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A NOTE ON THE UNIQUENESS OF ENTROPY SOLUTIONS TO FIRST ORDER QUASILINEAR EQUATIONS

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Abstract. In this note, we consider entropy solutions to scalar conservation laws with unbounded initial data. In particular, we offer an extension of Kružkhov's uniqueness proof (see [1]).

1. Introduction

We are concerned with the following Cauchy problem:

$$\begin{cases} u_t + \mathrm{div} F(u) = 0 \quad \text{in} \;\; S_T = \mathbb{R}^{\mathbb{N}} \times (\nvdash, \mathbb{T}) \\ \quad u(x,0) = u_0(x) \quad x \in \mathbb{R}^{\mathbb{N}}. \end{cases} \tag{1}$$

Here $F = (F_1, \cdots, F_N) \in [C^{0,1}(\mathbb{R})]^{\mathbb{N}}$, and $u_0 \in L^1_{loc}(\mathbb{R}^{\mathbb{N}})$. In particular, we are interested in the entropy solutions to (1). We say that $u \in L^\infty_{loc}(S_T)$ is an entropy solution to (1) if

$$\iint_{S_T} \mathrm{sign}(u-k)\,[(u-k)\phi_t + (F(u) - F(k)) \cdot D\phi]\; dx\,dt \;\geq 0, \tag{2}$$

for all $\phi \in C_0^\infty(S_T)$, $\phi \geq 0$, and all $k \in \mathbb{R}$, and there exists a set $\Gamma_0 \subseteq [0,T]$ of measure zero, such that for all compact sets $K \subseteq \mathbb{R}^{\mathbb{N}}$

$$\lim_{\substack{t \to 0^+ \\ t \notin \Gamma_0}} \|u(\cdot,t) - u_0\|_{1,K} = 0. \tag{3}$$

In [1], Kružkhov proves existence and uniqueness of an entropy solution to (1) when u_0 is bounded and F is sufficiently smooth. If $u_0, v_0 \in L^1(\mathbb{R}^{\mathbb{N}}) \cap \mathbb{L}^\infty(\mathbb{R}^{\mathbb{N}})$ with corresponding entropy solutions u, v respectively then

$$\int_{\mathbb{R}^{\mathbb{N}}} |u(x,t) - v(x,t)|\; dx \;\leq \int_{\mathbb{R}^{\mathbb{N}}} |u_0(x) - v_0(x)|\; dx$$

for a.e. $t \in [0,T]$ (see [1] equation 3.1). If $u_0 \in L^1(\mathbb{R}^{\mathbb{N}})$ (but not bounded) then there is a natural candidate for an entropy solution with this initial data. This note is motivated by the following two questions:

(i) Is this candidate an entropy solution?

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(ii) If it is an entropy solution then is it the unique entropy solution?
This note is a partial answer to the second of these two questions.

2. Main Result

In proving uniqueness Kružkhov proves the following Proposition:

Proposition 2.1. *If u and v are entropy solutions to (1) satisfying*

$$\left\| \frac{F(u)-F(v)}{u-v} \right\|_{\infty, S_T} \leq M$$

then $u = v$ almost everywhere in S_T.

The primary result of this note is the following improvement of Proposition 2.1.

Proposition 2.2. *If u and v are entropy solutions to (1) satisfying*

$$\left\| \frac{F(u(\cdot,t))-F(v(\cdot,t))}{u(\cdot,t)-v(\cdot,t)} \right\|_{\infty, B_\rho} \leq M(t,\rho) \tag{4}$$

where M satisfies

$$\lim_{\rho\to\infty} \left(\rho - \int_0^T M(t,\rho)\,dt \right) = \infty \tag{5}$$

then $u = v$ almost everywhere in S_T.

The advantage of Proposition 2.2 over Proposition 2.1 is that Proposition 2.2 allows for u_0 to become unbounded. Set $A(u) = (F_1'(u), \cdots, F_N'(u))$. Then one can easily verify that Proposition 2.2 implies the following.

Corollary 2.3. *There exists at most one entropy solution to (1) satisfying*

$$\|A(u(\cdot,t))\|_{\infty,\mathbb{R}^{\mathbb{N}}} \leq M(t)$$

where M satisfies

$$\int_0^T M(t)\,dt < \infty.$$

As an example we apply Corollary 2.3 to the Burger's equation, i.e., $N = 1$ and $F(u) = \frac{1}{2}u^2$.

Lemma 2.4. *If $u_0 \in L^p(\mathbb{R}^{\mathbb{N}})$ with $1 \leq p < \infty$ then there exists an entropy solution u of (1) satisfying*

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^{\mathbb{N}}} \leq \frac{(2\|u_0\|_p)^{\frac{p}{p+1}}}{t^{\frac{1}{p+1}}} \in L^1([0,1]). \tag{6}$$

Proof. Let u be the almost everywhere unique minimizer of

$$\psi(x,t,v) = \int_x^{x-vt} u_0(s)\,ds + \frac{tv^2}{2}.$$

For details see [2]. Then u is an entropy solution to (1). Now $\psi(x,t,0) = 0$ and by Holder's inequality

$$\begin{aligned}
\psi(x,t,v) &\geq -\|u_0\|_p \, |vt|^{\frac{p-1}{p}} + \frac{tv^2}{2} \\
&= |vt|^{\frac{p-1}{p}} \, (-\|u_0\|_p + \frac{t^{\frac{1}{p}} \, |v|^{\frac{p+1}{p}}}{2}) \\
&> 0
\end{aligned}$$

when

$$v > \frac{(2\|u_0\|_p)^{\frac{p}{p+1}}}{t^{\frac{1}{p+1}}}.$$

This implies (6). □

Thus from Corollary 2.3 the entropy solution to Burger's equation satisfying (6) is unique. One can prove a similar estimate for solutions when $F(u) = |u|^\alpha$ for any $\alpha > 1$.

Proof. (of Proposition 2.2). Let u and v be any two entropy solutions to (1). Let $J \in C_0^\infty(-1,1)$ satisfy:

$$\begin{cases} \int_{-1}^{1} J(x)\,dx = 1 \\ J \geq 0. \end{cases}$$

Any two entropy solutions u, v satisfy

$$\iint_{S_T} \operatorname{sign}(u-v)[(u-v)\phi_t + (F(u) - F(v)) \cdot D\phi]\,dx\,dt \geq 0 \tag{7}$$

for all $\phi \in C_0^\infty(S_T)$ with $\phi \geq 0$ (see [1] equation 3.7). Set

$$r(t,\rho) = \rho - \int_0^t M(\tau,\rho)d\tau.$$

Since $t \to r(t,\rho)$ is decreasing, (5) implies

$$\lim_{\rho\to\infty} r(t,\rho) = \infty \quad \text{for all} \quad 0 \leq t \leq T.$$

Select $R > 0$ such that if $\rho > R$ then $r(T,\rho) > 0$. Fix $\rho > R$. Set $r(t) = r(t,\rho)$. Let $0 < t_1 < t_2 < T$. Let $0 < \epsilon < \min\{t_1, T - t_2\}$. Consider the following test function:

$$\phi_\epsilon(x,t) = \epsilon^{-N-1} \int_{t-t_2}^{t-t_1} J\left(\frac{s}{\epsilon}\right)\,ds \int_{|x|-r(t)+\epsilon}^{\infty} J\left(\frac{s}{\epsilon}\right)\,ds\,.$$

Notice that

$$\operatorname{supp}\phi_\epsilon \subseteq \{(x,t) : t_1 - \epsilon < t < t_2 + \epsilon \ \text{ and } \ |x| < r(0) + \epsilon\},$$

so that ϕ_ϵ is an admissible test function for (7). Now we compute $\phi_{\epsilon,t}$ and $D\phi_\epsilon$.

$$\begin{aligned} \phi_{\epsilon,t} =& \epsilon^{-1}\left[J\left(\frac{t-t_1}{\epsilon}\right) - J\left(\frac{t-t_2}{\epsilon}\right)\right]\epsilon^{-N}\int_{|x|-r(t)+\epsilon}^{\infty} J\left(\frac{s}{\epsilon}\right)\,ds \\ &+ \left[M(t)\epsilon^{-N} J\left(\frac{|x| - r(t) + \epsilon}{\epsilon}\right)\right]\epsilon^{-1}\int_{t-t_2}^{t-t_1} J\left(\frac{s}{\epsilon}\right)\,ds\,. \end{aligned}$$

$$D\phi_\epsilon = \frac{x}{|x|}\left[\epsilon^{-N} J\left(\frac{|x| - r(t) + \epsilon}{\epsilon}\right)\right]\epsilon^{-1}\int_{t-t_2}^{t-t_1} J\left(\frac{s}{\epsilon}\right)\,ds\,.$$

Then using this test function in (7) yields

$$\begin{aligned}
&\iint_{S_T} |u-v|\, \epsilon^{-1}\left[J\left(\frac{t-t_1}{\epsilon}\right)-J\left(\frac{t-t_2}{\epsilon}\right)\right]\epsilon^{-N}\int_{|x|-r(t)+\epsilon}^{\infty} J\left(\frac{s}{\epsilon}\right)\, ds\\
&\quad\geq \iint_{S_T} \epsilon^{-N} J\left(\frac{|x|\,-r(t)+\epsilon}{\epsilon}\right)\epsilon^{-1}\int_{t-t_2}^{t-t_1} J\left(\frac{s}{\epsilon}\right)\, ds\\
&\quad\times\left[\operatorname{sign}(u-v)(F(u)-F(v))\cdot\frac{x}{|x|}+M(t,\rho)\,|u-v|\right] dx\, dt\\
&\quad\geq 0 \text{ because } M(t,\rho)\,|u-v|\geq|F(u)-F(v)|\,.
\end{aligned}$$

So by letting $\epsilon \to 0$ we obtain

$$\int_{B_{r(t_2)}} |u-v|\,(x,t_2)\, dx \,\leq \int_{B_{r(t_1)}} |u-v|\,(x,t_1)\, dx\,.$$

Since $r(t_1) < \rho$

$$\int_{B_{r(t_1)}} |u-v|\,(x,t_2)\, dx \,\leq \int_{B_{\rho}} |u-v|\,(x,t_1)\, dx\,.$$

Combining these with (3) yields

$$\int_{B_{r(t_2)}} |u-v|\,(x,t_2)\, dx \,\leq \int_{B_{\rho}} |u-v|\,(x,0)\, dx \,= 0.$$

Thus, for $0 < t_2 < T$, $u(x,t_2) = v(x,t_2)$ for $x \in B_{r(t_2)}$. Letting $\rho \to \infty$ and using condition (5), $u(\cdot,t_2) = v(\cdot,t_2)$. Since t_2 is arbitrary, $u = v$ in S_T. □

As another example we consider Burger's equation with initial data $u_0(x) = -x$. We show that in this situation (1) has a unique entropy solution in S_1 satisfying

$$\|u(\cdot,t)\|_{\infty,B_\rho} \leq \frac{\rho}{1-t}.$$

First note that

$$u(x,t) = \frac{-x}{1-t}$$

is one such solution. Suppose v is another. For $0 < t < 1$

$$r(t,\rho) = \rho - \int_0^t \frac{\rho}{1-\tau} d\tau = \rho(1+\ln(1-t)),$$

so Proposition 2.2 guarantees that $u = v$ in $S_{1-e^{-1}}$. By reapplying Proposition 2.2 this time beginning at $t = 1-e^{-1}$ we find that $u = v$ in $S_{1-e^{-2}}$. Finally, by repeating this argument n times, $u = v$ in $S_{1-e^{-n}}$. Thus $u = v$ in S_1.

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