1,2, \quad (u,v)_{\Gamma_j} := \int_{\Gamma_j} u(x)v(x)\mathrm{d}x, \quad j = 1,2.$$

For a Banach space X, $\|\cdot\|_X$ denotes the norm of X. For simplicity, we denote $\|\cdot\|_{L^2(\Omega_i)}$ and $\|\cdot\|_{L^2(\Gamma_j)}$ by $\|\cdot\|_{\Omega_i}$ and $\|\cdot\|_{\Gamma_j}$, respectively.

We use al the notation

$$(h * u)(t) := \int_0^t h(t-s)u(s)\mathrm{d}s,$$

$$(h \circ u)(t) := \int_0^t h(t-s)[u(t) - u(s)]\mathrm{d}s,$$

$$(h \diamond u)(t) := \int_0^t h(t-s)|u(t) - u(s)|^2\mathrm{d}s.$$

A direct calculation shows that

$$\begin{aligned}(h * u, u_t)_{\Gamma_2} = &-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_{\Gamma_2} (h \diamond u)(t)\mathrm{d}\Gamma - \Big(\int_0^t h(s)\mathrm{d}s\Big)\|u\|^2_{\Gamma_2}\Big] \\ &- \frac{1}{2}h(t)\|u\|^2_{\Gamma_2} + \frac{1}{2}\int_{\Gamma_2} (h' \diamond u)(t)\mathrm{d}\Gamma,\end{aligned} \tag{2.1}$$

and

$$\|(h \circ u)(t)\|^2_{\Gamma_2} \le \Big(\int_0^t |h(s)|\mathrm{d}s\Big)\int_{\Gamma_2} (|h| \diamond u)(t)\mathrm{d}\Gamma. \tag{2.2}$$

Differentiating (1.3), we arrive at the following Volterra equation

$$(1 + \|\nabla u\|^2_{\Omega_1})\frac{\partial u}{\partial \nu} + \frac{1}{g(0)}g' * (1 + \|\nabla u\|^2_{\Omega_1})\frac{\partial u}{\partial \nu} = -\frac{1}{g(0)}u_t.$$

Applying the Volterra's inverse operator, we obtain

$$(1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial \nu} = -\frac{1}{g(0)}(u_t + k * u_t),$$

where the resolvent kernel satisfies $k(t) + \frac{1}{g(0)}(g' * k)(t) = -\frac{1}{g(0)}g'(t)$. Denoting $r = 1/g(0)$, then the aforementioned equality can be written as

$$(1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial \nu} = -r[u_t + k(0)u - k(t)u_0 + (k' * u)(t)]. \tag{2.3}$$

which (2.3) implies (1.3).

For the resolvent kernel function k, as in [27, 28], we assume that

(H1) $k : \mathbb{R}^+ \to \mathbb{R}^+$ is a function of C^2 such that

$$k(0) > 0, \quad \lim_{t\to\infty} k(t) = 0, \quad k'(t) \le 0,$$

and there exists a non-increasing continuous function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$k''(t) \ge -\xi(t)k'(t), \quad \forall t \ge 0, \text{ and } \int_0^{+\infty} \xi(s)\mathrm{d}s = +\infty.$$

As in [26], we introduce the variable

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \quad (x,\rho,t) \in S_1 \times (0,1).$$

Then z satisfies

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in S_1 \times (0,1).$$

Therefore, problem (1.1)-(1.8) can be rewritten as

$$u_{tt} - (1+\|\nabla u\|_{\Omega_1}^2)\Delta u + \mu_1 u_t + \mu_2 z(x,1,t) = 0, \quad (x,t) \in S_1, \tag{2.4}$$

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in S_1 \times (0,1), \tag{2.5}$$

$$v_{tt} - (1+\|\nabla v\|_{\Omega_2}^2)\Delta v = 0, \quad (x,t) \in S_2, \tag{2.6}$$

$$v = 0, \quad (x,t) \in \partial S_0, \tag{2.7}$$

$$(1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial \nu} = (1+\|\nabla v\|_{\Omega_2}^2)\frac{\partial v}{\partial \nu},\ u = v, \quad (x,t) \in \partial S_1, \tag{2.8}$$

$$(1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial \nu} = -r[u_t + k(0)u - k(t)u_0 + (k' * u)(t)], \quad (x,t) \in \partial S_2, \tag{2.9}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega_1, \tag{2.10}$$

$$z(x,1,t) = f_0(x, t-\tau), \quad (x,t) \in \Omega_1 \times (0,\tau), \tag{2.11}$$

$$z(x,0,t) = u_t, \quad (x,t) \in S_1, \tag{2.12}$$

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \quad x \in \Omega_2. \tag{2.13}$$

Therefore, it is sufficient to consider problem (2.4)-(2.13), which is equivalent to (1.1)-(1.8).

Firstly, we present the definition of weak solution of (2.4)-(2.13).

Definition 2.1. Let the initial data $(u_0, v_0) \in H_0^2(\Omega_1) \times H_0^2(\Omega_2)$, $(u_1, v_1) \in V$, and $f_0 \in L^2(\Omega_1 \times (-\tau, 0))$ be given. Functions $(u, v, z) \in C(0,T; V \times L^2(\Omega_1 \times (0,1)))$

are called the weak solution of problem (2.4)-(2.13), if (u, v, z) satisfies the initial conditions $(u(0), v(0)) = (u_0, v_0)$, $z(x, 1, t) = f_0(x, t-\tau)$, for all $t \in (0, \tau)$, and

$$\begin{aligned}
&\int_{\Omega_1} u_{tt}\phi \mathrm{d}x + (1 + \|\nabla u\|_{\Omega_1}^2)\int_{\Omega_1} \nabla u\cdot\nabla\phi \mathrm{d}x + \mu_1\int_{\Omega_1} u_t\phi \mathrm{d}x + \mu_2\int_{\Omega_1} z(x,1,t)\phi \mathrm{d}x \\
&+ \int_{\Omega_2} v_{tt}\psi \mathrm{d}x + (1 + \|\nabla v\|_{\Omega_2}^2)\int_{\Omega_2} \nabla v\cdot\nabla\psi \mathrm{d}x \\
&= -r\int_{\Gamma_2} [u_t + k(0)u + (k' * u)(t) - k(t)u(0)]\phi \mathrm{d}x,
\end{aligned}$$

$$\int_{\Omega_1} \tau z_t\varphi \mathrm{d}x + \int_{\Omega_1} z_\rho\varphi \mathrm{d}x = 0,$$

for all $(\phi, \psi) \in V$, all $\varphi(x, \rho) \in L^2(\Omega_1 \times (0, 1))$, and all $t \in [0, \tau]$.

As for the well-posedness of solution to problem (2.4)-(2.13), by the Feado-Galerkin approximation technique, we obtain the following result.

Theorem 2.2. *Suppose that $\mu_2 \le \mu_1$ and (H1) holds. Then for $(u_0, v_0) \in H_0^2(\Omega_1)\times H_0^2(\Omega_2)$, $(u_1, v_1) \in V$, $f_0 \in L^2(\Omega_1 \times (-\tau, 0))$ satisfying the compatibility conditions*

$$(1 + \|\nabla u_0\|_{\Omega_1}^2)\frac{\partial u_0}{\partial\nu} + ru_1 = 0, \ \text{on } \Gamma_2, \quad v_0 = 0, \ \text{on } \Gamma_0,$$
$$u_0 = v_0, \ (1 + \|\nabla u_0\|_{\Omega_1}^2)\frac{\partial u_0}{\partial\nu} = (1 + \|\nabla v_0\|_{\Omega_2}^2)\frac{\partial v_0}{\partial\nu}, \quad \text{on } \Gamma_1,$$

there exists a unique weak solution (u, v, z) of problem (2.4)-(2.13) such that

$$(u, v) \in C((0, +\infty); V) \cap C^1((0, +\infty); L^2(\Omega_1) \times L^2(\Omega_2)),$$
$$z \in C((0, +\infty); L^2((0, 1) \times \Omega_1)).$$

To state the result of uniform decay rate for energy, we define the energy functional

$$\begin{aligned}
E(t) :=& \frac{1}{2}(\|u_t\|_{\Omega_1}^2 + \|v_t\|_{\Omega_2}^2 + \|\nabla u\|_{\Omega_1}^2 + \|\nabla v\|_{\Omega_2}^2) + \frac{1}{4}(\|\nabla u\|_{\Omega_1}^4 + \|\nabla v\|_{\Omega_2}^4) \\
&+ \frac{r}{2}k(t)\|u\|_{\Gamma_2}^2 - \frac{r}{2}\int_{\Gamma_2} (k' \diamond u)(t)\mathrm{d}\Gamma + \frac{\zeta}{2}\int_{\Omega_1}\int_0^1 |z(x,\rho,t)|^2 \mathrm{d}\rho \mathrm{d}x,
\end{aligned} \tag{2.14}$$

where ζ is a positive constant such that

$$\tau\mu_2 < \zeta < \tau(2\mu_1 - \mu_2). \tag{2.15}$$

Next, we establish a general decay estimate result.

Theorem 2.3. *Let (u, v, z) be the solution of (2.4)-(2.13), assuming $\mu_2 < \mu_1$ and (H1) holds. Then for $t_0 > 0$ large enough, there exist constants $C_0 > 0$ and $\varpi > 0$ such that*

(i) $E(t) \le C_0E(0)e^{-\varpi\int_0^t \xi(s)\mathrm{d}s}$ *for all $t \ge t_0$, if $u_0 = 0$ on Γ_2,*
(ii) *otherwise,* $E(t) \le C_0[E(0) + \|u_0\|_{\Gamma_2}^2\int_0^t k^2(s)e^{\varpi\int_0^s \xi(r)\mathrm{d}r}\mathrm{d}s]e^{-\varpi\int_0^t \xi(s)\mathrm{d}s}$ *for all $t \ge t_0$.*

Remark 2.4. The exponential decay and polynomial decay in previous literatures are special cases of the result in Theorem 2.3. In fact, if we take

$$k(t) = e^{-\sigma t}, \quad \sigma > 0, \quad \xi(t) = \sigma; \ k(t) = \frac{1}{(1+t)^\sigma}, \quad \sigma > 0, \quad \xi(t) = \frac{1+\sigma}{1+t};$$

$$k(t) = \frac{1}{\ln(\ln(3+t))}, \quad \xi(t) = \frac{1}{\ln(3+t)(3+t)},$$

then by the result of Theorem 2.3, the energy may decay exponentially, polynomially, and logarithmically, respectively.

3. Well-posedness

In this section, by using Feado-Galerkin approximation technique and some prior estimates, we establish the well-posedness of problem (2.4)-(2.13).

Proof of Theorem 2.2. We divide the proof into four steps.

Step 1. Feado-Galerkin approximation. Let $\{(\phi_j, \psi_j)\}_{j\in\mathbb{N}_+}$ be a basis for V, which is orthogonal in $L^2(\Omega_1) \times L^2(\Omega_2)$. For all $n \geq 1$, denoting $V_n := \operatorname{span}\{(\phi_1, \psi_1), (\phi_2, \psi_2), \dots, (\phi_n, \psi_n)\}$ and defining the sequence $\{\varphi_j(x, \rho)\}_{1\leq j\leq n}$ as follows:

$$\varphi_j(x, 0) = \phi_j(x).$$

Then we may extend $\varphi_j(x, 0)$ by $\varphi_j(x, \rho)$ over $L^2(\Omega_1 \times (0, 1))$ and denote $W_n = \operatorname{span} = \{\varphi_1, \varphi_2 \dots, \varphi_n\}$.

We define the approximations:

$$(u^{(n)}(x,t), v^{(n)}(x,t)) := \sum_{j=1}^{n} b_{jn}(t)(\phi_j(x), \psi_j(x)),$$
$$z^{(n)}(x,\rho,t) := \sum_{j=1}^{n} c_{jn}(t)\varphi_j(x,\rho),$$

where $(u^{(n)}, v^{(n)}, z^{(n)})$ are solutions to the following finite dimensional Cauchy problem:

$$\begin{aligned}
&\int_{\Omega_1} u_{tt}^{(n)}\phi_j \mathrm{d}x + (1 + \|\nabla u^{(n)}\|_{\Omega_1}^2)\int_{\Omega_1} \nabla u^{(n)} \cdot \nabla\phi_j \mathrm{d}x \\
&+ \mu_1 \int_{\Omega_1} u_t^{(n)}\phi_j \mathrm{d}x + \mu_2 \int_{\Omega_1} z^{(n)}(x,1,t)\phi_j \mathrm{d}x \\
&+ \int_{\Omega_2} v_{tt}^{(n)}\psi_j \mathrm{d}x + (1 + \|\nabla v^{(n)}\|_{\Omega_2}^2)\int_{\Omega_2} \nabla v^{(n)} \cdot \nabla\psi_j \mathrm{d}x \\
&= -r\int_{\Gamma_2} [u_t^{(n)} + k(0)u^{(n)} + (k' * u^{(n)})(t) - k(t)u_{0n}]\phi_j \mathrm{d}x,
\end{aligned} \tag{3.1}$$

$$\int_{\Omega_1} \tau z_t^{(n)}\varphi_j \mathrm{d}x + \int_{\Omega_1} z_\rho^{(n)}\varphi_j \mathrm{d}x = 0, \tag{3.2}$$

$$z^{(n)}(x,0,t) = u_t^{(n)}(x,t), \tag{3.3}$$

and

$$(u_{0n}, v_{0n}) = (u^{(n)}(0), v^{(n)}(0)) \to (u_0, v_0), \quad \text{in } H_0^2(\Omega_1) \times H_0^2(\Omega_2), \tag{3.4}$$

$$(u_{1n}, v_{1n}) = (u_t^{(n)}(0), v_t^{(n)}(0)) \to (u_1, v_1), \quad \text{in } V, \tag{3.5}$$

$$z_{0n} = z^{(n)}(x,1,t) \to f_0(x, t-\tau), \quad \text{in } L^2(\Omega_1 \times (-\tau, 0)). \tag{3.6}$$

According to the standard theory of ordinary differential equations, the finite dimensional problem (3.1)-(3.3) possesses a unique solution $(b_{jn}(t), c_{jn}(t))_{j=1,\dots,n}$ on

$[0, T_n)$, $T_n > 0$. The extension of these solutions to the whole interval $[0, T]$, for all $T > 0$, is a consequence of the first estimate which we are going to prove below.

Step 2. Energy estimates. A prior estimate I: Multiplying (3.1) by $b'_{jn}(t)$ and summing on j, then using (2.1) we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}(\|u_t^{(n)}\|_{\Omega_1}^2 + \|v_t^{(n)}\|_{\Omega_2}^2 + \|\nabla u^{(n)}\|_{\Omega_1}^2 + \|\nabla v^{(n)}\|_{\Omega_2}^2) \\
&+ \frac{1}{4}(\|\nabla u^{(n)}\|_{\Omega_1}^4 + \|\nabla v^{(n)}\|_{\Omega_2}^4)\Big\} \\
&= -\mu_1\|u_t^{(n)}\|_{\Omega_1}^2 - \mu_2\int_{\Omega_1} u_t^{(n)} z^{(n)}(x,1,t)\mathrm{d}x - r\|u_t^{(n)}\|_{\Gamma_2}^2 \\
&\quad + rk(t)\int_{\Gamma_2} u_{0n}u_t^{(n)}\mathrm{d}x - \frac{r}{2}\int_{\Gamma_2}(k'' \diamond u^{(n)})(t)\mathrm{d}\Gamma + \frac{r}{2}k'(t)\|u^{(n)}\|_{\Gamma_2}^2 \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\big[\frac{r}{2}\int_{\Gamma_2}(k' \diamond u^{(n)})(t)\mathrm{d}\Gamma - \frac{r}{2}k(t)\|u^{(n)}\|_{\Gamma_2}^2\big].
\end{aligned} \tag{3.7}$$

Multiplying (3.2) by $\frac{\zeta}{\tau}c'_{jn}(t)$ and integrating over $(0,1)$ on ρ and then summing on j, we obtain

$$\frac{\zeta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}\int_0^1 |z^{(n)}(x,\rho,t)|^2\mathrm{d}\rho\mathrm{d}x = -\frac{\zeta}{2\tau}[\|z^{(n)}(x,1,t)\|_{\Omega_1}^2 - \|u_t^{(n)}\|_{\Omega_1}^2], \tag{3.8}$$

where ζ is a positive constant such that

$$\tau\mu_2 \le \zeta \le \tau(2\mu_1 - \mu_2).$$

Combining (3.7) and (3.8), we can derive

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E^{(n)}(t) + r\|u_t^{(n)}\|_{\Gamma_2}^2 + \frac{r}{2}\int_{\Gamma_2}(k'' \diamond u^{(n)})(t)\mathrm{d}\Gamma - \frac{r}{2}k'(t)\|u^{(n)}\|_{\Gamma_2}^2 \\
&= -(\mu_1 - \frac{\zeta}{2\tau})\|u_t^{(n)}\|_{\Omega_1}^2 - \frac{\zeta}{2\tau}\|z^{(n)}(x,1,t)\|_{\Omega_1}^2 \\
&\quad - \mu_2\int_{\Omega_1} u_t^{(n)} z^{(n)}(x,1,t)\mathrm{d}x + rk(t)\int_{\Gamma_2} u_{0n}u_t^{(n)}\mathrm{d}\Gamma,
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
E^{(n)}(t) =& \frac{1}{2}\|u_t^{(n)}\|_{\Omega_1}^2 + \frac{1}{2}\|v_t^{(n)}\|_{\Omega_2}^2 + \frac{1}{2}\|\nabla u^{(n)}\|_{\Omega_1}^2 + \frac{1}{2}\|\nabla v^{(n)}\|_{\Omega_2}^2 \\
&+ \frac{1}{4}\|\nabla u^{(n)}\|_{\Omega_1}^4 + \frac{1}{4}\|\nabla v^{(n)}\|_{\Omega_2}^4 - \frac{r}{2}\int_{\Gamma_2}(k' \diamond u^{(n)})(t)\mathrm{d}\Gamma \\
&+ \frac{r}{2}k(t)\|u^{(n)}\|_{\Gamma_2}^2 + \frac{\zeta}{2}\int_{\Omega_1}\int_0^1 |z^{(n)}(x,\rho,t)|^2\mathrm{d}\rho\mathrm{d}x.
\end{aligned} \tag{3.10}$$

It follows from Young's inequality that

$$|-\mu_2\int_{\Omega_1} u_t^{(n)} z^{(n)}(x,1,t)\mathrm{d}x| \le \frac{\mu_2}{2}\|u_t^{(n)}\|_{\Omega_1}^2 + \frac{\mu_2}{2}\|z^{(n)}(x,1,t)\|_{\Omega_1}^2, \tag{3.11}$$

$$rk(t)\int_{\Gamma_2} u_{0n}u_t^{(n)}\mathrm{d}\Gamma \le \frac{r}{2}\|u_t^{(n)}\|_{\Gamma_2}^2 + \frac{rk^2(t)}{2}\|u_{0n}\|_{\Gamma_2}^2. \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E^{(n)}(t) \leq &-\frac{r}{2}\Big[\|u_t^{(n)}\|_{\Gamma_2}^2+\int_{\Gamma_2}(k''\diamond u^{(n)})(t)\mathrm{d}\Gamma-k'(t)\|u^{(n)}\|_{\Gamma_2}^2\Big]\\ &-(\mu_1-\frac{\mu_2}{2}-\frac{\zeta}{2\tau})\|u_t^{(n)}\|_{\Omega_1}^2\\ &-\big(\frac{\zeta}{2\tau}-\frac{\mu_2}{2}\big)\|z^{(n)}(x,1,t)\|_{\Omega_1}^2+\frac{rk^2(t)}{2}\|u_{0n}\|_{\Gamma_2}^2.\end{aligned}\tag{3.13}$$

Integrating (3.13) over $(0,t)$, $0<t\leq T$, and then using Gronwall's lemma and (3.4)-(3.6), we obtain the first estimate

$$\begin{aligned}&\|u_t^{(n)}\|_{\Omega_1}^2+\|v_t^{(n)}\|_{\Omega_2}^2+\|\nabla u^{(n)}\|_{\Omega_1}^2+\|\nabla v^{(n)}\|_{\Omega_2}^2+\|\nabla u^{(n)}\|_{\Omega_1}^4+\|\nabla v^{(n)}\|_{\Omega_2}^4\\ &+\|z^{(n)}(x,\rho,t)\|_{L^2(\Omega_1\times(0,1))}^2+\int_0^t\|u_t^{(n)}(s)\|_{\Gamma_2}^2\mathrm{d}s\leq L_1,\end{aligned}\tag{3.14}$$

where $L_1>0$ is a constant independent of n.

A prior estimate II: First of all, it can be deduced easily from the assumptions on initial data in Theorem 2.2 that $\|u_{tt}^{(n)}(0)\|_{\Omega_1}^2+\|v_{tt}^{(n)}(0)\|_{\Omega_2}^2\leq C$, where $C>0$ is independent of n. Differentiating (3.1) with respect to t and multiplying it by $b''_{jn}(t)$, and summing on j, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\|u_{tt}^{(n)}\|_{\Omega_1}^2+\frac{1}{2}\|v_{tt}^{(n)}\|_{\Omega_2}^2+\frac{1}{2}\|\nabla u_t^{(n)}\|_{\Omega_1}^2+\frac{1}{2}\|\nabla v_t^{(n)}\|_{\Omega_2}^2\\ &+\frac{1}{2}\|\nabla u^{(n)}\|_{\Omega_1}^2\|\nabla u_t^{(n)}\|_{\Omega_1}^2+\frac{1}{2}\|\nabla v^{(n)}\|_{\Omega_2}^2\|\nabla v_t^{(n)}\|_{\Omega_2}^2+\frac{rk(0)}{2}\|u_t^{(n)}\|_{\Gamma_2}^2\\ &+\Big(\int_{\Omega_1}\nabla u^{(n)}\cdot\nabla u_t^{(n)}\mathrm{d}x\Big)^2+\big(\int_{\Omega_2}\nabla v^{(n)}\cdot\nabla v_t^{(n)}\mathrm{d}x\big)^2\Big\}\\ &=3\|\nabla u_t^{(n)}\|_{\Omega_1}^2\int_{\Omega_1}\nabla u^{(n)}\cdot\nabla u_t^{(n)}\mathrm{d}x+3\|\nabla v_t^{(n)}\|_{\Omega_2}^2\int_{\Omega_2}\nabla v^{(n)}\cdot\nabla v_t^{(n)}\mathrm{d}x\\ &\quad-\mu_1\|u_{tt}^{(n)}\|_{\Omega_1}^2-\mu_2\int_{\Omega_1}u_{tt}^{(n)}z_t^{(n)}(x,1,t)\mathrm{d}x+rk'(t)\int_{\Gamma_2}u_{0n}u_{tt}^{(n)}\mathrm{d}\Gamma\\ &\quad-r\|u_{tt}^{(n)}\|_{\Gamma_2}^2-r\int_{\Gamma_2}(k''*u^{(n)})u_{tt}^{(n)}\mathrm{d}\Gamma-rk'(0)\int_{\Gamma_2}u^{(n)}u_{tt}^{(n)}\mathrm{d}\Gamma.\end{aligned}\tag{3.15}$$

Differentiating (3.2) with respect to t and multiplying it by $\frac{\zeta}{\tau}c''_{jn}(t)$, integrating over $(0,1)$ on ρ and then summing on j, we obtain

$$\frac{\zeta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}\int_0^1|z_t^{(n)}(x,\rho,t)|^2\mathrm{d}\rho\mathrm{d}x=-\frac{\zeta}{2\tau}\big[\|z_t^{(n)}(x,1,t)\|_{\Omega_1}^2-\|u_{tt}^{(n)}\|_{\Omega_1}^2\big].\tag{3.16}$$

Combining (3.15) and (3.16), we can derive

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_1^{(n)}(t) + r\|u_{tt}^{(n)}\|_{\Gamma_2}^2 \\
&= 3\|\nabla u_t^{(n)}\|_{\Omega_1}^2 \int_{\Omega_1} \nabla u^{(n)} \cdot \nabla u_t^{(n)} \mathrm{d}x + 3\|\nabla v_t^{(n)}\|_{\Omega_2}^2 \int_{\Omega_2} \nabla v^{(n)} \cdot \nabla v_t^{(n)} \mathrm{d}x \\
&\quad - (\mu_1 - \frac{\zeta}{2\tau})\|u_{tt}^{(n)}\|_{\Omega_1}^2 - \mu_2 \int_{\Omega_1} u_{tt}^{(n)} z_t^{(n)}(x,1,t)\mathrm{d}x \\
&\quad + rk'(t)\int_{\Gamma_2} u_{0n}u_{tt}^{(n)}\mathrm{d}\Gamma - r\int_{\Gamma_2}(k''*u^{(n)})u_{tt}^{(n)}\mathrm{d}\Gamma \\
&\quad - rk'(0)\int_{\Gamma_2} u^{(n)}u_{tt}^{(n)}\mathrm{d}\Gamma - \frac{\zeta}{2\tau}\|z_t^{(n)}(x,1,t)\|_{\Omega_1}^2,
\end{aligned}
\tag{3.17}
$$

where

$$
\begin{aligned}
E_1^{(n)}(t) =& \frac{1}{2}\|u_{tt}^{(n)}\|_{\Omega_1}^2 + \frac{1}{2}\|v_{tt}^{(n)}\|_{\Omega_2}^2 + \frac{1}{2}\|\nabla u_t^{(n)}\|_{\Omega_1}^2 + \frac{1}{2}\|\nabla v_t^{(n)}\|_{\Omega_2}^2 \\
&+ \frac{1}{2}\|\nabla u^{(n)}\|_{\Omega_1}^2\|\nabla u_t^{(n)}\|_{\Omega_1}^2 + \frac{1}{2}\|\nabla v^{(n)}\|_{\Omega_2}^2\|\nabla v_t^{(n)}\|_{\Omega_2}^2 \\
&+ \Big(\int_{\Omega_1}\nabla u^{(n)}\cdot\nabla u_t^{(n)}\mathrm{d}x\Big)^2 + \Big(\int_{\Omega_2}\nabla v^{(n)}\cdot\nabla v_t^{(n)}\mathrm{d}x\Big)^2 \\
&+ \frac{rk(0)}{2}\|u_t^{(n)}\|_{\Gamma_2}^2 + \frac{\zeta}{2}\int_{\Omega_1}\int_0^1 |z_t^{(n)}(x,\rho,t)|^2\mathrm{d}\rho\mathrm{d}x.
\end{aligned}
\tag{3.18}
$$

By Young's inequality, we obtain

$$
|-\mu_2\int_{\Omega_1} u_{tt}^{(n)}z_t^{(n)}(x,1,t)\mathrm{d}x| \le \frac{\mu_2}{2}\|u_{tt}^{(n)}\|_{\Omega_1}^2 + \frac{\mu_2}{2}\|z_t^{(n)}(x,1,t)\|_{\Omega_1}^2, \tag{3.19}
$$

$$
|rk'(t)\int_{\Gamma_2} u_{0n}u_{tt}^{(n)}\mathrm{d}\Gamma| \le \eta r\|u_{tt}^{(n)}\|_{\Gamma_2}^2 + \frac{r}{4\eta}(k'(t))^2\|u_{0n}\|_{\Gamma_2}^2, \tag{3.20}
$$

$$
\begin{aligned}
&\big|-r\int_{\Gamma_2}(k''*u^{(n)})(t)u_{tt}^{(n)}\mathrm{d}\Gamma\big| \\
&\le \eta r\|u_{tt}^{(n)}\|_{\Gamma_2}^2 + \frac{r(k'(t))^2}{4\eta}\|k''(t)\|_{L^1(0,+\infty)}\int_0^t k''(t-s)\|u^{(n)}(s)\|_{\Gamma_2}^2\mathrm{d}s,
\end{aligned}
\tag{3.21}
$$

$$
|-rk'(0)\int_{\Gamma_2} u^{(n)}u_{tt}^{(n)}\mathrm{d}\Gamma| \le \eta r\|u_{tt}^{(n)}\|_{\Gamma_2}^2 + \frac{r}{4\eta}(k'(0))^2\|u^{(n)}\|_{\Gamma_2}^2, \tag{3.22}
$$

where $0 < \eta < 1/3$ is a constant. Substituting (3.19)-(3.22) into (3.17), we can derive

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_1^{(n)}(t) + r(1-3\eta)\|u_{tt}^{(n)}\|_{\Gamma_2}^2 \\
&\le 3\|\nabla u^{(n)}\|_{\Omega_1}\|\nabla u_t^{(n)}\|_{\Omega_1}^3 + 3\|\nabla v^{(n)}\|_{\Omega_2}\|\nabla v_t^{(n)}\|_{\Omega_2}^3 \\
&\quad - (\mu_1 - \frac{\mu_2}{2} - \frac{\zeta}{2\tau})\|u_{tt}^{(n)}\|_{\Omega_1}^2 - (\frac{\zeta}{2\tau} - \frac{\mu_2}{2})\|z_t^{(n)}(x,1,t)\|_{\Omega_1}^2 \\
&\quad + \frac{r}{4\eta}(k'(t))^2\|k''(t-s)\|_{L^1(0,+\infty)}\int_0^t k''(t-s)\|u^{(n)}(s)\|_{\Gamma_2}^2\mathrm{d}s \\
&\quad + \frac{r}{4\eta}(k'(t))^2\|u_{0n}\|_{\Gamma_2}^2 + \frac{r}{4\eta}(k'(0))^2\|u^{(n)}\|_{\Gamma_2}^2.
\end{aligned}
\tag{3.23}
$$

Integrating (3.23) over $(0,t)$, $0 < t \le T$ and then using Gronwall's lemma, we obtain the second estimate

$$\begin{aligned} &\|u_{tt}^{(n)}\|_{\Omega_1}^2 + \|v_{tt}^{(n)}\|_{\Omega_2}^2 + \|\nabla u_t^{(n)}\|_{\Omega_1}^2 + \|\nabla v_t^{(n)}\|_{\Omega_2}^2 \\ &+ \|z_t^{(n)}(x,\rho,t)\|_{L^2(\Omega_1\times(0,1))}^2 + \int_0^t \|u_{tt}^{(n)}(s)\|_{\Gamma_2}^2 \mathrm{d}s \le L_2, \end{aligned} \tag{3.24}$$

where $L_2 > 0$ is a constant independent of n.

Step 3. Pass to the limit. It follows from the first prior estimate (3.14) and second prior estimate (3.24) that there exist subsequences of $\{u^{(n)}\}$, $\{v^{(n)}\}$, $\{z^{(n)}\}$ (we still denote the subsequences by $\{u^{(n)}\}$, $\{v^{(n)}\}$, $\{z^{(n)}\}$ for convenience) such that

$$\begin{gathered} (u^{(n)}, v^{(n)}) \to (u,v) \quad \text{strongly in } C(0,T;V), \\ (u_t^{(n)}, v_t^{(n)}) \to (u_t, v_t) \quad \text{strongly in } C(0,T;L^2(\Omega_1)\times L^2(\Omega_2)), \\ z^{(n)} \to z \quad \text{strongly in } C(0,T;L^2(\Omega_1\times(0,1))), \\ (u^{(n)}, v^{(n)}) \to (u,v) \quad \text{weak star in } L^\infty(0,T;V), \\ (u_t^{(n)}, v_t^{(n)}) \to (u_t, v_t) \quad \text{weak star in } L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2)), \\ (u_{tt}^{(n)}, v_{tt}^{(n)}) \to (u_{tt}, v_{tt}) \quad \text{weak star in } L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2)), \\ z^{(n)} \to z \quad \text{weak star in } L^\infty(0,T;L^2(\Omega_1\times(0,1))), \\ z_t^{(n)} \to z_t \quad \text{weak star in } L^\infty(0,T;L^2(\Omega_1\times(0,1))), \\ u_t^{(n)} \to u_t \quad \text{strongly in } L^2(0,T;L^2(\Gamma_2)). \end{gathered}$$

The above convergence results are sufficient to pass to the limit in the linear terms of (3.1) and (3.2). From the first estimate and taking the continuity of trace operator $\gamma_0 : H^1(\Omega_1) \to H^{\frac12}(\Gamma_2)$ into account, we have

$$\begin{gathered} \{u^{(n)}\} \quad \text{is bounded in } L^2(0,T;H^{\frac12}(\Gamma_2)), \\ \{u_t^{(n)}\} \quad \text{is bounded in } L^2(0,T;H^{\frac12}(\Gamma_2)), \\ \{u_{tt}^{(n)}\} \quad \text{is bounded in } L^2(0,T;L^2(\Gamma_2)). \end{gathered}$$

The second estimate (3.24) implies

$$\begin{gathered} (1+\|\nabla u^{(n)}\|_{\Omega_1}^2)u^{(n)} \to (1+\|\nabla u\|_{\Omega_1}^2)u \quad \text{strongly in } C(0,T;H_0^1(\Omega_1)), \\ (1+\|\nabla v^{(n)}\|_{\Omega_2}^2)v^{(n)} \to (1+\|\nabla v\|_{\Omega_2}^2)v \quad \text{strongly in } C(0,T;H_0^1(\Omega_2)). \end{gathered}$$

Thus we can pass to the limit in (3.1) and (3.2) to obtain

$$\begin{gathered} u_{tt} - (1+\|\nabla u\|_{\Omega_1}^2)\Delta u + \mu_1 u_t + \mu_2 z(x,1,t) = 0, \quad \text{in } L^2(0,\infty;L^2(\Omega_1)), \\ v_{tt} - (1+\|\nabla v\|_{\Omega_2}^2)\Delta v = 0, \quad \text{in } L^2(0,\infty;L^2(\Omega_2)), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } L^2(0,\infty;L^2(\Omega_1)), \\ (1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial \nu} = -r[u_t + k(0)u - k(t)u_0 + k' * u], \quad \text{in } L^2(0,\infty;H^{\frac12}(\Gamma_2)). \end{gathered}$$

Step 4. Uniqueness. Let $(\overline{u},\overline{v},\overline{z})$ and $(\widetilde{u},\widetilde{v},\widetilde{z})$ be two solutions of problem (2.4)-(2.13). Then $(\widehat{u},\widehat{v},\widehat{z})=(\overline{u}-\widetilde{u},\overline{v}-\widetilde{v},\overline{z}-\widetilde{z})$ satisfies

$$\begin{aligned}&\widehat{u}_{tt}-[(1+\|\nabla\overline{u}\|_{\Omega_1}^2)\Delta\overline{u}-(1+\|\nabla\widetilde{u}\|_{\Omega_1}^2)\Delta\widetilde{u}]\\&+\mu_1\widehat{u}_t+\mu_2\widehat{z}(x,1,t)=0,\quad (x,t)\in S_1,\end{aligned}\tag{3.25}$$

$$\tau\widehat{z}_t(x,\rho,t)+\widehat{z}_\rho(x,\rho,t)=0,\quad (x,t)\in S_1\times(0,1),\tag{3.26}$$

$$\widehat{v}_{tt}-[(1+\|\nabla\overline{v}\|_{\Omega_2}^2)\Delta\overline{v}-(1+\|\nabla\widetilde{v}\|_{\Omega_2}^2)\Delta\widetilde{v}]=0,\quad (x,t)\in S_2,\tag{3.27}$$

$$v=0,\quad (x,t)\in\partial S_0,\tag{3.28}$$

$$\begin{aligned}&(1+\|\nabla\overline{u}\|_{\Omega_1}^2)\frac{\partial\overline{u}}{\partial\nu}-(1+\|\nabla\widetilde{u}\|_{\Omega_1}^2)\frac{\partial\widetilde{u}}{\partial\nu}\\&=(1+\|\nabla\overline{v}\|_{\Omega_2}^2)\frac{\partial\overline{v}}{\partial\nu}-(1+\|\nabla\widetilde{v}\|_{\Omega_2}^2)\frac{\partial\widetilde{v}}{\partial\nu},\end{aligned}\tag{3.29}$$

$$\widehat{u}=\widehat{v},\quad (x,t)\in\partial S_1,\tag{3.30}$$

$$\begin{aligned}&[((1+\|\nabla\overline{u}\|_{\Omega_1}^2))\frac{\partial\overline{u}}{\partial\nu}-((1+\|\nabla\widetilde{u}\|_{\Omega_1}^2))\frac{\partial\widetilde{u}}{\partial\nu}]\\&=-r[\widehat{u}_t+k(0)\widehat{u}+(k'*\widehat{u})(t)],\quad (x,t)\in\partial S_2,\end{aligned}\tag{3.31}$$

$$\widehat{u}(x,0)=0,\quad \widehat{u}_t(x,0)=0,\quad x\in\Omega_1,\tag{3.32}$$

$$\widehat{z}(x,1,t)=0,\quad (x,t)\in\Omega_1\times(0,\tau),\tag{3.33}$$

$$\widehat{z}(x,0,t)=\widehat{u}_t,\quad (x,t)\in S_1,\tag{3.34}$$

$$\widehat{v}(x,0)=0,\quad \widehat{v}_t(x,0)=0,\quad x\in\Omega_2.\tag{3.35}$$

Multiplying (3.25) and (3.27) by $\widehat{u}_t$ and $\widehat{v}_t$, and integrating over Ω_1 and Ω_2, respectively, using (3.28)-(3.31) and (2.1), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\widehat{u}_t\|_{\Omega_1}^2+\|\widehat{v}_t\|_{\Omega_2}^2+\|\nabla\widehat{u}\|_{\Omega_1}^2+\|\nabla\widehat{v}\|_{\Omega_2}^2\\&+\|\nabla\overline{u}\|_{\Omega_1}^2\|\nabla\widehat{u}\|_{\Omega_1}^2+\|\nabla\overline{v}\|_{\Omega_2}^2\|\nabla\widehat{v}\|_{\Omega_2}^2\Big\}\\&=\frac{1}{2}\|\nabla\widehat{u}\|_{\Omega_1}^2\int_{\Omega_1}\nabla\overline{u}\cdot\nabla\overline{u}_t\mathrm{d}x-\int_{\Omega_1}\nabla\widehat{u}\cdot(\nabla\overline{u}+\nabla\widetilde{u})\mathrm{d}x\int_{\Omega_1}\nabla\widetilde{u}\cdot\nabla\widehat{u}_t\mathrm{d}x\\&\quad+\frac{1}{2}\|\nabla\widehat{v}\|_{\Omega_2}^2\int_{\Omega_2}\nabla\overline{v}\cdot\nabla\overline{v}_t\mathrm{d}x-\int_{\Omega_2}\nabla\widehat{v}\cdot(\nabla\overline{v}+\nabla\widetilde{v})\mathrm{d}x\int_{\Omega_2}\nabla\widetilde{v}\cdot\nabla\widehat{v}_t\mathrm{d}x\\&\quad-\mu_1\|\widehat{u}_t\|_{\Omega_1}^2-\mu_2\int_{\Omega_1}\widehat{u}_t\widehat{z}(x,1,t)\mathrm{d}x-r\|\widehat{u}_t\|_{\Gamma_2}^2-\frac{r}{2}\int_{\Gamma_2}(k''\diamond\widehat{u})(t)\mathrm{d}\Gamma\\&\quad+\frac{r}{2}k'(t)\|\widehat{u}\|_{\Gamma_2}^2+\frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{r}{2}\int_{\Gamma_2}(k'\diamond\widehat{u})(t)\mathrm{d}\Gamma-\frac{r}{2}k(t)\|\widehat{u}\|_{\Gamma_2}^2\Big].\end{aligned}\tag{3.36}$$

Multiplying (3.26) by $\frac{\zeta}{\tau}\widehat{z}(x,\rho,t)$ and integrating over $\Omega_1\times(0,1)$ on x and ρ, respectively, we obtain

$$\frac{\zeta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}\int_0^1|\widehat{z}(x,\rho,t)|^2\mathrm{d}\rho\mathrm{d}x=-\frac{\zeta}{2\tau}[\|\widehat{z}(x,1,t)\|_{\Omega_1}^2-\|\widehat{u}_t\|_{\Omega_1}^2],\tag{3.37}$$

Combining (3.36), (3.37) and using Young's inequality and estimates (3.14), (3.24), we can derive

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\widehat{u}_t\|_{\Omega_1}^2+\|\widehat{v}_t\|_{\Omega_2}^2+\|\nabla\widehat{u}\|_{\Omega_1}^2+\|\nabla\widehat{v}\|_{\Omega_2}^2+\|\nabla\overline{u}\|_{\Omega_1}^2\|\nabla\widehat{u}\|_{\Omega_1}^2\\ &+\|\nabla\overline{v}\|_{\Omega_2}^2\|\nabla\widehat{v}\|_{\Omega_2}^2-r\int_{\Gamma_2}(k'\diamond\widehat{u})\mathrm{d}\Gamma+rk(t)\|\widehat{u}\|_{\Gamma_2}^2\\ &+\zeta\int_{\Omega_1}\int_0^1|\widehat{z}(x,\rho,t)|^2\mathrm{d}\rho\mathrm{d}x\Big\}\\ &\le C(\|\nabla\widehat{u}\|_{\Omega_1}^2+\|\nabla\widehat{v}\|_{\Omega_2}^2). \end{aligned} \tag{3.38}$$

Integrating (3.38) over $(0,t)$, $0<t\le T$ and then using Gronwall's lemma, we obtain

$$\|\widehat{u}_t\|_{\Omega_1}^2+\|\widehat{v}_t\|_{\Omega_2}^2+\|\nabla\widehat{u}\|_{\Omega_1}^2+\|\nabla\widehat{v}\|_{\Omega_2}^2+\zeta\int_{\Omega_1}\int_0^1|\widehat{z}(x,\rho,t)|^2\mathrm{d}\rho\mathrm{d}x=0.$$

Hence, uniqueness follows.

With the above 4 steps, we obtain the well-posedness of solution for problem (1.1)-(1.8). □

4. Decay estimates

In this section, we consider the asymptotic behavior of problem (2.4)-(2.13). For the proof of Theorem 2.3, we need the following lemmas.

Lemma 4.1. *Let (u,v,z) be the solution of (2.4)-(2.13), then we have*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(t)\le&-\frac{r}{2}\|u_t\|_{\Gamma_2}^2-(\mu_1-\frac{\zeta}{2\tau}-\frac{\mu_2}{2})\|u_t\|_{\Omega_1}^2-(\frac{\zeta}{2\tau}-\frac{\mu_2}{2})\|z(x,1,t)\|_{\Omega_1}^2\\ &+\frac{r}{2}k^2(t)\|u_0\|_{\Gamma_2}^2-\frac{r}{2}\int_{\Gamma_2}(k''\diamond u)(t)\mathrm{d}\Gamma+\frac{r}{2}k'(t)\|u\|_{\Gamma_2}^2. \end{aligned} \tag{4.1}$$

Proof. Multiplying (2.4) and (2.6) by u_t and v_t, and integrating over Ω_1 and Ω_2, respectively, with the aid of (2.7)-(2.9) and (2.1), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}(\|u_t\|_{\Omega_1}^2+\|v_t\|_{\Omega_2}^2+\|\nabla u\|_{\Omega_1}^2+\|\nabla v\|_{\Omega_2}^2)+\frac{1}{4}(\|\nabla u\|_{\Omega_1}^4+\|\nabla v\|_{\Omega_2}^4)\Big\}\\ &=-\mu_1\|u_t\|_{\Omega_1}^2-\mu_2\int_{\Omega_1}u_tz(x,1,t)\mathrm{d}x-r\|u_t\|_{\Gamma_2}^2+rk(t)\int_{\Gamma_2}u_0u_t\mathrm{d}\Gamma\\ &\quad-\frac{r}{2}\int_{\Gamma_2}(k''\diamond u)\mathrm{d}\Gamma+\frac{r}{2}k'(t)\|u\|_{\Gamma_2}^2+\frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{r}{2}\int_{\Gamma_2}(k'\diamond u)\mathrm{d}\Gamma-\frac{r}{2}k(t)\|u\|_{\Gamma_2}^2\Big]. \end{aligned} \tag{4.2}$$

Multiplying (2.5) by $\frac{\zeta}{\tau}z(x,\rho,t)$ and integrating over $\Omega_1\times[0,1]$ with respect to x and ρ, respectively, we have

$$\frac{\zeta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}\int_0^1|z(x,\rho,t)|^2\mathrm{d}\rho\mathrm{d}x=-\frac{\zeta}{2\tau}[\|z(x,1,t)\|_{\Omega_1}^2-\|u_t\|_{\Omega_1}^2]. \tag{4.3}$$

Combining (4.2), (4.3) and (2.14), we can obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) =& -(\mu_1-\frac{\zeta}{2\tau})\|u_t\|^2_{\Omega_1}-\frac{\zeta}{2\tau}\|z(x,1,t)\|^2_{\Omega_1}\\&-\mu_2\int_{\Omega_1}u_tz(x,1,t)\mathrm{d}x-r\|u_t\|^2_{\Gamma_2}+rk(t)\int_{\Gamma_2}u_0u_t\mathrm{d}x\\&-\frac{r}{2}\int_{\Gamma_2}(k''\diamond u)(t)\mathrm{d}\Gamma+\frac{r}{2}k'(t)\|u\|^2_{\Gamma_2}.\end{aligned}\tag{4.4}$$

By Young's inequality, we obtain

$$|-\mu_2\int_{\Omega_1}u_tz(x,1,t)\mathrm{d}x|\le\frac{\mu_2}{2}\|u_t\|^2_{\Omega_1}+\frac{\mu_2}{2}\|z(x,1,t)\|^2_{\Omega_1},\tag{4.5}$$

$$rk(t)\int_{\Gamma_2}u_0u_t\mathrm{d}x\le\frac{r}{2}\|u_t\|^2_{\Gamma_2}+\frac{r}{2}k^2(t)\|u_0\|^2_{\Gamma_2}.\tag{4.6}$$

Substituting (4.5), (4.6) into (4.4), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t)\le& -\frac{r}{2}\|u_t\|^2_{\Gamma_2}-(\mu_1-\frac{\zeta}{2\tau}-\frac{\mu_2}{2})\|u_t\|^2_{\Omega_1}-(\frac{\zeta}{2\tau}-\frac{\mu_2}{2})\|z(x,1,t)\|^2_{\Omega_1}\\&+\frac{r}{2}k^2(t)\|u_0\|^2_{\Gamma_2}-\frac{r}{2}\int_{\Gamma_2}(k''\diamond u)(t)\mathrm{d}\Gamma+\frac{r}{2}k'(t)\|u\|^2_{\Gamma_2}.\end{aligned}\tag{4.7}$$

Then we can derive the result of Lemma 4.1. □

Remark 4.2. From the range of ζ, we can see that $\mu_1-\frac{\zeta}{2\tau}-\frac{\mu_2}{2}>0$ and $\frac{\zeta}{2\tau}-\frac{\mu_2}{2}>0$. However, since

$$\frac{r}{2}k^2(t)\|u_0\|^2_{\Gamma_2}\ge 0,$$

$E(t)$ may be not nonincreasing.

Now we define the functional

$$\Phi_1(t):=\int_{\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u]u_t\mathrm{d}x+\int_{\Omega_2}[(m\cdot\nabla v)+\big(\frac{N}{2}-\theta\big)v]v_t\mathrm{d}x,$$

where $0<\theta<1$ is a constant which will be determined later.

Lemma 4.3. *Let (u,v,z) be a solution of problem (2.4)-(2.13), then for $t_0>0$ large enough, there exist $\alpha_1>0$ such that*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1(t)\le& -(1-\theta)\|\nabla u\|^4_{\Omega_1}-\alpha_1\|\nabla u\|^2_{\Omega_1}+\big(\frac{1}{4\eta_1}-\theta\big)\|u_t\|^2_{\Omega_1}\\&+\frac{1}{4\eta_2}\|z(x,1,t)\|^2_{\Omega_1}+\big(\frac{R}{2}+\frac{r^2}{\eta_3}\big)\|u_t\|^2_{\Gamma_2}+\frac{r^2}{\eta_3}k^2(t)\|u_0\|^2_{\Gamma_2}\\&-\theta\|v_t\|^2_{\Omega_2}-(1-\theta)(1+\|\nabla v\|^2_{\Omega_2})\|\nabla v\|^2_{\Omega_2},\end{aligned}$$

where $\eta_i (i=1,2,3)$ are sufficiently small positive constants.

Proof. By (2.4) and integration by parts, we can derive

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u]u_t\mathrm{d}x\\
&=\int_{\Omega_1}(m\cdot\nabla u_t)u_t\mathrm{d}x-(1+\|\nabla u\|_{\Omega_1}^2)\int_{\Omega_1}\nabla(m\cdot\nabla u)\cdot\nabla u\mathrm{d}x\\
&+\big(\frac{N}{2}-\theta\big)\|u_t\|_{\Omega_1}^2-\big(\frac{N}{2}-\theta\big)(1+\|\nabla u\|_{\Omega_1}^2)\|\nabla u\|_{\Omega_1}^2\\
&+\int_{\partial\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u](1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial\nu}\mathrm{d}\Gamma\\
&-\mu_1\int_{\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u]u_t\mathrm{d}x\\
&-\mu_2\int_{\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u]z(x,1,t)\mathrm{d}x.
\end{aligned}
\tag{4.8}
$$

Noting that

$$
\int_{\Omega_1}(m\cdot\nabla u_t)u_t\mathrm{d}x=-\frac{N}{2}\|u_t\|_{\Omega_1}^2+\frac{1}{2}\int_{\partial\Omega_1}(m\cdot\nu)|u_t|^2\mathrm{d}\Gamma,
\tag{4.9}
$$

and

$$
\begin{aligned}
&-\int_{\Omega_1}\nabla(m\cdot\nabla u)\cdot\nabla u\mathrm{d}x\\
&=-\int_{\Omega_1}\Sigma_{i,j=1}^N\big[\frac{\partial}{\partial x_i}\big(m_j\frac{\partial u}{\partial x_j}\big)\frac{\partial u}{\partial x_i}\big]\mathrm{d}x\\
&=-\int_{\Omega_1}\Sigma_{i,j=1}^N\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial m_j}{\partial x_i}\mathrm{d}x-\frac{1}{2}\int_{\Omega_1}\Sigma_{i,j=1}^N\frac{\partial}{\partial x_j}\big(\frac{\partial u}{\partial x_i}\big)^2m_j\mathrm{d}x\\
&=(\frac{N}{2}-1)\|\nabla u\|_{\Omega_1}^2-\frac{1}{2}\int_{\partial\Omega_1}(m\cdot\nu)|\nabla u|^2\mathrm{d}\Gamma.
\end{aligned}
\tag{4.10}
$$

Substituting (4.9), (4.10) into (4.8) to deduce that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u]u_t\mathrm{d}x\\
&=-\theta\|u_t\|_{\Omega_1}^2-(1-\theta)(1+\|\nabla u\|_{\Omega_1}^2)\|\nabla u\|_{\Omega_1}^2+\frac{1}{2}\int_{\partial\Omega_1}(m\cdot\nu)|u_t|^2\mathrm{d}\Gamma\\
&-\frac{1}{2}(1+\|\nabla u\|_{\Omega_1}^2)\int_{\partial\Omega_1}(m\cdot\nu)|\nabla u|^2\mathrm{d}\Gamma\\
&+\int_{\partial\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u](1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial\nu}\mathrm{d}\Gamma\\
&-\mu_1\int_{\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u]u_t\mathrm{d}x\\
&-\mu_2\int_{\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u]z(x,1,t)\mathrm{d}x.
\end{aligned}
\tag{4.11}
$$

Similarly, using (2.6) and integration by parts, we can derive

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_2}[(m\cdot\nabla v)+\big(\frac{N}{2}-\theta\big)v]v_t\mathrm{d}x\\
&=-\theta\|v_t\|_{\Omega_2}^2-(1-\theta)(1+\|\nabla v\|_{\Omega_2}^2)\|\nabla v\|_{\Omega_2}^2+\frac{1}{2}\int_{\partial\Omega_2}(m\cdot\widetilde{\nu})|v_t|^2\mathrm{d}\Gamma\\
&-\frac{1}{2}(1+\|\nabla v\|_{\Omega_2}^2)\int_{\partial\Omega_2}(m\cdot\widetilde{\nu})|\nabla v|^2\mathrm{d}\Gamma\\
&+\int_{\partial\Omega_2}[(m\cdot\nabla v)+\big(\frac{N}{2}-\theta\big)v](1+\|\nabla v\|_{\Omega_2}^2)\frac{\partial v}{\partial\widetilde{\nu}}\mathrm{d}\Gamma,
\end{aligned}
\tag{4.12}
$$

where $\widetilde{\nu}$ denotes the outer normal vector pointing towards the exterior of Ω_2. Adding (4.11) to (4.12) and using transmission conditions (2.8), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1(t)=&-\theta\|u_t\|_{\Omega_1}^2-(1-\theta)(1+\|\nabla u\|_{\Omega_1}^2)\|\nabla u\|_{\Omega_1}^2\\
&+\frac{1}{2}\int_{\Gamma_2}(m\cdot\nu)|u_t|^2\mathrm{d}\Gamma-\frac{1}{2}(1+\|\nabla u\|_{\Omega_1}^2)\int_{\Gamma_2}(m\cdot\nu)|\nabla u|^2\mathrm{d}\Gamma\\
&+\int_{\Gamma_2}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u](1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial\nu}\mathrm{d}\Gamma\\
&-\mu_1\int_{\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u]u_t\mathrm{d}x\\
&-\mu_2\int_{\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u]z(x,1,t)\mathrm{d}x\\
&-\theta\|v_t\|_{\Omega_2}^2-(1-\theta)(1+\|\nabla v\|_{\Omega_2}^2)\|\nabla v\|_{\Omega_2}^2\\
&+\frac{1}{2}\int_{\Gamma_0}(m\cdot\widetilde{\nu})|v_t|^2\mathrm{d}\Gamma-\frac{1}{2}(1+\|\nabla v\|_{\Omega_2}^2)\int_{\Gamma_0}(m\cdot\widetilde{\nu})|\nabla v|^2\mathrm{d}\Gamma\\
&+\int_{\Gamma_0}[(m\cdot\nabla v)+\big(\frac{N}{2}-\theta\big)v](1+\|\nabla v\|_{\Omega_2}^2)\frac{\partial v}{\partial\widetilde{\nu}}\mathrm{d}\Gamma.
\end{aligned}
\tag{4.13}
$$

Since $\frac{\partial v}{\partial x_i}=\widetilde{\nu}_i\frac{\partial v}{\partial\widetilde{\nu}}$, $i=1,\dots,N$ and $m\cdot\widetilde{\nu}\le 0$ on Γ_0, we have

$$
\begin{aligned}
&-\frac{1}{2}(1+\|\nabla v\|_{\Omega_2}^2)\int_{\Gamma_0}(m\cdot\widetilde{\nu})|\nabla v|^2\mathrm{d}\Gamma+(1+\|\nabla v\|_{\Omega_2}^2)\int_{\Gamma_0}(m\cdot\nabla v)\frac{\partial v}{\partial\widetilde{\nu}}\mathrm{d}\Gamma\\
&=\frac{1}{2}(1+\|\nabla v\|_{\Omega_2}^2)\int_{\Gamma_0}(m\cdot\widetilde{\nu})\big|\frac{\partial v}{\partial\widetilde{\nu}}\big|^2\mathrm{d}\Gamma\le 0.
\end{aligned}
$$

Moreover, since $v=0$ on Γ_0, (4.13) can be rewritten as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1(t)=&-\theta\|u_t\|_{\Omega_1}^2-(1-\theta)(1+\|\nabla u\|_{\Omega_1}^2)\|\nabla u\|_{\Omega_1}^2\\
&+\frac{1}{2}\int_{\Gamma_2}(m\cdot\nu)|u_t|^2\mathrm{d}\Gamma-\frac{1}{2}(1+\|\nabla u\|_{\Omega_1}^2)\int_{\Gamma_2}(m\cdot\nu)|\nabla u|^2\mathrm{d}\Gamma\\
&+\int_{\Gamma_2}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u](1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial\nu}\mathrm{d}\Gamma\\
&-\mu_1\int_{\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u]u_t\mathrm{d}x\\
&-\mu_2\int_{\Omega_1}[(m\cdot\nabla u)+\big(\frac{N}{2}-\theta\big)u]z(x,1,t)\mathrm{d}x
\end{aligned}
$$

$$- \theta\|v_t\|_{\Omega_2}^2 - (1-\theta)(1+\|\nabla v\|_{\Omega_2}^2)\|\nabla v\|_{\Omega_2}^2. \tag{4.14}$$

It follows from Young's inequality that

$$\begin{aligned} &\big| - \mu_1 \int_{\Omega_1} [(m\cdot\nabla u) + \big(\frac{N}{2}-\theta\big)u]u_t \mathrm{d}x\big| \\ &\le 2[R^2 + \big(\frac{N}{2}-\theta\big)^2\lambda_1^2]\mu_1^2\eta_1\|\nabla u\|_{\Omega_1}^2 + \frac{1}{4\eta_1}\|u_t\|_{\Omega_1}^2, \end{aligned} \tag{4.15}$$

$$\begin{aligned} &\big| - \mu_2 \int_{\Omega_1} [(m\cdot\nabla u) + \big(\frac{N}{2}-\theta\big)u]z(x,1,t) \mathrm{d}x\big| \\ &\le 2[R^2 + \big(\frac{N}{2}-\theta\big)^2\lambda_1^2]\mu_2^2\eta_2\|\nabla u\|_{\Omega_1}^2 + \frac{1}{4\eta_2}\|z(x,1,t)\|_{\Omega_1}^2, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} &\big| \int_{\Gamma_2} [(m\cdot\nabla u) + \big(\frac{N}{2}-\theta\big)u](1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial \nu} \mathrm{d}\Gamma\big| \\ &\le 2R^2\eta_3\|\nabla u\|_{\Gamma_2}^2 + 2\big(\frac{N}{2}-\theta\big)^2\lambda^2\eta_3\|\nabla u\|_{\Omega_1}^2 + \frac{1}{4\eta_3}\big\|(1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial \nu}\big\|_{\Gamma_2}^2 \\ &\le 2R^2\eta_3\|\nabla u\|_{\Gamma_2}^2 + 2\big(\frac{N}{2}-\theta\big)^2\lambda^2\eta_3\|\nabla u\|_{\Omega_1}^2 \\ &\quad + \frac{r^2}{4\eta_3}\|u_t + k(0)u - k(t)u_0 + k' * u\|_{\Gamma_2}^2 \\ &\le 2R^2\eta_3\|\nabla u\|_{\Gamma_2}^2 + 2\big(\frac{N}{2}-\theta\big)^2\lambda^2\eta_3\|\nabla u\|_{\Omega_1}^2 + \frac{r^2}{\eta_3}\|u_t\|_{\Gamma_2}^2 + \frac{r^2}{\eta_3}k^2(t)\|u_0\|_{\Gamma_2}^2 \\ &\quad + \frac{r^2}{\eta_3}k^2(t)\|u\|_{\Gamma_2}^2 - \frac{r^2}{\eta_3}k(0)\int_{\Gamma_2}(k'\diamond u)\mathrm{d}\Gamma, \end{aligned} \tag{4.17}$$

where $\eta_i (i = 1,2,3)$ are sufficiently small positive constants and we have used inequality (2.2) and the following identity

$$\begin{aligned} (1+\|\nabla u\|_{\Omega_1}^2)\frac{\partial u}{\partial \nu} &= - r[u_t + k(0)u - k(t)u_0 + k' * u] \\ &= - r[u_t + k(t)u - k(t)u_0 - k' \circ u]. \end{aligned}$$

Besides, λ and λ_1 are the optimal constants of trace inequality and the first eigenvalue of $-\Delta$ with Dirichlet boundary condition, respectively, i.e. $\|u\|_{\Gamma_2}^2 \le \lambda\|\nabla u\|_{\Omega_1}^2$ and $\|u\|_{\Omega_1}^2 \le \lambda_1\|\nabla u\|_{\Omega_1}^2$, respectively.

Substituting (4.15)-(4.17) into (4.14), we can derive

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Phi_1(t) \le& \big(\frac{1}{4\eta_1} - \theta\big)\|u_t\|_{\Omega_1}^2\big(\frac{R}{2} + \frac{r^2}{\eta_3}\big)\|u_t\|_{\Gamma_2}^2 - (1-\theta)\|\nabla u\|_{\Omega_1}^4 \\ &- \Big\{\frac{1-\theta}{2} - 2[R^2 + \big(\frac{N}{2}-\theta\big)^2\lambda_1^2](\mu_1^2\eta_1 + \mu_2^2\eta_2) \\ &- 2\big(\frac{N}{2}-\theta\big)^2\lambda^2\eta_3\Big\}\|\nabla u\|_{\Omega_1}^2 - \big(\frac{\delta}{2} - 2R^2\eta_3\big)\|\nabla u\|_{\Gamma_2}^2 \\ &+ \frac{1}{4\eta_2}\|z(x,1,t)\|_{\Omega_1}^2 - [\frac{1-\theta}{2\lambda^2 k(t)} - \frac{r^2}{\eta_3}k(t)]k(t)\|u\|_{\Gamma_2}^2 \\ &+ \frac{r^2}{\eta_3}k^2(t)\|u_0\|_{\Gamma_2}^2 - \frac{r^2}{\eta_3}k(0)\int_{\Gamma_2}(k'\diamond u)(t)\mathrm{d}\Gamma \\ &- \theta\|v_t\|_{\Omega_2}^2 - (1-\theta)(1+\|\nabla v\|_{\Omega_2}^2)\|\nabla v\|_{\Omega_2}^2. \end{aligned} \tag{4.18}$$

Choosing η_i $(i = 1, 2, 3)$ small enough such that

$$\alpha_1 = \frac{1-\theta}{2} - 2[R^2 + \big(\frac{N}{2} - \theta\big)^2 \lambda_1^2](\mu_1^2\eta_1 + \mu_2^2\eta_2) - 2\big(\frac{N}{2} - \theta\big)^2 \lambda^2 \eta_3 > 0,$$
$$\frac{\delta}{2} - 2R^2\eta_3 > 0,$$

it follows from $\lim_{t\to\infty} k(t) = 0$ that our result holds for $t_0 > 0$ large enough. $\square$

Next, we define the functional

$$\Phi_2(t) := \tau \int_{\Omega_1} \int_0^1 e^{-\tau\rho} |z(x, \rho, t)|^2 \mathrm{d}\rho \mathrm{d}x.$$

Then we have the following lemma.

Lemma 4.4. *The functional Φ_2 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2(t) \le -C(\tau)[\int_{\Omega_1} \int_0^1 |z(x, \rho, t)|^2 \mathrm{d}\rho \mathrm{d}x + \|z(x, 1, t)\|^2_{\Omega_1}] + \|u_t\|^2_{\Omega_1},$$

where $C(\tau)$ is a positive constant only depending on τ.

Proof. We use the method introduced by [30] to prove this lemma. Taking the derivative of $\Phi_2(t)$ directly, and using (2.5), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2(t) =& 2\tau \int_{\Omega_1} \int_0^1 e^{-\tau\rho} z(x, \rho, t) z_t(x, \rho, t) \mathrm{d}\rho \mathrm{d}x \\
=& -2 \int_{\Omega_1} \int_0^1 e^{-\tau\rho} z(x, \rho, t) z_\rho(x, \rho, t) \mathrm{d}\rho \mathrm{d}x \\
=& -\int_{\Omega_1} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\rho}[e^{-\tau\rho} |z(x, \rho, t)|^2] \mathrm{d}\rho \mathrm{d}x - \tau \int_{\Omega_1} \int_0^1 e^{-\tau\rho} |z(x, \rho, t)|^2 \mathrm{d}\rho \mathrm{d}x \\
=& -\int_{\Omega_1} e^{-\tau} |z(x, 1, t)|^2 - |z(x, 0, t)|^2 \mathrm{d}x - \tau \int_{\Omega_1} \int_0^1 e^{-\tau\rho} |z(x, \rho, t)|^2 \mathrm{d}\rho \mathrm{d}x \\
\le& - C(\tau)[\int_{\Omega_1} \int_0^1 |z(x, \rho, t)|^2 \mathrm{d}\rho \mathrm{d}x + \|z(x, 1, t)\|^2_{\Omega_1}] + \|u_t\|^2_{\Omega_1},
\end{aligned}$$

where $C(\tau)$ is a positive constant only depending on τ. $\square$

Proof of Theorem 2.3. We define the Lyapunov functional

$$L(t) := M_1 E(t) + M_2 \Phi_1(t) + M_3 \Phi_2(t), \tag{4.19}$$

where M_i $(i = 1, 2, 3)$ are positive constants which will be determined later.

Differentiating $L(t)$ directly and using Lemma 4.1-Lemma 4.4, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}L(t) \le &- \big[\big(\mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_2}{2}\big)M_1 - \big(\frac{1}{4\eta_1} - \theta\big)M_2 - M_3\big]\|u_t\|_{\Omega_1}^2 \\
&- [(1-\theta)M_2]\|\nabla u\|_{\Omega_1}^4 - M_2\alpha_1\|\nabla u\|_{\Omega_1}^2 \\
&- M_3C(\tau)\int_{\Omega_1}\int_0^1 |z(x,\rho,t)|^2\,\mathrm{d}\rho\mathrm{d}x \\
&- \big[\big(\frac{\zeta}{2\tau} - \frac{\mu_2}{2}\big)M_1 - \frac{M_2}{4\eta_2} + M_3C(\tau)\big]\|z(x,1,t)\|_{\Omega_1}^2 \\
&- \big[\frac{M_1 r}{2} - \big(\frac{r^2}{\eta_3} + \frac{R}{2}\big)M_2\big]\|u_t\|_{\Gamma_2}^2 \\
&+ \big(\frac{M_1 r}{2} + \frac{M_2 r^2}{\eta_3}\big)k^2(t)\|u_0\|_{\Gamma_2}^2 \\
&- \frac{M_1 r}{2}\int_{\Gamma_2}(k'' \diamond u)(t)\mathrm{d}\Gamma + \frac{M_1 r}{2}k'(t)\|u\|_{\Gamma_2}^2 \\
&- M_2\theta\|v_t\|_{\Omega_2}^2 - M_2(1-\theta)(1+\|\nabla v\|_{\Omega_2}^2)\|\nabla v\|_{\Omega_2}^2.
\end{aligned}
\tag{4.20}
$$

Choosing $M_i > 0$, $(i = 1,2,3)$ such that

$$
\begin{gathered}
\big(\mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_2}{2}\big)M_1 - \big(\frac{1}{4\eta_1} - \theta\big)M_2 - M_3 > 0, \\
\big(\frac{\zeta}{2\tau} - \frac{\mu_2}{2}\big)M_1 - \frac{M_2}{4\eta_2} + M_3C(\tau) > 0, \\
\frac{M_1 r}{2} - \big(\frac{r^2}{\eta_3} + \frac{R}{2}\big)M_2 > 0,
\end{gathered}
$$

and since $E(t)$ is equivalent to

$$
\begin{aligned}
&\|u_t\|_{\Omega_1}^2 + \|v_t\|_{\Omega_2}^2 + \|\nabla u\|_{\Omega_1}^2 + \|\nabla v\|_{\Omega_2}^2 + \frac{1}{4}\|\nabla u\|_{\Omega_1}^4 + \frac{1}{4}\|\nabla v\|_{\Omega_2}^4 \\
&+ k(t)\|u\|_{\Gamma_2}^2 - \int_{\Gamma_2}(k' \diamond u)\mathrm{d}\Gamma + \int_{\Omega_1}\int_0^1 |z(x,\rho,t)|^2\mathrm{d}\rho\mathrm{d}x,
\end{aligned}
$$

we know that there exist positive constants β_1, β_2, β_3, such that

$$
\frac{\mathrm{d}}{\mathrm{d}t}L(t) \le -\beta_1 E(t) + \beta_2 k^2(t)\|u_0\|_{\Gamma_2}^2 - \beta_3\int_{\Gamma_2}(k' \diamond u)(t)\mathrm{d}\Gamma, \quad \forall t \ge t_0. \tag{4.21}
$$

Multiplying (4.21) by $\xi(t)$ and using (H) and (4.1), we obtain

$$
\begin{aligned}
\xi(t)\frac{\mathrm{d}}{\mathrm{d}t}L(t) &\le -\beta_1\xi(t)E(t) + \beta_2\xi(t)k^2(t)\|u_0\|_{\Gamma_2}^2 - \beta_3\xi(t)\int_{\Gamma_2}(k' \diamond u)(t)\mathrm{d}\Gamma \\
&\le -\beta_1\xi(t)E(t) + \beta_2\xi(t)k^2(t)\|u_0\|_{\Gamma_2}^2 + \beta_3\int_{\Gamma_2}(k'' \diamond u)(t)\mathrm{d}\Gamma \\
&\le -\beta_1\xi(t)E(t) + \beta_2\xi(t)k^2(t)\|u_0\|_{\Gamma_2}^2 \\
&\quad + \beta_3[-\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}t}E(t) + k^2(t)\|u_0\|_{\Gamma_2}^2].
\end{aligned}
\tag{4.22}
$$

Noting that $\xi'(t) \le 0$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}[\xi(t)L(t) + \frac{2\beta_3}{r}E(t)] \le -\beta_1\xi(t)E(t) + [\beta_2\xi(0) + \beta_3]k^2(t)\|u_0\|_{\Gamma_2}^2. \tag{4.23}
$$

Now, define the functional

$$\mathcal{L}(t) := \xi(t)L(t) + \frac{2\beta_3}{r}E(t), \quad \forall t \geq t_0.$$

Then it is easy to verify that $\mathcal{L}(t)$ is equivalent to $E(t)$ and there exist constants $\gamma_1,\ \gamma_2 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leq -\gamma_1\xi(t)\mathcal{L}(t) + \gamma_2 k^2(t)\|u_0\|_{\Gamma_2}^2, \quad \forall t \geq t_0. \tag{4.24}$$

Case 1: If $u_0 = 0$ on Γ_2, inequality (4.24) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leq -\gamma_1\xi(t)\mathcal{L}(t).$$

Integrating this inequality from 0 to t, we have

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\gamma_1\int_0^t \xi(s)\mathrm{d}s}, \quad \forall t \geq t_0. \tag{4.25}$$

It follows from the equivalence relation between $\mathcal{L}(t)$ and $E(t)$ that there exists a constant $C > 0$ such that

$$E(t) \leq CE(0)e^{-\gamma_1\int_0^t \xi(s)\mathrm{d}s}, \quad \forall t \geq t_0.$$

Case 2: If $u_0 \neq 0$ on Γ_2, we set

$$\mathcal{F}(t) = \mathcal{L}(t) - \gamma_2\|u_0\|_{\Gamma_2}^2 e^{-\gamma_1\int_0^t \xi(s)\mathrm{d}s}\int_0^t k^2(s)e^{\gamma_1\int_0^s \xi(r)\mathrm{d}r}\mathrm{d}s, \quad \forall t \geq t_0. \tag{4.26}$$

Then by calculating directly, and using (4.24) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) \leq -\gamma_1\xi(t)\mathcal{F}(t), \quad \forall t \geq t_0.$$

Integrating this inequality over $(0, t)$, we have

$$\mathcal{F}(t) \leq \mathcal{F}(0)e^{-\gamma_1\int_0^t \xi(s)\mathrm{d}s}, \quad \forall t \geq t_0. \tag{4.27}$$

Combining (4.26) and (4.27), we have

$$\mathcal{L}(t) \leq \Big[\mathcal{L}(0) + \gamma_2\|u_0\|_{\Gamma_2}^2\int_0^t k^2(s)e^{\gamma_1\int_0^s \xi(r)\mathrm{d}r}\mathrm{d}s\Big]e^{-\gamma_1\int_0^t \xi(s)\mathrm{d}s}, \quad \forall t \geq t_0. \tag{4.28}$$

It follows from the equivalence relation between $\mathcal{L}(t)$ and $E(t)$ that there exists a constant $C > 0$ such that

$$E(t) \leq C\Big[E(0) + \gamma_2\|u_0\|_{\Gamma_2}^2\int_0^t k^2(s)e^{\gamma_1\int_0^s \xi(r)\mathrm{d}r}\mathrm{d}s\Big]e^{-\gamma_1\int_0^t \xi(s)\mathrm{d}s}, \quad \forall t \geq t_0.$$

The proof of Theorem 2.3 is complete. □

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Zhiqing Liu
School of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao 266061, China
Email address: Lzhiqing1005@163.com

Cunchen Gao
School of Mathematical Sciences, Ocean University of China, Qingdao 266100, China
Email address: ccgao123@126.com

Zhong Bo Fang (corresponding author)
School of Mathematical Sciences, Ocean University of China, Qingdao 266100, China
Email address: fangzb7777@hotmail.com