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INITIAL-BOUNDARY LAYER ASSOCIATED WITH THE 3-D BOUSSINESQ SYSTEM FOR RAYLEIGH-BÉNARD CONVECTION

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Abstract. This article concerns the initial-boundary layer effects of the 3-D incompressible Boussinesq system for Rayleigh-Bénard convection with ill-prepared initial data. We consider a non-slip boundary condition for the velocity field and inhomogeneous Dirichlet boundary condition for the temperature. By means of multi-scale analysis and matched asymptotic expansion methods, we establish an accurate approximating solution for the viscous and diffusive Boussinesq system. We also study the convergence of the infinite Prandtl number limit.

1. Introduction

In atmospheric fronts and oceanographic circulation, fluid phenomena with heat transfer have received much attentions (see [11, 20, 22, 26]). Here we deal with the Rayleigh-Bénard convection setting of a horizontal layer of fluid confined by two parallel planes a distance h apart and heated at the bottom plane at temperature T_2 and cooled at the top plane at temperature $T_1 < T_2$. In presence of the gravity force, hot fluid at the bottom rises while cool fluid on top sinks. The dynamic model consists of 3D incompressible Navier-Stokes equation via a buoyancy force proportional to the temperature coupled with the heat advection-diffusion of temperature [1, 4, 32, 33]. We consider the Boussinesq system with rotation for Rayleigh-Bénard convection [2, 9, 17, 31].

$$
\begin{gathered}
\partial_t u + (u \cdot \nabla) u + \nabla p + 2\Omega e_3 \times u = \nu \Delta u + g\alpha e_3 T, \\
\nabla \cdot u = 0, \\
\partial_t T + u \cdot \nabla T = \kappa \Delta T, \\
u|_{z=0,h} = 0, \\
T|_{z=0} = T_2, \quad T|_{z=h} = T_1.
\end{gathered}
$$

The unknown functions $u = (u_1, u_2, u_3)^T$, p and T represent the vector velocity field, the scalar pressure and the scalar temperature of the fluid, respectively. ν and κ are the kinematic viscosity and the thermal diffusion coefficient, respectively. Ω is the rotation rate. e_3 denotes the unit upward vector. As usual, $e_3 := (0, 0, 1)^T$.

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g denotes the gravity acceleration constant. α stands for the thermal expansion coefficient. For simplicity, we impose the periodicity in the horizontal directions.

The mathematical analysis of the nondimensional form has attracted much attention. Wang [32] studied the infinite Prandtl number limit and derived the convergence of Boussinesq system for Rayleigh-Bénard convection to the infinite Prandtl number system (see, also [33]). This singular limit problem was also investigated by Shi et al. [27], in which they considered initial layer problem by an effective approximating expansion and rigorously proved the convergence of the Boussinesq system to the infinite Prandtl number system.

In this article, we study a simplification of Boussinesq system by using the Boussinesq approximation and non-dimensionalization,

$$\begin{gathered}\epsilon[\partial_t u^\epsilon + (u^\epsilon \cdot \nabla)u^\epsilon] + \nabla p^\epsilon + \frac{1}{Ek} e_3 \times u^\epsilon = \Delta u^\epsilon + Rae_3 T^\epsilon, \\ (x,y,z,t) \in X \times (0,S),\end{gathered} \tag{1.1}$$

$$\nabla \cdot u^\epsilon = 0, \quad (x,y,z,t) \in X \times (0,S), \tag{1.2}$$

$$\partial_t T^\epsilon + u^\epsilon \cdot \nabla T^\epsilon = \epsilon \Delta T^\epsilon, \quad (x,y,z,t) \in X \times (0,S), \tag{1.3}$$

where $X := \mathbb{T}^2 \times [0,1]$, $\mathbb{T}^2 = (R^1/2\pi)^2$ is the torus in R^2, $S > 0$. $\epsilon = 1/\sqrt{Pr}$, $Pr = \nu/\kappa$ is the Prandtl number, $Ek = \nu/(2\Omega h^2)$ is the Ekman number and

$$Ra = \frac{g\alpha(T_2 - T_1)h^3}{\nu\kappa}$$

is the Rayleigh number.

We suplement the above system with no-slip boundary conditions for the vector velocity and inhomogeneous Dirichlet boundary conditions for the temperature:

$$u^\epsilon|_{z=0,1} = 0, \quad (x,y,t) \in \mathbb{T}^2 \times (0,S), \tag{1.4}$$

$$T^\epsilon|_{z=0} = a_0(x,y,t), \quad T^\epsilon|_{z=1} = a_1(x,y,t), \quad (x,y,t) \in \mathbb{T}^2 \times (0,S). \tag{1.5}$$

To the above system we impose the initial conditions

$$u^\epsilon(t=0) = u_0^\epsilon(x,y,z), \quad T^\epsilon(t=0) = T_0^0(x,y,z), \quad (x,y,z) \in X. \tag{1.6}$$

This nondimensional form (1.1)–(1.6) of Boussinesq system is different from those in [27, 32, 33]. Motivated by results on the existence and the regularities of the suitable global weak solution in [32, 33], and the related models, see [7, 18, 21], the system (1.1)–(1.6) has also suitable weak solution, standard Galerkin approximation procedure implies the existence of weak solution.

In this paper, we are interested in the behavior of the system (1.1)–(1.6) with the infinite Prandtl number limit. As the Prandtl number tends to infinity, it is clear that (1.1) and (1.3) become Stokes-hyperbolic coupled equations (instead of parabolic-parabolic coupled equations), the boundary $z = 0, 1$ become characteristic for the temperature due to the non-slip boundary condition $u^\epsilon|_{z=0,1} = 0$. Then, the boundary condition of temperature should be dropped. This leads to the appearance of the boundary layers. The initial layer of velocity arises from ill-prepared initial data. Compared with the studies in [27, 32, 33], the problem in this paper becomes more complicated due to the appearance of boundary layers and initial layer. This perturbed problems have been studied in many other works see for instance, [3, 10, 13, 23, 24, 25, 28, 35, 36, 37] and references therein.

The present work is mainly motivated by [8, 14, 16, 27, 30, 34]. Firstly, we derive the appearance of boundary layers and initial layer in detail. Secondly, we construct an approximation solution to the original equation as the combination of inner, initial and boundary expansions. We mainly use the matched asymptotic expansion methods of singular perturbation theory [5, 12] and the multi-scale approach [12, 20]. Finally, we consider the convergence of (1.1)–(1.6) to the infinite Prandtl number system as the Prandtl number approaches infinity.

The rest of this article is organized as follows. The derivation of initial and boundary layers is given in Section 2. The main results are stated in Section 3. In Section 4, we establish the approximating solution. The proof of main results is shown in Section 5.

2. Derivation of initial and boundary layers

In this section, by employing the singular perturbation theory [5, 12] and the Stokes operator [15, 17, 19], we consider the behavior of the solution when the Prandtl number tends to ∞, i.e., ϵ tends to 0.

Letting $\epsilon = 0$ in (1.1)–(1.6), we obtain

$$\nabla p^0 + \frac{1}{Ek} e_3 \times u^0 = \Delta u^0 + Ra e_3 T^0, \tag{2.1}$$

$$\nabla \cdot u^0 = 0, \tag{2.2}$$

$$\partial_t T^0 + \left(u^0 \cdot \nabla\right) T^0 = 0, \tag{2.3}$$

$$u^0|_{z=0,1} = 0. \tag{2.4}$$

Here we consider the initial data

$$T^0(t=0) = T_0^0(x,y,z),$$

where $T_0^0(x,y,z)$ is the initial value of $T^\epsilon(x,y,z)$.

We first study the boundary conditions of T^0. Restricting (2.3) to $z = 0, 1$ and then plugging (2.4) into the resulting equation, we have

$$\partial_t T^0|_{z=0,1} = 0. \tag{2.5}$$

The compatibility conditions for (1.5)–(1.6) yield that

$$T_0^0(x,y,z)|_{z=0} = a_0(x,y,t=0), \quad T_0^0(x,y,z)|_{z=1} = a_1(x,y,t=0). \tag{2.6}$$

The combination of (2.5) and (2.6) implies

$$T^0|_{z=0} = a_0(x,y,t=0),\ T^0|_{z=1} = a_1(x,y,t=0). \tag{2.7}$$

Moreover, comparing (1.5) and (2.7), we obtain that

$$\lim_{\epsilon\to 0} a_0(x,y,t) \neq a_0(x,y,t=0), \quad \lim_{\epsilon\to 0} a_1(x,y,t) \neq a_1(x,y,t=0),$$

i.e.,

$$\lim_{\epsilon\to 0}(T^\epsilon|_{z=0}) \neq T^0|_{z=0}, \quad \lim_{\epsilon\to 0}(T^\epsilon|_{z=1}) \neq T^0|_{z=1}.$$

This leads to the appearance of the boundary layers of the scalar temperature.

Then we turn to derive the initial conditions of u^0. Because of the singularity of perturbation, generally speaking, the limit of $u_0^\epsilon(x,y,z)$ as $\epsilon \to 0$ can not be satisfied by the velocity $u^0(t=0)$ in the limit system. Restricting (2.1), (2.2) and (2.4) to $t = 0$ gives

$$\nabla p^0(t=0) + \frac{1}{Ek} e_3 \times u^0(t=0) = \Delta u^0(t=0) + Ra e_3 T^0(t=0),$$

$$\nabla \cdot u^0(t=0) = 0,$$

$$u^0|_{z=0,1}(t=0)=0.$$

By solving above system, we know that the value of $u^0(t=0)$ is determined by the initial data of the temperature, while $\lim_{\epsilon\to 0} u_0^\epsilon$ can be given arbitrarily and independently of $T^0(t=0)$, so $\lim_{\epsilon\to 0} u_0^\epsilon \neq u^0(t=0)$. This leads to the appearance of an initial layer of the vector velocity. Hence, the infinite Prandtl number limit of the initial and boundary value problem (1.1)–(1.6) is a singular problem involving boundary layers and an initial layer.

3. Main results

Let $(u^\epsilon, p^\epsilon, T^\epsilon)$ be the global weak solution of (1.1)–(1.6) in the Leray's sense. We assume that the initial data has an expansion up to the 0 order as follows

$$(u^\epsilon, T^\epsilon)(t=0) = (u_0^0 + u_{0E}^\epsilon, T_0^0)(x,y,z), \tag{3.1}$$

where u_0^0 and T_0^0 are all $C^\infty(X)$ functions, u_{0E}^ϵ denotes the remainders except for 0 order term for the initial data of u^ϵ, $u_{0E}^\epsilon \in C^\infty(X)$ satisfies

$$\|u_{0E}^\epsilon(x,y,z)\|_{L^2(X)} \le C\sqrt{\epsilon}. \tag{3.2}$$

Theorem 3.1. *Assume that* (3.1) *holds. Also, assume that* u_0^0 *and* $T_0^0 \in C^\infty(X)$, $a_0, a_1 \in C^\infty(\mathbb{T}^2 \times [0,+\infty))$ *satisfy some compatibility conditions as* (2.6) *and* $(u_a^\epsilon, p_a^\epsilon, T_a^\epsilon)$ *is an approximate solution of the system* (1.1)–(1.6). *Then, as* $\epsilon \to 0$, *for any* $0 < S < \infty$, *we have*

$$\|(u^\epsilon - u_a^\epsilon, T^\epsilon - T_a^\epsilon)\|_{L^\infty(0,S;L^2(X))} \le C\epsilon^{1/4}, \tag{3.3}$$

$$\begin{aligned} &\|\nabla(u^\epsilon - u_a^\epsilon)\|^2_{L^2(0,S;L^2(X))} + \epsilon\|\nabla(T^\epsilon - T_a^\epsilon)\|^2_{L^2(0,S;L^2(X))} \\ &\le C\sqrt{\epsilon}, \|u^\epsilon - u_a^\epsilon\|_{L^2(0,S;H^1(X))} \le C\epsilon^{1/4}, \end{aligned} \tag{3.4}$$

where $H^1(X) = W^{1,2}(X)$, *for some positive constants* C *independent of* ϵ.

The functions u_a^ϵ, T_a^ϵ, and p_a^ϵ are defined in Section 4. By a standard method [6, 19, 29], we formulate any m^{th}, $m = 0,1,2,\dots$, order compatibility conditions.

4. Approximate solution

In this section, we construct the approximate solution including the inner expansion away from $z = 0,1$, initial layer expansion near $t = 0$ and the boundary layers expansion near $z = 0,1$. Some useful properties of approximating solution are also derived. It is easy to see that

$$\begin{gathered} (u^\epsilon, p^\epsilon, T^\epsilon)(x,y,z,t) \sim \sum_{i=0}^\infty (\sqrt{\epsilon})^i (u^{In,i}(x,y,z,t) + u^{I,i}(x,y,z,\tau), \\ p^{In,i}(x,y,z,t) + p^{I,i}(x,y,z,\tau), \\ T^{In,i}(x,y,z,t) + f(z)T_+^{B,i}(x,y,Z,t) + h(z)T_-^{B,i}(x,y,\overline{Z},t)), \end{gathered}$$

where $\tau = t/\epsilon$ is the fast time variable, $Z = \frac{z}{\sqrt{\epsilon}}$ and $\overline{Z} = \frac{1-z}{\sqrt{\epsilon}}$ are the fast space variables. ϵ and $\sqrt{\epsilon}$ are the lengths of the initial layer and boundary layers, respectively. $(u^{In,i}, p^{In,i}, T^{In,i})(x,y,z,t)$ are the inner functions for the velocity field, pressure and temperature field, respectively, independent of ϵ. $(u^{I,i}, p^{I,i})(x,y,z,\tau)$ are the initial layer functions near $t = 0$ for the velocity field and pressure, respectively. The initial layer functions satisfy that $u^{I,i}, p^{I,i}$ decay to zero exponentially, as $\tau \to \infty$. $T_+^{B,i}(x,y,Z,t)$ represents the lower boundary layer function of temperature field near $z=0$. $T_-^{B,i}(x,y,\overline{Z},t)$ stands for the upper boundary layer function of temperature field near $z=1$. The boundary layer functions satisfy that $T_+^{B,i}$ decays to zero exponentially, as $Z \to \infty$, and $T_-^{B,i}$ decays to zero exponentially, as $\overline{Z} \to \infty$.

Here $f(z)$ and $h(z)$ are smooth C^2 cut-off functions [23], near $z=0$, we define

$$f(z) = \begin{cases} 1, & z \in [0, \frac{1}{4}], \\ 0, & z \in [\frac{1}{2}, 1]. \end{cases}$$

Similarly, near $z=1$, we define

$$h(z) = \begin{cases} 0, & z \in [0, \frac{1}{4}], \\ 1, & z \in [\frac{1}{2}, 1], \end{cases}$$

which, in turn, imply that $f(0) = h(1) = 1$ and

$$f(1) = f'(1) = f''(1) = f'(0) = f''(0) = h(0) = h'(0) = h''(0) = h'(1) = h''(1) = 0.$$

We assume that the asymptotic expansion of the system (1.1)–(1.6) including initial and boundary corrections is of the form

$$\begin{aligned} &(u_a^\epsilon, p_a^\epsilon, T_a^\epsilon)(x,y,z,t) \\ &= \Big(u^{In,0}(x,y,z,t) + u^{I,0}(x,y,z,\tau), p^{In,0}(x,y,z,t) \\ &\quad + p^{I,0}(x,y,z,\tau), T^{In,0}(x,y,z,t) + f(z)T_+^{B,0}(x,y,Z,t) \\ &\quad + h(z)T_-^{B,0}(x,y,\overline{Z},t)\Big). \end{aligned} \tag{4.1}$$

Moreover, to match the boundary and initial conditions (1.4)–(1.6), we impose the following restrictions

$$(u^{In,0} + u^{I,0})|_{z=0,1} = 0, \tag{4.2}$$

$$T^{In,0}|_{z=0} + T_+^{B,0}|_{Z=0} = a_0(x,y,t), \quad T^{In,0}|_{z=1} + T_-^{B,0}|_{\overline{Z}=0} = a_1(x,y,t), \tag{4.3}$$

$$\begin{gathered} u^{In,0}(t=0) + u^{I,0}(\tau=0) = u_0^0, \\ (T^{In,0} + f(z)T_+^{B,0} + h(z)T_-^{B,0})(t=0) = T_0^0, \end{gathered} \tag{4.4}$$

We discuss the construction of the inner, initial layer and boundary layers functions

$$(u_a^\epsilon, p_a^\epsilon, T_a^\epsilon) := (u_{In}^\epsilon, p_{In}^\epsilon, T_{In}^\epsilon) + (u_I^\epsilon, p_I^\epsilon, T_B^\epsilon), \tag{4.5}$$

where

$$(u_{In}^\epsilon, p_{In}^\epsilon, T_{In}^\epsilon) = (u^{In,0}, p^{In,0}, T^{In,0}), \tag{4.6}$$

$$(u_I^\epsilon, p_I^\epsilon) = (u^{I,0}, p^{I,0}), \tag{4.7}$$

$$T_B^\epsilon = T^{B,0} = f(z)T_+^{B,0} + h(z)T_-^{B,0}. \tag{4.8}$$

First, we study inner expansion away from the boundary $z=0$ and $z=1$ in Section 4.1. Then, we study the initial layer expansion near $t=0$ and lower boundary layer expansion near $z=0$ in Section 4.2 and the upper boundary layer expansion near $z=1$ can be used by the similar method in Section 4.3. Finally, we consider the approximating solution in Section 4.4.

4.1. **Inner expansion.** Away from the boundary $z = 0$ and $z = 1$, from (4.1), the solution to (1.1)–(1.5) has the expansion

$$(u^\epsilon, p^\epsilon, T^\epsilon)(x, y, z, t) \sim \sum_{i=0}^{\infty} (\sqrt{\epsilon})^i (u^{In,i}, p^{In,i}, T^{In,i})(x, y, z, t).$$

First, inserting above expansion into (1.1)–(1.5) and using direct calculations, we obtain

$$\begin{gathered}
\sum_{i=0}^{\infty} (\sqrt{\epsilon})^i \Big(\epsilon[\partial_t u^{In,i} + \sum_{j=0}^{i} u^{In,j} \cdot \nabla u^{In,i-j}] + \nabla p^{In,i} \\
+ \frac{1}{Ek} e_3 \times u^{In,i} - \Delta u^{In,i} - Rae_3 T^{In,i} \Big) = 0, \\
\sum_{i=0}^{\infty} (\sqrt{\epsilon})^i \nabla \cdot u^{In,i} = 0, \\
\sum_{i=0}^{\infty} (\sqrt{\epsilon})^i \Big(\partial_t T^{In,i} + \sum_{j=0}^{i} u^{In,j} \cdot \nabla T^{In,i-j} - \epsilon \Delta T^{In,i} \Big) = 0, \\
\sum_{i=0}^{\infty} (\sqrt{\epsilon})^i u^{In,i}|_{z=0,1} = 0, \\
\sum_{i=0}^{\infty} (\sqrt{\epsilon})^i T^{In,i}(t = 0) = T_0^0 .
\end{gathered}$$

Then $(u^\epsilon_{In}, p^\epsilon_{In}, T^\epsilon_{In})$ satisfies

$$\epsilon[\partial_t u^\epsilon_{In} + (u^\epsilon_{In} \cdot \nabla) u^\epsilon_{In}] + \nabla p^\epsilon_{In} + \frac{1}{Ek} e_3 \times u^\epsilon_{In} = \Delta u^\epsilon_{In} + Rae_3 T^\epsilon_{In} + R^\epsilon_{In,u}, \tag{4.9}$$

$$\nabla \cdot u^\epsilon_{In} = 0, \tag{4.10}$$

$$\partial_t T^\epsilon_{In} + (u^\epsilon_{In} \cdot \nabla) T^\epsilon_{In} = \epsilon \Delta T^\epsilon_{In} + R^\epsilon_{In,T}, \tag{4.11}$$

$$u^\epsilon_{In}|_{z=0,1} = 0, \tag{4.12}$$

$$T^\epsilon_{In}(t = 0) = T_0^0, \tag{4.13}$$

where the remainders are

$$\begin{aligned}
R^\epsilon_{In,u} = - \sum_{i=1}^{\infty} (\sqrt{\epsilon})^i (\epsilon[\partial_t u^{In,i} + \sum_{j=0}^{i} u^{In,j} \cdot \nabla u^{In,i-j}] + \nabla p^{In,i} \\
+ \frac{1}{Ek} e_3 \times u^{In,i} - \Delta u^{In,i} - Rae_3 T^{In,i}),
\end{aligned}$$

and

$$R^\epsilon_{In,T} = - \sum_{i=1}^{\infty} (\sqrt{\epsilon})^i \Big(\partial_t T^{In,i} + \sum_{j=0}^{i} u^{In,j} \cdot \nabla T^{In,i-j} - \epsilon \Delta T^{In,i} \Big).$$

So we know that $R^\epsilon_{In,u}$ and $R^\epsilon_{In,T}$ satisfy the estimates

$$\|(R^\epsilon_{In,u}, R^\epsilon_{In,T})\|_{L^\infty(0,S;H^s(X))} \le C\sqrt{\epsilon}, \tag{4.14}$$

for $S > 0$ and $s \ge 1$. Denote C by a positive constant, independent of ϵ.

We now set the coefficient of $O\big((\sqrt{\epsilon})^0\big)$ in (4.9)–(4.11) as zero and use the initial and boundary conditions (4.12)–(4.13). We obtain a system for $(u^{In,0}, p^{In,0}, T^{In,0})$,

$$\nabla p^{In,0} + \frac{1}{Ek} e_3 \times u^{In,0} = \Delta u^{In,0} + Rae_3 T^{In,0}, \tag{4.15}$$

$$\nabla \cdot u^{In,0} = 0, \tag{4.16}$$

$$\partial_t T^{In,0} + (u^{In,0} \cdot \nabla) T^{In,0} = 0, \tag{4.17}$$

$$u^{In,0}|_{z=0,1} = 0, \tag{4.18}$$

$$T^{In,0}(t=0) = T_0^0(x,y,z). \tag{4.19}$$

The rotating system (4.15)–(4.19) has stationary Stokes equations via a buoyancy force proportional to temperature coupled with heat advection of the temperature. Hence, the existence of the smooth solutions is the same to those of the incompressible Stokes equations. Since the proof is basic, we omit the details.

Proposition 4.1. *Assume that $T_0^0 \in C^\infty(X)$ satisfies some compatibility conditions like* (2.6). *There is a unique and global $C^\infty(X \times [0, +\infty))$ smooth solution to the system* (4.15)–(4.19).

Now we turn to the construction of the initial layer and lower boundary layer function.

4.2. **Initial layer and lower boundary layer expansion.** We now derive the systems satisfying the initial layer and lower boundary layer function, which is divided into six steps.

Step 1. Near $t = 0, z = 0$, $f(z) = 1$ and $h(z) = 0$, equation (4.8) turns into

$$T_B^\epsilon = T^{B,0} = T_+^{B,0}(x,y,Z,t). \tag{4.20}$$

Step 2. Inserting (4.5) into (1.1)–(1.3), then using direct calculation yields

$$\begin{aligned}
&\epsilon[\partial_t u_a^\epsilon + (u_a^\epsilon \cdot \nabla)u_a^\epsilon] + \nabla p_a^\epsilon + \frac{1}{Ek} e_3 \times u_a^\epsilon - \Delta u_a^\epsilon - Rae_3 T_a^\epsilon \\
&= \epsilon[\partial_t(u_{In}^\epsilon + u_I^\epsilon) + ((u_{In}^\epsilon + u_I^\epsilon) \cdot \nabla)(u_{In}^\epsilon + u_I^\epsilon)] + \nabla(p_{In}^\epsilon + P_I^\epsilon) \\
&\quad + \frac{1}{Ek} e_3 \times (u_{In}^\epsilon + u_I^\epsilon) - \Delta(u_{In}^\epsilon + u_I^\epsilon) - Rae_3(T_{In}^\epsilon + T_B^\epsilon) \\
&= R_{In,u}^\epsilon + \epsilon[\partial_t u_I^\epsilon + (u_{In}^\epsilon \cdot \nabla)u_I^\epsilon + u_I^\epsilon \cdot \nabla(u_{In}^\epsilon + u_I^\epsilon)] \\
&\quad + \nabla p_I^\epsilon + \frac{1}{Ek} e_3 \times u_I^\epsilon - \Delta u_I^\epsilon - Rae_3 T_B^\epsilon,
\end{aligned} \tag{4.21}$$

$$\nabla \cdot u_a^\epsilon = \nabla \cdot (u_{In}^\epsilon + u_I^\epsilon) = 0, \tag{4.22}$$

$$\begin{aligned}
&\partial_t T_a^\epsilon + (u_a^\epsilon \cdot \nabla)T_a^\epsilon - \epsilon \Delta T_a^\epsilon \\
&= \partial_t(T_{In}^\epsilon + T_B^\epsilon) + ((u_{In}^\epsilon + u_I^\epsilon) \cdot \nabla)(T_{In}^\epsilon + T_B^\epsilon) - \epsilon\Delta(T_{In}^\epsilon + T_B^\epsilon) \\
&= R_{In,T}^\epsilon + u_I^\epsilon \cdot \nabla T_{In}^\epsilon + \partial_t T_B^\epsilon + (u_{In}^\epsilon \cdot \nabla)T_B^\epsilon - \epsilon\Delta T_B^\epsilon + u_I^\epsilon \cdot \nabla T_B^\epsilon.
\end{aligned} \tag{4.23}$$

Step 3. We deduce after plugging (4.7), (4.20) into (4.21) that

$$\begin{aligned}
&\epsilon[\partial_t u_a^\epsilon + (u_a^\epsilon\cdot\nabla)u_a^\epsilon] + \nabla p_a^\epsilon + \frac{1}{Ek}e_3\times u_a^\epsilon - \Delta u_a^\epsilon - Rae_3T_a^\epsilon \\
&= R_{In,u}^\epsilon + \epsilon[\partial_t u_I^\epsilon + (u_{In}^\epsilon\cdot\nabla)u_I^\epsilon + u_I^\epsilon\cdot\nabla(u_{In}^\epsilon + u_I^\epsilon)] \\
&\quad + \nabla p_I^\epsilon + \frac{1}{Ek}e_3\times u_I^\epsilon - \Delta u_I^\epsilon - Rae_3T_B^\epsilon \\
&= R_{In,u}^\epsilon + (\partial_\tau u^{I,0} + \nabla p^{I,0} + \frac{1}{Ek}e_3\times u^{I,0} - \Delta u^{I,0}) \\
&\quad + \epsilon[(u^{In,0}\cdot\nabla)u^{I,0} + (u^{I,0}\cdot\nabla)(u^{I,0}+u^{In,0})] - Rae_3T_+^{B,0}.
\end{aligned} \tag{4.24}$$

Step 4. Now we set the coefficient of order $O\big((\sqrt{\epsilon})^0\big)$ in (4.24) as zero and use (4.6), (4.7), (4.16), the boundary conditions (4.2), (4.18) and initial condition (4.4). Then we have the following system for the initial layer function $(u^{I,0}, p^{I,0})(x,y,z,\tau)$

$$\partial_\tau u^{I,0} + \frac{1}{Ek}e_3\times u^{I,0} + \nabla p^{I,0} = \Delta u^{I,0}, \tag{4.25}$$
$$\nabla\cdot u^{I,0} = 0, \tag{4.26}$$
$$u^{I,0}|_{z=0,1} = 0, \tag{4.27}$$
$$u^{I,0}|_{\tau=0} = u_0^0 - u^{O,0}(t=0), \tag{4.28}$$
$$u^{I,0}\to 0, \quad \text{as } \tau\to+\infty. \tag{4.29}$$

As in [27], we obtain the property of the initial layer function.

Proposition 4.2. *Let the assumptions of Theorem 3.1 hold. There is a unique and smooth solution* $\big(u^{I,0}, p^{I,0}\big)$ *of* (4.25)–(4.29) *satisfying the exponential decay to zero as* $\tau\to\infty$,

$$\|u^{I,0}(\cdot,\tau)\|_{H^s(X)} \le Ce^{-\lambda\tau},$$

for some positive constants C,λ *and any* $s\ge 1$.

Step 5. We deduce after plugging (4.7), (4.20) into (4.23) that

$$\begin{aligned}
&\partial_t T_a^\epsilon + (u_a^\epsilon\cdot\nabla)T_a^\epsilon - \epsilon\Delta T_a^\epsilon \\
&= R_{In,T}^\epsilon + u_I^\epsilon\cdot\nabla T_{In}^\epsilon + \partial_t T_B^\epsilon + (u_{In}^\epsilon\cdot\nabla)T_B^\epsilon - \epsilon\Delta T_B^\epsilon + u_I^\epsilon\cdot\nabla T_B^\epsilon \\
&= R_{In,T}^\epsilon + (u^{I,0}\cdot\nabla)T^{In,0} + \partial_t T_+^{B,0} + (u^{In,0}\cdot\nabla)T_+^{B,0} - \epsilon\Delta T_+^{B,0} + (u^{I,0}\cdot\nabla)T_+^{B,0},
\end{aligned}$$

where

$$\begin{aligned}
&\partial_t T_+^{B,0} + (u^{In,0}\cdot\nabla)T_+^{B,0} - \epsilon\Delta T_+^{B,0} \\
&= \partial_t T_+^{B,0} + \big(u_1^{In,0}\partial_x + u_2^{In,0}\partial_y\big)T_+^{B,0} + \frac{1}{\sqrt{\epsilon}}u_3^{In,0}\partial_Z T_+^{B,0} \\
&\quad - \epsilon(\partial_{xx}+\partial_{yy})T_+^{B,0} - \partial_{ZZ}T_+^{B,0} \\
&= \partial_t T_+^{B,0} + \Big[(u_1^{In,0}(x,y,0,t) + \sqrt{\epsilon}\partial_z u_1^{In,0}(z=0)Z + \cdots)\partial_x \\
&\quad + (u_2^{In,0}(x,y,0,t) + \sqrt{\epsilon}\partial_z u_2^{In,0}(z=0)Z + \cdots)\partial_y\Big]T_+^{B,0} \\
&\quad + \frac{1}{\sqrt{\epsilon}}\Big(u_3^{In,0}(x,y,0,t) + \sqrt{\epsilon}\partial_z u_3^{In,0}(z=0)Z + \cdots\Big)\partial_Z T_+^{B,0} \\
&\quad - \epsilon(\partial_{xx}+\partial_{yy})T_+^{B,0} - \partial_{ZZ}T_+^{B,0},
\end{aligned} \tag{4.30}$$

where $u^{In,0} = \big(u_1^{In,0}, u_2^{In,0}, u_3^{In,0}\big)$. Here the Taylor series expansion is used

$$u^{In,0}(x,y,z,t) = u^{In,0}(x,y,\sqrt{\epsilon}Z,t) = u^{In,0}(x,y,0,t) + \sqrt{\epsilon}\partial_z u^{In,0}(z=0)Z + \cdots .$$

Step 6. Now we set the coefficients of order $O\big((\sqrt{\epsilon})^k\big)$ $(k=-1,0)$ in (4.30) equal to zero. First, collecting $O\big((\sqrt{\epsilon})^{-1}\big)$ terms we have

$$u_3^{In,0}(z=0)\partial_Z T_+^{B,0} = 0,$$

from which, together with (2.4), we cannot obtain more properties of $\partial_Z T_+^{B,0}$.

Next, collecting $O((\sqrt{\epsilon})^0)$ terms we obtain

$$\partial_t T_+^{B,0} + Z\partial_z u_3^{In,0}(z=0)\partial_Z T_+^{B,0} = \partial_{ZZ} T_+^{B,0}. \tag{4.31}$$

Restricting (2.2) to $z=0$, one gets

$$\nabla \cdot u^{In,0}|_{z=0} = 0,$$

from which, with (2.4), we have

$$\partial_z u_3^{In,0}(z=0) = 0. \tag{4.32}$$

Plugging (4.32) into (4.31), we obtain

$$\partial_t T_+^{B,0} = \partial_{ZZ} T_+^{B,0}. \tag{4.33}$$

Finally, by using the boundary condition (2.7) (4.3) and the initial condition (4.4), equation (4.19) yields

$$T_+^{B,0}|_{Z=0} = a_0(x,y,t) - a_0(x,y,t=0), \tag{4.34}$$

$$T_+^{B,0}(t=0) = 0. \tag{4.35}$$

Thus, the lower boundary layer function $T_+^{B,0}(x,y,Z,t)$ satisfies the system

$$\begin{gathered}
\partial_t T_+^{B,0} = \partial_{ZZ} T_+^{B,0},\\
T_+^{B,0}|_{Z=0} = a_0(x,y,t) - a_0(x,y,t=0),\\
T_+^{B,0}(t=0) = 0,\\
T_+^{B,0} \to 0, \ \text{as } Z \to +\infty.
\end{gathered}$$

Now we state the property of the lower boundary layer function.

Proposition 4.3. *Let the assumptions of Theorem 3.1 hold. Then there exists a unique and smooth solution $T_+^{B,0}(x,y,Z,t)$ of the above system satisfying*

$$\|\big(T_+^{B,0}, Z\partial_Z T_+^{B,0}\big)\|_{L^\infty(0,S;L^2(X))} \le C\epsilon^{1/4},$$

for any $t \in [0,S]$, positive constant C independent of ϵ.

Proof. The proof in a straightforward and consists of two steps.

Step 1. First we estimate $\|T_+^{B,0}(x,y,Z,t)\|_{L^\infty(0,S;L^2(X))}$. Multiplying (4.33) by $T_+^{B,0}(x,y,Z,t)$ and integrating over X, we obtain

$$\frac{1}{2}\frac{d}{dt}\|T_+^{B,0}\|^2_{L^2(X)} = \int_X \partial_{ZZ} T_+^{B,0} T_+^{B,0}\, dx\, dy\, dz =: J_1. \tag{4.36}$$

For J_1 we have

$$
\begin{aligned}
J_1 &= \int_{\mathbb{T}^2}\int_0^1 \partial_{ZZ}T_+^{B,0}T_+^{B,0}\,dz\,dy\,dx \\
&= \int_{\mathbb{T}^2}\int_0^\sigma \partial_{ZZ}T_+^{B,0}T_+^{B,0}\,dz\,dy\,dx \\
&= \sqrt{\epsilon}\int_{\mathbb{T}^2}\int_0^{\frac{\sigma}{\sqrt{\epsilon}}} \partial_{ZZ}T_+^{B,0}T_+^{B,0}\,dZ\,dy\,dx \\
&= \sqrt{\epsilon}\int_{\mathbb{T}^2}\big(\partial_Z T_+^{B,0}T_+^{B,0}\big)\big|_{Z=0}^{Z=\frac{\sigma}{\sqrt{\epsilon}}}\,dy\,dx - \sqrt{\epsilon}\int_{\mathbb{T}^2}\int_0^{\frac{\sigma}{\sqrt{\epsilon}}}\big(\partial_Z T_+^{B,0}\big)^2\,dZ\,dy\,dx \\
&= \sqrt{\epsilon}\int_{\mathbb{T}^2}\big(\partial_Z T_+^{B,0}T_+^{B,0}\big)\big|_{Z=0}^{Z=\frac{\sigma}{\sqrt{\epsilon}}}\,dy\,dx - \int_{\mathbb{T}^2}\int_0^{\sigma}\big(\partial_Z T_+^{B,0}\big)^2\,dz\,dy\,dx \\
&= \sqrt{\epsilon}\int_{\mathbb{T}^2}\big(\partial_Z T_+^{B,0}T_+^{B,0}\big)\big|_{Z=0}^{Z=\frac{\sigma}{\sqrt{\epsilon}}}\,dy\,dx - \int_X\big(\partial_Z T_+^{B,0}\big)^2\,dx\,dy\,dz,
\end{aligned}
\tag{4.37}
$$

where we have used integration by parts. Here σ is a sufficiently small positive constant. Then, inserting the estimates derived in (4.37) into (4.36) leads to the inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|T_+^{B,0}\|_{L^2(X)}^2 + \int_X\big(\partial_Z T_+^{B,0}\big)^2\,dx\,dy\,dz \\
&\le \big|\sqrt{\epsilon}\int_0^{2\pi}\int_0^{2\pi}\big(\partial_Z T_+^{B,0}T_+^{B,0}\big)\big|_{Z=0}^{Z=\frac{\sigma}{\sqrt{\epsilon}}}\,dy\,dx\big| \\
&\le C\sqrt{\epsilon}.
\end{aligned}
\tag{4.38}
$$

Finally, integrating (4.38) with respect to t over $[0,t]$, for any $t\in[0,S]$ and any fixed $S>0$, together with (4.35), yields

$$
\|T_+^{B,0}\|_{L^2(X)}^2 + 2\int_0^t\int_X\big(\partial_Z T_+^{B,0}\big)^2\,dx\,dy\,dz d\xi \le C\sqrt{\epsilon},
$$

which implies

$$
\|T_+^{B,0}\|_{L^\infty(0,S;L^2(X))} \le C\epsilon^{1/4}.
$$

Step 2. We now prove an estimate for $\|Z\partial_Z T_+^{B,0}\|_{L^\infty(0,S;L^2(X))}$. First, multiplying (4.33) by $\partial_t T_+^{B,0}$ and integrating it over X, by integrating by parts, yields

$$
\begin{aligned}
0 &= \int_X\Big(\partial_t T_+^{B,0}\Big)^2\,dx\,dy\,dz - \int_X \partial_t T_+^{B,0}\partial_{ZZ}T_+^{B,0}\,dx\,dy\,dz \\
&= \int_X\Big(\partial_t T_+^{B,0}\Big)^2\,dx\,dy\,dz + \frac{1}{2}\frac{d}{dt}\int_X\big(\partial_Z T_+^{B,0}\big)^2\,dx\,dy\,dz \\
&\quad - \int_X \partial_Z(\partial_t T_+^{B,0}\partial_Z T_+^{B,0})\,dx\,dy\,dz.
\end{aligned}
\tag{4.39}
$$

Let us consider the last term on the right-hand side of (4.39), one has

$$\begin{aligned}
&-\int_X \partial_Z(\partial_t T_+^{B,0}\partial_Z T_+^{B,0})\,dx\,dy\,dz\\
&=-\int_{\mathbb{T}^2}\int_0^\sigma \partial_Z(\partial_t T_+^{B,0}\partial_Z T_+^{B,0})\,dz\,dy\,dx\\
&=-\sqrt{\epsilon}\int_{\mathbb{T}^2}\int_0^{\frac{\sigma}{\sqrt{\epsilon}}} \partial_Z(\partial_t T_+^{B,0}\partial_Z T_+^{B,0})\,dZ\,dy\,dx\\
&=-\sqrt{\epsilon}\int_{\mathbb{T}^2}\big(\partial_t T_+^{B,0}\partial_Z T_+^{B,0}\big)\big|_{Z=0}^{Z=\frac{\sigma}{\sqrt{\epsilon}}}\,dy\,dx.
\end{aligned}\tag{4.40}$$

Substituting the above estimate into (4.39), we obtain

$$\begin{aligned}
&\int_X \big(\partial_t T_+^{B,0}\big)^2\,dx\,dy\,dz+\frac{1}{2}\frac{d}{dt}\int_X\big(\partial_Z T_+^{B,0}\big)^2\,dx\,dy\,dz\\
&=\sqrt{\epsilon}\int_{\mathbb{T}^2}\big(\partial_t T_+^{B,0}\partial_Z T_+^{B,0}\big)\big|_{Z=0}^{Z=\frac{\sigma}{\sqrt{\epsilon}}}\,dy\,dx\\
&\le\big|\sqrt{\epsilon}\int_{\mathbb{T}^2}\big(\partial_t T_+^{B,0}\partial_Z T_+^{B,0}\big)\big|_{Z=0}^{Z=\frac{\sigma}{\sqrt{\epsilon}}}\,dy\,dx\big|\\
&\le C\sqrt{\epsilon},
\end{aligned}$$

which leads to

$$\|\partial_t T_+^{B,0}\|_{L^\infty(0,S;L^2(X))}\le C\epsilon^{1/4}.\tag{4.41}$$

Then, one multiplies (4.33) by $Z^2\partial_t T_+^{B,0}$ and integrates it over X, integrating by parts we obtain

$$\begin{aligned}
0&=\int_X Z^2\Big(\partial_t T_+^{B,0}\Big)^2\,dx\,dy\,dz-\int_X Z^2\partial_t T_+^{B,0}\partial_{ZZ}T_+^{B,0}\,dx\,dy\,dz\\
&=\int_X Z^2(\partial_t T_+^{B,0})^2\,dx\,dy\,dz-\int_X\partial_Z(Z^2\partial_t T_+^{B,0}\partial_Z T_+^{B,0})\,dx\,dy\,dz\\
&\quad+\int_X\partial_Z(Z^2\partial_t T_+^{B,0})\partial_Z T_+^{B,0}\,dx\,dy\,dz\\
&=\int_X Z^2(\partial_t T_+^{B,0})^2\,dx\,dy\,dz-\int_X\partial_Z(Z^2\partial_t T_+^{B,0}\partial_Z T_+^{B,0})\,dx\,dy\,dz\\
&\quad+\frac{1}{2}\frac{d}{dt}\int_X Z^2(\partial_Z T_+^{B,0})^2\,dx\,dy\,dz+2\int_X Z\partial_Z T_+^{B,0}\partial_t T_+^{B,0}\,dx\,dy\,dz.
\end{aligned}$$

his can be reduced to

$$\begin{aligned}
&\int_X Z^2\big(\partial_t T_+^{B,0}\big)^2\,dx\,dy\,dz+\frac{1}{2}\frac{d}{dt}\int_X Z^2\big(\partial_Z T_+^{B,0}\big)^2\,dx\,dy\,dz\\
&=\int_{\mathbb{T}^2}\int_0^\sigma\partial_Z(Z^2\partial_t T_+^{B,0}\partial_Z T_+^{B,0})\,dz\,dy\,dx-2\int_X Z\partial_Z T_+^{B,0}\partial_t T_+^{B,0}\,dx\,dy\,dz\\
&=\sqrt{\epsilon}\int_{\mathbb{T}^2}\int_0^{\frac{\sigma}{\sqrt{\epsilon}}}\partial_Z(Z^2\partial_t T_+^{B,0}\partial_Z T_+^{B,0})\,dZ\,dy\,dx-2\int_X Z\partial_Z T_+^{B,0}\partial_t T_+^{B,0}\,dx\,dy\,dz\\
&=\sqrt{\epsilon}\int_{\mathbb{T}^2}(Z^2\partial_t T_+^{B,0}\partial_Z T_+^{B,0})|_{Z=0}^{Z=\frac{\sigma}{\sqrt{\epsilon}}}\,dy\,dx-2\int_X Z\partial_Z T_+^{B,0}\partial_t T_+^{B,0}\,dx\,dy\,dz\\
&\le|\sqrt{\epsilon}\int_{\mathbb{T}^2}(Z^2\partial_t T_+^{B,0}\partial_Z T_+^{B,0})(Z=\frac{\sigma}{\sqrt{\epsilon}})\,dy\,dx|
\end{aligned}$$

$$
\begin{aligned}
&+ |2\int_X Z\partial_Z T_+^{B,0}\partial_t T_+^{B,0}\,dx\,dy\,dz| \\
&\le C\sqrt{\epsilon} + \eta_1\|Z\partial_Z T_+^{B,0}\|^2_{L^2(X)} + C(\eta_1)\|\partial_t T_+^{B,0}\|^2_{L^2(X)},
\end{aligned} \tag{4.42}
$$

where we apply Hölder inequality and Young inequality. η_1 is a small constant, $C(\eta_1) > 0$ is a constant independent of ϵ.

Finally, plugging (4.41) into (4.42) and using Gronwall's inequality that

$$\|Z\partial_t T_+^{B,0}\|_{L^2(0,S;L^2(X))} + \|Z\partial_Z T_+^{B,0}\|_{L^\infty(0,S;L^2(X))} \le C\epsilon^{1/4}.$$

This completes the proof of $\|Z\partial_Z T_+^{B,0}\|_{L^\infty(0,S;L^2(X))} \le C\epsilon^{1/4}$. □

4.3. **Initial layer and upper boundary layer expansion.** Near $z = 1$, $f(z) = 0$ and $h(z) = 1$, instead of $T_B^\epsilon = T^{B,0} = T_+^{B,0}(x, y, Z, t)$ in (4.8), we obtain $T_B^\epsilon = T^{B,0} = T_-^{B,0}(x, y, \overline{Z}, t)$. The upper boundary layer function $T_-^{B,0}\big(x, y, \overline{Z}, t\big)$ has corresponding results with minor difference in some equations.

$$
\begin{gathered}
\begin{aligned}
&\epsilon[\partial_t u_a^\epsilon + (u_a^\epsilon\cdot\nabla)u_a^\epsilon] + \nabla p_a^\epsilon + \frac{1}{Ek}e_3\times u_a^\epsilon - \Delta u_a^\epsilon - Rae_3T_a^\epsilon \\
&- \Delta(u_{In}^\epsilon + u_I^\epsilon) - Rae_3(T_{In}^\epsilon + T_B^\epsilon) \\
&= R_{In,u}^\epsilon + \epsilon[\partial_t u_I^\epsilon + (u_{In}^\epsilon\cdot\nabla)u_I^\epsilon + u_I^\epsilon\cdot\nabla(u_{In}^\epsilon + u_I^\epsilon)] \\
&\quad + \nabla p_I^\epsilon + \frac{1}{Ek}e_3\times u_I^\epsilon - \Delta u_I^\epsilon - Rae_3T_B^\epsilon,
\end{aligned}\\
\nabla\cdot u_a^\epsilon = \nabla\cdot(u_{In}^\epsilon + u_I^\epsilon) = 0,\\
\begin{aligned}
&\partial_t T_a^\epsilon + (u_a^\epsilon\cdot\nabla)T_a^\epsilon - \epsilon\Delta T_a^\epsilon \\
&= R_{In,T}^\epsilon + u_I^\epsilon\cdot\nabla T_{In}^\epsilon + \partial_t T_B^\epsilon + (u_{In}^\epsilon\cdot\nabla)T_B^\epsilon - \epsilon\Delta T_B^\epsilon + u_I^\epsilon\cdot\nabla T_B^\epsilon.
\end{aligned}
\end{gathered}
$$

By a similar method, we have

$$
\begin{aligned}
&\partial_t T_a^\epsilon + (u_a^\epsilon\cdot\nabla)T_a^\epsilon - \epsilon\Delta T_a^\epsilon \\
&= R_{In,T}^\epsilon + (u^{I,0}\cdot\nabla)T^{In,0} + \partial_t T_-^{B,0} + (u^{In,0}\cdot\nabla)T_-^{B,0} - \epsilon\Delta T_-^{B,0} + (u^{I,0}\cdot\nabla)T_-^{B,0},
\end{aligned}
$$

where

$$
\begin{aligned}
&\partial_t T_-^{B,0} + (u^{In,0}\cdot\nabla)T_-^{B,0} - \epsilon\Delta T_-^{B,0} \\
&= \partial_t T_-^{B,0} + (u_1^{In,0}\partial_x + u_2^{In,0}\partial_y)T_-^{B,0} - \frac{1}{\sqrt{\epsilon}}u_3^{In,0}\partial_{\overline{Z}}T_-^{B,0} \\
&\quad - \epsilon(\partial_{xx} + \partial_{yy})T_-^{B,0} - \partial_{\overline{Z}\overline{Z}}T_-^{B,0} \\
&= \partial_t T_-^{B,0} + [(u_1^{In,0}(x, y, 1, t) - \sqrt{\epsilon}\partial_z u_1^{In,0}(z = 1)\overline{Z} + \cdots)\partial_x \\
&\quad + (u_2^{In,0}(x, y, 1, t) - \sqrt{\epsilon}\partial_z u_2^{In,0}(z = 1)\overline{Z} + \cdots)\partial_y]T_-^{B,0} \\
&\quad - \frac{1}{\sqrt{\epsilon}}(u_3^{In,0}(x, y, 1, t) - \sqrt{\epsilon}\partial_z u_3^{In,0}(z = 1)\overline{Z} + \cdots)\partial_{\overline{Z}}T_-^{B,0} \\
&\quad - \epsilon(\partial_{xx} + \partial_{yy})T_-^{B,0} - \partial_{\overline{Z}\overline{Z}}T_-^{B,0}.
\end{aligned}
$$

Here, we also use the Taylor series expansion

$$u^{In,0}(x, y, z, t) = u^{In,0}(x, y, 1-\sqrt{\epsilon}\,\overline{Z}, t) = u^{In,0}(x, y, 1, t) - \sqrt{\epsilon}\partial_z u^{In,0}(z = 1)\overline{Z} + \cdots.$$

Applying a similar approach to the one in Section 4.2, we find that the upper boundary layer function $T_-^{B,0}(x, y, \overline{Z}, t)$. It satisfies

$$\partial_t T_-^{B,0} = \partial_{\overline{Z}\overline{Z}}T_-^{B,0},$$

$$
\begin{gathered}
T_-^{B,0}|_{\overline{Z}=0} = a_1(x,y,t) - a_1(x,y,t=0),\\
T_-^{B,0}(t=0) = 0,\\
T_-^{B,0} \to 0, \quad \text{as } \overline{Z} \to +\infty.
\end{gathered}
$$

Now we state the property of the upper boundary layer function.

Proposition 4.4. *Let the assumptions of Theorem 3.1 hold. Then there exists a unique and smooth solution $T_-^{B,0}(x,y,\overline{Z},t)$ to the above system satisfying*

$$
\|(T_-^{B,0}, \overline{Z}\partial_{\overline{Z}} T_-^{B,0})\|_{L^\infty(0,S;L^2(X))} \le C\epsilon^{1/4},
$$

for any $t \in [0,S]$, C is a positive constant independent of ϵ.

4.4. **Approximate solution.** Now we study approximate solution (4.1). The combination of inner function, initial and boundary layers expansions in Sections 4.1–4.3 implies

$$
\begin{aligned}
&\epsilon[\partial_t u_a^\epsilon + (u_a^\epsilon \cdot \nabla) u_a^\epsilon] + \nabla p_a^\epsilon + \frac{1}{Ek} e_3 \times u_a^\epsilon - \Delta u_a^\epsilon - Rae_3 T_a^\epsilon\\
&= R_{In,u}^\epsilon + (\partial_\tau u^{I,0} + \nabla p^{I,0} + \frac{1}{Ek} e_3 \times u^{I,0} - \Delta u^{I,0})\\
&\quad + \epsilon[(u^{In,0} \cdot \nabla) u^{I,0} + (u^{I,0} \cdot \nabla)(u^{I,0} + u^{In,0})] - Rae_3 T^{B,0}\\
&= R_{In,u}^\epsilon + \epsilon[(u^{In,0} \cdot \nabla) u^{I,0} + (u^{I,0} \cdot \nabla)(u^{I,0} + u^{In,0})] - Rae_3 T^{B,0}\\
&=: R_{In,u}^\epsilon + R_{C,u}^\epsilon,
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_t T_a^\epsilon + (u_a^\epsilon \cdot \nabla) T_a^\epsilon - \epsilon \Delta T_a^\epsilon\\
&= R_{In,T}^\epsilon + (u^{I,0} \cdot \nabla) T^{In,0} + \partial_t T^{B,0} + (u^{In,0} \cdot \nabla) T^{B,0} - \epsilon \Delta T^{B,0} + (u^{I,0} \cdot \nabla) T^{B,0}\\
&= R_{In,T}^\epsilon + (u^{I,0} \cdot \nabla) T^{In,0} + (u^{In,0} \cdot \nabla) T^{B,0} - \epsilon(\partial_{xx} + \partial_{yy}) T^{B,0} + (u^{I,0} \cdot \nabla) T^{B,0}\\
&=: R_{In,T}^\epsilon + R_{C,T}^\epsilon,
\end{aligned}
$$

where $T^{B,0} = f(z)T_+^{B,0} + h(z)T_-^{B,0}$, the remainders $R_{C,u}^\epsilon$ and $R_{C,T}^\epsilon$, caused by the initial layer and the boundary layer, are given exactly by

$$
R_{C,u}^\epsilon = \epsilon[(u^{I,0} \cdot \nabla) u^{In,0} + (u^{In,0} \cdot \nabla) u^{I,0} + (u^{I,0} \cdot \nabla) u^{I,0}] - Rae_3 T^{B,0}, \tag{4.43}
$$

$$
R_{C,T}^\epsilon = (u^{I,0} \cdot \nabla) T^{In,0} + (u^{In,0} \cdot \nabla) T^{B,0} - \epsilon(\partial_{xx} + \partial_{yy}) T^{B,0} + (u^{I,0} \cdot \nabla) T^{B,0}. \tag{4.44}
$$

Therefore, $(u_a^\epsilon, p_a^\epsilon, T_a^\epsilon)$ solves the initial-boundary problem

$$
\begin{aligned}
&\epsilon[\partial_t u_a^\epsilon + (u_a^\epsilon \cdot \nabla) u_a^\epsilon] + \nabla p_a^\epsilon + \frac{1}{Ek} e_3 \times u_a^\epsilon\\
&= \Delta u_a^\epsilon + Rae_3 T_a^\epsilon + R_{In,u}^\epsilon + R_{C,u}^\epsilon,
\end{aligned} \tag{4.45}
$$

$$
\nabla \cdot u_a^\epsilon = 0, \tag{4.46}
$$

$$
\partial_t T_a^\epsilon + (u_a^\epsilon \cdot \nabla) T_a^\epsilon = \epsilon \Delta T_a^\epsilon + R_{In,T}^\epsilon + R_{C,T}^\epsilon, \tag{4.47}
$$

$$
u_a^\epsilon|_{z=0,1} = 0, \tag{4.48}
$$

$$
T_a^\epsilon|_{z=0} = a_0(x,y,t),\ T_a^\epsilon|_{z=1} = a_1(x,y,t), \tag{4.49}
$$

$$
(u_a^\epsilon, T_a^\epsilon)(t=0) = (u_0^0, T_0^0), \tag{4.50}
$$

where the remainders $R^\epsilon_{In,u}$ and $R^\epsilon_{In,T}$ satisfy the estimate (4.14), and $R^\epsilon_{C,u}, R^\epsilon_{C,T}$ defined by (4.43) and (4.44) respectively satisfy

$$\|R^\epsilon_{C,u}(t)\|_{L^2(X)} \le C\epsilon e^{-\lambda\tau} + C\epsilon^{1/4}, \tag{4.51}$$

$$\|R^\epsilon_{C,T}(t)\|_{L^2(X)} \le Ce^{-\lambda\tau} + C\epsilon^{1/4}, \tag{4.52}$$

for some positive constants C and λ. The estimates (4.51), (4.52) can be obtained by (4.43)–(4.44) and Propositions 4.2–4.4.

5. Proof of main results

In this section, we use the classical L^2-energy method to prove Theorem 3.1. Without loss of generality, in the following, we denote C by a positive generic constant independent of ϵ. Noting that C may depend upon S for any fixed $S > 0$. We divide the proof into six steps.

Step 1. We define the error functions

$$u^\epsilon_e = u^\epsilon - u^\epsilon_a,\ p^\epsilon_e = p^\epsilon - p^\epsilon_a, \quad T^\epsilon_e = T^\epsilon - T^\epsilon_a,$$

which satisfies

$$\begin{aligned} &\epsilon[\partial_t u^\epsilon_e + (u^\epsilon_a\cdot\nabla)u^\epsilon_e + (u^\epsilon_e\cdot\nabla)(u^\epsilon_a + u^\epsilon_e)] + \nabla p^\epsilon_e + \frac{1}{Ek}e_3\times u^\epsilon_e \\ &= \Delta u^\epsilon_e + Rae_3T^\epsilon_e - R^\epsilon_{In,u} - R^\epsilon_{C,u}, \end{aligned} \tag{5.1}$$

$$\nabla\cdot u^\epsilon_e = 0, \tag{5.2}$$

$$\partial_t T^\epsilon_e + (u^\epsilon_a\cdot\nabla)T^\epsilon_e + (u^\epsilon_e\cdot\nabla)(T^\epsilon_a + T^\epsilon_e) = \epsilon\Delta T^\epsilon_e - R^\epsilon_{In,T} - R^\epsilon_{C,T}, \tag{5.3}$$

$$u^\epsilon_e|_{z=0,1} = 0, \tag{5.4}$$

$$T^\epsilon_e|_{z=0,1} = 0, \tag{5.5}$$

$$(u^\epsilon_e, T^\epsilon_e)(t=0) = (u^\epsilon_{0E}, 0), \tag{5.6}$$

where $R^\epsilon_{In,u}$, $R^\epsilon_{In,u}$, $R^\epsilon_{C,u}$ and $R^\epsilon_{C,T}$ are the remainders, u^ϵ_{0E} is defined in Section 3.

Step 2. Testing the velocity equation (5.1) by u^ϵ_e and integrating over X with respect to (x,y,z), we obtain

$$\begin{aligned} &\int_X \Big(\epsilon[\partial_t u^\epsilon_e + (u^\epsilon_a\cdot\nabla)u^\epsilon_e + (u^\epsilon_e\cdot\nabla)(u^\epsilon_a + u^\epsilon_e)] + \nabla p^\epsilon_e \\ &\quad + \frac{1}{Ek}e_3\times u^\epsilon_e\Big)u^\epsilon_e\,dx\,dy\,dz \\ &= \int_X \left(\Delta u^\epsilon_e + Rae_3T^\epsilon_e - R^\epsilon_{In,u} - R^\epsilon_{C,u}\right)u^\epsilon_e\,dx\,dy\,dz. \end{aligned} \tag{5.7}$$

First, by the divergence formula, divergence theorem, (4.46), (5.2) and the boundary condition (4.48), (5.4), we deal with the left-hand side terms of (5.7).

$$\int_X \epsilon\partial_t u^\epsilon_e u^\epsilon_e\,dx\,dy\,dz = \frac{\epsilon}{2}\frac{d}{dt}\|u^\epsilon_e\|^2_{L^2(X)},$$

$$\begin{aligned} &\int_X \epsilon(u^\epsilon_a\cdot\nabla)u^\epsilon_e u^\epsilon_e\,dx\,dy\,dz \\ &= \int_X \epsilon\nabla\cdot\Big(u^\epsilon_a\frac{(u^\epsilon_e)^2}{2}\Big)\,dx\,dy\,dz - \int_X \epsilon\nabla\cdot u^\epsilon_a\frac{(u^\epsilon_e)^2}{2}\,dx\,dy\,dz = 0, \end{aligned}$$

$$
\begin{aligned}
&\int_X \epsilon(u_e^\epsilon\cdot\nabla)(u_a^\epsilon+u_e^\epsilon)u_e^\epsilon\,dx\,dy\,dz\\
&=\int_X \epsilon(u_e^\epsilon\cdot\nabla)u_a^\epsilon u_e^\epsilon\,dx\,dy\,dz+\int_X \epsilon\nabla\cdot\Big(u_e^\epsilon\frac{(u_e^\epsilon)^2}{2}\Big)\,dx\,dy\,dz\\
&\quad-\int_X \epsilon\nabla\cdot u_e^\epsilon\frac{(u_e^\epsilon)^2}{2}\,dx\,dy\,dz\\
&=\int_X \epsilon\,(u_e^\epsilon\cdot\nabla)\,u_a^\epsilon u_e^\epsilon\,dx\,dy\,dz\\
&\le\Big|\int_X \epsilon\,(u_e^\epsilon\cdot\nabla)\,u_a^\epsilon u_e^\epsilon\,dx\,dy\,dz\Big|\\
&\le\epsilon\|\nabla u_a^\epsilon\|_{L^\infty(X)}\|u_e^\epsilon\|_{L^2(X)}^2,
\end{aligned}
$$

$$
\int_X \nabla p_e^\epsilon u_e^\epsilon\,dx\,dy\,dz=\int_X\nabla\cdot(p_e^\epsilon u_e^\epsilon)\,dx\,dy\,dz-\int_X\nabla\cdot u_e^\epsilon p_e^\epsilon\,dx\,dy\,dz=0,
$$

$$
\int_X\frac{1}{Ek}e_3\times u_e^\epsilon u_e^\epsilon\,dx\,dy\,dz=\int_X\frac{1}{Ek}(-u_{2e}^\epsilon,u_{1e}^\epsilon,0)(u_{1e}^\epsilon,u_{2e}^\epsilon,u_{3e}^\epsilon)^T\,dx\,dy\,dz=0.
$$

Next, we deal with the right-hand side terms of (5.7):

$$
\begin{aligned}
\int_X\Delta u_e^\epsilon u_e^\epsilon\,dx\,dy\,dz&=\int_X\sum_{i=1}^3(\partial_{xx}+\partial_{yy}+\partial_{zz})u_{ie}^\epsilon u_{ie}^\epsilon\,dx\,dy\,dz\\
&=\int_0^1\int_0^{2\pi}\sum_{i=1}^3\Big(\partial_x u_{ie}^\epsilon u_{ie}^\epsilon|_{x=0}^{x=2\pi}-\int_0^{2\pi}(\partial_x u_{ie}^\epsilon)^2dx\Big)\,dy\,dz\\
&\quad+\int_0^1\int_0^{2\pi}\sum_{i=1}^3\Big(\partial_y u_{ie}^\epsilon u_{ie}^\epsilon|_{y=0}^{y=2\pi}-\int_0^{2\pi}(\partial_y u_{ie}^\epsilon)^2dy\Big)\,dx\,dz\\
&\quad+\int_{\mathbb{T}^2}\sum_{i=1}^3\Big(\partial_z u_{ie}^\epsilon u_{ie}^\epsilon|_{z=0}^{z=1}-\int_0^1(\partial_z u_{ie}^\epsilon)^2dz\Big)\,dx\,dy\\
&=-\int_X(\nabla u_e^\epsilon)^2\,dx\,dy\,dz,
\end{aligned}
$$

$$
\begin{aligned}
\int_X Ra e_3 T_e^\epsilon u_e^\epsilon\,dx\,dy\,dz&\le\Big|\int_X Ra e_3T_e^\epsilon u_e^\epsilon\,dx\,dy\,dz\Big|\\
&\le\eta_2\|u_e^\epsilon\|_{L^2(X)}^2+C(\eta_2)Ra^2\|T_e^\epsilon\|_{L^2(X)}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_X(R_{In,u}^\epsilon+R_{I,u}^\epsilon)u_e^\epsilon\,dx\,dy\,dz&\le\Big|\int_X(R_{In,u}^\epsilon+R_{C,u}^\epsilon)u_e^\epsilon\,dx\,dy\,dz\Big|\\
&\le\eta_3\|u_e^\epsilon\|_{L^2(X)}^2+C(\eta_3)\|R_{In,u}^\epsilon+R_{C,u}^\epsilon\|_{L^2(X)}^2\\
&\le\eta_3\|u_e^\epsilon\|_{L^2(X)}^2+C(\eta_3)\Big(C\epsilon^2e^{-2\lambda\tau}+C\epsilon^{1/2}\Big),
\end{aligned}
$$

where we have used Hölder inequality, Young inequality and estimates (4.14), (4.51). Here $\eta_i>0$, $(i=2,3)$ are small constants, $C(\eta_i)>0$ is a constant which is independent of ϵ.

Then, putting the above equations into (5.7), we obtain that

$$\begin{aligned}&\frac{\epsilon}{2}\frac{d}{dt}\|u_e^\epsilon\|_{L^2(X)}^2+\|\nabla u_e^\epsilon\|_{L^2(X)}^2\\&\le \epsilon\|\nabla u_a^\epsilon\|_{L^\infty(X)}\|u_e^\epsilon\|_{L^2(X)}^2+\eta_2\|u_e^\epsilon\|_{L^2(X)}^2+C(\eta_2)Ra^2\|T_e^\epsilon\|_{L^2(X)}^2\\&\quad+\eta_3\|u_e^\epsilon\|_{L^2(X)}^2+C(\eta_3)\Big(C\epsilon^2e^{-2\lambda\tau}+C\epsilon^{1/2}\Big).\end{aligned}$$

With the help of the Poincaré inequality, restricting ϵ to be sufficiently small such that $\epsilon\|\nabla u_a^\epsilon\|_{L^\infty(X)}\le C\epsilon\le\frac14$ and taking η_2, η_3 to be sufficiently small ($\eta_2+\eta_3=1/4$) but independent of ϵ, one obtains

$$\begin{aligned}&\frac{\epsilon}{2}\frac{d}{dt}\|u_e^\epsilon\|_{L^2(X)}^2+\frac12\|\nabla u_e^\epsilon\|_{L^2(X)}^2\\&\le C(\eta_2)Ra^2\|T_e^\epsilon\|_{L^2(X)}^2+C(\eta_3)\Big(C\epsilon^2e^{-2\lambda\tau}+C\epsilon^{1/2}\Big),\end{aligned}\tag{5.8}$$

which implies

$$\begin{aligned}&\epsilon\frac{d}{dt}\|u_e^\epsilon\|_{L^2(X)}^2+\|u_e^\epsilon\|_{L^2(X)}^2\\&\le 2C(\eta_2)Ra^2\|T_e^\epsilon\|_{L^2(X)}^2+2C(\eta_3)\Big(C\epsilon^2e^{-2\lambda\tau}+C\epsilon^{1/2}\Big),\end{aligned}$$

i.e.,

$$\begin{aligned}&\frac{d}{dt}(e^{\frac{t}{\epsilon}}\|u_e^\epsilon\|_{L^2(X)}^2)\\&\le\Big[2C(\eta_2)Ra^2\|T_e^\epsilon\|_{L^2(X)}^2+2C(\eta_3)\Big(C\epsilon^2e^{-2\lambda\tau}+C\epsilon^{1/2}\Big)\Big]\epsilon^{-1}e^{\frac{t}{\epsilon}}.\end{aligned}\tag{5.9}$$

Integrating (5.9) with respect to t over $[0,t]$ for $t\in[0,S]$ and any fixed $S>0$, we have

$$\begin{aligned}&\|u_e^\epsilon(t)\|_{L^2(X)}^2\le\|u_e^\epsilon(t=0)\|_{L^2(X)}^2+2C(\eta_2)Ra^2\|T_e^\epsilon(t)\|_{L^\infty(0,t;L^2(X))}^2\\&+2C(\eta_3)C\epsilon^{1/2}.\end{aligned}\tag{5.10}$$

Step 3. By performing the L^2-inner product of temperature error equation (5.3) with T_e^ϵ and integrating over X with respect to (x,y,z), we obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\|T_e^\epsilon\|_{L^2(X)}^2\\&=\int_X\epsilon\Delta T_e^\epsilon T_e^\epsilon\,dx\,dy\,dz-\int_X(R_{In,T}^\epsilon+R_{C,T}^\epsilon)T_e^\epsilon\,dx\,dy\,dz\\&\quad-\int_X(u_a^\epsilon\cdot\nabla)\,T_e^\epsilon T_e^\epsilon\,dx\,dy\,dz-\int_X(u_e^\epsilon\cdot\nabla)\,(T_a^\epsilon+T_e^\epsilon)T_e^\epsilon\,dx\,dy\,dz\\&=:J_3+J_4+J_5+J_6.\end{aligned}\tag{5.11}$$

Using Green's first formula and the boundary condition (5.5), we have

$$\begin{aligned}J_3&=\oint\oint_\Gamma\epsilon T_e^\epsilon\frac{\partial T_e^\epsilon}{\partial n}dS-\epsilon\int_X|\nabla T_e^\epsilon|^2\,dx\,dy\,dz\\&=-\epsilon\int_X|\nabla T_e^\epsilon|^2\,dx\,dy\,dz,\end{aligned}\tag{5.12}$$

where Γ denotes the boundary surface.

In view of Hölder inequality, Young inequality and (4.14), (4.52), we have the estimate

$$\begin{aligned} |J_4| &\le \eta_4 \|T_e^\epsilon\|^2_{L^2(X)} + C(\eta_4) \|(R^\epsilon_{In,T} + R^\epsilon_{C,T})\|^2_{L^2(X)} \\ &\le \eta_4 \|T_e^\epsilon\|^2_{L^2(X)} + C(\eta_4)\big(Ce^{-2\lambda\tau} + C\epsilon^{1/2}\big), \end{aligned} \tag{5.13}$$

where $\eta_4 > 0$ is a small constant, and $C(\eta_4) > 0$ is a constant which is independent of ϵ.

It follows from the divergence formula, divergence theorem, (4.46) and the boundary condition (4.48) that

$$\begin{aligned} J_5 &= -\int_X u_a^\epsilon \cdot \nabla\Big(\frac{(T_e^\epsilon)^2}{2}\Big)\, dx\, dy\, dz \\ &= -\int_X \nabla \cdot \Big(u_a^\epsilon \frac{(T_e^\epsilon)^2}{2}\Big)\, dx\, dy\, dz + \int_X \nabla \cdot u_a^\epsilon \frac{(T_e^\epsilon)^2}{2}\, dx\, dy\, dz = 0. \end{aligned} \tag{5.14}$$

Similarly,

$$\begin{aligned} J_6 &= -\int_X (u_e^\epsilon \cdot \nabla)\, T_a^\epsilon T_e^\epsilon\, dx\, dy\, dz - \int_X (u_e^\epsilon \cdot \nabla) T_e^\epsilon T_e^\epsilon\, dx\, dy\, dz \\ &= -\int_X (u_e^\epsilon \cdot \nabla) T_a^\epsilon T_e^\epsilon\, dx\, dy\, dz \\ &\le \Big| -\int_X (u_e^\epsilon \cdot \nabla) T_a^\epsilon T_e^\epsilon\, dx\, dy\, dz \Big| \\ &\le \eta_5 \|u_e^\epsilon\|^2_{L^2(X)} + C(\eta_5) \|\nabla T_a^\epsilon\|^2_{L^\infty(X)} \|T_e^\epsilon\|^2_{L^2(X)}, \end{aligned} \tag{5.15}$$

where we used Hölder inequality and Young inequality. $\eta_5 > 0$ is a small constant, and $C(\eta_5) > 0$ is a constant which is independent of ϵ.

Finally, using (5.12)–(5.15) in (5.11) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|T_e^\epsilon\|^2_{L^2(X)} + \epsilon \int_X |\nabla T_e^\epsilon|^2\, dx\, dy\, dz \\ &\le \eta_4 \|T_e^\epsilon\|^2_{L^2(X)} + C(\eta_4)\Big(Ce^{-2\lambda\tau} + C\epsilon^{1/2}\Big) \\ &\quad + \eta_5 \|u_e^\epsilon\|^2_{L^2(X)} + C(\eta_5)\|\nabla T_a^\epsilon\|^2_{L^\infty(X)} \|T_e^\epsilon\|^2_{L^2(X)}. \end{aligned} \tag{5.16}$$

Step 4. Combining (5.8) and (5.16) yields

$$\begin{aligned} &\frac{\epsilon}{2}\frac{d}{dt}\|u_e^\epsilon\|^2_{L^2(X)} + \frac{1}{2}\int_X (\nabla u_e^\epsilon)^2\, dx\, dy\, dz + \frac{1}{2}\frac{d}{dt}\|T_e^\epsilon\|^2_{L^2(X)} + \epsilon \int_X |\nabla T_e^\epsilon|^2\, dx\, dy\, dz \\ &\le \eta_5 \|u_e^\epsilon\|^2_{L^2(X)} + \eta_4 \|T_e^\epsilon\|^2_{L^2(X)} \\ &\quad + \big[C(\eta_2) Ra^2 + C(\eta_5)\|\nabla T_a^\epsilon\|^2_{L^\infty(X)}\big] \|T_e^\epsilon\|^2_{L^2(X)} \\ &\quad + C(\eta_3)\Big(C\epsilon^2 e^{-2\lambda\tau} + C\epsilon^{1/2}\Big) + C(\eta_4)\Big(Ce^{-2\lambda\tau} + C\epsilon^{1/2}\Big) \\ &\le \eta_5 \|u_e^\epsilon\|^2_{L^2(X)} + C_1 \|T_e^\epsilon\|^2_{L^2(X)} + C_2\big[e^{-2\lambda\tau}(\epsilon^2 + 1) + \epsilon^{1/2}\big], \end{aligned}$$

where $C_1 = \eta_4 + C(\eta_2) Ra^2 + C(\eta_5)\|\nabla T_a^\epsilon\|^2_{L^\infty(X)}$ and $C_2 = (C(\eta_3) + C(\eta_4))C$.

Using the Poincaré inequality and restricting η_5 to be sufficiently small independent of ϵ we have

$$\begin{aligned} &\frac{d}{dt}\Big(\epsilon\|u^\epsilon_{err}\|^2_{L^2(X)}+\|T^\epsilon_{err}\|^2_{L^2(X)}\Big)+\Big(\|\nabla u^\epsilon_{err}\|^2_{L^2(X)}+\epsilon\|\nabla T^\epsilon_{err}\|^2_{L^2(X)}\Big)\\ &\le C\Big(\epsilon\|u^\epsilon_{err}\|^2_{L^2(X)}+\|T^\epsilon_{err}\|^2_{L^2(X)}\Big)+C\big[e^{-2\lambda\tau}(\epsilon^2+1)+\epsilon^{1/2}\big]. \end{aligned} \tag{5.17}$$

Step 5. It follows from (3.2), (5.6) and Gronwall's inequality that

$$\begin{aligned} \epsilon\|u^\epsilon_e\|^2_{L^2(X)}+\|T^\epsilon_e\|^2_{L^2(X)} &\le e^{\int_0^t C d\xi}\big[\epsilon\|u^\epsilon_e(t=0)\|^2_{L^2(X)}+\|T^\epsilon_e(t=0)\|^2_{L^2(X)}\\ &\quad+\int_0^t C(e^{-2\lambda\tau}(\epsilon^2+1)+\epsilon^{1/2})d\xi\big]\\ &\le C\sqrt{\epsilon}, \end{aligned} \tag{5.18}$$

which leads to

$$\epsilon\|u^\epsilon_e\|^2_{L^\infty(0,S;L^2(X))}\le C\epsilon^{1/2},$$

and

$$\|T^\epsilon_e\|_{L^\infty(0,S;L^2(X))}\le C\epsilon^{1/4}. \tag{5.19}$$

Using this inequality in (5.10) yields

$$\|u^\epsilon_e\|_{L^\infty(0,S;L^2(X))}\le C\epsilon^{1/4}. \tag{5.20}$$

Step 6. Using (5.18) in (5.17) and integrating (5.17) with respect to t over $[0,t]$ yields

$$\|\nabla u^\epsilon_e\|^2_{L^2(0,S;L^2(X))}+\epsilon\|\nabla T^\epsilon_e\|^2_{L^2(0,S;L^2(X))}\le C\sqrt{\epsilon}. \tag{5.21}$$

This, with (5.19) and (5.20) imply

$$\|u^\epsilon_e\|_{L^2(0,S;H^1(X))}\le C\epsilon^{1/4}. \tag{5.22}$$

Collecting estimates (5.19)–(5.22), we complete the proof of Theorem 3.1.

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