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Neumann and periodic boundary-value problems for quasilinear ordinary differential equations with a nonlinearity in the derivative *

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Abstract

We present sufficient conditions for the existence of solutions to Neumann and periodic boundary-value problems for some class of quasilinear ordinary differential equations. We also show that this condition is necessary for certain nonlinearities. Our results involve the p-Laplacian, the mean-curvature operator and nonlinearities blowing up.

1 Introduction

The semilinear boundary-value problems

$$u''(t) + g(u'(t)) + h(u(t)) = f(t) \quad t \in (0,T)\,, \tag{1.1}$$
$$u(0) = u(T)\,, \quad u'(0) = u'(T)\,, \tag{1.2}$$

and

$$u''(t) + g(u'(t)) = f(t) \quad t \in (0,T)\,, \tag{1.3}$$
$$u'(0) = 0\,, \quad u'(T) = 0\,, \tag{1.4}$$

have been extensively studied by many authors (see e.g. [4, 6, 16]). In this paper we extend their results to the quasilinear boundary-value problems:

$$\big(\varphi(u'(t))\big)' + g(u'(t)) + h(u(t)) = f(t) \quad t \in (0,T)\,, \tag{1.5}$$

subject to (1.2) or to (1.4).

Overall, we will assume the following:

(P) φ is an increasing homeomorphism of I_1 onto I_2, where $I_1, I_2 \subset \mathbb{R}$ are open intervals containing zero and $\varphi(0) = 0$,

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(G) g is a continuous real function,

(H) h is a continuous, bounded real function having limits in $\pm\infty$:

$$h(-\infty) := \lim_{\xi\to-\infty} h(\xi) < \lim_{\xi\to+\infty} h(\xi) =: h(+\infty)\,.$$

We also need to impose some of the following assumptions to prove several particular results.

(P') the inverse of φ (denoted by φ_{-1}) is continuously differentiable,

(P") $\varphi_{-1}'(0) > 0$,

(PH) φ is odd and there exist $c, \delta > 0$ and $p > 1$ such that for all $z \in (-\delta, \delta) \cap \operatorname{Dom}\varphi :\ c|z|^{p-1} \le |\varphi(z)|$,

(G') g is a continuously differentiable real function.

The results presented involve blow-up-type nonlinearities such as $\varphi(z) = \tan(z)$, bounded nonlinearities of the type $\varphi(z) = \arctan(z)$, the p-Laplacian when $\varphi(z) = |z|^{p-2}z, 1 < p < \infty$, and include of course the classical results ([4, 6, 16]) where φ is considered to be an identity mapping on $\mathbb{R}$. As for the Neumann conditions, we obtain new results even in the semilinear case with '$\varphi = $ Identity' on $\mathbb{R}$. To extend known results for (1.5)–(1.2) or (1.5)–(1.4), we mainly combine and modify methods from [4, 6, 16].

Since boundary-value problems (BVPs) of the type (1.5)–(1.2) and (1.5)–(1.4) appear in a wide variety of applications, our extension of known results is interesting not only from the theoretical point of view but also from a practical one. For instance the p-Laplacian with $p \neq 2$ arises in the study of nonlinear diffusion. Blow-up-type nonlinearity $\varphi(z) = z/\sqrt{1-z^2}$ comes from the modeling of mechanical oscillations with relativistic correction. As an application of bounded nonlinearity let us mention the mean curvature (or capillary surface) operator $\varphi(z) = z/\sqrt{1+z^2}$, which occurs in problems of mathematical physics.

Related topics are the subjects of studies by many authors. Among others let us mention the recent paper [14] in which the authors study the existence of periodic solutions of systems of quasilinear ordinary differential equations (ODEs) at resonance. Let us also mention papers [9, 10]; even thought the results of those papers are of different nature, some of the techniques used are similar. The resonance problems for the p-Laplacian in partial derivatives are treated in [2] and [3].

The paper is organized as follows. We begin by stating our main results concerning the existence of the solution (Theorems 1-4) and discuss their applicability to various problems arising from physics in Section 2. Proofs of Theorems 1–3 can be found in Section 3. A proof of Theorem 4 is presented in Section 4. Finally, we present some results concerning the uniqueness of the solution in Section 5.

Before proceeding to the results we shall establish the notation that is used throughout the paper:

For measurable functions defined on $[0,T]$ we define:

$$L^p := \Big\{u : \|u\|_{L^p} := \Big(\int_0^T |u(t)|^p dt\Big)^{\frac{1}{p}} < \infty\Big\},\ 1 \le p < +\infty\,,$$
$$L^\infty := \Big\{u : \|u\|_\infty := \inf_{\substack{\mathcal{N} \subset (0,T) \\ \operatorname{meas}\mathcal{N} = 0}} \ \sup_{t\in(0,T)\setminus\mathcal{N}} |u(t)| < \infty\Big\}.$$

For functions with domain $[0,T]$, with distributional derivatives, we define:

$$W^{1,p} := \{u \in L^p : u' \in L^p\}\,,$$
$$W^{1,p}_T := \{u \in W^{1,p} : u(0) = u(T)\}\,.$$

For a given $k \ge 0$, let

$$C^k_T := \{u : u \in C^k[0,T], u(0) = u(T), u'(0) = u'(T), \dots, u^{(k-1)}(0) = u^{(k-1)}(T)\}\,,$$

where $C^k[0,T]$ is the standard space of k-times continuously differentiable functions defined on $[0,T]$, endowed with the norm $\|u\|_{C^k} := \sum_{i=0}^k \|u^{(i)}\|_\infty$ ($C[0,T] := C^0[0,T]$, $C_T := C^0_T$). By $C_0^\infty(0,T)$ we denote the set of functions u defined on $(0,T)$, possessing derivatives of any order in $(0,T)$ and such that the closure of $\{t \in (0,T) :\ u(t) \ne 0\}$ is a subset $(0,T)$.

To formulate our results, we use the decomposition

$$f = \widetilde{f} + \overline{f}\,, \quad \text{with } \overline{f} = \frac{1}{T}\int_0^T f(t)\,dt \tag{1.6}$$

and the following function spaces

$$\widetilde{C}[0,T] := \Big\{u \in C[0,T] : \textstyle\int_0^T u(t)dt = 0\Big\}\,,$$

$\widetilde{C}_T := C_T \cap \widetilde{C}[0,T]$, $\widetilde{C}^1[0,T] := C^1[0,T] \cap \widetilde{C}[0,T]$, $\widetilde{C}^1_T := C^1_T \cap \widetilde{C}[0,T]$.

Since we deal with the quasilinear equation, we shall work with a more general concept of solutions of ODE than that of the classical solutions.

Definition 1.1 A function $u \in C^1[0,T]$ is called a solution of (1.5)–(1.2) or (1.5)–(1.4) if $\varphi(u') \in C^1[0,T]$, u satisfies (1.5) pointwise in $(0,T)$ and $u(0) = u(T)$, $u'(0) = u'(T)$ or $u'(0) = 0$, $u'(T) = 0$, respectively.

2 Main results

In the following we will consider the range of φ to be an interval (a,b), where a and b could be $-\infty$ and $+\infty$, respectively.

Let us consider (1.5)–(1.2) and (1.5)–(1.4) with $h \equiv 0$. Substituting $w = \varphi(u')$ in (1.5) we obtain the semilinear first order equation:

$$w'(t) + g\big(\varphi_{-1}\big(w(t)\big)\big) = f(t)\,. \tag{2.1}$$

Now it remains to find how this substitution affects the boundary conditions.

Taking into account

$$u(T) - u(0) = \int_0^T u'(t)dt = \int_0^T \varphi_{-1}\big(w(t)\big)dt$$

and the fact that φ is an increasing homeomorphism, the periodic conditions (1.2) are transformed as follows:

$$\int_0^T \varphi_{-1}\big(w(t)\big)dt = 0\,, \quad w(0) = w(T)\,. \tag{2.2}$$

Analogously, an equivalent form of the Neumann conditions (1.4) reads

$$w(0) = 0\,, \quad w(T) = 0\,. \tag{2.3}$$

Theorem 2.1 *Let the assumptions* (P), (G) *be satisfied and* $h \equiv 0$. *Then, for any* $\widetilde{f} \in \widetilde{C}_T : \|\widetilde{f}\|_{L^2} < \sqrt{\frac{3}{T}}\min\{-a, b\}$, *there exists precisely one* $s(\widetilde{f}) \in \mathbb{R}$ *such that (2.1)–(2.2) has a solution if and only if*

$$\overline{f} = s(\widetilde{f})\,.$$

In this case the periodic boundary-value problem for (1.5) has a family of solutions $u_c(t) = u(t) + c$, *where* $u(t) = \int_0^t \varphi_{-1}\big(w(\widetilde{f}, \tau)\big)d\tau$, $c \in \mathbb{R}$ *is arbitrary and* $w(\widetilde{f}, \cdot)$ *is the solution of (2.1)–(2.2). Moreover, the mapping* $s : \widetilde{C}_T \to \mathbb{R}$, $\widetilde{f} \mapsto s(\widetilde{f})$ *is continuous and the absolute value of* $s(\widetilde{f})$ *satisfy*

$$|s(\widetilde{f})| \leq \max\left\{ |g(\xi)| \,:\, \varphi_{-1}\left(-\sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}\right) \leq \xi \leq \varphi_{-1}\left(\sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}\right)\right\}. \tag{2.4}$$

Remark 2.1 Let us observe that if φ is an homeomorphism of interval (α, β), $\alpha < 0 < \beta$ onto $\mathbb{R}$, then (2.4) yields an uniform bound for $s(\widetilde{f})$:

$$|s(\widetilde{f})| \leq \max_{\xi\in[\alpha,\beta]} |g(\xi)|\,.$$

Theorem 2.2 *Suppose that* (P),(P'),(G) *are satisfied and* $h \equiv 0$. *Then the assertion of Theorem 2.1 is valid and* $u \in C^2_T$. *If also* (G') *holds true then the solution* $w(\widetilde{f}, \cdot)$ *of (2.1)–(2.2) is unique. The mappings* $w : \widetilde{C}_T \to C^1_T, \widetilde{f} \mapsto w(\widetilde{f}, \cdot)$ *and* $s : \widetilde{C}_T \to \mathbb{R}, \widetilde{f} \mapsto s(\widetilde{f})$ *are continuously differentiable at any* $\widetilde{f} \in \widetilde{C}[0, T] : \widetilde{f} \not\equiv 0$. *If* (P") *is satisfied then* w *and* s *are continuously differentiable also at* $\widetilde{f} \equiv 0$.

Theorem 2.3 *Let the assumptions* (P), (G) *be satisfied and* $h \equiv 0$. *Then, for any* $\widetilde{f} \in \widetilde{C}[0,T]$: $\|\widetilde{f}\|_{L^2} < \sqrt{\frac{3}{T}} \min\{-a, b\}$, *there exists precisely one* $s(\widetilde{f}) \in \mathbb{R}$ *such that (2.1)–(2.3) has a solution if and only if* $\overline{f} = s(\widetilde{f})$. *In this case the Neumann boundary-value problem for (1.5) has a family of solutions* $u_c(t) = u(t) + c$, *where* $u(t) = \int_0^t \varphi_{-1}\big(w(\widetilde{f}, \tau)\big) d\tau$, $c \in \mathbb{R}$ *is arbitrary and* $w(\widetilde{f}, \cdot)$ *is the solution of (2.1)–(2.3). The mapping* $s : \widetilde{C}[0,T] \to \mathbb{R}, \widetilde{f} \mapsto s(\widetilde{f})$ *is continuous and the absolute value of* $s(\widetilde{f})$ *is estimated by (2.4). If* (P') *is satisfied then* $u \in C^2_T$. *If also* (G') *holds true then mappings* $w : \widetilde{C}_T \to C^1_T$, $\widetilde{f} \mapsto w(\widetilde{f}, \cdot)$ *and* s *are continuously differentiable at any point* $\widetilde{f} \in \widetilde{C}[0,T]$.

Remark 2.2 Let us see that we do not need the assumption (P") in Theorem 2.3 in order to get that $w : \widetilde{f} \mapsto w(\widetilde{f}, \cdot)$ and $s : \widetilde{f} \mapsto s(\widetilde{f})$ are continuously differentiable at $\widetilde{f} \equiv 0$, as we do in Theorem 2.2. The reason is the following one: The boundary conditions (2.3) of the first order semilinear problem corresponding to the Neumann BVP are linear. Thus, they provide more regularity than the nonlinear ones (2.2) corresponding to the periodic BVP.

Since the results considering (1.5)–(1.2) and (1.5)–(1.4), respectively, have the same formal structure, we need to formulate them for the BVP (1.5)–(1.2) only. However, the statement, corresponding to the BVP (1.5)–(1.4), can be obtained by replacing the expressions standing in front of brackets with the bracketed ones in Theorem 2.4. We keep this notation in Section 4, when formulating Lemmas 4, 6, 7 used in the proof of Theorem 4. Note, that in the following theorem we suppose that φ is an odd mapping, so that $b = -a$.

Theorem 2.4 *Assume* (P), (PH), (G), (H) *and*

$$\sqrt{\frac{3}{T}} b - \sqrt{T} \sup_{\xi \in \mathbb{R}} |h(\xi)| > 0 \,. \tag{2.5}$$

Then for any $\widetilde{f} \in \widetilde{C}_T$ *(* $\widetilde{f} \in \widetilde{C}[0,T]$ *) satisfying*

$$\|\widetilde{f}\|_{L^2} < \sqrt{\frac{3}{T}} b - \sqrt{T} \sup_{\xi \in \mathbb{R}} |h(\xi)| \tag{2.6}$$

the BVP (1.5)–(1.2) ((1.5)–(1.4)) has a solution if

$$s(\widetilde{f}) + h(-\infty) < \overline{f} < s(\widetilde{f}) + h(+\infty) \,, \tag{2.7}$$

where $s(\widetilde{f})$ *is given by Theorem 2.1 (Theorem 2.3).*

Suppose, moreover, that

$$h(-\infty) < h(\xi) < h(+\infty) \quad \textit{for all } \xi \in \mathbb{R} \,.$$

Then, if (2.7) is false and $\widetilde{f} \in \widetilde{C}_T$ *(* $\widetilde{f} \in \widetilde{C}[0,T]$ *) satisfies (2.6), the BVP (1.5)–(1.2) ((1.5)–(1.4)) does not admit any solution.*

Remark 2.3 As one can expect, an analogous result to Theorem 2.4 holds true also for h satisfying
$$h(+\infty) < h(-\infty)\,.$$
In this case, for any $\widetilde{f} \in \widetilde{C}_T$ ($\widetilde{f} \in \widetilde{C}[0,T]$) satisfying (2.6), the sufficient condition on $\overline{f}$ reads as follows:
$$s(\widetilde{f}) + h(+\infty) < \overline{f} < s(\widetilde{f}) + h(-\infty)\,. \tag{2.8}$$
If moreover h satisfies
$$h(+\infty) < h(\xi) < h(-\infty) \quad \text{for all } \xi \in \mathbb{R}$$
then (2.8) is also necessary.

This is in full agreement with the semilinear case for the periodic BVP studied in [6].

Remark 2.4 Notice, that if the range of φ is $\mathbb{R}$ then $b = +\infty$ and the conditions (2.5) and (2.6) are satisfied identically. On the other hand, if $b < +\infty$ then one can apply Theorem 2.4 provided that $T < \frac{\sqrt{3}b}{\sup_{\xi\in\mathbb{R}} |h(\xi)|}$. Observe that the bigger T is, the weaker condition (2.6) is.

Remark 2.5 Since we treat the problems (1.5)–(1.2) and (1.5)–(1.4) using transformed problems (2.1)–(2.2) and (2.1)–(2.3), respectively, we need suitable additional condition, which ensure that solution w of (2.1)–(2.2) or (2.1)–(2.3), respectively, satisfies $w(t) \in \operatorname{Im}\varphi$ for all $t \in [0,T]$. One of such a possible condition is $\|\widetilde{f}\|_{L^2} < \sqrt{\frac{3}{T}} \min\{-a, b\}$ (which is a consequence of (3.3)) provided $h \equiv 0$.

However, we would like to point out that the restrictive condition on the 'greatness' of $\widetilde{f}$ is given not only by the specific limitation of the employed method, but that it also arises directly from the nature of the problem. Let us consider (1.5)–(1.2) or (1.5)–(1.4), respectively. Taking into account the boundary conditions (1.2) or (1.4), respectively, there exists $t_0 \in [0,T] : u'(t_0) = 0$. Hence integrating (1.5) we find:
$$\varphi(u'(t)) = \int_{t_0}^{t} \Big(\widetilde{f}(\tau) + \overline{f} - g\left(u'(\tau)\right) - h\left(u(\tau)\right)\Big)\, d\tau\,,$$
which requires
$$a < \int_{t_0}^{t} \Big(\widetilde{f}(\tau) + \overline{f} - g\left(u'(\tau)\right) - h\left(u(\tau)\right)\Big)\, d\tau < b\,. \tag{2.9}$$
Integrating (1.5) from 0 to T we also find
$$\overline{f} = \frac{1}{T}\int_0^T \left(g(u'(t)) + h(u(t))\right)\, dt\,,$$

which implies

$$\inf_{\xi\in\mathbb{R}} g(\xi) + \inf_{\xi\in\mathbb{R}} h(\xi) \le \overline{f} \le \sup_{\xi\in\mathbb{R}} g(\xi) + \sup_{\xi\in\mathbb{R}} h(\xi)\,. \tag{2.10}$$

Suppose, in addition, that g is bounded. If $\overline{f}$ does not satisfy (2.10) or one of the following two inequalities holds:

$$\sup_{t_0\in[0,T]} \inf_{t\in[0,T]} \left\{ \int_{t_0}^{t} \left(\widetilde{f}(\tau) + \overline{f} - \inf_{\xi\in\mathbb{R}} g(\xi) - \inf_{\xi\in\mathbb{R}} h(\xi) \right) d\tau \right\} < a\,, \tag{2.11}$$

$$\inf_{t_0\in[0,T]} \sup_{t\in[0,T]} \left\{ \int_{t_0}^{t} \left(\widetilde{f}(\tau) + \overline{f} - \max_{\xi\in\mathbb{R}} g(\xi) - \max_{\xi\in\mathbb{R}} h(\xi) \right) d\tau \right\} > b\,, \tag{2.12}$$

then the BVP (1.5)–(1.2) or (1.5)–(1.4), respectively, does not admit any solution with $f = \widetilde{f} + \overline{f}$. Compare also with results in [11].

Remark 2.6 It is worth noting that if we consider the periodic BVP for quasilinear ODE:

$$(\varphi(u'))' + \lambda u' + h(u) = f, \quad t \in (0,T)\,,$$

where $\lambda \in \mathbb{R}$, then the condition (2.7) from Theorem 2.4 is reduced to

$$h(-\infty) < \overline{f} < h(+\infty)\,. \tag{2.13}$$

It means that we obtain the Landesman-Lazer condition as a particular result.

Here we present a proof. If $g(z) \equiv \lambda z$, where $\lambda \in \mathbb{R}$, then integrating (2.1) from 0 to T and using boundary conditions (2.2) we find $0 = \int_0^T f(t)dt = (T\overline{f})$. Since (2.1)–(2.2) has a solution if and only if $\overline{f} = s(\widetilde{f})$ (see Theorem 2.1), we have that $s(\widetilde{f}) \equiv 0$. Consequently, if $g(z) \equiv \lambda z$ then the condition (2.7) is reduced to (2.13).

Note, that in order to get $s(\widetilde{f}) \equiv 0$ for the Neumann boundary-value problem, we have to impose $g(z) \equiv 0$.

Example 2.1 Consider $\varphi(z) = m_0 z / \sqrt{1 - \frac{z^2}{c^2}}$, $m_0 > 0$ and $g, h \in C(\mathbb{R},\mathbb{R})$, h has finite limits $h(-\infty) < h(+\infty)$. Then it follows from Theorem 2.4 that the periodic BVP for

$$\left(\frac{m_0 u'(t)}{\sqrt{1 - \frac{(u')^2(t)}{c^2}}} \right)' + g\big(u'(t)\big) + h\big(u(t)\big) = \widetilde{f}(t) + \overline{f}$$

has a solution for every couple $\widetilde{f}, \overline{f}$ satisfying condition (2.7):

$$h(-\infty) + s(\widetilde{f}) < \overline{f} < h(+\infty) + s(\widetilde{f})\,,$$

where $s : \widetilde{C}_T \to \mathbb{R}$ comes from Theorem 2.1.

Let us remark that the conditions (2.5) and (2.6) are satisfied for any $\widetilde{f} \in \widetilde{C}_T$. This follows from the fact that the range of φ is $\mathbb{R}$ (i.e. $b = +\infty$).

It is worth noting that this problem arises from the relativistic dynamics and describes damped oscillations.

Example 2.2 With respect to Theorem 2.1 for all $\widetilde{f} \in C_T$ there exists unique $\overline{f}$ such that the periodic boundary-value problem for

$$u''(t) + k\left(u'(t)\right)^2 = \widetilde{f}(t) + \overline{f}$$

has a solution. This boundary-value problem describes a steady-state process of gas burning in jets of rockets (cf. [17]).

Example 2.3 Consider the following periodic boundary-value problem:

$$\begin{gathered}\left(u'(t) - (u'(t))^3\right)' + u'(t) + \arctan(u(t)) = \widetilde{f}(t) + \overline{f} \quad \text{in} \quad (0,T)\,,\\ u(0) = u(\pi), \quad u'(0) = u'(\pi)\,.\end{gathered}$$

Let $\sqrt{\frac{3}{T}}\frac{2\sqrt{3}}{9} - \frac{\pi}{2}\sqrt{T} > 0$. Then, for all $\widetilde{f} \in \widetilde{C}_T$: $\|\widetilde{f}\|_{L^2} < \sqrt{\frac{3}{T}}\frac{2\sqrt{3}}{9} - \frac{\pi}{2}\sqrt{T}$, this problem has a solution if and only if $\overline{f} \in \mathbb{R}$ satisfy

$$-\frac{\pi}{2} < \overline{f} < \frac{\pi}{2}\,. \tag{2.14}$$

To prove this result, we can use Theorem 2.4. Indeed, φ is odd homeomorphism of $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ onto $(-\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9})$, so that $b = -a = \frac{2\sqrt{3}}{9}$. Moreover,

$$\lim_{z \pm\infty} \arctan z = \pm\frac{\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < \arctan z < \frac{\pi}{2}, \text{for all } z \in \mathbb{R}\,.$$

Hence the condition (2.14) follows from (2.7), where $s(\widetilde{f}) \equiv 0$, because $g(z) \equiv z$ (see Remark 2.6).

This problem describes forced oscillations of voltage u in an electrical circuit with *ferro-resonance* (see e.g. [12]). Such nonlinear circuits are used in radiotechnics.

Table 1 illustrates the applicability of Theorems 2.1–2.4 for some particular nonlinearities φ. The symbol A (applicable) indicates that the corresponding assumption is satisfied and thus using an appropriate Theorem one obtains the desired conclusion; on the other hand NA (not applicable) indicates that the corresponding assumption is not satisfied and that the selected Theorem is not applicable in that case). We also relate our results to the results already known.

$\varphi(z)$	Periodic, $g(s)=s^3$				Neumann, $g(s)=s^3$			both, $h(s)=\arctan s$	
	Thm 1	Theorem 2			Theorem 3			Theorem 4	
	(P)+(G)	+(P')	+(G')	+(P”)	(P)+(G)	+(P')	+(G')	(P)+(G)+(PH)	
$\lvert z\rvert^{p-2}z$, $1<p<2$	A	A		NA	A			A	
z	A : [6, Thm 1], [4, Thm 3.4], [16, Thm 4]				A : [4, Thm 3.3], [16, Thm 1]			A [6, Thm 2]#	
$\lvert z\rvert^{p-2}z$, $p>2$	A	NA			A	NA		A	
$\frac{z}{\sqrt{1-z^2}}$	A	A			A			A	
$\frac{z}{\sqrt{1+z^2}}$	A*	A*			A*			$T<\frac{2\sqrt{3}}{\pi}$ A†	$T>\frac{2\sqrt{3}}{\pi}$ NA
$\exp\left(\frac{1}{\lvert z\rvert}\right)z$	A	A		NA	A			NA	

Table 1: Legend: # With periodic conditions only,
* Provided that $\|\widetilde{f}\|_{L^2}<\sqrt{\frac{3}{T}}$, † Assuming that $\|\widetilde{f}\|_{L^2}<\sqrt{\frac{3}{T}}-\frac{\pi}{2}\sqrt{T}$.

3 Proofs of Theorems 2.1 – 2.3

We start this section with the following lemma:

Lemma 3.1 *Let $f\in C[0,T]$, $q:J_1\to J_2$ be continuous, where J_1,J_2 are nonempty intervals and J_1 contains zero in its interior. Let $w\in C^1[0,T]$, $w(0)=w(T)$ and w satisfies*

$$w'(t)+q(w(t))=f(t),\quad t\in(0,T) \tag{3.1}$$

then

$$|w(t)-w(s)|\le|t-s|^{\frac{1}{2}}\|\widetilde{f}\|_{L^2}\,. \tag{3.2}$$

Moreover, if there exists $t_0\in[0,T]$ such that $w(t_0)=0$ then

$$\|w\|_C\le\sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}\,. \tag{3.3}$$

Remark 3.1 The previous lemma will be used to estimate solutions of (2.1)–(2.2) or (2.1)–(2.3), respectively. Indeed, the second of (2.2) yields $w(0)=w(T)$ directly. On the other hand, the first of (2.2), the continuity of w and the fact, that φ satisfies (P) imply that there exists $t_0\in[0,T]$ such that $w(t_0)=0$. In the case of (2.3) we have $w(0)=w(T)=0$.

Proof Using the Cauchy–Schwarz inequality it is easy to show that

$$|w(t)-w(s)|=\left|\int_s^t w'(\tau)d\tau\right|\le|t-s|^{\frac{1}{2}}\|w'\|_{L^2}\,. \tag{3.4}$$

Multiplying both sides of equation (2.1) by w', integrating from 0 to T, splitting f as in (1.6) and using $w(0) = w(T)$, we find that $\|w'\|^2_{L^2} = \left(\widetilde{f}, w'\right)_{L^2}$ (where $\left(\widetilde{f}, w'\right)_{L^2} := \int_0^T \widetilde{f} w'$ is the scalar product in L^2). The Cauchy-Schwarz inequality yields $\|w'\|_{L^2} \le \|\widetilde{f}\|_{L^2}$, which together with (3.4) establishes (3.2).

Now we are going to estimate $\|w\|_C$. Decompose the function w as $w = \widetilde{w} + \overline{w}$, where $\int_0^T \widetilde{w}(t)dt = 0$ and $\overline{w} \in \mathbb{R}$. Since $\widetilde{w} \in C^1[0,T] \subset W^{1,2}[0,T]$ and $w(0) = w(T)$, the Sobolev inequality [15, Proposition 1.3] yields $\|\widetilde{w}\|_C \le \sqrt{\frac{T}{12}}\|w'\|_{L^2} \le \sqrt{\frac{T}{12}}\|\widetilde{f}\|_{L^2}$. Since we assume that there exists a $t_0 \in [0,T]$ such that $w(t_0) = 0$, it shall be $|\overline{w}| \le \|\widetilde{w}\|_C$. Hence $\|w\|_C \le 2\sqrt{\frac{T}{12}}\|\widetilde{f}\|_{L^2} = \sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}$, which is the desired inequality (3.3). ♢

Now we proceed to proofs of Theorems 2.1–2.3.

Proof of Theorem 2.1 We divide the proof into six steps:
Step 1 *Let q be a continuously differentiable and bounded real function. We prove that for all $\widetilde{f} \in \widetilde{C}_T$ there exists $\overline{f} \in \mathbb{R}$ such that the equation*

$$w'(t) + q\big(w(t)\big) = \widetilde{f}(t) + \overline{f} \tag{3.5}$$

has a solution $w \in C^1[0,T]$ satisfying (2.2).

Let $\widetilde{f} \in \widetilde{C}_T$ be given. Since q is continuously differentiable and bounded function, the solution (denoted by $w(t, \overline{f}, \alpha)$) to the initial-value problem

$$\begin{gathered} w'(t) + q\big(w(t)\big) = \widetilde{f}(t) + \overline{f}, \\ w(0) = \alpha \end{gathered} \tag{3.6}$$

exists on $(0,T)$, it is unique and is continuously differentiable with respect to parameters α and $\overline{f}$ (see e.g [5]). Let $M = \sup_{y\in\mathbb{R}} |q(y)|$. Taking $\overline{f} > M + \varepsilon$ and integrating (3.5) from 0 to T, we find $w(T, \overline{f}, \alpha) > \alpha$. On the other hand, if $\overline{f} < -M - \varepsilon$ then we get $w(T, \overline{f}, \alpha) < \alpha$. Since w depends continuously on $\overline{f}$, for all $\alpha \in \mathbb{R}$, there exists $\overline{f}_\alpha \in \mathbb{R}$ such that $w(T, \overline{f}_\alpha, \alpha) = \alpha$. Moreover, the partial derivative with respect to the parameter $\overline{f}$, $w_{\overline{f}}(t, \overline{f}, \alpha)$, is a solution to the linear initial-value problem

$$\begin{gathered} z'(t) = 1 - q'\big(w(t, \overline{f}, \alpha)\big)z(t)\,, \\ z(0) = 0\,. \end{gathered} \tag{3.7}$$

The explicit formula of the solution of (3.7) yields $w_{\overline{f}}(T, \overline{f}, \alpha) > 0$, which means that $w(T, \cdot, \alpha)$ is increasing and $\overline{f}_\alpha$ is unique. Thus, we can define the mapping $\psi_1 : \mathbb{R} \to [-M, M]$ by $\alpha \mapsto \overline{f}_\alpha$. As $w_{\overline{f}}(T, \overline{f}, \alpha) > 0$, by the abstract implicit function theorem (see [1, Theorem 2.2.3]), we find that $\psi_1 : \mathbb{R} \to [-M, M]$ is continuous.

Rewrite (3.6) as an integral equation:

$$w(t)=\int_0^t \Big(\widetilde{f}(\tau)-q(w(\tau))\Big)\,d\tau+\overline{f}t+\alpha\,.$$

Taking $\alpha>\int_0^T|\widetilde{f}(\tau)|d\tau+2TM$, we see that $w(t,\overline{f},\alpha)>0$ for all $t\in[0,T]$ and $\overline{f}\in[-M,M]$. Consequently $\int_0^T\varphi_{-1}\big(w(\tau,\overline{f},\alpha)\big)d\tau>0$. Conversely, if $\alpha<-\int_0^T|\widetilde{f}(\tau)|d\tau-2TM$ then $w(t,\overline{f},\alpha)<0$ for all $t\in[0,T]$ and $\overline{f}\in[-M,M]$. Hence $\int_0^T\varphi_{-1}\big(w(\tau,\overline{f},\alpha)\big)d\tau<0$. Since φ_{-1} is a continuous real function, the mapping $\alpha\mapsto\int_0^T\varphi_{-1}\big(w(t,\overline{f},\alpha)\big)dt$ is continuous and there exists at least one $\alpha_{\overline{f}}\in\mathbb{R}$ such that $\int_0^T\varphi_{-1}\big(w(t,\overline{f},\alpha)\big)dt=0$. Moreover the partial derivative of $w(t,\overline{f},\alpha)$ with respect to initial condition α, $w_\alpha(t,\overline{f},\alpha)$, is the solution of the linear initial-value problem

$$\begin{gathered}v'(t)+q'\big(w(t,\overline{f},\alpha)\big)v(t)=0\,,\\ v(0)=1\,.\end{gathered}\tag{3.8}$$

Taking into account explicit formula of the solution of (3.8), it is easy to verify that $w_\alpha(t,\overline{f},\alpha)>0$ for all $t\in[0,T]$ and $\overline{f},\alpha\in\mathbb{R}$. Thus, for all $\overline{f}\in\mathbb{R}$ there exists unique $\alpha=\alpha_{\overline{f}}$ such that $\int_0^T\varphi_{-1}\big(w(t,\overline{f},\alpha_{\overline{f}})\big)dt=0$. By the same reason as above, the mapping $\psi_2:\mathbb{R}\to\mathbb{R}$ defined by $\overline{f}\mapsto\alpha_{\overline{f}}$ is continuous.

Let us consider the continuous mapping $\psi_1\circ\psi_2:[-M,M]\to[-M,M]$. Due to the Brouwer fixed point theorem there exists at least one $\overline{f}_0\in[-M,M]$ such that $\psi_1\big(\psi_2(\overline{f}_0)\big)=\overline{f}_0$. Hence the function $w(t,\overline{f}_0,\psi_2(\overline{f}_0))$ is a solution of the equation (3.5) subject to (2.2).

Step 2 *We show that if* (P), (G) *are satisfied then, for all* $\widetilde{f}\in\widetilde{C}_T$*, there exists at least one* $\overline{f}\in\mathbb{R}$ *such that the boundary-value problem for the equation*

$$w'(t)+g\big(\varphi_{-1}\big(w(t)\big)\big)=\widetilde{f}(t)+\overline{f}\tag{3.9}$$

subject to (2.2) has at least one solution.

Let w be the solution of (3.9)–(2.2). Integrating (3.9) from 0 to T and then dividing by T, we obtain

$$\frac{1}{T}\int_0^T g\big(\varphi_{-1}(w(t))\big)dt=\overline{f}\,.$$

With respect to (3.3) (consider $q:=g\circ\varphi_{-1}$ in Lemma 3.1) the following a priori bound on w results: $\|w\|_C\le\sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}$. Hence we estimate $\overline{f}$ as follows:

$$|\overline{f}|\le\sup\left\{|g(\xi)|:\varphi_{-1}\left(-\sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}\right)\le\xi\le\varphi_{-1}\left(\sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}\right)\right\}.\tag{3.10}$$

As $\|w\|_C\le\sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}$, the restriction of $g\circ\varphi_{-1}$ on $I=\Big[-\sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}\,,\sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}\Big]$ is essential in further considerations. Note that the assumption

$\|\widetilde{f}\|_{L^2} \le \sqrt{\frac{T}{3}}\min\{|a|, b\}$ imply $I \subset (a,b)$ $(=\operatorname{Dom}\varphi_{-1})$; hence $g\circ\varphi_{-1}$ is well defined on I. Let us introduce the following sequence of functions $\gamma_n(x) := n\int_I \varrho\left(\frac{x-y}{n}\right) g\big(\varphi_{-1}(y)\big)dy$, where $\varrho : \mathbb{R}\to\mathbb{R}$ is the regularization kernel given by

$$\varrho(x) := \begin{cases} c_0 e^{\frac{1}{|x|^2-1}} & \text{for } |x| < 1\,, \\ 0 & \text{for } |x| \ge 1\,, \end{cases} \tag{3.11}$$

where c_0 is a constant such that $\int_{-1}^{1}\varrho(x)dx = 1$. It is easy to verify that $\{\gamma_n\}_{n=1}^{\infty}$ converges to $g\circ\varphi_{-1}$ uniformly on I, that γ_n is continuously differentiable for any $n\in\mathbb{N}$ and that $\|\gamma_n\|_{C(I)} \le \|g\circ\varphi_{-1}\|_{C(I)}$ ($C(I)$ denotes the space of continuous real functions defined on I).

From the first step we know that there exist w_n and $\overline{f}_n$ such that

$$w_n'(t) + \gamma_n\big(w_n(t)\big) = \widetilde{f}(t) + \overline{f}_n$$

and

$$\int_0^T \varphi_{-1}\big(w_n(t)\big)\,dt = 0, \quad w_n(0) = w_n(T).$$

Utilizing (3.3) and $\|\gamma\|_{C(I)} \le \|g\circ\varphi_{-1}\|_{C(I)} < +\infty$ we find that $\{\overline{f}_n\}_{n=1}^{\infty}$ is a bounded sequence. By (3.3), w_n are equibounded and, by (3.2), w_n are equicontinuous (we apply Lemma 3.1 with $q := \gamma_n$ for each $n\in\mathbb{N}$, employing the fact that resulting inequalities do not contain q). Hence we can select subsequences $w_{n_k}, \overline{f}_{n_k}$ such that $w_{n_k}\to w$ in C_T and $\overline{f}_{n_k}\to\overline{f}$. Since

$$w_{n_k}(t) = w_{n_k}(0) + \int_0^t \Big(\widetilde{f}(\tau) + \overline{f}_{n_k} - \gamma_{n_k}(w_{n_k}(\tau))\Big)\,d\tau\,,$$

passing to the limit we obtain

$$w(t) = w(0) + \int_0^t \Big(\widetilde{f}(\tau) + \overline{f} - g\big(\varphi_{-1}(w(\tau))\big)\Big)\,d\tau\,.$$

As the integrand in the right-hand-side is continuous, $w\in C_T^1$ and satisfies (3.9) in $(0,T)$. Thus Step 2 is over.

Step 3 *We prove that if $\overline{f}_1, \overline{f}_2 \in \mathbb{R}$ and the equation (2.1) with $f = \widetilde{f} + \overline{f}_i, i = 1,2$ has a solution satisfying (2.2), then $\overline{f}_1 = \overline{f}_2$.* Conversely, assume that $\overline{f}_1 > \overline{f}_2$ and there exists $w_i \in C_T^1, i = 1,2$ such that $w_i'(t) + g\left(\varphi_{-1}(w_i(t))\right) = \widetilde{f}(t) + \overline{f}_i$, $i = 1,2$ subject to (2.2). Then from (1.2) it follows that the function $\varphi_{-1}\big(w_1(t)\big) - \varphi_{-1}\big(w_2(t)\big)$ is a T-periodic function with mean value zero, so that there exist t_0 and $\delta_1 > 0$ such that $\varphi_{-1}\big(w_1(t_0)\big) - \varphi_{-1}\big(w_2(t_0)\big) = 0$ and $\varphi_{-1}\big(w_1(t)\big) - \varphi_{-1}\big(w_2(t)\big) < 0$ for all $t_0 < t < t_0 + \delta_1$ (note that as we consider the periodic problem, we can shift the problem suitably in t if it is necessary). Taking into account that φ is an increasing homeomorphism, we get $w_1(t_0) = w_2(t_0)$ and $w_1(t) - w_2(t) < 0$ for all $t_0 < t < t_0 + \delta_1$. Since $w_1(t_0) = w_2(t_0)$, $\overline{f}_1 > \overline{f}_2$ and since the functions $g\left(\varphi_{-1}(w_1(\cdot))\right)$ and $g\left(\varphi_{-1}(w_2(\cdot))\right)$ are continuous, there exists $\delta_2 > 0$ such that $|g(\varphi_{-1}(w_1(t))) - g(\varphi_{-1}(w_2(t)))| < (\overline{f}_1 - \overline{f}_2)/2$ for any $t_0 < t < t_0 + \delta_2$. Taking $\delta = \min\{\delta_1, \delta_2\}$, we arrive at

$$(w_1 - w_2)'(t) = \overline{f}_1 - \overline{f}_2 + g(\varphi_{-1}(w_1(t))) - g(\varphi_{-1}(w_2(t))) > 0$$

for all $t_0 < t < t_0 + \delta$. However, $w_1(t) - w_2(t) < 0$ for all $t_0 < t < t_0 + \delta$, a contradiction.

Step 4 *Estimate (2.4).*

The estimate (2.4) is a direct consequence of (3.10).

Step 5 *Continuity of s.*

The proof of the continuity of s is analogous to that one in [6] (page 256, step 5) and so is omitted.

Step 6 *Completion of the proof of Theorem 2.1.*

Taking into account the relation between the solution u of the problem (1.5)–(1.2) with $h \equiv 0$ and the solution w of the problem (2.1)–(2.2): $u' = \varphi_{-1}(w)$, the assertion of Theorem 2.1 follows. ♢

Proof of Theorem 2.2 Since $u' = \varphi_{-1}(w)$, using (P') we get $u \in C^2_T$.

Assumptions (P') and (G') imply that $g \circ \varphi_{-1} : (a, b) \to \mathbb{R}$ is a continuously differentiable real function. Let q be a continuously differentiable and bounded real function satisfying $q(y) = g\big(\varphi_{-1}(y)\big)$ for all $|y| \leq \sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}$.

Taking into account a priori bound (3.3) and our definition of q, any solution of (2.1)–(2.2) satisfies (3.5)–(2.2) and vice versa. Thus we are reduced to prove the uniqueness of the solution to (3.5)–(2.2). Due to Theorem 2.1, for each $\widetilde{f} \in \widetilde{C}_T : \|\widetilde{f}\|_{C_T} < \sqrt{\frac{3}{T}} \min\{|a|, b\}$, there exists precisely one $s(\widetilde{f}) \in \mathbb{R}$ such that (2.1)–(2.2) possesses solution; $s(\widetilde{f})$ is also the unique value corresponding to $\widetilde{f}$ such that (3.5)–(2.2) has solution. Since q satisfies assumptions of the Step 1 of the proof of Theorem 2.1, the initial value $w(0) = \alpha$ is uniquely determined by $\alpha = \phi_2(s(\widetilde{f}))$. Now, as q is a continuously differentiable function, the solution of the initial value problem (3.6), with $\overline{f} = s(\widetilde{f}), \alpha = \phi_2(\overline{f})$, is unique (see [5]) and so is that of (3.5)–(2.2) with $\overline{f} = s(\widetilde{f})$.

What remains to prove is that w and s are continuously differentiable with respect to $\widetilde{f}$. Following the idea of the proof of Theorem 3.4 in [4] we define $G : C_T \times \mathbb{R} \times \mathbb{R} \times \widetilde{C}_T \to C_T \times \mathbb{R} \times \mathbb{R}$ by the formula

$$G(w, \overline{f}, \alpha, \widetilde{f}) := \begin{pmatrix} w(t) - \overline{f}t - \int_0^t \widetilde{f}(\tau)d\tau + \int_0^t q\big(w(\tau)\big)d\tau - \alpha \\ \int_0^T \varphi_{-1}\big(w(\tau)\big)d\tau \\ w(T) - \alpha \end{pmatrix}. \tag{3.12}$$

As in [4] one can check that the operator equation

$$G(w, \overline{f}, \alpha, \widetilde{f}) = 0 \tag{3.13}$$

is equivalent to the boundary-value problem (3.5)–(2.2). As a consequence of this fact we obtain that for all $\widetilde{f}_0 \in \widetilde{C}_T$ there exists a unique triple $(w_0, \overline{f}_0, \alpha_0) \in C^1_T \times \mathbb{R} \times \mathbb{R}$ such that $G(w_0, \overline{f}_0, \alpha_0, \widetilde{f}_0) = 0$.

Now applying the abstract implicit function theorem we conclude the proof. To do this we shall prove: *The assumptions of the implicit function theorem (see [1, Theorem 2.2.3]) are satisfied at any point* $(w_0, \overline{f}_0, \alpha_0, \widetilde{f}_0) \in C^1_T \times \mathbb{R} \times \mathbb{R} \times \widetilde{C}_T$, $\widetilde{f}_0 \not\equiv 0$ *(and if* (P") *also for* $\widetilde{f}_0 \equiv 0$), *at which the equation (3.13) holds.*

One can see that the operator G and the partial Fréchet derivatives $G_{(w,\overline{f},\alpha)}$ and $G_{\widetilde{f}}$ (the first partial Fréchet derivatives of G with respect to $(w, \overline{f}, \alpha)$ and $\overline{f}$, respectively) are continuous in the neighbourhood of $(w_0, \overline{f}_0, \alpha_0, \widetilde{f}_0)$. By a direct calculation, we get that the partial Fréchet derivative $G_{(w,\overline{f},\alpha)}(w_0, \overline{f}_0, \alpha_0, \widetilde{f}_0)$: $C^1_T \times \mathbb{R} \times \mathbb{R} \to\ C^1_T \times \mathbb{R} \times \mathbb{R}$, is given by the formula:

$$(\omega, \sigma, \kappa) \mapsto \begin{pmatrix} \omega(t) + \int_0^t q'\big(w_0(\tau)\big)\omega(\tau)d\tau - \sigma t - \kappa \\ \int_0^T \varphi_{-1}\big(w_0(\tau)\big)\omega(\tau)d\tau \\ \omega(T) - \kappa \end{pmatrix}. \tag{3.14}$$

Let $(\widetilde{\phi}, \beta, r) \in \widetilde{C}_T \times \mathbb{R} \times \mathbb{R}$. Then the operator equation

$$G_{(w,\overline{f},\alpha)}\Big(w_0, \overline{f}_0, \alpha_0, \widetilde{f}_0\Big)(\omega, \sigma, \kappa) = \Big(\int_0^t \widetilde{\phi}(\tau)\, d\tau\ ,\ \beta\ ,\ r\Big)^T$$

is equivalent to the initial-value problem

$$\begin{gathered} \omega'(t) + q'\big(w_0(t)\big)\omega(t) = \sigma + \widetilde{\phi}(t)\,, \\ \omega(0) = \kappa\,, \end{gathered} \tag{3.15}$$

subjected to the additional conditions

$$\int_0^T \varphi'_{-1}\big(w_0(\tau)\big)\omega(\tau)d\tau = \beta\,, \quad \omega(T) = \kappa + r. \tag{3.16}$$

Let $A(t) := e^{\int_0^t q'(w_0(\tau))d\tau} > 0$. The solution of the linear initial value problem (3.15) has the explicit form:

$$\omega(t) = \kappa\frac{1}{A(t)} + \frac{1}{A(t)}\int_0^t \widetilde{\phi}(s)A(s)ds + \sigma\frac{1}{A(t)}\int_0^t A(s)ds\,. \tag{3.17}$$

Substituting (3.17) into (3.16) we obtain system of two linear equations for σ and κ. This system can be uniquely solved if corresponding determinant D is different from zero. By a straightforward calculation we obtain:

$$\begin{aligned} \mathrm{D} &= \frac{1}{A(T)}\Big[\int_0^T A(s)ds\int_0^T \frac{\varphi'_{-1}\left(w_0(t)\right)}{A(t)}dt \\ &\quad - \int_0^T \frac{\varphi'_{-1}(w_0(t))}{A(t)}\int_0^t A(s)\,ds\,dt\Big] + \int_0^T \frac{\varphi'_{-1}(w_0(t))}{A(t)}\int_0^t A(s)\,ds\,dt \\ &= \frac{1}{A(T)}\int_0^T A(s)\int_0^s \frac{\varphi'_{-1}(w_0(t))}{A(t)}\,dt\,ds + \int_0^T \frac{\varphi'_{-1}\big(w_0(t)\big)}{A(t)}\int_0^t A(s)\,ds\,dt\,. \end{aligned} \tag{3.18}$$

Since $\varphi_{-1} : I_2 \to I_1$ is an increasing homeomorphism, it follows that $\varphi'_{-1}(z) > 0$ a.e. in I_2. By a contradiction, one can show that if $\widetilde{f} \not\equiv 0$ then w_0 satisfying (2.1)–(2.2) is not a constant function. Thus $\varphi'_{-1}(w_0(t)) > 0$ on some subset of $[0, T]$ of positive measure, which taking into account (3.18) implies D > 0.

Now let us consider $\widetilde{f} \equiv 0$. We have to impose $\varphi'(0) > 0$ to conclude $D > 0$. The reason consists in the fact that $w_0 \equiv 0$ is the unique solution of (2.1)–(2.2) with $\widetilde{f} \equiv 0$.

We proved that $G_{(w,\overline{f},\alpha)}(w_0, \overline{f}_0, \alpha_0) : C^1_T \times \mathbb{R} \times \mathbb{R} \to C^1_T \times \mathbb{R} \times \mathbb{R}$ is bijective linear mapping. Then the inverse of $G_{(w,\overline{f},\alpha)}(w_0, \overline{f}_0, \alpha_0)$ is continuous due to the Banach open mapping theorem (see [18]). Thus $G_{(w,\overline{f},\alpha)}(w_0, \overline{f}_0, \alpha_0)$ is an isomorphism of $C^1_T \times \mathbb{R} \times \mathbb{R}$ onto itself and the assumptions of the implicit function theorem [1, Theorem 2.2.3] are satisfied. This completes the proof of Theorem 2.2. $\diamondsuit$

Proof of Theorem 2.3 At first we perform the proof under the assumptions (P), (P') and (G'). Considering (3.3) we can define continuously differentiable and bounded function q satisfying $q(y) = g\big(\varphi_{-1}(y)\big)$ for all $|y| \le \sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}$.

Since the related first order problem (2.1)–(2.3) is the same as that one considered in [4], the existence of the solution and the differentiability of $w(\widetilde{f})$ and $s(\widetilde{f})$ follows from [4, Theorem 3.4].

The assumption (P') can be omitted and the assumption (G') can be replaced by (G) using the same argument as in the Step 2 of the proof of Theorem 2.1. From [16, Theorem 3] we obtain the uniqueness of $\overline{f}$ corresponding to a fixed $\widetilde{f}$. The continuity of $s(\widetilde{f})$ can be proved in the same manner as in Step 5 of the proof of Theorem 2.1. A priori bound (2.4) is a consequence of (3.3) (cf. Step 4 in the proof of Theorem 2.1). $\diamondsuit$

4 Proof of Theorem 2.4

To prove Theorem 2.4, we need the following comparison results:

Lemma 4.1 *Let* $v'(t) + g\Big(\varphi_{-1}\big(v(t)\big)\Big) \le \alpha + \widetilde{f}(t)$ *on* $[0, T]$ *and* v *satisfies (2.2). Then* $\alpha \ge s(\widetilde{f})$ *(where* $s(\widetilde{f})$ *comes from Theorem 2.1).*

Proof Assume conversely that $\alpha < s(\widetilde{f})$. Let w satisfies (2.1) and (2.2). Since

$$\varphi_{-1}(v(0)) - \varphi_{-1}(w(0)) = \varphi_{-1}(v(T)) - \varphi_{-1}(w(T))$$

and

$$\int_0^T \big(\varphi_{-1}\big(v(t)\big) - \varphi_{-1}\big(w(t)\big)\big)dt = 0\,,$$

there exists a $t_0 \in [0,T)$ and $\delta > 0$ such that $\varphi_{-1}\big(v(t_0)\big) - \varphi_{-1}\big(w(t_0)\big) = 0$ and $\varphi_{-1}\big(v(t)\big) - \varphi_{-1}\big(w(t)\big) > 0$ for all $t_0 < t < t_0 + \delta$ (note that we consider periodic problem so if necessary we can shift suitably the problem in t). Taking into account that φ is an increasing homeomorphism, we get $v(t_0) = w(t_0)$ and $v(t) > w(t)$ for all $t_0 < t < t_0 + \delta$. On the other hand, there exists $\delta' > 0$ such that

$$(v' - w')(t) \le \alpha - s(\widetilde{f}) - g\Big(\varphi_{-1}\big(v(t)\big)\Big) + g\Big(\varphi_{-1}\big(w(t)\big)\Big) < 0$$

for all $t : |t - t_0| < \delta'$. Since $v(t_0) = w(t_0)$, we obtain $v(t) - w(t) < 0$ for all $t_0 < t < t_0 + \delta'$, which is a contradiction. ◇

Lemma 4.2 *Let $v'(t) + g\Big(\varphi_{-1}\big(v(t)\big)\Big) \le \alpha + \widetilde{f}(t)$ on $[0,T]$ and v satisfies (2.3), then $\alpha \ge s(\widetilde{f})$ (here $s(\widetilde{f})$ is taken from Theorem 2.3).*

Proof Suppose the contrary, i.e. $\alpha < s(\widetilde{f})$. Let w satisfies (2.1)–(2.3), then

$$v'(0) \le -g\Big(\varphi_{-1}\big(v(0)\big)\Big) + \widetilde{f}(0) + \alpha < -g\Big(\varphi_{-1}\big(w(0)\big)\Big) + \widetilde{f}(0) + s(\widetilde{f}) = w'(0).$$

From here and (2.3) we get that there exists $\varepsilon > 0$, such that $v(t) < w(t)$ for all $t \in (0, \varepsilon)$.

If $v(t) < w(t)$ for all $t \in (0,T]$ then $v(T) < w(T)$ and either v or w can not satisfy (2.3) at T. Thus $\varepsilon \le T$ and there exists $t_0 \in (0,T]$ such that

$$v(t) < w(t) \text{ for all } t \in (0, t_0) \text{ and } v(t_0) = w(t_0)\,.$$

For $t \in (0, t_0)$ we have

$$\frac{v(t) - v(t_0)}{t - t_0} > \frac{w(t) - w(t_0)}{t - t_0}\,,$$

i.e. $v'(t_0) \ge w'(t_0)$. On the other hand, we have

$$\begin{aligned} w'(t_0) &= -g\big(\varphi_{-1}\big(w(t_0)\big)\big) + \widetilde{f}(t_0) + s(\widetilde{f}) \\ &= -g\big(\varphi_{-1}\big(v(t_0)\big)\big) + \widetilde{f}(t_0) + s(\widetilde{f}) \\ &> -g\big(\varphi_{-1}\big(v(t_0)\big)\big) + \widetilde{f}(t_0) + \alpha \ge v'(t_0)\,, \end{aligned}$$

which is a contradiction. ◇

Remark 4.1 It is possible to show that the assertions of Lemmas 4.1 and 4.2 hold true also with inverted inequalities. This is in agreement with the semilinear periodic problem studied in [6]. These inequalities are used to prove the 'dual' version of Theorem 2.4 with $h(+\infty) < h(-\infty)$.

Lemma 4.3 *Let φ be an increasing homeomorphism of $\mathbb{R}$ onto itself and there exist $c, C > 0$ and $p > 1$: $c|z|^{p-1} \le |\varphi(z)| \le C(|z|^{p-1} + 1)$ for all $z \in \mathbb{R}$. Then, for any $y \in C_T$ ($y \in C[0,T]$), there exists exactly one $u \in C^1[0,T]$ with $\varphi(u') \in C^1[0,T]$ and satisfying*

$$\big(\varphi(u'(t))\big)' - \varphi(u(t)) = y(t) \quad t \in (0,T) \tag{4.1}$$

subject to (1.2) ((1.4)).

Proof For the sake of brevity we present the proof only for the periodic conditions. In the case of the Neumann conditions, the proof is analogous.

At first we prove that, for any given $y \in C_T$, there exists precisely one weak solution of (4.1)–(1.2), where the weak solution of (4.1)–(1.2) is any function $u \in W^{1,p}_T$ such that the following identity

$$\int_0^T \{\varphi(u')v' + \varphi(u)v\} = -\int_0^T yv \tag{4.2}$$

is satisfied for each $v \in W^{1,p}_T$. Then, using an regularity argument, we show that this function u is smooth enough and satisfies (4.1)–(1.2) in the sense indicated in the assertion of the lemma.

Existence and uniqueness of the weak solution. Let us define $\psi : W^{1,p}_T \to L^{p'}$ (where $p' := p/(p-1)$) by $u \mapsto \psi(u)$ if and only if $\langle \psi(u), v\rangle = \int_0^T \varphi(u')v'$, $u, v \in W^{1,p}_T$ for all $v \in W^{1,p}_T$. Since $|\varphi(z)| \le C(|z|^{p-1} + 1)$ for all $z \in \mathbb{R}$, it is easy to verify that ψ is a continuous operator; this follows from the Nemitskii theorem (see [1, Theorem 1.2.2]). From the fact that φ is an increasing function we get strict monotonicity of ψ. Since any monotone continuous operator is also hemicontinuous (see [18]), we get that ψ is a hemicontinuous one. The assumption $|\varphi(z)| \ge c|z|^{p-1}$ for all $z \in \mathbb{R}$ implies that ψ is weakly coercive. Now the existence and uniqueness of the weak solution of (4.1)–(1.2) follows from [18, Theorem 32.H].

Regularity. Suppose that u is a weak solution of (4.1)–(1.2). We show that $u \in C^1_T$, $\varphi(u') \in C^1[0,T]$ and that (4.1) holds pointwise in $(0,T)$. Integrating by parts we can rewrite the equation (4.2) into the form:

$$\int_0^T \left(\varphi(u'(t)) - \int_0^t \big[\varphi\big(u(\tau)\big) + y(\tau)\big]d\tau\right) v'(t)dt + \left[\int_0^t \big(\varphi\big(u(\tau)\big) + y(\tau)\big)d\tau\, v(t)\right]_0^T = 0\,. \tag{4.3}$$

Let us define a function $M : [0,T] \to \mathbb{R}$,

$$t \mapsto \varphi(u') - \int_0^t \big[\varphi\big(u(\tau)\big) + y(\tau)\big]d\tau.$$

It is easy to see that $M \in L^{p'}$ and from (4.3) we get

$$\int_0^T M(t)v'(t)dt = 0 \text{ for all } v \in C_0^\infty(0,T)\,.$$

Hence

$$\int_0^T \frac{\delta M}{\delta t} v = 0 \text{ for all } v \in C_0^\infty(0,T),$$

where $\frac{\delta M}{\delta t}$ denotes the distributional derivative of M. Since $M \in L^{p'} \hookrightarrow L^1$ and $\frac{\delta M}{\delta t} = 0$, we obtain that $M(t) = k$ a.e. in $[0,T]$ and $k \in \mathbb{R}$. Thus

$$\varphi(u'(t)) - \int_0^t \big[\varphi\big(u(\tau)\big) + y(\tau)\big]d\tau - k = 0 \quad \text{a.e. in} \quad [0,T]\,. \tag{4.4}$$

Since φ is an increasing homeomorphism of $\mathbb{R}$ onto itself, we can rewrite the previous equation into the following form

$$u'(t) - \varphi_{-1}\left(\int_0^t \big[\varphi\big(u(\tau)\big) + y(\tau)\big]d\tau - k\right) = 0\,. \tag{4.5}$$

Now let us define a function $F : \mathbb{R} \times [0,T] \to \mathbb{R}$,

$$(z,t) \mapsto z - \varphi_{-1}\left(\int_0^t \big[\varphi\big(u(\tau)\big) + y(\tau)\big]d\tau - k\right)\,.$$

It is possible to show that F is continuous on $\mathbb{R} \times [0,T]$. Moreover, $F(\cdot, t_0)$ is an increasing function for all $t_0 \in [0,T]$, and

$$\lim_{z \to -\infty} F(z,t_0) = -\infty, \qquad \lim_{z \to +\infty} F(z,t_0) = +\infty\,.$$

Hence for each $t \in [0,T]$ there exists exactly one $z(t) \in \mathbb{R}$, such that

$$F(z(t),t) = 0\,.$$

Since $\frac{\partial F}{\partial z}$ is continuous and $\frac{\partial F}{\partial z} = 1 \neq 0$, we can apply the implicit function theorem to show that $z(\cdot) \in C[0,T]$. From (4.5) we get

$$F(u'(t),t) = 0 \quad \text{a.e. in} \quad [0,T]\,.$$

Thus we arrive at

$$z(t) = u'(t) \quad \text{a.e. in} \quad [0,T]\,.$$

Since $u \in W_T^{1,p}$ is absolutely continuous, this identity holds true for all $t \in [0,T]$ and thus $u \in C^1[0,T] \cap W_T^{1,p}$.

Now it remains to prove that $\varphi(u') \in C^1[0,T]$. Let us define $G : \mathbb{R} \times [0,T] \to \mathbb{R}$,

$$(z,t) \mapsto z - \int_0^t \Big(\varphi(u(\tau)) - y(\tau)\Big)d\tau - k\,.$$

The function G is continuous on $\mathbb{R} \times [0,T]$. Moreover, for all $t_0 \in [0,T]$, $G(\cdot, t_0)$ is an increasing function and $\lim_{z\to\pm\infty} G(z,t_0) = \pm\infty$. Hence for each $t \in [0,T]$ there exists exactly one $z(t)$ such that

$$G\big(z(t),t\big) = 0\,.$$

Since $u \in C^1[0,T]$ and $y \in C[0,T]$, the partial derivatives $\frac{\partial G}{\partial z}$ and $\frac{\partial G}{\partial t}$ are continuous; moreover $\frac{\partial G}{\partial z} = 1 \neq 0$. Then the implicit function theorem yields $z(t) \in C^1[0,T]$. Taking into account (4.4) and the fact that $u \in C^1[0,T]$, we see that $z(t) = \varphi(u')$ for all $t \in [0,T]$; thus $\varphi(u') \in C^1[0,T]$. Now, since $u \in C^1[0,T] \cap W_T^{1,p}$ and $\varphi(u') \in C^1[0,T]$, integrating (4.2) by parts we show that u satisfies (4.1) in $(0,T)$ and that $u'(0) = u'(T)$. This concludes the proof. $\diamondsuit$

Now we can define a solution operator $K : C_T \to C_T^1$,

$$y \mapsto K(y)\,, \tag{4.6}$$

where $K(y)$ is the solution of (4.1)–(1.2). Analogously we can define a solution operator $K' : C[0,T] \to C^1[0,T]$ corresponding to the Neumann problem (4.1)–(1.4).

Lemma 4.4 *Let K and K' be defined as above. Then K is compact as a mapping between C_T and C_T^1 and K' is compact as a mapping between $C[0,T]$ and $C^1[0,T]$.*

Proof We prove the compactness of K. The proof of the compactness of K' is analogous.

Let us consider the following sequence of equations:

$$(\varphi(u_n'(t)))' - \varphi(u_n(t)) = y_n(t)\,, \tag{4.7}$$

subject to (1.2), where $\{y_n\}_{n=1}^\infty \subset C[0,T]$ is bounded. We are going to show that one can select a convergent subsequence from $\{u_n\}_{n=1}^\infty \subset C_T^1$. Multiplying the equation (4.7) by u_n, integrating from 0 to T, integrating the first term in the left-hand-side by parts and using the periodic conditions (1.2), we obtain

$$-\int_0^T \big\{\varphi(u_n'(t))u_n'(t) + \varphi(u_n(t))u_n(t)\big\}dt = \int_0^T y_n(t)u_n(t)dt\,. \tag{4.8}$$

Since we assume that $|\varphi(z)| \geq c|z|^{p-1}$ for all $z \in \mathbb{R}$, we find that

$$\|u_n\|^p_{W_T^{1,p}} \leq \frac{1}{c}\int_0^T \big\{\varphi(u_n'(t))u_n'(t) + \varphi(u_n(t))u_n(t)\big\}dt\,.$$

Using this we estimate the terms on the left-hand side of (4.8). The right hand-side of (4.8) is estimated by the Hölder inequality. Therefore we find that

$$\|u_n\|^p_{W_T^{1,p}} \leq \frac{1}{c}\|y_n\|_{L^{p'}}\|u_n\|_{L^p} \leq \frac{1}{c}\|y_n\|_{L^{p'}}\|u_n\|_{W_T^{1,p}}\,,$$

which implies

$$\|u_n\|_{W_T^{1,p}} \le \frac{1}{c}\left(\|y_n\|_{L^{p'}}\right)^{\frac{1}{p-1}} . \tag{4.9}$$

Thus $\{u_n\}_{n=1}^{\infty}$ is bounded in $W_T^{1,p}$. Since $W_T^{1,p}$ is compactly imbedded in C_T, we can select $\{u_{n_k}\}_{k=1}^{\infty}$ such that $u_{n_k} \to w$ in C_T. Let $h_k(t) = y_{n_k}(t) - \varphi\big(u_{n_k}(t)\big)$. One can see that there exists $\beta > 0$ such that $\|h_k\|_C \le \beta$. Hence $\|\left(\varphi\left(u'_{n_k}\right)\right)'\|_C \le \beta$. It follows from (1.2) that there exists $t_0^k \in [0,T]$, such that $u'_{n_k}(t_0^k) = 0$. From (4.7) we obtain $\varphi(u'_{n_k}(t)) = \int_{t_0^k}^t h_k(\tau)d\tau$ and consequently $\|\varphi(u'_{n_k}(t))\|_C \le T\beta$. Thus $\varphi(u'_{n_k}(t))$ is bounded in C_T^1 norm. Due to the compact imbedding of C_T^1 into C_T we can select $\varphi(u'_{n_{k_j}}) \to v$ in C_T. Since $u'_{n_{k_j}} = \varphi_{-1}\big(\varphi(u'_{n_{k_j}})\big)$ and φ_{-1} is continuous, $u'_{n_{k_j}} \to \varphi_{-1}(v)$ in C_T. On the other hand, $u_{n_{k_j}}$ can be written in the form

$$u_{n_{k_j}}(t) = u_{n_{k_j}}(0) + \int_0^t \varphi_{-1}\left(\varphi(u'_{n_{k_j}}(\tau))\right) d\tau .$$

Since $u_{n_{k_j}} \to w$ as $n_{k_j} \to$ in C_T, we find that $w(t) = w(0) + \int_0^t \varphi_{-1}\left(v(\tau)\right) d\tau$. As the integrand is continuous, differentiating the former equation we get $w' = \varphi_{-1}(v)$. Thus $u_{n_{k_j}} \to w$ in C_T^1. This ends the proof. $\diamondsuit$

Remark 4.2 The proof of the previous lemma is based on the ideas from [7].

Let $g, h, \widetilde{f}, T$ be as in Theorem 2.4 and let $z_0 := 2\sup_{\xi\in\mathbb{R}} |g(\xi)| + |h(+\infty)| + |h(-\infty)| + 2\sup_{\xi\in\mathbb{R}} |h(\xi)| + \|\widetilde{f}\|_C$. We define a function $l : \mathbb{R} \to \mathbb{R}$ by

$$l(z) := \begin{cases} z & \text{for } 0 \le z \le z_0 , \\ z_0 & \text{for } z > z_0 , \end{cases} \tag{4.10}$$

for $z \ge 0$ and $l(z) = -l(-z)$ for $z < 0$. We also define:

$$\Gamma(p,c,F) := \left|\sqrt{\frac{T}{3}}\frac{F}{c}\right|^{p-1} \tag{4.11}$$

for any $p > 1$, $c > 0$, $F \ge 0$.

Lemma 4.5 *Let φ be an increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ and there exist $c, C > 0$ and $p > 1$: $c|z|^{p-1} \le |\varphi(z)| \le C\left(|z|^{p-1} + 1\right)$. Let* (G) *be satisfied and g be bounded. Then, for any $\lambda \in [0,1]$ and $|\overline{f}| \le \sup_{\xi\in\mathbb{R}} |h(\xi)| + \sup_{\xi\in\mathbb{R}} |g(\xi)|$, all solutions of*

$$\big(\varphi(u')\big)' + \lambda g(u') + l(u) = \lambda(\widetilde{f} + \overline{f}) \tag{4.12}$$

subject to (1.2) ((1.4)) are a priori bounded by $\|u\|_{C^1} \le (1 + 2T)\,\Gamma\left(p, c, \|\widetilde{f}\|_{L^2} + \sqrt{T}z_0\right) + z_0$.

Proof We rewrite the equation (4.12) as

$$\big(\varphi(u')\big)' + \lambda g(u') = \lambda(\widetilde{f} + \overline{f}) - l(u)\,.$$

Then it follows from (3.3) (consider $q := \lambda g$ in Lemma 3.1) that any solution u of (4.12)–(1.2) satisfy

$$\|\varphi(u')\|_C \le \sqrt{\frac{T}{3}}\,\left\|\lambda\widetilde{f} - l(u)\right\|_{L^2}\,.$$

Taking into account the assumption $|\varphi(z)| \ge c|z|^{p-1}$, the following inequality

$$c|u'|^{p-1} \le |\varphi(u')| \le \sqrt{\frac{T}{3}}\,\left\|\lambda\widetilde{f} - l(u)\right\|_{L^2}$$

is satisfied for any $t \in [0,T]$. This implies

$$\|u'\|_C \le \left(\frac{1}{c}\sqrt{\frac{T}{3}}\,\left\|\lambda\widetilde{f} - l(u)\right\|_{L^2}\right)^{\frac{1}{p-1}}\,.$$

Since $\|\lambda\widetilde{f} - l(u)\|_{L^2} \le \|\widetilde{f}\|_{L^2} + \sqrt{T}z_0$ (recall that $\|\lambda\widetilde{f}\|_{L^2} = |\lambda|\|\widetilde{f}\|_{L^2}$, where $\lambda \in [0,1]$ and $\|l(u)\|_{L^2} \le \sqrt{T}\sup_{\xi\in\mathbb{R}}|l(\xi)| = \sqrt{T}z_0$), we get

$$\|u'\|_C \le \Gamma\left(p, c, \|\widetilde{f}\|_{L^2} + \sqrt{T}z_0\right)\,. \tag{4.13}$$

Let us split the function u as $u = \widetilde{u} + \overline{u}$, where $\overline{u} = \frac{1}{T}\int_0^T u(t)dt$. As $\|\widetilde{u}\|_C \le \int_0^T |u'| \le T\,\|u'\|_C$, from (4.13) we have

$$\|\widetilde{u}\|_C \le T\,\Gamma\left(p, c, \|\widetilde{f}\|_{L^2} + \sqrt{T}z_0\right)\,. \tag{4.14}$$

Now rewrite (4.12) as

$$\big(\varphi(u')\big)' = \lambda\left(\widetilde{f} + \overline{f} - g(u')\right) - l(u)$$

and suppose that $\overline{u} > z_0 + T\,\Gamma\left(p, c, \|\widetilde{f}\|_{L^2} + \sqrt{T}z_0\right)$. Then (4.14) implies $u(t) > z_0$ for all $t \in [0,T]$ and, by the definition of l, we have that $l(u) = z_0$ for all $t \in [0,T]$. Hence we can rewrite (4.12) as

$$(\varphi(u'))' = \lambda\left(\widetilde{f} + \overline{f} - g(u')\right) - z_0\,.$$

Integrating from 0 to T, using $g(u') < \sup_{\xi\in\mathbb{R}}|g(\xi)|$ and $\widetilde{f} < \|\widetilde{f}\|_C$ we get that

$$\varphi(u'(T)) < \left(\lambda(\|\widetilde{f}\|_C + \overline{f} + \sup_{\xi\in\mathbb{R}}|g(\xi)|) - z_0\right)T + \varphi(u'(0))\,.$$

Since $\lambda \in [0,1]$ and $z_0 > \sup_{\xi\in\mathbb{R}} |g(\xi)| + \|\widetilde{f}\|_C + \overline{f}$, we find that $\varphi(u'(T)) < \varphi(u'(0))$, which contradicts the periodic conditions. Analogously as above, the possibility $\overline{u} < -z_0 - T\,\Gamma\left(p, c, \|\widetilde{f}\|_{L^2} + \sqrt{T}z_0\right)$ leads to a contradiction. Hence $|\overline{u}| < z_0 + T\,\Gamma\left(p, c, \|\widetilde{f}\|_{L^2} + \sqrt{T}z_0\right)$. Now, taking into account $\|u\|_{C^1} \le \|\widetilde{u}\|_C + |\overline{u}| + \|u'\|_C$, the assertion follows. $\diamondsuit$

Lemma 4.6 *Assume that φ is an odd increasing homeomorphism of $\mathbb{R}$ onto itself, there exist $c, C > 0$ and $p > 1$: $c|z|^{p-1} \le |\varphi(z)| \le C(|z|^{p-1} + 1)$ for all $z \in \mathbb{R}$, g satisfies* (G) *and is bounded. Let us define $V : C^1_T \to C^1_T$ ($C^1[0,T] \to C^1[0,T]$) by*

$$u \mapsto u - K\big(-\varphi(u) - g(u') - l(u) + f\big)\,,$$

where the operator K is defined by (4.6) for the periodic (the Neumann) BVP. Then there exists $R_0 > 0$ such that, if $V(u) = 0$ then $\|u\|_{C^1} \le R_0$. Moreover the Leray–Schauder degree $\deg(V, B(0,R), 0)$ *is well defined and non-zero if $B(0,R) = \{u \in C^1_T$ ($C^1[0,T]$): $\|u\|_{C^1} < R\}$ with any $R > R_0$.*

Proof Let us define $U : C^1_T \times [0,1] \to C^1_T$ by

$$(u,\lambda) \to u - K\big(-\varphi(u) - \lambda g(u') - l(u) + \lambda f\big)\,.$$

Since, by Lemma 4.4, $K : C_T \to C^1_T$ is compact and

$$-\varphi(u) - \lambda g(u') - l(u) + \lambda f$$

is an continuous operator from C^1_T to C_T , $U(\cdot,\lambda)$ is a compact perturbation of the identity of C^1_T onto itself for all $\lambda \in [0,1]$. Furthermore, it follows that $V = U(\cdot,1)$ from the definition of U.

It is easy to see that the operator equation $U(\cdot,\lambda) = 0$ is equivalent to the equation (4.12):

$$\big(\varphi(u')\big)' + \lambda g(u') + l(u) = \lambda f$$

subject to (1.2). Due to Lemma 4.5, for any $\lambda \in [0,1]$, all solutions of (4.12)–(1.2) satisfy $\|u\|_{C^1} \le (1+2T)\Gamma\left(p, c, \|\widetilde{f}\|_{L^2} + \sqrt{T}z_0\right) + z_0 =: R_0$.

Hence the degree is well defined for every ball $B(0,R)$ with radius $R > R_0$. Since φ and l are odd functions, $U(\cdot,0)$ is odd mapping; consequently

$$\deg(M(\cdot,0), B(0,R), 0) \ne 0\,.$$

Finally, the homotopy invariance property of the degree implies that

$$\deg(U(\cdot,\lambda), B(0,R), 0) \ne 0 \text{ for all } \lambda \in [0,1].$$

$\diamondsuit$

Now, we prove Theorem 2.4. The method follows the idea of the proof of Theorem 2 in [6]. For the sake of brevity we will present a proof for the periodic conditions. The argument for the Neumann conditions is analogous.

Proof of Theorem 2.4 The proof is divided into three steps:
Step 1 *First, we will prove the theorem under additional assumptions:* $\operatorname{Dom}\varphi = \mathbb{R}$, *there exists* $c, C > 0$ *and* $p > 1$ *such that* $c|z|^{p-1} \leq |\varphi(z)| \leq C(|z|^{p-1} + 1)$ *for all* $z \in \mathbb{R}$ *and* g *is bounded.*

We use the degree argument. Define the operator $N : C^1_T \times \mathbb{R} \to C^1_T$ by

$$(u, \lambda) \mapsto u - K\left(-\varphi(u) - g(u') - \lambda h(u) - (1-\lambda)l(u) + \widetilde{f} + \overline{f}\right),$$

where $K : C_T \to C^1_T$ is introduced after Lemma 4.3 and $l : \mathbb{R} \to \mathbb{R}$ is defined by (4.10). Now we can rewrite the boundary-value problem (1.5)–(1.2) into an equivalent operator equation $N(u, 1) = 0$.

From the definition of N it follows that $N(\cdot, \lambda)$ is a compact perturbation of the identity of C^1_T onto itself for all $\lambda \in [0, 1]$. We start our homotopy argument with $\lambda = 0$. Then $N(\cdot, 0) = V$, where V is defined in Lemma 4.6. Hence, by Lemma 4.6, there exists $R_0 > 0$ such that if u is a solution of $N(u, 0) = 0$ then $\|u\|_{C^1} \leq R_0$; moreover, $\deg(N(\cdot, 0), B(0, R), 0) \neq 0$ provided that $R > R_0$. We show later that there exists $R'_0 > R_0 > 0$ such that, for all $\lambda \in [0, 1]$, every solution u of the operator equation $N(u, \lambda) = 0$ satisfies $\|u\|_{C^1} < R'_0$. Then $\deg(N(\cdot, \lambda), B(0, R'_0), 0)$ is well defined for all $\lambda \in [0, 1]$ and due to the homotopy invariance property $\deg(N(\cdot, \lambda), B(0, R'_0), 0) = \deg(N(\cdot, 0), B(0, R'_0), 0) \neq 0$ for all $\lambda \in [0, 1]$.

Since $\|u\|_{C^1} = \|u\|_C + \|u'\|_C$, we shall prove that both $\|u\|_C$ and $\|u'\|_C$ are estimated by some constants independent of λ. The equation $N(u, \lambda) = 0$ is equivalent to

$$(\varphi(u'))' + g(u') = f - \lambda h(u) - (1-\lambda)l(u)$$

subject to (1.2).

An argument similar to that in the proof of Lemma 4.5 shows that we can find $A > 0$, independent of λ, such that $\|u'\|_C \leq A$. Therefore $\|\widetilde{u}\|_C \leq AT$ and it remains to prove that there exists a constant $j > 0$, independent of λ such that $|\overline{u}| \leq j$. Let $\gamma := \frac{1}{2}\left(h(+\infty) + s(\widetilde{f}) - \overline{f}\right)$; by (2.7) $\gamma > 0$. Hence we can find $m > 0$ (independent of λ) such that

$$(1-\lambda)h(y) + \lambda l(y) > h(+\infty) - \gamma \tag{4.15}$$

for all $y \geq m$ and $0 \leq \lambda \leq 1$. Assuming that $\overline{u} > m + AT$, we find

$$(\varphi(u'))' + g(u') = \overline{f} + \widetilde{f} - (1-\lambda)h(u) - \lambda l(u) < \overline{f} + \widetilde{f} - h(+\infty) + \gamma = s(\widetilde{f}) - \gamma + \widetilde{f}.$$

Taking $\alpha = s(\widetilde{f}) - \gamma$ we see from Lemma 4.1 that $s(\widetilde{f}) - \gamma \geq s(\widetilde{f})$, a contrary to $\gamma > 0$. Thus $\overline{u} \leq m + AT$. Similarly, we exclude the possibility $\overline{u} \geq -m - AT$. Hence $\|u\|_C \leq AT + m =: j$ and $\|u\|_{C^1} \leq A(1+T) + m$, where A and m do not depend on λ. So that the desired radius is any $R'_0 > \max\{A(1+T) + m, R_0\}$.
Step 2 *Now we remove the additional assumptions on* φ *and* g.

Let us take fixed $\widetilde{f} \in C_T$ and define

$$M := \varphi_{-1}\Big(\sqrt{\frac{T}{3}}\big(\|\widetilde{f}\|_{L^2} + \sqrt{T}\sup_{\xi\in\mathbb{R}}|h(\xi)|\big)\Big) > 0\,.$$

Note that (2.6) imply that $0 < \varphi(M) < b \ (= -a)$. As a consequence $[-M, M] \subset \operatorname{Dom}\varphi \ (= I_1)$. Define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(y) := \begin{cases} \varphi(y) & \text{if } |y| \le M\,, \\ \varphi\big(M\operatorname{sgn}(y)\big) + y|y|^{p-2} - M^{p-1}\operatorname{sgn}(y) & \text{otherwise} \end{cases}$$

and $q : \mathbb{R} \to \mathbb{R}$ by

$$q(y) := \begin{cases} g(y) & \text{if } \ |y| \le M\,, \\ g(M\ \operatorname{sgn}(y)) & \text{otherwise}\,. \end{cases}$$

Observe that both ψ and q satisfy conditions of Step 1.

Thus, due to Step 1, the following problem

$$\begin{gathered} \big(\psi(y')\big)' + q(y') = \widetilde{f} + \overline{f} - h(y)\,, \\ y(0) = y(T)\,, \quad y'(0) = y'(T) \end{gathered} \tag{4.16}$$

has a solution provided that

$$r(\widetilde{f}) + h(-\infty) < \overline{f} < r(\widetilde{f}) + h(+\infty)\,, \tag{4.17}$$

where $r(\widetilde{f})$ is the (only admissible) value of the integral $\frac{1}{T}\int_0^T f(t)dt$ such that the associated first order problem:

$$\begin{gathered} v' + q\left(\psi_{-1}(v)\right) = f\,, \\ \textstyle\int_0^T \psi_{-1}(v(t))dt\,, \quad v(0) = v(T) \end{gathered} \tag{4.18}$$

has a solution (note that the existence and uniqueness of such $r(\widetilde{f})$ follows from Theorem 2.1 by setting $\varphi := \psi$ and $g := q$ in Theorem 2.1).

Using (3.3) we estimate v by $\|v\|_C \le \sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2} < \varphi(M)$, which, by definition of ψ, implies $\psi_{-1}(v) = \varphi_{-1}(v)$. Then the definition of q and the fact that $M > \varphi_{-1}\left(\sqrt{\frac{T}{3}}\|\widetilde{f}\|_{L^2}\right)$ yield $q(\varphi_{-1}(v)) = g(\varphi_{-1}(v))$. So that any solution to (4.18) satisfies also

$$\begin{gathered} v' + g(\varphi_{-1}(v)) = \widetilde{f} + s(\widetilde{f})\,, \\ \textstyle\int_0^T \varphi_{-1}(v(t))dt = 0\,, \quad v(0) = v(T) \end{gathered} \tag{4.19}$$

and vice versa. Integrating equations in (4.18) and (4.19),respectively, from 0 to T and taking into account corresponding boundary conditions, we arrive at

$$s(\widetilde{f}) = \frac{1}{T}\int_0^T q\left(\psi_{-1}(v(t))\right)\,dt = \frac{1}{T}\int_0^T q\left(\varphi_{-1}(v(t))\right)\,dt = r(\widetilde{f})\,.$$

Thus (4.17) becomes (2.7):

$$s(\widetilde{f}) + h(-\infty) < \overline{f} < s(\widetilde{f}) + h(+\infty)\,.$$

Now let us look at the equation (4.16) again. Using (3.3) we estimate the norm of $\varphi(y')$ by

$$\|\varphi(y')\|_C \leq \sqrt{\frac{T}{3}}\left(\|\widetilde{f}\|_{L^2} + \sqrt{T}\sup_{\xi\in\mathbb{R}}|h(\xi)|\right);$$

hence $\|y'\|_C \leq M$. So that $\psi(y') = \varphi(y')$ and any solution of (4.16) satisfy also (1.5)–(1.2). Since (4.16) possesses solution provided that (2.7), so does (1.5)–(1.2). This concludes Step 2.

Step 3 Assume that $h(-\infty) < h(\xi) < h(+\infty)$ for all $\xi \in \mathbb{R}$. An argument similar to that of [6, Theorem 4] shows that the condition (2.7) is necessary for the existence of a solution to (1.5)–(1.2). Details are left to the reader. We note that one should use Lemma 4.1 instead of Lemma 2 from [6]. (Lemma 4.2 shall be used in the case of the BVP (1.5)–(1.2)). ♢

5 Some remarks about uniqueness

In this section we would like to present one result concerning the uniqueness of the solution to the BVPs studied in the precedent sections. At first we formulate and prove two more general statements:

Proposition 5.1 *Suppose that $F : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous, $F(0,\cdot,\cdot) = F(2\pi,\cdot,\cdot)$ and*

$$\xi > \eta \quad \textit{implies} \quad F(\cdot,\xi,\cdot) > F(\cdot,\eta,\cdot)\,, \tag{5.1}$$

then there exists at most one solution of the BVP:

$$\begin{gathered}(\varphi(u'))' = F(t,u,u')\,, \quad t\in(0,T)\\ u(0)=u(2\pi), \quad u'(0)=u'(2\pi)\,.\end{gathered} \tag{5.2}$$

Proof If $u_1,\ u_2$ are two distinct solutions of (5.2) then

$$(\varphi(u_1'))' - (\varphi(u_2'))' = F(t,u_1,u_1') - F(t,u_2,u_2')\,. \tag{5.3}$$

Set $z = u_1 - u_2$; obviously $z \in C[0,T]$ and hence z attains its maximum value, say at t_M on [0,T]. Without lost of generality, we can suppose that $u_1(t_M) > u_2(t_M)$. This implies $\max_{t\in[0,T]} z(t) > 0$. Since $z(0) = z(T)$, we may assume $t_M \in [0,T)$. Taking into account that $z \in C^1[0,T]$ and that z satisfies (1.2), it follows that $z'(t_M) = 0$, which is equivalent to $u_1'(t_M) = u_2'(t_M)$.

Upon integrating from t_M to t, the equation (5.3) becomes

$$\varphi\left(u_1'(t)\right) - \varphi\left(u_2'(t)\right) = \int_{t_M}^{t}\left(F\left(\tau,u_1(\tau),u_1'(\tau)\right) - F\left(\tau,u_2(\tau),u_2'(\tau)\right)\right)d\tau\,.$$

Considering $u_1(t_M) > u_2(t_M)$, $u_1'(t_M) = u_2'(t_M)$, the continuity of $F\left(t,u_1(t),u_1'(t)\right) - F\left(t,u_2(t),u_2'(t)\right)$ and (5.1), we find that there exists $\delta > 0$ such that

$$\varphi(u_1'(t)) - \varphi(u_2'(t)) > 0 \text{ for all } t\in(t_M, t_M+\delta)\,.$$

Hence $u_1'(t) > u_2'(t)$ for all $t \in (t_M, t_M + \delta)$, which implies $z'(t) > 0$ for all $t \in (t_M, t_M + \delta)$, a contrary to $z(t_M) \geq z(t)$ for all $t \in [0, T]$. $\diamondsuit$

Proposition 5.2 *Suppose that $F : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies (5.1). Then the BVP*

$$\begin{gathered}(\varphi(u'))' = F(t, u, u'), \quad t \in (0, T) \\ u'(0) = 0, \quad u'(T) = 0\end{gathered} \tag{5.4}$$

admits at most one solution.

Proof The only difference to the proof of Proposition 5.1 is that we have to exclude the possibility $t_M = T$.

Suppose that $z = u_1 - u_2$ attains its maximum at T. Integrating (5.4) we find that

$$\varphi(u_1'(t)) - \varphi(u_2'(t)) = -\int_t^T \Big(F\left(\tau, u_1(\tau), u_1'(\tau)\right) - F\left(\tau, u_2(\tau), u_2'(\tau)\right) \Big) \, d\tau \,.$$

Taking into account $u_1(T) > u_2(T)$, $u_1'(T) = u_2'(T) = 0$, the continuity of $F(t, u_1(t), u_1'(t)) - F(t, u_2(t), u_2'(t))$ and (5.1), we find that there exists $\delta > 0$ such that

$$\varphi(u_1'(t)) - \varphi(u_2'(t)) < 0 \text{ for all } t \in (T - \delta, T) \,.$$

Hence $u_1'(t) - u_2'(t) < 0$ for all $t \in (T - \delta, T)$, which implies $z'(t) < 0$ for all $t \in (T - \delta, T)$, a contrary to $z(T) \geq z(t)$ for all $t \in [0, T]$. $\diamondsuit$

Taking a particular function $F(t, u, v) = -h(u) - g(v) + f$, where $h \in C(\mathbb{R}, \mathbb{R})$, is bounded and strictly decreasing, $g \in C(\mathbb{R}, \mathbb{R})$ and $f \in C[0, T]$, we obtain uniqueness for the BVPs (1.5)–(1.2) and (1.5)–(1.4) studied previously:

Theorem 5.3 *Suppose* (P), (G), *$f \in C_T$ ($f \in C[0, T]$), $h \in C(\mathbb{R}, \mathbb{R})$ and moreover impose that h is strictly decreasing. Then the BVP (1.5)–(1.2) ((1.5)–(1.4)) admits at most one solution.*

We would like to point out that the monotonicity assumption, h being decreasing, was essential to prove the uniqueness. The following simple example shows that there are no analogous statements to the previous theorem assuming h increasing, even not for semilinear BVP:

Example 5.1 Consider BVP:

$$\begin{gathered}u''(t) + h(u(t)) = \cos 2t \,, \qquad t \in (0, T) \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi)\end{gathered} \tag{5.5}$$

with

$$h(s) \begin{cases} s & \text{for } |s| \leq 1 \\ (2 - \frac{1}{|s|})\operatorname{sgn}(s) & \text{for } |s| > 1 \end{cases}$$

then $u = -\frac{1}{3}\cos(t) + c \sin(t + \phi)$, with parameters $|c| \leq \frac{2}{3}$ and $\phi \in [0, 2\pi)$, is a two dimensional subset of the set of all solutions to (5.5).

For some particular results in this direction (the semilinear problem) we refer to [6].

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References

[1] Ambrosetti, A., and Prodi, G., *A Primer of Nonlinear Analysis*, Cambridge University Press, Cambridge, 1993.

[2] Arcoya, D., and Orsina, L., Landesman–Lazer conditions and quasilinear elliptic equations, Nonlinear Anal., **28** (1997), 1623-1632.

[3] Boccardo, L., Drábek, P., and Kučera, M., Landasman-Lazer conditions for strongly nonlinear boundary-value problems, Comment. Math. Univ. Carolin., **30** (1989), 411-427.

[4] Cañada, A., and Drábek, P., On semilinear problems with nonlinearities depending only on derivatives, SIAM J. Math. Anal., **27** (1996), 543-557.

[5] Coddington E. A., Levinson, N., *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

[6] Dancer, E. N., On the ranges of certain damped nonlinear differential equation, Ann. Mat. Pura Appl., **119** (1979), 281-295.

[7] del Pino, M., *Acerca de un problema cuasilineal de secundo orden*, Diploma Thesis, Universidad de Chile, Santiago de Chile, 1987.

[8] Fučík, S., *Solvability of Nonlinear Equation and Boundary Value Problems*, D. Reidel Publishing Company, Dordrecht, 1980.

[9] García-Huidobro, M., Manásevich, R., and Zanolin, F., A Fredholm-like Result for Strongly Nonlinear Second Order ODE's, *J. Differential Equations*, **114** (1994), 132-167.

[10] García-Huidobro, M., Manásevich, R., and Zanolin, F., Strongly Nonlinear Second Order ODE's with Unilateral Conditions, *Differential and Integral Equations*, **6** (1993), 1057-1078.

[11] Kannan, R., and Seikkala, S., Existence of solutions to a periodic boundary-value problem, in *'Function Spaces, Differential Operators and Nonlinear Analysis'* (Vesa Mustonen, Jiří Rákosník, editors), Mathematical Institute AS ČR, Prague, 1999, 149-156.

[12] Kotek, Z., Kubík, S., *Nonlinear circuits* (in Czech), SNTL, Prague, 1962.

[13] Landesman, E. M., and Lazer, A. C., Nonlinear perturbations of linear elliptic boundary-value problems at resonance, J. Math. Mech., **19** (1970), 609-623.

[14] Manásevich, R., and Mawhin, J., Periodic solutions for nonlinear systems with p-Laplacian-like operators, J. Differential Equations, **145** (1998), 367-393.

[15] Mawhin, J., Willem, M., *Critical point theory and Hamiltonian systems*, Springer Verlag, 1989.

[16] Mawhin, J., Some remarks on semilinear problems at resonance where the nonlinearity depends only on the derivatives, Acta Math. Inform. Univ. Ostraviensis, **2** (1994), 61-69.

[17] Ames, W. F., *Nonlinear partial differential equations in engineering*, Academic press, New York, 1965.

[18] Zeidler, E., *Nonlinear Functional Analysis and its Applications I*, Springer-Verlag, Berlin, 1986.

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