Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 99, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

AVERY FIXED POINT THEOREM APPLIED TO HAMMERSTEIN INTEGRAL EQUATIONS

PAUL W. ELOE, JEFFREY T. NEUGEBAUER

Abstract. We apply a recent Avery et al. fixed point theorem to the Hammerstein integral equation

$$x(t) = \int_{T_1}^{T_2} G(t,s) f(x(s))\,\mathrm{d}s, \quad t \in [T_1, T_2].$$

Under certain conditions on G, we show the existence of positive and positive symmetric solutions. Examples are given where G is a convolution kernel and where G is a Green's function associated with different boundary-value problem.

1. Introduction

Let $T_1, T_2 \in \mathbb{R}$ with $T_1 < T_2$. Consider the Hammerstein integral equation

$$x(t) = \int_{T_1}^{T_2} G(t,s) f(x(s))\,\mathrm{d}s, \quad t \in [T_1, T_2], \tag{1.1}$$

where $f \in C([0,\infty),[0,\infty))$. We show that if G satisfies certain conditions, a fixed point theorem due to Avery, Anderson, and Henderson can be applied to show the existence of nonnegative solutions of (1.1).

In recent years, multiple researchers have applied various methods from fixed point theory to general Hammerstein integral equations. In [17], Cabada, Cid, and Infante apply fixed point index theory to a Hammerstein integral equation and then give an example where the kernel is a Green's function for a second-order system of ordinary differential equations. Figueroa and Tojo [22] use general cones and fixed point index theory to show the existence of concave solutions of a Hammerstein integral equation. As an example, they show the existence of concave solutions of a second-order boundary-value problem. This work is particularly motivated by the recent work of Webb [32], in which he considers a general Hammerstein integral equation, assumes the kernel satisfies basic properties, and applies fixed point index theory to obtain sufficient conditions for fixed points. He then applies these results to many boundary-value problems. For more examples of recent work on Hammerstein integral equations, see [16, 18, 23, 26, 31] and the references therein.

Recently, Avery et al. have been developing extensions of the Leggett-Williams fixed point theorem [25] to allow for more flexibility in the conditions required for

---

2010 *Mathematics Subject Classification.* 47H10, 34A08, 34B15, 34B27, 45G10.
Key words and phrases. Hammerstein integral equation; boundary-value problem; fractional boundary-value problem.

Submitted November 28, 2018. Published August 13, 2019.

the existence of a fixed point of an operator. In [14], an extension was given that does not require either of the functional boundaries to be invariant with respect to the functional wedge. This fixed point theorem has been applied to several different boundary-value problems [1, 2, 9, 15]. In this paper, the results from previous applications are generalized and extended. The hypotheses on G match with the properties of G when G is the Green's function associated with the boundary-value problems in the aforementioned papers. We also obtain new applications to integral equations with convolution type kernels, a fractional boundary-value problem, and an ordinary differential equation satisfying Lidstone boundary conditions.

While this article is concerned with integral equations on the real line, these results could be extended to time scales. This extension would generalize the results on time scales in [27] and on difference equations [3, 29]. Generally, applications of Avery type fixed point theorems (see, for example, [4, 5, 6, 7, 8, 10, 11, 12, 13]) take advantage of the properties of G mentioned in this paper. Therefore, this work could be extended to apply these Avery fixed point theorems.

1.1. **Definitions.**

In this subsection, we present definitions that will be used throughout the rest of this article.

Definition 1.1. Let E be a real Banach space. A nonempty closed convex set $\mathcal{P} \subset E$ is called a cone provided:

- (i) $x \in \mathcal{P}$, $\lambda \geq 0$ implies $\lambda x \in \mathcal{P}$;
- (ii) $x \in \mathcal{P}$, $-x \in \mathcal{P}$ implies $x = 0$.

Definition 1.2. A map α is said to be a nonnegative continuous concave functional on a cone $\mathcal{P}$ of a real Banach space E if $\alpha : \mathcal{P} \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0, 1]$. Similarly we say the map β is a nonnegative continuous convex functional functional on a cone $\mathcal{P}$ of a real Banach space E if $\beta : \mathcal{P} \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0, 1]$.

2. Fixed point theorem

We first define sets that are integral to the fixed point theorem. Let α and ψ be nonnegative continuous concave functionals on $\mathcal{P}$, and let δ and β be nonnegative continuous convex functionals on $\mathcal{P}$. We define the sets

$$\begin{gathered}
A = A(\alpha, \beta, a, d) = \{x \in \mathcal{P} : a \leq \alpha(x) \text{ and } \beta(x) \leq d\}, \\
B = B(\delta, b) = \{x \in A : \delta(x) \leq b\}, \\
C = C(\psi, c) = \{x \in A : c \leq \psi(x)\}.
\end{gathered}$$

The following fixed point theorem is attributed to Anderson, Avery, and Henderson [14] and is an extension of the original Leggett-Williams fixed point theorem [25].

Theorem 2.1. *Suppose $\mathcal{P}$ is a cone in a real Banach space E, α and ψ are nonnegative continuous concave functionals on $\mathcal{P}$, δ and β are nonnegative continuous convex functionals on $\mathcal{P}$, and for nonnegative real numbers a, b, c, and d, the sets A, B, and C are defined as above. Furthermore, suppose A is a bounded subset of*

$\mathcal{P}$, $T : A \to \mathcal{P}$ is a completely continuous operator, and that the following conditions hold:

- (A1) $\{x \in A : c < \psi(x)$ *and* $\delta(x) < b\} \neq \emptyset$, $\{x \in \mathcal{P} : \alpha(x) < a$ *and* $d < \beta(x)\} = \emptyset$;
- (A2) $\alpha(Tx) \geq a$ *for all* $x \in B$;
- (A3) $\alpha(Tx) \geq a$ *for all* $x \in A$ *with* $\delta(Tx) > b$;
- (A4) $\beta(Tx) \leq d$ *for all* $x \in C$; *and*
- (A5) $\beta(Tx) \leq d$ *for all* $x \in A$ *with* $\psi(Tx) < c$.

Then T has a fixed point $x^* \in A$.

3. Positive solutions of the Hammerstein equation

We make the following assumptions on G.

- (A6) $G \in C([T_1, T_2] \times [T_1, T_2], [0, \infty))$ and $G(t, s) \not\equiv 0$.
- (A7) For each s, if $t_1, t_2 \in [T_1, T_2]$ with $t_1 \leq t_2$, then $G(t_1, s) \leq G(t_2, s)$.
- (A8) There exists a $k > 0$ such that for any $y, w \in [T_1, T_2]$ with $y \leq w$,
$$(y - T_1)^k G(w, s) \leq (w - T_1)^k G(y, s).$$

We point out that assumption (A8) implies
$$(y - T_1)^k \int_{T_1}^{T_2} G(w, s)\,\mathrm{d}s \leq (w - T_1)^k \int_{T_1}^{T_2} G(y, s)\,\mathrm{d}s. \tag{3.1}$$

Let $\mathcal{B} = C([T_1, T_2], \mathbb{R})$ be the Banach Space composed of continuous functions defined from $[T_1, T_2]$ into $\mathbb{R}$ with the norm
$$\|x\| = \max_{t \in [T_1, T_2]} |x(t)|.$$

We define the operator $T : \mathcal{B} \to \mathcal{B}$ by
$$Tx(t) = \int_{T_1}^{T_2} G(t, s) f(x(s))\,\mathrm{d}s, \quad t \in [T_1, T_2].$$
Then x is a solution of (1.1) if and only if x is a fixed point of T.

We define the cone $\mathcal{P} \subset \mathcal{B}$ by
$$\begin{aligned} \mathcal{P} = \big\{&x \in \mathcal{B} : x \text{ is nonnegative, nondecreasing, and} \\ &(y - T_1)^k x(w) \leq (w - T_1)^k x(y) \text{ for all } y, w \in [T_1, T_2] \text{ with } y \leq w\big\}. \end{aligned}$$

Theorem 3.1. *The operator $T : \mathcal{P} \to \mathcal{P}$ and is completely continuous.*

Proof. Let $x \in \mathcal{P}$. By (A6), for any $t \in [T_1, T_2]$,
$$Tx(t) = \int_{T_2}^{T_1} G(t, s) f(x(s))\,\mathrm{d}s \geq 0.$$
So T is nonnegative. By (A7), for $t_1, t_2 \in [T_1, T_2]$ with $t_1 \leq t_2$,
$$\begin{aligned} Tx(t_1) &= \int_{T_2}^{T_1} G(t_1, s) f(x(s))\,\mathrm{d}s \\ &\leq \int_{T_2}^{T_1} G(t_2, s) f(x(s))\,\mathrm{d}s \\ &= Tx(t_2). \end{aligned}$$
So T is nondecreasing.

By (A8) and (3.1), if $y, w \in [T_1, T_2]$ with $y \le w$, then

$$\begin{aligned}(y-T_1)^k Tx(w) &= (y-T_1)^k \int_{T_1}^{T_2} G(w,s) f(x(s))\,\mathrm{d}s \\ &\le (w-T_1)^k \int_{T_1}^{T_2} G(y,s) f(x(s))\,\mathrm{d}s \\ &= (w-T_1)^k Tx(y).\end{aligned}$$

So $T : \mathcal{P} \to \mathcal{P}$. A standard application of the Arzelà-Ascoli theorem shows T is completely continuous. $\square$

For fixed $\tau, \mu, \nu \in [T_1, T_2]$, define the nonnegative concave functionals α and ψ to be

$$\alpha(x) = \min_{t\in[\tau,T_2]} x(t) = x(\tau), \quad \psi(x) = \min_{t\in[\mu,T_2]} x(t) = x(\mu),$$

and the nonnegative convex functionals δ and β to be

$$\delta(x) = \max_{t\in[T_1,\nu]} x(t) = x(\nu), \quad \beta(x) = \max_{t\in[T_1,T_2]} x(t) = x(T_2).$$

Notice by (A6), the values τ, μ, and ν can be chosen so that $\int_\tau^\nu G(\tau,s)\,\mathrm{d}s > 0$, $\int_{T_1}^{T_2} G(\mu,s)\,\mathrm{d}s > 0$, and $\int_{T_1}^{T_2} G(\nu,s)\,\mathrm{d}s > 0$.

Theorem 3.2. *Assume* (A6)–(A8) *hold. Choose $\tau, \mu, \nu \in [T_1, T_2]$ with $T_1 < \tau \le \mu < \nu \le T_2$, $\int_\tau^\nu G(\tau,s)\,\mathrm{d}s > 0$, $\int_{T_1}^{T_2} G(\mu,s)\,\mathrm{d}s > 0$, and $\int_{T_1}^{T_2} G(\nu,s)\,\mathrm{d}s > 0$. Let d and m be positive reals with $0 < m < \big(\frac{\mu-T_1}{T_2-T_1}\big)^k d$ and suppose $f : [0,\infty) \to [0,\infty)$ is continuous and satisfies the conditions:*

(i) *$f(w) \ge \frac{(\tau-T_1)^k d}{(T_2-T_1)^k \int_\tau^\nu G(\tau,r)\,\mathrm{d}r}$ for $w \in \big[\big(\frac{\tau-T_1}{T_2-T_1}\big)^k d, \big(\frac{\nu-T_1}{T_2-T_1}\big)^k d\big]$;*
(ii) *$f(w)$ is decreasing for $0 \le w \le m$ and $f(m) \ge f(w)$ for $m \le w \le d$; and*
(iii) *$\int_{T_1}^{\mu} G(T_2,s) f\big(\frac{(s-T_1)^k}{(\mu-T_1)^k} m\big)\,\mathrm{d}s \le d - f(m)\int_\mu^{T_2} G(T_2,s)\,\mathrm{d}s$.*

Then (1.1) *has at least one positive solution $x^* \in A\big(\alpha, \beta, \big(\frac{\tau-T_1}{T_2-T_1}\big)^k d, d\big)$.*

Proof. Define

$$a = \Big(\frac{\tau-T_1}{T_2-T_1}\Big)^k d, \quad b = \Big(\frac{\nu-T_1}{T_2-T_1}\Big)^k d, \quad c = \Big(\frac{\mu-T_1}{T_2-T_1}\Big)^k d.$$

Notice that if $x \in A \subset \mathcal{P}$, then $\|x\| = x(T_2) = \beta(x) \le d$. So A is bounded.

First, we show (A1) holds. Let

$$K \in \Big(\frac{(\mu-T_1)^k d}{(T_2-T_1)^k \int_{T_1}^{T_2} G(\mu,s)\,\mathrm{d}s}, \frac{(\nu-T_1)^k d}{(T_2-T_1)^k \int_{T_1}^{T_2} G(\nu,s)\,\mathrm{d}s}\Big),$$

which, by (3.1), is well-defined. We define

$$x_K(t) = K \int_{T_1}^{T_2} G(t,s)\,\mathrm{d}s.$$

So $x_K \in \mathcal{P}$,

$$\alpha(x_K) = K \int_{T_1}^{T_2} G(\tau,s)\,\mathrm{d}s$$

$$
\begin{aligned}
&> \frac{(\mu - T_1)^k d \int_{T_1}^{T_2} G(\tau, s)\,\mathrm{d}s}{(T_2 - T_1)^k \int_{T_1}^{T_2} G(\mu, s)\,\mathrm{d}s} \\
&\geq \frac{(\tau - T_1)^k d \int_{T_1}^{T_2} G(\mu, s)\,\mathrm{d}s}{(T_2 - T_1)^k \int_{T_1}^{T_2} G(\mu, s)\,\mathrm{d}s} \\
&= \Big(\frac{\tau - T_1}{T_2 - T_1}\Big)^k d = a,
\end{aligned}
$$

and

$$
\begin{aligned}
\beta(x_K) &= K \int_{T_1}^{T_2} G(T_2, s)\,\mathrm{d}s \\
&< \frac{(\nu - T_1)^k d \int_{T_1}^{T_2} G(T_2, s)\,\mathrm{d}s}{(T_2 - T_1)^k \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s} \\
&\leq \frac{(T_2 - T_1)^k d \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s}{(T_2 - T_1)^k \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s} = d.
\end{aligned}
$$

So $x_K \in A$. Now

$$
\begin{aligned}
\psi(x_K) &= K \int_{T_1}^{T_2} G(\mu, s)\,\mathrm{d}s \\
&> \frac{(\mu - T_1)^k d \int_{T_1}^{T_2} G(\mu, s)\,\mathrm{d}s}{(T_2 - T_1)^k \int_{T_1}^{T_2} G(\mu, s)\,\mathrm{d}s} \\
&= \frac{(\mu - T_1)^k d}{(T_2 - T_1)^k} = c,
\end{aligned}
$$

and

$$
\begin{aligned}
\delta(x_K) &= K \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s \\
&< \frac{(\nu - T_1)^k d \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s}{(T_2 - T_1)^k \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s} \\
&= \frac{(\nu - T_1)^k d}{(T_2 - T_1)^k} = b.
\end{aligned}
$$

So $x_K \in \{x \in A : c < \psi(x) \text{ and } \delta(x) < b\}$, and $\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset$. If $x \in \mathcal{P}$ and $\beta(x) > d$, then

$$
\alpha(x) = x(\tau) \geq \frac{(\tau - T_1)^k}{(T_2 - T_1)^k} x(T_2) = \Big(\frac{\tau - T_1}{T_2 - T_1}\Big)^k \beta(x) > \Big(\frac{\tau - T_1}{T_2 - T_1}\Big)^k d = a.
$$

So $\{x \in \mathcal{P} : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$. Thus (A1) holds.

Next, we show (A2) holds. Let $x \in B$. Then $\delta(x) \leq b$. By (i),

$$
\begin{aligned}
\alpha(Tx) &= \int_{T_1}^{T_2} G(\tau, s) f(x(s))\,\mathrm{d}s \\
&\geq \int_{\tau}^{\nu} G(\tau, s) f(x(s))\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\ge \int_\tau^v G(\tau,s)\Big(\frac{(\tau-T_1)^k d}{(T_2-T_1)^k\int_\tau^\nu G(\tau,r)\,\mathrm{d}r}\Big)\,\mathrm{d}s\\
&= \Big(\frac{\tau-T_1}{T_2-T_1}\Big)^k d = a.
\end{aligned}
$$

Now, we show that (A3) holds. Let $x\in A$ with $\delta(Tx)>b$. Then, by (3.1),

$$
\begin{aligned}
\alpha(Tx) &= \int_{T_1}^{T_2} G(\tau,s)f(x(s)\,\mathrm{d}s\\
&\ge \frac{(\tau-T_1)^k}{(\nu-T_1)^k}\int_{T_1}^{T_2} G(\nu,s)f(x(s))\,\mathrm{d}s\\
&= \frac{(\tau-T_1)^k}{(\nu-T_1)^k}\delta(Tx)\\
&> \frac{(\tau-T_1)^k}{(\nu-T_1)^k}\Big(\frac{\nu-T_1}{T_2-T_1}\Big)^k d\\
&= \Big(\frac{\tau-T_1}{T_2-T_1}\Big)^k d = a,
\end{aligned}
$$

Here, we show that (A4) holds. Let $x\in C$. Then $\psi(x)=x(\mu)>c$. So for $t\in[T_1,\mu]$,

$$
x(t)\ge \frac{(t-T_1)^k}{(\mu-T_1)^k}x(\mu)\ge \frac{(t-T_1)^k}{(\mu-T_1)^k}c\ge \frac{(t-T_1)^k}{(\mu-T_1)^k}m.
$$

Then by (ii) and (iii),

$$
\begin{aligned}
\beta(Tx) &= \int_{T_1}^{T_2} G(T_2,s)f(x(s))\,\mathrm{d}s\\
&= \int_{T_1}^{\mu} G(T_2,s)f(x(s))\,\mathrm{d}s + \int_{\mu}^{T_2} G(T_2,s)f(x(s))\,\mathrm{d}s\\
&\le \int_{T_1}^{\mu} G(T_2,s)f\Big(\frac{(s-T_1)^k}{(\mu-T_1)^k}m\Big)\,\mathrm{d}s + \int_{\mu}^{T_2} G(T_2,s)f(m)\,\mathrm{d}s \le d.
\end{aligned}
$$

So (A4) holds.

Finally, we show that (A5) holds. Let $x\in A$ with $\psi(Tx)<c$. So

$$
\begin{aligned}
\beta(Tx) &= \int_{T_1}^{T_2} G(T_2,s)f(x(s))\,\mathrm{d}s\\
&\le \frac{(T_2-T_1)^k}{(\mu-T_1)^k}\int_{T_1}^{T_2} G(\mu,s)f(x(s))\,\mathrm{d}s\\
&= \frac{(T_2-T_1)^k}{(\mu-T_1)^k}\psi(Tx)\\
&< \frac{(T_2-T_1)^k}{(\mu-T_1)^k}c\\
&= \frac{(T_2-T_1)^k}{(\mu-T_1)^k}\Big(\frac{\mu-T_1}{T_2-T_1}\Big)^k d = d.
\end{aligned}
$$

So (A5) holds.

Thus T has a fixed point $x^*\in A$ which is a positive solution of (1.1). $\square$

4. Positive solutions of integral equations and boundary-value problems

Example 4.1. Consider the integral equation with kernel of convolution type

$$x(t) = \int_{T_1}^{T_2} K(t-s) f(x(s))\,\mathrm{d}s. \tag{4.1}$$

If $K \in C([T_1 - T_2, T_2 - T_1], [0,\infty))$, if $K(t_1) \le K(t_2)$ for $t_1, t_2 \in [T_1 - T_2, T_2 - T_1]$ with $t_1 \le t_2$, and if there exists a $k > 0$ such that for any $y, w \in [T_1, T_2]$ with $y \le w$,

$$(y - T_1)^k K(w - s) \le (w - T_1)^k K(y - s),$$

then Theorem 3.2 can be applied to show the existence of a positive solution of (4.1).

Note when $k = 1$, (A8) is equivalent to concavity in the traditional sense. So if $K \in C^{(2)}([T_1 - T_2, T_2 - T_1], [0,\infty))$ with $K'(t) \ge 0$ for $t \in [T_1 - T_2, T_2 - T_1]$ and $K''(t) \le 0$ for $t \in [T_1 - T_2, T_2 - T_1]$, then Theorem 3.2 can be applied to show the existence of a positive solution of (4.1).

Specifically, consider the integral equation

$$x(t) = \int_0^{\pi/4} \sin\big(t - s + \frac{\pi}{4}\big) f(x(s))\,\mathrm{d}s. \tag{4.2}$$

Note that $\sin\big(t + \frac{\pi}{4}\big) \ge 0$, $\frac{d}{dt}\sin\big(t + \frac{\pi}{4}\big) = \cos\big(t + \frac{\pi}{4}\big) \ge 0$, and $\frac{d^2}{dt^2}\sin\big(t + \frac{\pi}{4}\big) = -\sin\big(t + \frac{\pi}{4}\big) \le 0$ for $t \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. So Theorem 3.2 can be applied to show the existence of a positive solution of (4.2).

Example 4.2. The Green's function associated with the boundary-value problem

$$x'' + f(x) = 0, \quad t \in (0,1), \tag{4.3}$$

$$x(0) = 0, \quad x'(1) = 0, \tag{4.4}$$

given by $G(t,s) = \min\{t,s\}$, satisfies (A6)–(A8) with $k = 1$. Since solutions of (4.3), (4.4) are solutions of the integral equation

$$x(t) = \int_0^1 G(t,s) f(x(s))\,\mathrm{d}s, \quad t \in [0,1],$$

Theorem 3.2 can be used to show the existence of positive solutions of the boundary-value problem (4.3), (4.4). In this case, Theorem 3.2 is equivalent to [15, Theorem 5].

Example 4.3. Consider the $2n$th order differential equation

$$(-1)^n x^{(2n)} = f(x), \quad t \in (0,1), \tag{4.5}$$

satisfying the boundary conditions

$$x^{(2i)}(0) = 0, \quad x^{(2i+1)}(1) = 0, \quad i = 0, 1, \dots, n \tag{4.6}$$

If $G(t,s) = \min\{t,s\}$, by letting $G_1(t,s) = G(t,s)$, we can recursively define, for $j = 2, \dots, n$,

$$G_j(t,s) = \int_0^1 G(t,r) G_{j-1}(r,s)\,\mathrm{d}r.$$

As a result, $G_n(t,s)$ is the Green's function corresponding to $(-1)^n x^{(2n)} = 0$, (4.6). Thus, $x(t)$ is a solution of (4.5), (4.6) if and only if

$$x(t) = \int_0^1 G_n(t,s) f(x(s))\,\mathrm{d}s.$$

Since $G(t,s) \geq 0$, $G_n(t,s) \geq 0$. So G satisfies (A6).

For (A7), let $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$. Now $G(t_1,s) \leq G(t_2,s)$, so

$$\begin{aligned} G_n(t_1,s) &= \int_0^1 G(t_1,r) G_{n-1}(r,s)\,\mathrm{d}r \\ &\leq \int_0^1 G(t_2,r) G_{n-1}(r,s)\,\mathrm{d}r \\ &= G(t_2,s). \end{aligned}$$

So G_n satisfies (A7).

It is also known that $G(t,s)$ satisfies (A8) with $k=1$. Thus,

$$\begin{aligned} yG_n(w,s) &= \int_0^1 yG(w,r) G_{n-1}(r,s)\,\mathrm{d}r \\ &\leq \int_0^1 wG(y,r) G_{n-1}(r,s)\,\mathrm{d}r \\ &= wG_n(y,s). \end{aligned}$$

Thus (A8) is satisfied with $k=1$.

Thus G_n satisfies (A6)–(A8) with $k=1$. Since solutions of (4.5), (4.6) must solve the integral equation

$$x(t) = \int_0^1 G(t,s) f(x(s))\,\mathrm{d}s,$$

Theorem 3.2 can be applied to show the existence of a positive solution of (4.5), (4.6).

Corollary 4.4. *Let $\tau, \mu, \nu \in [0,1]$ with $0 < \tau \leq \mu < \nu \leq 1$. Let d and m be positive reals with $0 < m < \mu d$ and suppose $f : [0,\infty) \to [0,\infty)$ is continuous and satisfies the conditions:*

- (i) *$f(w) \geq \frac{\tau d}{\int_\tau^\nu G_n(\tau,r)\,\mathrm{d}r}$ for $w \in [\tau d, \nu d]$;*
- (ii) *$f(w)$ is decreasing for $0 \leq w \leq m$ and $f(m) \geq f(w)$ for $m \leq w \leq d$; and*
- (iii) *$\int_0^\mu G_n(1,s) f(\frac{s}{\mu} m)\,\mathrm{d}s \leq d - f(m) \int_\mu^1 G_n(1,s)\,\mathrm{d}s$.*

Then (4.5), (4.6) has at least one positive solution $x^* \in A(\alpha, \beta, \tau d, d)$.

Example 4.5. Consider, for $n \in \mathbb{N}$, $n \geq 3$, $n-1 < \alpha \leq n$, $\beta \in [1, n-1]$, the fractional differential equation

$$D_{0+}^{\alpha} x + f(x) = 0, \quad t \in (0,1), \tag{4.7}$$

satisfying the boundary conditions

$$x^{(i)}(0) = 0, \quad i = 0, \dots, n-2, \quad D_{0+}^{\beta} x(1) = 0, \tag{4.8}$$

where D_{0+}^{α}, D_{0+}^{β} are the Riemann-Liouville fractional derivatives of order α and β. For a detailed view of fractional calculus, see the books by Diethelm [19], Kilbas, Srivastava, and Trujillo [24], Miller and Ross [28], or Podlubny [30].

The Green's function for $-D^{\alpha}_{0+} = 0$ satisfying the boundary conditions (4.8) is

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s < t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \le t \le s < 1. \end{cases} \tag{4.9}$$

Therefore, x is a solution of (4.7), (4.8) if and only if x solves the integral equation

$$x(t) = \int_0^1 G(t,s) f(x(s))\,\mathrm{d}s, \quad t \in [0,1].$$

In [20], it is shown that G satisfies (A6) and (A7). The argument used in [21] with $\alpha > 2$ shows $G(t,s)$ has the property that

$$y^{\alpha-1} G(w,s) \le w^{\alpha-1} G(y,s)$$

for all $y, w \in [0,1]$ with $y \le w$. So (A8) holds with $k = \alpha - 1$. Thus Theorem 3.2 can be applied to give the existence of a positive solution of (4.7), (4.8).

Corollary 4.6. *Let $\tau, \mu, \nu \in [0,1]$ with $0 < \tau \le \mu < \nu \le 1$. Let d and m be positive reals with $0 < m < \mu^{\alpha-1} d$ and suppose $f : [0,\infty) \to [0,\infty)$ is continuous and satisfies the conditions:*

- (i) *$f(w) \ge \frac{\tau^{\alpha-1} d}{\int_\tau^\nu G(\tau,r)\,\mathrm{d}r}$ for $w \in [\tau^{\alpha-1} d, \nu^{\alpha-1} d]$;*
- (ii) *$f(w)$ is decreasing for $0 \le w \le m$ and $f(m) \ge f(w)$ for $m \le w \le d$; and*
- (iii) *$\int_0^\mu G(1,s) f\big(\frac{s^{\alpha-1}}{\mu^{\alpha-1}} m\big)\,\mathrm{d}s \le d - f(m) \int_\mu^1 G(1,s)\,\mathrm{d}s$.*

Then (4.7), (4.8) has at least one positive solution $x^* \in A(\alpha, \beta, \tau^{\alpha-1} d, d)$.

5. Positive symmetric solutions of the Hammerstein equation

Define $\bar{T} = \frac{T_1+T_2}{2}$. Define the cone

$$\begin{aligned} \mathcal{K} = \big\{ & x \in \mathcal{B} : x(T_2 - t + T_1) = x(t) \text{ for all } t \in [T_1, T_2], \\ & x \text{ is nonnegative on } [T_1,T_2], \text{ nondecreasing on } [T_1, \bar{T}], \text{ and} \\ & (y - T_1)^k x(w) \le (w - T_1)^k x(y) \text{ for all } y, w \in [T_1, T_2] \text{ with } y \le w \big\}. \end{aligned}$$

We need the following additional assumptions.

- (A9) Let $t_1, t_2 \in [T_1, \bar{T}]$ with $t_1 \le t_2$.
  - (i) If $t_2 \le s \le T_2 - t_2 + T_1$, then $G(t_1, s) \le G(t_2, s)$.
  - (ii) If $s \le t_2$, then $G(t_1,s) + G(T_2 - t_1 + T_1, s) \le G(t_2, s) + G(T_2 - t_2 + T_1, s)$.
- (A10) For all $t, s \in [T_1, T_2]$,
$$G(T_2 - t + T_1, T_2 - s + T_1) = G(t,s).$$

Lemma 5.1. *Assume* (A6), (A8)–(A10). *Then the operator $T : \mathcal{K} \to \mathcal{K}$ and is completely continuous.*

Proof. Let $x \in \mathcal{K}$. By (A6), for any $t \in [T_1, T_2]$,

$$Tx(t) = \int_{T_2}^{T_1} G(t,s) f(x(s))\,\mathrm{d}s \ge 0.$$

So T is nonnegative on $[T_1, T_2]$.

By (A10), if $t \in [T_1, T_2]$, then, by making the substitution $\sigma = T_2 - s + T_1$,

$$Tx(T_2 - t + T_1) = \int_{T_1}^{T_2} G(T_2 - t + T_1, s) f(x(s))\,\mathrm{d}s$$

$$\begin{aligned}
&= -\int_{T_2}^{T_1} G(T_2 - t + T_1, T_2 - \sigma + T_1) f(x(T_2 - \sigma + T_1))\,\mathrm{d}\sigma \\
&= \int_{T_2}^{T_1} G(t, \sigma) f(x(\sigma))\,\mathrm{d}\sigma \\
&= Tx(t).
\end{aligned}$$

Next, for $t_1, t_2 \in [T_1, \bar{T}]$ with $t_1 \le t_2$,

$$\begin{aligned}
Tx(t_1) &= \int_{T_1}^{T_2} G(t_1, s) f(x(s))\,\mathrm{d}s \\
&= \int_{T_1}^{t_2} G(t_1, s) f(x(s))\,\mathrm{d}s + \int_{t_2}^{T_2 - t_2 + T_1} G(t_1, s) f(x(s))\,\mathrm{d}s \\
&\quad + \int_{T_2 - t_2 + T_1}^{T_2} G(t_1, s) f(x(s))\,\mathrm{d}s.
\end{aligned}$$

By (A9) (i),

$$\int_{t_2}^{T_2 - t_2 + T_1} G(t_1, s) f(x(s))\,\mathrm{d}s \le \int_{t_2}^{T_2 - t_2 + T_1} G(t_2, s) f(x(s))\,\mathrm{d}s.$$

By (A9) (ii) and by (A10),

$$\begin{aligned}
&\int_{T_1}^{t_2} G(t_1, s) f(x(s))\,\mathrm{d}s + \int_{T_2 - t_1 + T_1}^{T_2} G(t_1, s) f(x(s))\,\mathrm{d}s \\
&= \int_{T_1}^{t_2} G(t_1, s) f(x(s))\,\mathrm{d}s + \int_{T_1}^{t_2} G(t_1, T_2 - s + T_1) f(x(s))\,\mathrm{d}s \\
&= \int_{T_1}^{t_2} [G(t_1, s) + G(T_2 - t_1 + T_1, s)] f(x(s))\,\mathrm{d}s \\
&\le \int_{T_1}^{t_2} [G(t_2, s) + G(T_2 - t_2 + T_1, s)] f(x(s))\,\mathrm{d}s \\
&= \int_{T_1}^{t_2} G(t_2, s) f(x(s))\,\mathrm{d}s + \int_{T_2 - t_2 + T_1}^{T_2} G(t_2, s) f(x(s))\,\mathrm{d}s.
\end{aligned}$$

Thus

$$\begin{aligned}
Tx(t_1) &= \int_{T_1}^{t_2} G(t_1, s) f(x(s))\,\mathrm{d}s + \int_{t_2}^{T_2 - t_2 + T_1} G(t_1, s) f(x(s))\,\mathrm{d}s \\
&\quad + \int_{T_2 - t_2 + T_1}^{T_2} G(t_1, s) f(x(s))\,\mathrm{d}s \\
&\le \int_{T_1}^{t_2} G(t_2, s) f(x(s))\,\mathrm{d}s + \int_{t_2}^{T_2 - t_2 + T_1} G(t_2, s) f(x(s))\,\mathrm{d}s \\
&\quad + \int_{T_2 - t_2 + T_1}^{T_2} G(t_2, s) f(x(s))\,\mathrm{d}s \\
&= Tx(t_2).
\end{aligned}$$

So Tx is nondecreasing on $[T_1, \bar{T}]$.

By (A8) and (3.1), if $y, w \in [T_1, T_2]$ with $y \le w$, then

$$\begin{aligned}(y-T_1)^k Tx(w) &= (y-T_1)^k \int_{T_1}^{T_2} G(w,s) f(x(s))\,\mathrm{d}s\\ &\le (w-T_1)^k \int_{T_1}^{T_2} G(y,s) f(x(s))\,\mathrm{d}s\\ &= (w-T_1)^k Tx(y).\end{aligned}$$

So $T : \mathcal{K} \to \mathcal{K}$. A standard application of the Arzelà-Ascoli theorem shows T is completely continuous. □

For fixed $\tau, \mu, \nu \in [T_1, \bar{T}]$, define the nonnegative concave functionals α and ψ to be

$$\alpha(x) = \min_{t\in[\tau,\bar{T}]} x(t) = x(\tau), \quad \psi(x) = \min_{t\in[\mu,\bar{T}]} x(t) = x(\mu),$$

and the nonnegative convex functionals δ and β to be

$$\delta(x) = \max_{t\in[T_1,\nu]} x(t) = x(\nu), \quad \beta(x) = \max_{t\in[T_1,\bar{T}]} x(t) = x(\bar{T}).$$

Theorem 5.2. *Assume* (A6), (A8)–(A10) *hold. Choose* $\tau, \mu, \nu \in [T_1, \bar{T}]$ *with* $T_1 < \tau \le \mu < \nu \le \bar{T}$, $\int_\tau^\nu G(\tau,s)\,\mathrm{d}s > 0$, $\int_{T_1}^{T_2} G(\mu,s)\,\mathrm{d}s > 0$, *and* $\int_{T_1}^{T_2} G(\nu,s)\,\mathrm{d}s > 0$. *Let* d *and* m *be positive reals with* $0 < m < \big(\frac{\mu-T_1}{\bar{T}-T_1}\big)^k d$ *and suppose* $f : [0,\infty) \to [0,\infty)$ *is continuous and satisfies the conditions:*

(i) $f(w) \ge \frac{(\tau-T_1)^k d}{(\bar{T}-T_1)^k \int_\tau^\nu G(\tau,r)\,\mathrm{d}r}$ *for* $w \in \big[\big(\frac{\tau-T_1}{\bar{T}-T_1}\big)^k d, \big(\frac{\nu-T_1}{\bar{T}-T_1}\big)^k d\big]$;

(ii) $f(w)$ *is decreasing for* $0 \le w \le m$ *and* $f(m) \ge f(w)$ *for* $m \le w \le d$; *and*

(iii) $2\int_{T_1}^{\mu} G(\bar{T},s) f\big(\frac{(s-T_1)^k}{(\mu-T_1)^k} m\big)\,\mathrm{d}s \le d - 2f(m)\int_\mu^{\bar{T}} G(\bar{T},s)\,\mathrm{d}s$.

Then (1.1) *has at least one positive solution* $x^* \in A\big(\alpha, \beta, \big(\frac{\tau-T_1}{\bar{T}-T_1}\big)^k d, d\big)$.

Proof. Define

$$a = \Big(\frac{\tau-T_1}{\bar{T}-T_1}\Big)^k d, \quad b = \Big(\frac{\nu-T_1}{\bar{T}-T_1}\Big)^k d, \quad c = \Big(\frac{\mu-T_1}{\bar{T}-T_1}\Big)^k d.$$

Note that if $x \in A \subset \mathcal{P}$, then $\|x\| = x(\bar{T}) = \beta(x) \le d$. So A is bounded.

First, we show that (A1) holds. Let

$$K \in \Big(\frac{(\mu-T_1)^k d}{(\bar{T}-T_1)^k \int_{T_1}^{T_2} G(\mu,s)\,\mathrm{d}s}, \frac{(\nu-T_1)^k d}{(\bar{T}-T_1)^k \int_{T_1}^{T_2} G(\nu,s)\,\mathrm{d}s}\Big),$$

which, by (3.1), is well-defined. We define

$$x_K(t) = K\int_{T_1}^{T_2} G(t,s)\,\mathrm{d}s.$$

So $x_K \in \mathcal{P}$,

$$\begin{aligned}\alpha(x_K) &= K\int_{T_1}^{T_2} G(\tau,s)\,\mathrm{d}s\\ &> \frac{(\mu-T_1)^k d \int_{T_1}^{T_2} G(\tau,s)\,\mathrm{d}s}{(\bar{T}-T_1)^k \int_{T_1}^{T_2} G(\mu,s)\,\mathrm{d}s}\end{aligned}$$

$$\begin{aligned}
&\geq \frac{(\tau - T_1)^k d \int_{T_1}^{T_2} G(\mu, s)\,\mathrm{d}s}{(\bar{T} - T_1)^k \int_{T_1}^{T_2} G(\mu, s)\,\mathrm{d}s} \\
&= \Big(\frac{\tau - T_1}{\bar{T} - T_1}\Big)^k d = a,
\end{aligned}$$

and

$$\begin{aligned}
\beta(x_K) &= K \int_{T_1}^{T_2} G\left(\bar{T}, s\right)\,\mathrm{d}s \\
&< \frac{(\nu - T_1)^k d \int_{T_1}^{T_2} G\left(\bar{T}, s\right)\,\mathrm{d}s}{(\bar{T} - T_1)^k \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s} \\
&\leq \frac{(\bar{T} - T_1)^k d \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s}{(\bar{T} - T_1)^k \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s} = d.
\end{aligned}$$

So $x_K \in A$. Now

$$\begin{aligned}
\psi(x_K) &= K \int_{T_1}^{T_2} G(\mu, s)\,\mathrm{d}s \\
&> \frac{(\mu - T_1)^k d \int_{T_1}^{T_2} G(\mu, s)\,\mathrm{d}s}{(\bar{T} - T_1)^k \int_{T_1}^{T_2} G(\mu, s)\,\mathrm{d}s} \\
&= \frac{(\mu - T_1)^k d}{(\bar{T} - T_1)^k} = c,
\end{aligned}$$

and

$$\begin{aligned}
\delta(x_K) &= K \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s \\
&< \frac{(\nu - T_1)^k d \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s}{(\bar{T} - T_1)^k \int_{T_1}^{T_2} G(\nu, s)\,\mathrm{d}s} \\
&= \frac{(\nu - T_1)^k d}{(\bar{T} - T_1)^k} = b.
\end{aligned}$$

So $x_K \in \{x \in A : c < \psi(x) \text{ and } \delta(x) < b\}$, and $\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset$.

If $x \in \mathcal{P}$ and $\beta(x) > d$, then

$$\alpha(x) = x(\tau) \geq \frac{(\tau - T_1)^k}{(\bar{T} - T_1)^k} x(T_2) = \Big(\frac{\tau - T_1}{\bar{T} - T_1}\Big)^k \beta(x) > \Big(\frac{\tau - T_1}{\bar{T} - T_1}\Big)^k d = a.$$

So $\{x \in \mathcal{P} : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$. Thus (A1) holds.

Next, we show that (A2) holds. Let $x \in B$. Then $\delta(x) \leq b$. By (i),

$$\begin{aligned}
\alpha(Tx) &= \int_{T_1}^{T_2} G(\tau, s) f(x(s))\,\mathrm{d}s \\
&\geq \int_{\tau}^{\nu} G(\tau, s) f(x(s))\,\mathrm{d}s \\
&\geq \int_{\tau}^{v} G(\tau, s) \Big(\frac{(\tau - T_1)^k d}{(\bar{T} - T_1)^k \int_{\tau}^{\nu} G(\tau, r)\,\mathrm{d}r}\Big)\,\mathrm{d}s
\end{aligned}$$

$$= \Big(\frac{\tau - T_1}{\bar{T} - T_1}\Big)^k d = a.$$

Now, we show that (A3) holds. Let $x \in A$ with $\delta(Tx) > b$. Then, by (3.1),

$$\begin{aligned}
\alpha(Tx) &= \int_{T_1}^{T_2} G(\tau, s) f(x(s) \,\mathrm{d}s \\
&\geq \frac{(\tau - T_1)^k}{(\nu - T_1)^k} \int_{T_1}^{T_2} G(\nu, s) f(x(s)) \,\mathrm{d}s \\
&= \frac{(\tau - T_1)^k}{(\nu - T_1)^k} \delta(Tx) \\
&> \frac{(\tau - T_1)^k}{(\nu - T_1)^k} \Big(\frac{\nu - T_1}{\bar{T} - T_1}\Big)^k d \\
&= \Big(\frac{\tau - T_1}{\bar{T} - T_1}\Big)^k d = a,
\end{aligned}$$

Penultimately, we show that (A4) holds. Let $x \in C$. Then $\psi(x) = x(\mu) > c$. So for $t \in [T_1, \mu]$,

$$x(t) \geq \frac{(t - T_1)^k}{(\mu - T_1)^k} x(\mu) \geq \frac{(t - T_1)^k}{(\mu - T_1)^k} c \geq \frac{(t - T_1)^k}{(\mu - T_1)^k} m.$$

Then by (ii) and (iii),

$$\begin{aligned}
\beta(Tx) &= \int_{T_1}^{T_2} G\left(\bar{T}, s\right) f(x(s)) \,\mathrm{d}s \\
&= 2 \int_{T_1}^{\bar{T}} G\left(\bar{T}, s\right) f(x(s)) \,\mathrm{d}s \\
&= 2 \int_{T_1}^{\mu} G\left(\bar{T}, s\right) f(x(s)) \,\mathrm{d}s + 2 \int_{\mu}^{\bar{T}} G\left(\bar{T}, s\right) f(x(s)) \,\mathrm{d}s \\
&\leq 2 \int_{T_1}^{\mu} G(\bar{T}, s) f\Big(\frac{(s - T_1)^k}{(\mu - T_1)^k} m\Big) \,\mathrm{d}s + 2 \int_{\mu}^{\bar{T}} G(\bar{T}, s) f(m) \,\mathrm{d}s \leq d.
\end{aligned}$$

So (A4) holds.

Finally, we show that (A5) holds. Let $x \in A$ with $\psi(Tx) < c$. So

$$\begin{aligned}
\beta(Tx) &= \int_{T_1}^{T_2} G\left(\bar{T}, s\right) f(x(s)) \,\mathrm{d}s \\
&\leq \frac{(\bar{T} - T_1)^k}{(\mu - T_1)^k} \int_{T_1}^{T_2} G(\mu, s) f(x(s)) \,\mathrm{d}s \\
&= \frac{(\bar{T} - T_1)^k}{(\mu - T_1)^k} \psi(Tx) \\
&< \frac{(\bar{T} - T_1)^k}{(\mu - T_1)^k} c \\
&= \frac{(\bar{T} - T_1)^k}{(\mu - T_1)^k} \Big(\frac{\mu - T_1}{\bar{T} - T_1}\Big)^k d = d.
\end{aligned}$$

So (A5) holds.

Thus T has a fixed point $x^* \in A$ which is a positive solution of (1.1). $\square$

6. Positive symmetric solutions of boundary value problems

Example 6.1. In [2], the Green's function corresponding to the Dirichlet problem

$$x'' = f(x), \quad t \in (0,1), \tag{6.1}$$

$$x(0) = 0, \quad x(1) = 0, \tag{6.2}$$

given by

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases}$$

is shown to satisfy (A6) and (A8) with $k = 1$, and (A10).

We verify (A9). Let $t_1, t_2 \in [0, \frac{1}{2}]$ with $t_1 \le t_2$. Suppose $t_2 \le s \le 1 - t_2$. Then

$$G(t_1, s) = t_1(1-s) < t_2(1-s) = G(t_2, s).$$

Next, suppose $s \le t_2$. Notice $1 - t_1 \ge 1 - t_2 \ge \frac{1}{2} \ge s$. Notice

$$G(t_2, s) + G(1 - t_2, s) = s(1 - t_2) + s\,(1 - (1 - t_2)) = s.$$

First, suppose $s \le t_1$. Then

$$\begin{aligned} G(t_1, s) + G(1 - t_1, s) &= s(1 - t_1) + s\,(1 - (1 - t_1)) \\ &= s \\ &= G(t_2, s) + G(1 - t_2, s). \end{aligned}$$

Next, suppose $t_1 \le s \le t_2$. Then

$$\begin{aligned} G(t_1, s) + G(1 - t_1, s) &= t_1(1 - s) + s\,(1 - (1 - t_1)) \\ &= t_1 \\ &\le s \\ &= G(t_2, s) + G(1 - t_2, s). \end{aligned}$$

So G satisfies (A9). Since solutions of (6.1), (6.2) must solve the integral equation

$$x(t) = \int_0^1 G(t,s) f(x(s))\,\mathrm{d}s,$$

Theorem 5.2 can be applied to show the existence of positive symmetric solutions of the boundary-value problem (6.1), (6.2). In this case, Theorem 5.2 is equivalent to [2, Theorem 3.5].

Example 6.2. In [9], it is shown that the Green's function corresponding to the two point problem

$$x^{(4)} = f(x), \quad t \in (0,1), \tag{6.3}$$

$$x^{(i)}(0) = 0, \quad x^{(i)}(1) = 0, \quad i = 0, 1, \tag{6.4}$$

given by

$$G(t,s) = \frac{1}{6} \begin{cases} t^2(1-s)^2(3(s-t) + 2(1-s)t), & 0 \le t \le s \le 1, \\ s^2(1-t)^2(3(t-s) + 2(1-t)s), & 0 \le s \le t \le 1, \end{cases}$$

satisfies (A6) and (A8) with $k = 2$, and (A10).

Note that for $t \in [0, \frac{1}{2}]$ and $t \le s$, we have

$$\frac{\partial}{\partial t} G(t,s) = \frac{1}{2} t(1-s)^2\,(2s(1-t) - t)$$

$$\geq \frac{1}{2}t(1-s)^2\Big(2s\big(1-\frac{1}{2}\big)-s\Big)=0.$$

So for $t_1, t_2 \in [0, \frac{1}{2}]$ with $t_1 \leq t_2 \leq s \leq 1-t_2$, $G(t_1, s) \leq G(t_2, s)$. Now for $t \in [0, \frac{1}{2}]$, $s \leq t$,

$$G(t,s)+G(1-t,s)=3s^2t-3s^2t^2-s^3.$$

Here

$$\begin{aligned}\frac{\partial}{\partial t}\left(G(t,s)+G(1-t,s)\right) &= 3s^2-6s^2t\\ &= 3s^2(1-2t)\geq 0.\end{aligned}$$

So if $s \leq t_1 \leq t_2 \leq \frac{1}{2}$, $G(t_1,s)+G(1-t_1,s) \leq G(t_2,s)+G(1-t_2,s)$. If $t \leq s \leq \frac{1}{2}$, then

$$G(t,s)+G(1-t,s)=3st^2-3s^2t^2-t^3.$$

So

$$\begin{aligned}\frac{\partial}{\partial t}\left(G(t,s)+G(1-t,s)\right) &= 6st-6s^2t-3t^2\\ &= 3t(2s-2s^2-t)\\ &\geq 3t(2s-2s^2-s)\\ &= 3t(s(1-2s))\geq 0.\end{aligned}$$

Therefore, if $t_1 \leq s \leq t_2 \leq \frac{1}{2}$, then

$$G(t_1,s)+G(1-t_1,s)\leq G(s,s)+G(1-s,s)\leq G(t_2,s)+G(1-t_2,s).$$

So (A9) is satisfied.

Now solutions of (6.3), (6.4) must solve the integral equation

$$x(t)=\int_0^1 G(t,s)f(x(s))\,\mathrm{d}s.$$

Thus, like in [9], Theorem 5.2 can be used to prove the existence of positive symmetric solutions of the given boundary-value problem. In fact, Theorem 5.2 is equivalent to [9, Theorem 3.4] for the given Green's function.

Example 6.3. Consider the $2n$th order differential equation

$$(-1)^n x^{(2n)} = f(x), \quad t\in(0,1), \tag{6.5}$$

satisfying the Lidstone boundary conditions

$$x^{(2i)}(0)=0, \quad x^{(2i)}(1)=0, \quad i=0,1,\dots,n-1. \tag{6.6}$$

If

$$G(t,s)=\begin{cases} t(1-s), & 0\leq t\leq s\leq 1,\\ s(1-t), & 0\leq s\leq t\leq 1,\end{cases}$$

by letting $G_1(t,s)=G(t,s)$, we can recursively define, for $j=2,\dots,n$,

$$G_j(t,s)=\int_0^1 G(t,r)G_{j-1}(r,s)\,\mathrm{d}r.$$

As a result, $G_n(t,s)$ is the Green's function corresponding to $(-1)^n x^{(2n)}=0$, (6.6). Thus, $x(t)$ is a solution of (6.5), (6.6) if and only if

$$x(t)=\int_0^1 G_n(t,s)f(x(s))\,\mathrm{d}s.$$

It is known that $G_n(t,s) \geq 0$ and $G_n(1-t,1-s) = G_n(t,s)$. So G satisfies (A6) and (A10).

It is also known that $G(t,s)$ satisfies (A8) with $k=1$. Thus,

$$\begin{aligned} yG_n(w,s) &= \int_0^1 yG(w,r)G_{n-1}(r,s)\,\mathrm{d}r \\ &\leq \int_0^1 wG(y,r)G_{n-1}(r,s)\,\mathrm{d}r \\ &= wG_n(y,s). \end{aligned}$$

Thus (A8) is satisfied with $k=1$.

We prove directly that G_2 satisfies (A9). Let $t \in [0,\frac{1}{2}]$. Then for $t \leq s \leq 1-t$,

$$\begin{aligned} G_2(t,s) &= \int_0^1 G(t,r)G(r,s)\,\mathrm{d}r \\ &= \int_0^t r(1-t)r(1-s)\,\mathrm{d}r + \int_t^s t(1-r)r(1-s)\,\mathrm{d}r + \int_s^1 t(1-r)s(1-r)\,\mathrm{d}r \\ &= \frac{1}{6}(1-s)t(2s-s^2-t^2). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t}G_2(t,s) &= \frac{1}{6}(1-s)(2s-s^2-3t^2) \\ &= \frac{1}{6}(1-s)(1-(1-s)^2-3t^2) \\ &\geq \frac{1}{6}(1-s)(1-(1-t)^2-3t^2) \\ &= \frac{1}{6}(1-s)(2t(1-2t)) \geq 0. \end{aligned}$$

So for $t_1, t_2 \in [0,\frac{1}{2}]$ with $t_2 \leq s \leq 1-t_2$, we have $G_2(t_1,s) \leq G(t_2,s)$.

Now for $t \in [0,1/2]$, $s \leq t$,

$$\begin{aligned} G_2(t,s) + G_2(1-t,s) &= \int_0^1 [G(t,r) + G(1-t,r)]G(r,s)\,\mathrm{d}r \\ &= \int_0^s [r(1-t)+rt]r(1-s)\,\mathrm{d}r + \int_s^t [r(1-t)+rt]s(1-r)\,\mathrm{d}r \\ &\quad + \int_t^{1-t} [t(1-r)+rt]s(1-r)\,\mathrm{d}r \\ &\quad + \int_{1-t}^1 [t(1-r)+(1-t)(1-r)]s(1-r)\,\mathrm{d}r \\ &= -\frac{1}{6}s^3 + \frac{1}{2}st - \frac{1}{2}st^2. \end{aligned}$$

Here

$$\begin{aligned} \frac{\partial}{\partial t}\left(G_2(t,s) + G_2(1-t,s)\right) &= \frac{1}{2}s - st \\ &= s\big(\frac{1}{2} - t\big) \geq 0. \end{aligned}$$

So if $s \le t_1 \le t_2 \le \frac{1}{2}$, then $G_2(t_1,s)+G_2(1-t_1,s) \le G_2(t_2,s)+G_2(1-t_2,s)$. If $t_1 \le s \le 1/2$, then

$$
\begin{aligned}
&G_2(t_1,s)+G_2(1-t_1,s)\\
&=\int_0^1 [G(t,r)+G(1-t,r)]G(r,s)\,\mathrm{d}r\\
&=\int_0^t [r(1-t)+rt]r(1-s)\,\mathrm{d}r+\int_t^s [t(1-r)+rt]r(1-s)\,\mathrm{d}r\\
&\quad+\int_s^{1-t} [t(1-r)+rt]s(1-r)\,\mathrm{d}r\\
&\quad+\int_{1-t}^1 [t(1-r)+(1-t)(1-r)]s(1-r)\,\mathrm{d}r\\
&=\frac{1}{2}st-\frac{1}{2}s^2t-\frac{1}{6}t^3.
\end{aligned}
$$

Now

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(G_2(t,s)+G_2(1-t,s)\right) &= \frac{1}{2}(s-s^2-t^2)\\
&\ge \frac{1}{2}(s-s^2-s^2)\\
&= \frac{1}{2}(s(1-2s)) \ge 0.
\end{aligned}
$$

Therefore, if $t_1 \le s \le t_2 \le \frac{1}{2}$, then

$$
G(t_1,s)+G(1-t_1,s) \le G(s,s)+G(1-s,s) \le G(t_2,s)+G(1-t_2,s).
$$

So (A9) is satisfied.

Since G_2 satisfies (A9), this implies for $t_1,t_2 \in [0,\frac{1}{2}]$ with $t_1 \le t_2$ and for $t_2 \le s \le 1-t_2$,

$$
\int_0^1 G(t_1,r)G(r,s)\,\mathrm{d}r \le \int_0^1 G(t_2,r)G(r,s)\,\mathrm{d}r,
$$

and for $s \le t_2$,

$$
\int_0^1 [G(t_1,r)+G(1-t_1,r)]G(r,s)\,\mathrm{d}r \le \int_0^1 [G(t_2,r)+G(1-t_2,r)]G(r,s)\,\mathrm{d}r.
$$

Let $t_1,t_2 \in [0,\frac{1}{2}]$ with $t_1 \le t_2$. For $t_2 \le s \le 1-t_2$,

$$
\begin{aligned}
G_n(t_1,s) &= \int_0^1 G(t_1,r)G_{n-1}(r,s)\,\mathrm{d}r\\
&= \int_0^1 G(t_1,r)\Big[\int_0^1 G(r,u)G_{n-2}(u,s)\,\mathrm{d}u\Big]\,\mathrm{d}r\\
&= \int_0^1 G_{n-2}(u,s)\Big[\int_0^1 G(t_1,r)G(r,u)\,\mathrm{d}r\Big]\,\mathrm{d}u\\
&\le \int_0^1 G_{n-2}(u,s)\Big[\int_0^1 G(t_2,r)G(r,u)\,\mathrm{d}r\Big]\,\mathrm{d}u\\
&= \int_0^1 G(t_2,r)\Big[\int_0^1 G(r,u)G_{n-2}(u,s)\,\mathrm{d}u\Big]\,\mathrm{d}r
\end{aligned}
$$

$$\begin{aligned}&= \int_0^1 G(t_1,r)G_{n-1}(r,s)\,\mathrm{d}r\\ &= G_n(t_2,s).\end{aligned}$$

Also, for $s \le t_2$,

$$\begin{aligned}G_n(t_1,s)+G_n(1-t_1,s) &= \int_0^1 [G(t_1,r)+G(1-t_1,r)]G_{n-1}(r,s)\,\mathrm{d}r\\ &= \int_0^1 [G(t_1,r)+G(1-t_1,r)]\Big[\int_0^1 G(r,u)G_{n-2}(u,s)\,\mathrm{d}u\Big]\,\mathrm{d}r\\ &= \int_0^1 G_{n-2}(u,s)\Big[\int_0^1 [G(t_1,r)+G(1-t_1,r)]G(r,u)\,\mathrm{d}r\Big]\,\mathrm{d}u\\ &\le \int_0^1 G_{n-2}(u,s)\Big[\int_0^1 [G(t_2,r)+G(1-t_2,r)]G(r,u)\,\mathrm{d}r\Big]\,\mathrm{d}u\\ &= \int_0^1 [G(t_2,r)+G(1-t_2,r)]\Big[\int_0^1 G(r,u)G_{n-2}(u,s)\,\mathrm{d}u\Big]\,\mathrm{d}r\\ &= \int_0^1 [G(t_2,r)+G(1-t_2,r)]G_{n-1}(r,s)\,\mathrm{d}r\\ &= G_n(t_2,r)+G_n(1-t_2,r).\end{aligned}$$

So G_n satisfies (A9).

Thus G_n satisfies (A6) and (A8) with $k=1$, (A9), and (A10). Since solutions of (6.5), (6.6) must solve the integral equation

$$x(t) = \int_0^1 G_n(t,s)f(x(s))\,\mathrm{d}s,$$

Theorem 5.2 can be applied to show the existence of positive symmetric solutions of (6.5), (6.6).

Corollary 6.4. *Let $\tau,\mu,\nu \in [0,1/2]$ with $0<\tau\le\mu<\nu\le 1/2$. Let d and m be positive reals with $0<m<2d$ and suppose $f:[0,\infty)\to[0,\infty)$ is continuous and satisfies the conditions:*

(i) *$f(w) \ge \frac{2\tau d}{\int_\tau^\nu G_n(\tau,r)\,\mathrm{d}r}$ for $w\in[2\tau d, 2\nu d]$;*
(ii) *$f(w)$ is decreasing for $0\le w\le m$ and $f(m)\ge f(w)$ for $m\le w\le d$; and*
(iii) *$2\int_0^\mu G_n(\frac12,s)f\big(\frac{s}{\mu}m\big)\,\mathrm{d}s \le d - 2f(m)\int_\mu^{1/2} G_n(\frac12,s)\,\mathrm{d}s$.*

Then (6.5), (6.6) has at least one positive symmetric solution $x^*\in A(\alpha,\beta,2\tau d,d)$.

References

[1] Abdulmalik Al Twaty, Paul W. Eloe; *Concavity of solutions of a 2n-th order problem with symmetry*, Opuscula Math. **33** (2013), no. 4, 603–613. MR 3108837

[2] Abdulmalik A. Altwaty, Paul Eloe; *The role of concavity in applications of Avery type fixed point theorems to higher order differential equations*, J. Math. Inequal. **6** (2012), no. 1, 79–90. MR 2934568

[3] D. R. Anderson, R. I. Avery, J. Henderson, X. Liu, J. W. Lyons; *Existence of a positive solution for a right focal discrete boundary value problem*, J. Difference Equ. Appl. **17** (2011), no. 11, 1635–1642. MR 2846504

[4] Douglas Anderson, Richard Avery, Johnny Henderson, Xueyan Liu; *Multiple fixed point theorems utilizing operators and functionals*, Commun. Appl. Anal. **16** (2012), no. 3, 377–387. MR 3051303

[5] Douglas R. Anderson, Richard I. Avery, Johnny Henderson; *Functional expansion-compression fixed point theorem of Leggett-Williams type*, Electron. J. Differential Equations **2010** (2010), no. 63, 1–9. MR 2651744

[6] Douglas R. Anderson, Richard I. Avery, Johnny Henderson, Xueyan Liu; *Operator type expansion-compression fixed point theorem*, Electron. J. Differential Equations **2011** (2011), no. 42, 1–11. MR 2788661

[7] Douglas R. Anderson, Richard I. Avery, Johnny Henderson, Xueyan Liu; *Fixed point theorem utilizing operators and functionals*, Electron. J. Qual. Theory Differ. Equ. **2012** (2012), no. 12, 1–16. MR 2889754

[8] Richard Avery, Douglas Anderson, Johnny Henderson; *Some fixed point theorems of Leggett-Williams type*, Rocky Mountain J. Math. **41** (2011), no. 2, 371–386. MR 2794444

[9] Richard Avery, Paul Eloe, Johnny Henderson; *A Leggett-Williams type theorem applied to a fourth order problem*, Commun. Appl. Anal. **16** (2012), no. 4, 579–588. MR 3051839

[10] Richard Avery, Johnny Henderson, Donal O'Regan; *Functional compression-expansion fixed point theorem*, Electron. J. Differential Equations **2008** (2008), no. 22, 1–12.

[11] Richard I. Avery, Douglas R. Anderson, Johnny Henderson; *An extension of the compression-expansion fixed point theorem of functional type*, Electron. J. Differential Equations **2016** (2016), no. 253, 1–12. MR 3578274

[12] Richard I. Avery, Douglas R. Anderson, Johnny Henderson; *Generalization of the functional omitted ray fixed point theorem*, Comm. Appl. Nonlinear Anal. **25** (2018), no. 1, 39–51. MR 3791735

[13] Richard I. Avery, John R. Graef, Xueyan Liu; *Compression fixed point theorems of operator type*, J. Fixed Point Theory Appl. **17** (2015), no. 1, 83–97. MR 3392983

[14] Richard I. Avery, Johnny Henderson, Douglas R. Anderson; *A topological proof and extension of the Leggett-Williams fixed point theorem*, Comm. Appl. Nonlinear Anal. **16** (2009), no. 4, 39–44. MR 2591327

[15] Richard I. Avery, Johnny Henderson, Douglas R. Anderson; *Existence of a positive solution to a right focal boundary value problem*, Electron. J. Qual. Theory Differ. Equ. **2010** (2010), no. 5, 1–6. MR 2577158

[16] Daria Bugajewska, Gennaro Infante, Piotr Kasprzak; *Solvability of Hammerstein integral equations with applications to boundary value problems*, Z. Anal. Anwend. **36** (2017), no. 4, 393–417. MR 3713050

[17] Alberto Cabada, José Ángel Cid, Gennaro Infante; *A positive fixed point theorem with applications to systems of Hammerstein integral equations*, Bound. Value Probl. **2014** (2014), no. 254, 1–10. MR 3286705

[18] Filomena Cianciaruso, Gennaro Infante, Paolamaria Pietramala; *Solutions of perturbed Hammerstein integral equations with applications*, Nonlinear Anal. Real World Appl. **33** (2017), 317–347. MR 3543125

[19] Kai Diethelm; *The analysis of fractional differential equations*, Lecture Notes in Mathematics, vol. 2004, Springer-Verlag, Berlin, 2010, An application-oriented exposition using differential operators of Caputo type. MR 2680847

[20] Paul W. Eloe, Jeffrey W. Lyons, Jeffrey T. Neugebauer; *An ordering on Green's functions for a family of two-point boundary value problems for fractional differential equations*, Commun. Appl. Anal. **19** (2015), 453–462.

[21] Paul W. Eloe, Jeffrey T. Neugebauer; *Concavity in fractional calculus*, Filomat **32** (2018), no. 9, 3123–3128. MR 3898885

[22] Rubén Figueroa, F. Adrián F. Tojo; *Fixed points of Hammerstein-type equations on general cones*, Fixed Point Theory **19** (2018), no. 2, 571–585. MR 3821784

[23] Gennaro Infante, Feliz Minhós; *Nontrivial solutions of systems of Hammerstein integral equations with first derivative dependence*, Mediterr. J. Math. **14** (2017), no. 6, Art. 242, 18. MR 3735472

[24] Anatoly A. Kilbas, Hari M. Srivastava, Juan J. Trujillo; *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006. MR 2218073

[25] Richard W. Leggett, Lynn R. Williams; *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J. **28** (1979), no. 4, 673–688. MR 542951

[26] Cristina Lois-Prados, Rosana Rodríguez-López; *A generalization of krasnosel'skii compression fixed point theorem by using star convex sets*, Proc. Roy. Soc. Edinburgh Sect. A (2019), 1—27.

[27] J. W. Lyons, J. T. Neugebauer; *Existence of a positive solution for a right focal dynamic boundary value problem*, Nonlinear Dyn. Syst. Theory **14** (2014), no. 1, 76–83. MR 3222127

[28] Kenneth S. Miller, Bertram Ross; *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1993. MR 1219954

[29] Jeffrey T. Neugebauer, Charley L. Seelbach; *Positive symmetric solutions of a second-order difference equation*, Involve **5** (2012), no. 4, 497–504. MR 3069051

[30] Igor Podlubny; *Fractional differential equations*, Mathematics in Science and Engineering, vol. 198, Academic Press, Inc., San Diego, CA, 1999, An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. MR 1658022

[31] Radu Precup, Jorge Rodríguez-López; *Multiplicity results for operator systems via fixed point index*, Results Math. **74** (2019), no. 1, Art. 25, 14. MR 3897520

[32] J. R. L. Webb; *New fixed point index results and nonlinear boundary value problems*, Bull. Lond. Math. Soc. **49** (2017), no. 3, 534–547. MR 3723638

Paul W. Eloe
Department of Mathematics, University of Dayton, Dayton, OH 45469, USA
Email address: `peloe1@udayton.edu`

Jeffrey T. Neugebauer
Department of Mathematics and Statistics, Eastern Kentucky University, Richmond, KY 40475, USA
Email address: `Jeffrey.Neugebauer@eku.edu`