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# EXISTENCE AND UNIQUENESS OF TRAVELLING WAVEFRONTS FOR A BIO-REACTOR EQUATIONS WITH DISTRIBUTED DELAYS

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Abstract. We consider the diffusive single species growth in a plug flow reactor model with distributed delay. For small delay, existence and uniqueness of such wavefronts are proved when the convolution kernel assumes the strong generic delay kernel. The approaches used in this paper are geometric singular perturbation theory and the center manifold theorem.

## 1. Introduction

There has been considerable interest recently in the system of reaction-diffusion equations

$$\begin{aligned} S_t &= \varepsilon S_{xx} - \alpha S_x - f(S)u \\ u_t &= u_{xx} - \alpha u_x + (f(S) - k)u, \end{aligned} \tag{1.1}$$

as a mathematical model to study some problems in biology and chemical reaction. Most recently (1.1) has been derived in [1] to study a single population microbial growth for a limiting nutrient in a flow reactor, where $\alpha > 0$ is the flow velocity, $S(x,t)$ and $u(x,t)$ are the concentrations of nutrient and microbial population at position $x$ and time $t$, respectively. We refer readers to [1, 7, 10] and the references therein for further details of model description. To best describe this phenomenon [10], we consider an infinitely long flow reactor. Suppose that the amount $S^0$ of nutrient is input at a constant velocity $\alpha$ at one end of the flow reactor, says at $x = -\infty$. On the other hand, assume that the nutrient uptake function $f$ satisfies $f(0) = 0$, $f' > 0$ and $f(S^0) > k$ (see [10]), where $k > 0$ is the cell death rate. We naturally expect that the nutrient should be sufficient for growth upstream of the pulse and be depleted below the level at which bacteria can grow downstream of the pulse. Hence one many expect that a hump-shaped bacteria population density $u(x,t)$ moves towards the other end of reactor. That is, we expect that there are constants $c$, $S_0$ with $f(S_0) < k$, and nonnegative travelling wavefronts $S(x,t) = S(x+ct)$ and $u(x,t) = u(x+ct)$ satisfy

$$S(-\infty) = S^0, \quad u(-\infty) = 0, \quad S(+\infty) = S_0 < S^0, \quad u(+\infty) = 0. \tag{1.2}$$

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The equations satisfied by $S(z)$ and $u(z)$, where $z=x+ct$ are

$$\begin{aligned} 0&=\varepsilon S''-(\alpha+c)S'-f(S)u\\ 0&=u''-(\alpha+c)u'+(f(S)-k)u. \end{aligned}\tag{1.3}$$

From (1.2) and (1.3), we get $S_0$ must satisfy

$$(\alpha+c)(S^0-S_0)=k\int_{-\infty}^{+\infty}u(z)dz. \tag{1.4}$$

The question of the existence of travelling wavefronts of (1.1) and (1.2) has recently been solved that can be summarized as follows.

**Proposition 1.1.** *Assume $\varepsilon\ge 0$, $k>0$ are constants, and suppose that $f$ satisfies $f(0)=0$, $f'>0$ and $f(S^*)=k$ for some positive number $S^*$. Then, given $S^0>S^*$ and there is a unique $S_0\in(0,S^*)$ such that (1.1) has a travelling wavefronts $S(x+ct),u(x+ct)$ satisfying the boundary condition (1.2) if and only if $c+\alpha\ge C^*:=\sqrt{4(f(S^0)-k)}$. Moreover, $S(z)$ is strictly decreasing and $u(z)$ is strictly positive for $z\in\mathbb{R}$.*

The objective of the present paper is to address the question of the existence and uniqueness of travelling wavefronts solution of the following more general version of the system (1.1) with $\varepsilon=0$,

$$\begin{aligned} S_t&=-\alpha S_x-f(S)(g*u)\\ u_t&=u_{xx}-\alpha u_x+(f(S)-k)(g*u), \end{aligned}\tag{1.5}$$

where the convolution $g*u$ is defined by

$$(g*u)(x,t)=\int_{-\infty}^{t}g(t-s)u(x,s)ds \tag{1.6}$$

and the kernel $g:[0,\infty)\to[0,\infty)$ satisfies

$$g(t)\ge 0,\quad \forall t\ge 0\quad \text{and}\quad \int_0^\infty g(t)dt=1. \tag{1.7}$$

The delay kernel $g$ are frequently of the form

$$g(t)=\delta(t-\tau),\quad g(t)=\frac{1}{\tau}e^{-t/\tau},\quad g(t)=\frac{t}{\tau^2}e^{-t/\tau}. \tag{1.8}$$

In each of these cases, the parameter $\tau>0$ measures the delay. The first of these kernels gives rise to a model having a discrete time-delay, where $\delta$ denotes Dirac's delta function. The other two kernels in (1.8) are called weak and strong generic delay kernels. The "weak" case $g(t)=\frac{1}{\tau}e^{-t/\tau}$ reflects the idea that the importance of the past decreases exponentially the further one looks into the past. The "strong" case $g(t)=\frac{t}{\tau^2}e^{-t/\tau}$ can be regarded as a smoothed out version of the discrete delay case $g(t)=\delta(t-\tau)$. This strong kernel implies that a particular time in the past, namely $\tau$ time units ago, is more important than any other since kernel achieves its unique maximum at $t=\tau$.

The remaining part of this paper is organized as follows. Section 2 is devote to some preliminary discussion mainly focus on the particular case of the kernel. In Section 3, we establish the existence and uniqueness of travelling wavefronts solutions when $\tau$ is small. Geometrical singular perturbation theory and the center manifold theory play a major role in the proofs.

## 2. Preliminaries

The purpose of this section is to establish propositions that will serve main proof of the existence and uniqueness of the travelling wavefront.

At first, the results on travelling fronts for the non-delay equation is needed. (1.3) with $\varepsilon = 0$ can read as a system of first-order equations

$$\begin{aligned} u'' &= (c+\alpha)u' - (f(S)-k)u \\ S' &= -f(S)u/(c+\alpha) \end{aligned} \tag{2.1}$$

which is equivalent to (see [10])

$$\begin{aligned} u' &= (c+\alpha)[-G(S^0) + u + G(S)] \\ S' &= -f(S)u/(c+\alpha) \end{aligned} \tag{2.2}$$

where

$$G(S) = S - k\int_{S^*}^{S} \frac{1}{f(s)}ds, \quad S > 0. \tag{2.3}$$

The function $G$ satisfies

$$G(0_+) = G(+\infty) = +\infty, \quad G(S^*) = S^*, \quad G'(S) = \frac{f(S)-k}{f(S)}, \quad G''(S) > 0. \tag{2.4}$$

The following lemma yields the existence of a travelling wavefront solution of the non-delay equation (2.2).

**Lemma 2.1** ([10]). *If $c+\alpha \geq C^* := \sqrt{4(f(S^0)-k)}$, then in the $(S,u)$ phase plane, a heteroclinic connection exists between the critical points $(S,u) = (S^0, 0)$ and $(S_0, 0)$ for $S^0 > S^*$, $G(S_0) = G(S^0)$ and $S(\cdot)$ is strictly decreasing and $u(\cdot)$ is positive and unimodal.*

We return now to the delay equation (1.5). The travelling wavefronts is a solution of the form $S(x,t) = S(z)$, $u(x,t) = u(z)$, where $z = x + ct$ and $c > 0$ is called wave speed, satisfies

$$\begin{aligned} 0 &= -(\alpha+c)S' - f(S)(g*u) \\ 0 &= u'' - (\alpha+c)u' + (f(S)-k)(g*u) \end{aligned} \tag{2.5}$$

with

$$(g*u)(z) = \int_0^{\infty} g(w)u(z-cw)dw. \tag{2.6}$$

We shall seek leftward-moving waves, thus we take

$$S(-\infty) = S^0, \quad S(+\infty) = S_0, \quad u(-\infty) = 0, \quad u(+\infty) = 0. \tag{2.7}$$

Next, we shall analyze (2.5) for travelling wavefronts in the particular case when the kernel $g$ is the third of (1.8), the strong generic delay case. The corresponding calculations for the weak kernel are similar and will be omitted. Recall that the parameter $\tau$ measures the delay. It is useful to reference (2.5) as

$$\begin{aligned} u'' &= (c+\alpha)u' - (f(S)-k)(g*u) \\ S' &= -\frac{f(S)}{(c+\alpha)}(g*u). \end{aligned} \tag{2.8}$$

Thus

$$g(t) = \frac{t}{\tau^2}e^{-t/\tau}, \quad \tau > 0,$$

and we define
$$p(z)=(g*u)(z)=\int_0^\infty \frac{t}{\tau^2}e^{-t/\tau}u(z-ct)dt.$$
Differentiating with respect to $z$, we obtain
$$\frac{dp}{dz}=\frac{1}{c\tau}(p-q),$$
where
$$q(z)=\int_0^\infty \frac{1}{\tau}e^{-t/\tau}u(z-ct)dt.$$
Similarly,
$$\frac{dq}{dz}=\frac{1}{c\tau}(q-u).$$
If we further denote $u'=v$, then (2.8) with the kernel given above can be replaced by the system
$$\begin{gathered}
u'=v\\
v'=(c+\alpha)v-(f(S)-k)p\\
S'=-\frac{f(S)}{(c+\alpha)}p\\
c\tau p'=p-q\\
c\tau q'=q-u,
\end{gathered}\tag{2.9}$$
Note that if $\tau=0$, then (2.9) reduces to
$$\begin{gathered}
u'=v\\
v'=(c+\alpha)v-(f(S)-k)u\\
S'=-f(S)u/(c+\alpha),
\end{gathered}\tag{2.10}$$
the autonomous ordinary differential system for travelling wavefronts solutions of (1.5) in the non-delay case.

For $\tau>0$, (2.9) defines a system of ODEs whose solutions evolve in the five-dimensional $(u,v,S,p,q)$ phase space. In this phase space, $E=\{(0,0,S,0,0)\}$ is the one-dimensional manifold of critical for (2.9). A travelling wavefronts solution of the (2.8) will exist if among the solutions of (2.9), there exists a heteroclinic connection between the two critical points in $E$.

Then, we will show that (2.9) has travelling wavefronts for sufficiently small $\tau>0$ by the geometric singular perturbation theory and the center manifold theorem. Note that when $\tau=0$, system (2.9) does not define a dynamical system in $\mathbb{R}^5$. This problem may be overcome by the transformation $z=\tau\eta$, under which the system becomes
$$\begin{gathered}
\dot u=\tau v\\
\dot v=\tau[(c+\alpha)v-(f(S)-k)p]\\
\dot S=\tau[-\frac{f(S)}{(c+\alpha)}p]\\
c\dot p=p-q\\
c\dot q=q-u,
\end{gathered}\tag{2.11}$$
where a dot on top of a variable denotes differentiation with respect to $\eta$. We refer to (2.9) as the slow system and (2.11) as the fast system. The two are equivalent when $\tau>0$.

Consider the fast system (2.11), for $\tau = 0$, then the flow of that system is confined to the set

$$M_0 = \{(u, v, S, p, q) \in \mathbb{R}^5 : p = u,\ q = u\}, \tag{2.12}$$

which is, therefore, a three-dimensional invariant manifold for (2.9). $E \subset M_0$ and an easy calculation shows that the eigenvalues of the Jacobian, on setting $\tau = 0$, has 3 zero eigenvalues corresponding the tangent space of $M_0$ and two same positive eigenvalues, namely $\frac{1}{c}$. Thus, $M_0$ is normally hyperbolic manifold.

According to Fenichel's Invariant Manifold Theorem (see [4], [8]), there exist a locally invariant three-dimensional manifold $M_\tau$ with $\tau$ is sufficiently small. It can be written in the form

$$M_\tau = \big\{(u, v, S, p, q) \in \mathbb{R}^5 : p = u + \widetilde{h}_1(u, v, S, \tau),\ q = u + \widetilde{h}_2(u, v, S, \tau)\big\}, \tag{2.13}$$

where the functions $\widetilde{h}_1$, $\widetilde{h}_2$ are smooth functions defined on a compact domain, and satisfies $\widetilde{h}_1(u, v, S, 0) = \widetilde{h}_2(u, v, S, 0) = 0$ and thus that

$$\widetilde{h}_1(u, v, S, \tau) = \tau\overline{h}_1(u, v, S, \tau),\ \widetilde{h}_2(u, v, S, \tau) = \tau\overline{h}_2(u, v, S, \tau). \tag{2.14}$$

Since $\tau$ is small, $\widetilde{h}_1, \widetilde{h}_2$ can be expanded into the form of Taylor series about $\tau$, and $\overline{h}_1, \overline{h}_2$ can express as

$$\begin{gathered}\overline{h}_1(u, v, S, \tau) = \overline{h}_1^1(u, v, S) + \tau\overline{h}_1^2(u, v, S) + \dots,\\ \overline{h}_2(u, v, S, \tau) = \overline{h}_2^1(u, v, S) + \tau\overline{h}_2^2(u, v, S) + \dots.\end{gathered} \tag{2.15}$$

By substituting (2.14) into (2.9), we see that $\overline{h}_1, \overline{h}_2$ must satisfy

$$c(v+\tau(\frac{\partial\overline{h}_1}{\partial u}v+\frac{\partial\overline{h}_1}{\partial v}((c+\alpha)v-(f(S)-k)(u+\tau\overline{h}_1))-\frac{\partial\overline{h}_1}{\partial S}\frac{f(S)}{(c+\alpha)}(u+\tau\overline{h}_1))) = \overline{h}_1-\overline{h}_2$$

and

$$c(v+\tau(\frac{\partial\overline{h}_2}{\partial u}v+\frac{\partial\overline{h}_2}{\partial v}((c+\alpha)v-(f(S)-k)(u+\tau\overline{h}_1))-\frac{\partial\overline{h}_2}{\partial S}\frac{f(S)}{(c+\alpha)}(u+\tau\overline{h}_1))) = \overline{h}_2$$

Substitute (2.15) into the above two equations and comparing powers of $\tau$ yields, we obtain

$$\begin{gathered}\overline{h}_1(u, v, S, \tau) = 2cv + 3\tau c^2((c+\alpha)v - (f(S) - k)u) + \dots,\\ \overline{h}_2(u, v, S, \tau) = cv + \tau c^2((c+\alpha)v - (f(S) - k)u) + \dots.\end{gathered} \tag{2.16}$$

We study the flow of (2.9) restricted to $M_\tau$ and show that it has a travelling front solution. The slow system (2.9) restricted to $M_\tau$ is given by

$$\begin{gathered}u' = v\\ v' = (c+\alpha)v - (f(S) - k)(u + \tau\overline{h}_1(u, v, S, \tau))\\ S' = -\frac{f(S)}{c+\alpha}(u + \tau\overline{h}_1(u, v, S, \tau)).\end{gathered} \tag{2.17}$$

which is equal to

$$\begin{gathered}u' = v\\ v' = (c+\alpha)v - (f(S) - k)u + \tau h_1(u, v, S, \tau)\\ S' = -\frac{f(S)}{c+\alpha}u + \tau h_2(u, v, S, \tau),\end{gathered} \tag{2.18}$$

where $h_1(u,v,S,\tau) = -(f(S)-k)\overline{h}_1(u,v,S,\tau)$, $h_2(u,v,S,\tau) = -\frac{f(S)}{c+\alpha}\overline{h}_1(u,v,S,\tau)$. Note that when $\tau = 0$, this system reduces to the corresponding system for the non-delay (2.10). It is easily verified that for $\tau > 0$, system (2.18) still has one-dimensional manifold of critical $E = (0,0,S)$.

## 3. Main results

In this section, we discuss the existence and uniqueness of travelling wavefronts solutions of (1.5)

The ideas of the following proof are similar to those of Smith and Zhao [10] who were considering the question of persistence of travelling wavefronts solutions in an equation with a fourth-order spatial derivative but no time delay.

Note that system (2.10) is equivalent to (2.1). According to Lemma 2.1 and [10], for $0 < S_0 < S^*$ and $c+\alpha > 0$, the positive branch of the one-dimensional stable manifold of $(0,0,S_0)$ for system (2.10), $W_0^s(S_0)$, connect to $(0,0,S^0)$, where $G(S_0) = G(S^0) > S^*$. We want to show that for $\tau > 0$ but very small, the positive branch of one-dimensional stable manifold of $(0,0,S_0)$ for system (2.18), $W_\tau^s(S_0)$, also connects to $(0,0,S^0)$. We may describe the local stable manifold as the forward orbit $\{x_\tau(z) : z \geq 0\}$ of (2.18) through a point $x_\tau := x_\tau(0)$ on the local stable manifold, which depends continuously on $\tau$, and by a compact piece of the global stable manifold we mean $\{x_\tau(z) : z \geq -Z\}$ ($Z \gg 1$), with endpoint $x_\tau(-Z)$. We expect that such a compact piece of $W_\tau^S(S_0)$ has endpoint nearby $(0,0,S^0)$ for small $\tau > 0$. The next result indicates what happens to the backward orbit through this endpoint.

**Lemma 3.1.** *Given $S^0 > S^*$ and $\delta_0 > 0$, there exists $\tau_0$, $\delta > 0$ such that if $\xi = (u,v,S)$ satisfies $|\xi - (0,0,S^0)| < \delta$ and $0 \leq \tau < \tau_0$, then the solution of starting at $\xi$, $x^\tau(z) = (u^\tau(z), v^\tau(z), S^\tau(z))$, satisfies $|x^\tau(z) - (0,0,S^0)| < \delta_0$ for all $z < 0$, and there exist $\beta^\tau = (0,0,S^\tau)$ such that $x^\tau(z) \to \beta^\tau$ as $z \to -\infty$.*

*Proof.* Appending an equation for $\tau$ to (2.18), we shall argument the system (2.18) with equation for $\tau$.

$$\begin{gathered} u' = v \\ v' = (c+\alpha)v - (f(S)-k)u + \tau h_1(u,v,S,\tau) \\ S' = -\frac{f(S)}{c+\alpha}u + \tau h_2(u,v,S,\tau) \\ \tau' = 0 \end{gathered} \tag{3.1}$$

We apply the center manifold theory in [2] to the time reversed system (3.1). Note that this four-dimensional system has the two-dimensional manifold of critical given by $N = \{u = v = 0\}$. Focus on one of steady states $N^0 = (0,0,S^0,0)$. A change of variables $S \to S_1$ given by

$$S_1 = S - S^0 + \frac{f(S^0)}{r}(u - \frac{v}{c+\alpha}), \quad r = f(S^0) - k. \tag{3.2}$$

Translates $N^0$ to the origin and de-couples the linear parts of the time reversed system (3.1). Then resulting system is

$$\begin{aligned} u' &= -v \\ v' &= ru - (c+\alpha)v + (f(S) - f(S^0))u - \tau h_1(u,v,S,\tau) \\ S_1' &= \frac{1}{(c+\alpha)r}(f(S) - f(S^0))u - \tau h_2(u,v,S,\tau) \\ \tau' &= 0 \end{aligned} \tag{3.3}$$

where $S$ is determined by (3.2). We let $x = (S_1, \tau)$ and $y = (u, v)$, then (3.3) has the form

$$\begin{aligned} x' &= Ax + f(x,y) \\ y' &= By + g(x,y), \end{aligned} \tag{3.4}$$

where $A$ is the zero matrix and $B = \begin{pmatrix} 0 & -1 \\ r & -C \end{pmatrix}$ where $C = c+\alpha$, all the eigenvalues of $B$ have negative real parts,

$$f(x,y) = f(u,v,S_1,\tau) = \begin{pmatrix} \frac{f(S)-f(S^0)}{Cr}u - \tau h_2(u,v,S,\tau) \\ 0 \end{pmatrix}$$

and

$$g(x,y) = g(u,v,S_1,\tau) = \begin{pmatrix} 0 \\ (f(S) - f(S^0))u - \tau h_1(u,v,S,\tau)) \end{pmatrix}.$$

We have $f(0,0) = 0$, $f'(0,0) = 0$, and $g(0,0) = 0$, $g'(0,0) = 0$. [2, Theorem 1] asserts there exists a center manifold for (3.3), but we already know that the center manifold which is unique in our case, is just the manifold of critical $N$ (see [10]). The dynamical on $N$ is trivial:

$$\begin{aligned} S_1' &= 0 \\ \tau' &= 0, \end{aligned} \tag{3.5}$$

Since critical point $(S_1, \tau) = (0,0)$ is stable for the dynamics on $N$, from [2, throem 2], we get that the origin is stable for (3.3). Furthermore, by the second assertion of [2, Theorem 2], a solution $(u(z), v(z), S_1(z), \tau)$ of (3.3) which start $(0,0,S_1^0,\tau)$ near the origin, such that as $z \to +\infty$,

$$u(z) = O(e^{-\gamma z}), \quad v(z) = O(e^{-\gamma z}), \quad S_1(z) = S_1^0 + O(e^{-\gamma z}),$$

where $\gamma > 0$. Thus, we get $S(z) = S^0 + S_1^0 + O(e^{-\gamma z})$. This is exactly what we assert above. □

Now we prove the main results in this section.

**Theorem 3.2.** *Let $S_0$ satisfy $0 < S_0 < S^*$ and let $S^0 > S^*$ satisfy $G(S^0) = G(S_0)$. If $\tau > 0$ is sufficiently small and $c+\alpha > 0$ the system (1.5) has a unique travelling wavefronts solution $(S(z), u(z))(z = x + ct)$ connecting $(S^0, 0)$ and $(S_0, 0)$ with $u(z) > 0$ for $z \approx +\infty$.*

*Proof.* For $0 < S_0 < S^*$, by [8, Fenichel Invariant Manifold Theorem 2], "compact pieces" of the positive branch of the one-dimensional stable manifold of $(0,0,S_0)$ for (2.18), $W_\tau^S(S_0)$, lie within $O(\tau)$ of, and are diffeomorphic to $W_0^S(S_0)$. But $W_0^S(S_0)$ connects $(0,0,S_0)$ to $(0,0,S^0)$ by Lemma 2.1. If $S^0$ satisfies $G(S^0) = G(S_0)$, $\delta_0$ satisfies $0 < \delta_0 < \frac{1}{2}(S^0 - S^*)$ and $\delta > 0$ is as in Lemma 3.1, then there exists $\tau_1 > 0$, such that for all $\tau \in [0, \tau_1)$, a compact piece of $W_\tau^S(S_0)$ has end point with in distance $\delta$ of $(0,0,S^0)$. We can assume that $\tau_1 < \tau_0$ of Lemma 3.1 and so, according to Lemma 3.1, the backward continuation of the compact piece of $W_\tau^S(S_0)$ is asymptotic to a point $\beta^\tau = (0,0,S^\tau)$ satisfying $|\beta^\tau - (0,0,S^0)| < \delta_0$. Thus, we have shown the existence of a heteroclinc orbit for (2.18) connecting $(0,0,S_0)$ to $\beta^\tau$. That is there exists a heteroclinc orbit of (2.9) connecting $(0,0,\widehat{S},0,0)$ to $(0,0,S_0,0,0)$ where $\beta^\tau = (0,0,\widehat{S})$.

Next, we prove that $\widehat{S} = S^0$. As in (1.4), we have

$$(\alpha + c)(\widehat{S} - S_0) = k\int_{-\infty}^{+\infty} (g * u)(z)dz = k\int_{-\infty}^{+\infty} p(z)dz. \tag{3.6}$$

From then third equation of (2.9) we find that

$$p = -\frac{(c+\alpha)S'}{f(S)},$$

which, substituting into (3.6) and integrating, lead to

$$G(\widehat{S}) = G(S_0), \tag{3.7}$$

where $G$ is defined by (2.3), by (2.4) we get that $\widehat{S}(\tau) = S^0$. Consequently, the heteroclinc orbit of (2.9) connecting $(0,0,S^0,0,0)$ to $(0,0,S_0,0,0)$ □

**Remark 3.3.** The travelling wave solution described in the Theorem 3.2 depends on $\tau$ and $c + \alpha$.

Note that we make no assertions about the signs of $u$ and $S'$. In the next theory, we take up these issues.

**Theorem 3.4.** *Let $S_0$ satisfies $0 < S_0 < S^*$ and $c + \alpha > C^* := \sqrt{4(f(S^0) - k)}$. If $\tau > 0$ is sufficiently small, then the travelling wavefronts solution described in Theorem 3.2, $(S(x+ct), u(x+ct))$, has the property that $S(\cdot)$ is strictly decreasing and $u(\cdot)$ is positive and unimodal.*

*Proof.* For $\tau = 0$, we can get (2.10) is the same as (2.1) and from the second equation of (2.1), $u(z), u'(z)$ satisfies

$$\frac{u'(z)}{u(z)} = -\frac{u'(z)}{S'(z)}\frac{f(S(z))}{c+\alpha}.$$

Letting $z \to -\infty$, from [2, Corollary 2.1], the ratio approaches

$$2(\frac{r^0}{c+\alpha})[\frac{1}{1+(1-\chi)^{\frac{1}{2}}}] < 2\frac{r^0}{c+\alpha} < \frac{c+\alpha}{2},$$

where $\chi = \frac{4(f(S^0)-k)}{(c+\alpha)^2}$ and we use $\frac{(c+\alpha)^2}{4} > r^0 = f(S^0) - k$ in the last inequality. If $Z$ is sufficiently large, then $u(-Z), u'(-Z) > 0$ and

$$\frac{u'(-Z)}{u(-Z)} < \frac{c+\alpha}{2}.$$

By continuity, for $\tau > 0$ sufficiently small, we have that $u > 0$ along the part of the heteroclinic orbit which lies outside the small $\delta-$neighborhood of $(0,0,S^0)$ identified in Lemma 3.1. By choosing $Z$ larger if necessary, we can assume that

$u(z), v(z), S(z)$ belongs to the $\delta-$neighborhood of $(0,0,S^0)$ for $z < -Z$, that $u(z) > 0$ for $-Z \le z < \infty$, $v(-Z) > 0$ and that

$$\frac{v(-Z)}{u(-Z)} < \frac{c+\alpha}{2}. \tag{3.8}$$

We wish to show that $u(z) > 0$ for all $z$. Therefore, it is only necessary to consider $(u(z), v(z), S(z))$ for $z \le -Z$.

It is useful to reverse "time" by setting $z \to -z$, then we consider the heteroclinic orbit for $(u(z), v(z), S(z))$ for $z \ge Z$, which belongs to the $\delta-$neighborhood of $(0,0,S^0)$. Now we replacing $(u,v)$ in (2.18) by polar coordinates $(\rho,\theta)$, then we get

$$\begin{gathered}
\rho^2\theta' = -(c+\alpha)uv + ru^2 + v^2 - \tau u h_1 \\
\rho\rho' = -(1-r)uv - (c+\alpha)v^2 - \tau v h_1 \\
S' = \frac{f(S)}{c+\alpha}u - \tau h_2,
\end{gathered} \tag{3.9}$$

where $r = f(S) - k$ depend on $S(z)$. We are interested in (3.9) for $z \ge Z$ where $S(z) - S^0$ is so small that $\frac{(c+\alpha)^2}{4} - r > 0$. By (3.8), we see that $(u(Z), v(Z))$ belong to the open first quadrant and that

$$0 < \theta(Z) = \tan^{-1}(\frac{v(Z)}{u(Z)}) < \theta_0 := \tan^{-1}(\frac{c+\alpha}{2}).$$

If $\theta(z) = 0$(i.e., $v = 0$), the first equation of (3.9) become $\rho^2\theta' = ru^2 - \tau u h_1$ substituting (3.8) into it and we get $\rho^2\theta' = [r - 3\tau^2c^2r^2]u^2 + O(\tau^3)$, for $\tau$ is sufficiently small, the sign of $\theta'(z)$ depend on $r$, thus $\theta'(z) > 0$ whenever $\theta(z) = 0$. If $\theta(z) = \theta_0$(i.e., $v = \frac{c+\alpha}{2}u$), the first equation of (3.9) become $\rho^2\theta' = -(\frac{(c+\alpha)^2}{4} - r)u^2 - \tau u h_1$ the same way we get the sign of $\theta'(z)$ depend on $-(\frac{(c+\alpha)^2}{4} - r)$, thus $\theta'(z) < 0$ whenever $\theta(z) = \theta_0$. Thus, $0 \le \theta(z) \le \theta_0$ for all $z \ge Z$ and, in particular, $u(z) > 0$ for $z \ge Z$. Thus, $u(z) > 0$ for all $z$.

For the third equation of (2.18) substituting (2.16) into it and we get

$$S' = -f(S)u/(c+\alpha) - \tau v f(S) + O(\tau^2) = -f(S)(\frac{u}{c+\alpha} + \tau v) + O(\tau^2)$$

Since $u > 0$, $\tau$ is sufficiently small, we have $S' < 0$ for all $z \in \mathbb{R}$.

From [2], we can get (2.9) is equals to

$$\begin{gathered}
u' = (c+\alpha)[-G(S^0) + u + G(S)] \\
S' = -\frac{f(S)}{c+\alpha}p \\
c\tau p' = p - q \\
c\tau q' = q - u
\end{gathered} \tag{3.10}$$

where $G$ is defined by (2.3). It has two critical points $(0, S^0, 0, 0)$ and $(0, S_0, 0, 0)$. The linearized matrix $J$ of system (3.10) is

$$J(u,S,p,q) = \begin{pmatrix} c+\alpha & -(c+\alpha)G'(S) & 0 & 0 \\ 0 & \frac{f'(S)}{c+\alpha}p & -\frac{f(S)}{c+\alpha} & 0 \\ 0 & 0 & \frac{1}{c\tau} & -\frac{1}{c\tau} \\ -\frac{1}{c\tau} & 0 & 0 & \frac{1}{c\tau} \end{pmatrix} \tag{3.11}$$

The eigenvalues $\lambda$ of this matrix at critical points satisfy

$$(c\tau)^2\lambda^4 - ((c\tau)^2(c+\alpha) + 2c\tau)\lambda^3 + (1 + 2c\tau(c+\alpha))\lambda^2 - (c+\alpha)\lambda - (f(S) - k) = 0$$

At critical point $(0, S^0, 0, 0)$, for $S^0 > S^*$, sufficiently small $\tau$ and $c > 0$, this equation has four positive real part. At critical point $(0, S_0, 0, 0)$, for $S_0 < S^*$, this equation has there positive real part and one negative real part. Then the heteroclinic orbit of (3.10) approaches $(0, S_0, 0, 0)$ tangent to the eigenvector corresponding to the negative eigenvalue $\lambda_-$. An easy calculation of the eigenvector on $u - S$ phase plane leads to its slope:

$$\frac{\triangle u}{\triangle S} = \frac{(c+\alpha)G'(S)}{\lambda_- - (c+\alpha)} > 0$$

Since $S' < 0$, then $v = u'$ is negative when $z$ is very close to $+\infty$. As $u(\pm\infty) = 0$, $v = u'$ admits at least one zero. By the first equation of (3.10) and (2.17), it follows that

$$\begin{aligned} v' &= u'' \\ &= (c+\alpha)G'(S)S' = (k - f(S))p \\ &= (k - f(S))u + O(\tau^2) \quad \text{whenever } v(z) = 0. \end{aligned} \tag{3.12}$$

Let $z_0$ be the largest zero of $v$. Then $v'(z_0) \le 0$ and $v < 0$, hold for any $z > z_0$. Suppose $v'(z_0) = 0$, for $\tau$ is sufficiently small, then (3.12) implies that $v''(z_0) = -f'(S)S'(z_0)u(z_0) + O(\tau^2) > 0$, hence $v(z_0) = 0$ is the local minimum of $v(z)$ around $z_0$, which contradicts the choice of $z_0$. Hence, we get $v'(z_0) < 0$ i.e., $S(z_0) > S^*$. Since $S'(z) < 0$, for all $z \in \mathbb{R}$, we get $S(z) > S^*$ hold for any $z \in (-\infty, z_0)$, hence $v' < 0$ hold for any $z \in (-\infty, z_0)$. Thus, $v(z)$ admits no zero in $(-\infty, z_0)$ and $v > 0$. So $v = u'$ has precisely one zero $z_0$ and $v(z) > 0$ for all $z \in (-\infty, z_0)$ and $v(z) < 0$ for all $z \in (z_0, \infty)$. Hence $u(z)$ is positive and unimodal. □

**Remark 3.5.** We have considered travelling wavefronts of a plug flow reactor model (1.1) with $\varepsilon = 0$, and with distributed delay in the form of an integral convolution in time, mainly using strong generic kernel. It should certainly be applicable in principle to (1.1) with $\varepsilon > 0$ involving time delay

$$\begin{aligned} S_t &= \varepsilon S_{xx} - \alpha S_x - f(S)(g * u) \\ u_t &= u_{xx} - \alpha u_x + (f(S) - k)(g * u) \end{aligned} \tag{3.13}$$

and other coupled system.

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