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CLASSICAL-REGULAR SOLVABILITY OF INITIAL BOUNDARY VALUE PROBLEMS OF NONLINEAR WAVE EQUATIONS WITH TIME-DEPENDENT DIFFERENTIAL OPERATOR AND DIRICHLET BOUNDARY CONDITIONS

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Abstract. This article concerns the nonlinear wave equation

$$\begin{gathered} u_{tt}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\big\{a_{ij}(t,x)\frac{\partial u}{\partial x_j}\big\}+c(t,x)u+\lambda u \\ +\mathbf{F}'\big(|u|^2\big)u+\zeta u=0, \quad t\in[0,\infty),\ x\in\bar\Omega \\ u(0,x)=\varphi,\quad u_t(0,x)=\psi,\quad u|_{\partial\Omega}=0. \end{gathered}$$

Essentially this article ascertains and proves the important mapping property

$$\mathbf{M}: D\big(\mathbf{A}^{(k_0''+1)/2}(0)\big)\to D\big(\mathbf{A}^{k_0''/2}(0)\big),\quad D(\mathbf{A}(0))=H_0^1(\Omega)\cap H^2(\Omega),$$

as well as the associated Lipschitz condition

$$\begin{aligned} &\|\mathbf{A}^{k_0''/2}(0)(\mathbf{M}u-\mathbf{M}v)\| \\ &\le k\Big(\|\mathbf{A}^{(k_0''+1)/2}(0)u\|+\|\mathbf{A}^{(k_0''+1)/2}(0)v\|\Big)\|\mathbf{A}^{(k_0''+1)/2}(0)(u-v)\|, \end{aligned}$$

where

$$\begin{gathered} \mathbf{A}(t):=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\big\{a_{ij}(t,x)\frac{\partial}{\partial x_j}\big\}+c(t,x)+\lambda, \\ \mathbf{M}u:=\mathbf{F}'\big(|u|^2\big)u+\zeta u, \\ k''\in\mathbb{N},\quad k''>\frac{n}{2}+1,\quad k_0'':=\min\{k''\}, \end{gathered}$$

and $k(\cdot)\in C^0_{\rm loc}\big(\mathbb{R}^+,\mathbb{R}^{++}\big)$ is monotonically increasing. Here are $\mathbb{R}^+=[0,\infty)$, $\mathbb{R}^{++}=(0,\infty)$. This mapping property is true for the dimensions $n\le 5$. But we investigate only the case $n=5$ because the problem is already solved for $n\le 4$, however, without the mapping property.

With the proof of the mapping property and the associated Lipschitz condition, the problem becomes considerably comparable with a paper from von Wahl, who investigated the same problem as Cauchy problem and solved it for the dimensions $n\le 6$, i.e. without boundary condition. In the case of the Cauchy problem there are no difficulties with regard to the mapping property.

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1. Introduction

This article concerns the classical solvability of the nonlinear wave equation

$$\begin{aligned} & u_{tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\big\{a_{ij}(t,x)\frac{\partial u}{\partial x_j}\big\} + c(t,x)u + \lambda u \\ & + \mathbf{F}'\big(|u|^2\big)u + \zeta u = 0, \quad t \in [0,\infty),\ x \in \bar{\Omega} \\ & u(0,x) = \varphi, \quad u_t(0,x) = \psi, \quad u|_{\partial\Omega} = 0. \end{aligned} \tag{1.1}$$

von Wahl considered in his thesis [10] the above problem as Cauchy problem and solved it for the dimensions $n \leq 6$. In contrast to Cauchy problem, the above mentioned mapping property is the real problem in our case of boundary value problem. von Wahl [11, 12] solved problem (1.1) for dimensions $n = 3, 4$. The proofs therein avoid the mapping property and they are complicated therefore. In this article, we prove this mapping property and the associated Lipschitz condition for the dimension $n = 5$ (Theorem 4.4), which means that it holds for lower dimensions also. With that we solve problem (1.1) completely because the rest is more or less similar to [10] or follows from the abstract part of that (i.e. the solution in abstract Hilbert space). Our treatment is based on Hilbert space methods. Furthermore, the discussion in this Introduction consider only initial boundary value problems with Dirichlet boundary condition.

The paper by Sather [6] belongs to the beginnings, in which the author obtained classical solutions for dimension $n = 3$ and the condition $\mathbf{A}(t) = -\Delta$ (i.e. $\mathbf{A}(t)$ is constant) and $Mu = u^3$ and the region $\Omega \subset \mathbb{R}^3$ is bounded. This boundedness is a consequence of the application of the Galerkin proceeding for the existence of the solution. This proceeding depends on the eigenfunctions of the Laplace operator Δ, which (the eigenfunctions) does not exist if Ω is unbounded.

Pecher [5] investigate a similar problem type and obtained classical solutions for dimensions $n \leq 5$ under different conditions on the nonlinearity as ours and with $\mathbf{A}(t) = \mathbf{A}$ (i.e. $\mathbf{A}(t)$ is constant).

Brenner and von Wahl [2] obtained also classical solutions for the same problem in dimensions $n = 5, 6, 7$, but under the condition that the differential operators $\mathbf{A}(t)$ are constant. Scarpellini and Uesaka [7, 9] generalized the problem area and obtained via different methods strong solutions for special cases of our problem.

The present results include those in [4], in which the authors considered $\mathbf{F}(s) \leq 0$. That means that an a priori estimate for the energy norm fails. The authors use the Galerkin method that yields a unique global weak solution for the problem with bounded Ω and small initial data as well as $\mathbf{A}(t) = -\Delta$. All the solutions mentioned above are real valued, while ours are complex valued as in [10].

2. Preliminaries

In this chapter, we give an insight in our strategy and prepare some of the needed tools. Our approach is based on Hilbert space methods. That means a treatment of the problem first local (i.e. in some interval $[t_0, t_0 + T_0] \subset \mathbb{R}^+$) in abstract Hilbert space H. Later we set $H = L^2(\Omega)$. The problem in the abstract Hilbert space H means with $k \in \mathbb{N}$:

$$\begin{aligned} & u''(t) + \mathbf{A}(t)u(t) + \mathbf{M}u(t) = 0, \quad t \in [t_0, t_0 + T_0] \\ & u(t_0) = \varphi \in D(\mathbf{A}^{(k+1)/2}(0)), \quad u'(t_0) = \psi \in D(\mathbf{A}^{k/2}(0)). \end{aligned} \tag{2.1}$$

Here, $\mathbf{A}(t)$, $t \in \mathbb{R}^+$ is a family of self-adjoint, positive definite operators in H with a constant domain of definition (i.e. $D(\mathbf{A}(t)) = D(\mathbf{A}(0))$. We assume further that for each $\nu \in \mathbb{N}$ there exists a constant $L(\nu)$ such that

$$\|(\mathbf{A}^{\nu/2}(t) - \mathbf{A}^{\nu/2}(s))\mathbf{A}^{-\nu/2}(r)\| \leq L(\nu)|t-s|, \quad t, s, r \in \mathbb{R}^+, \tag{2.2}$$

which implies the relations

$$\|\mathbf{A}^{\nu/2}(t)\mathbf{A}^{-\nu/2}(0)\|,\ \|\mathbf{A}^{\nu/2}(0)\mathbf{A}^{-\nu/2}(t)\| \leq (1+L(\nu))t$$

as well as for $u \in D(\mathbf{A}^{\nu/2}(0))$

$$\|\mathbf{A}^{\nu/2}(0)u\| = \|\mathbf{A}^{\nu/2}(0)\mathbf{A}^{-\nu/2}(t)\mathbf{A}^{\nu/2}(t)u\| \leq (1+L(\nu)t)\|\mathbf{A}^{\nu/2}(t)u\|, \quad t \in \mathbb{R}^+.$$

The $k \in \mathbb{N}$ above represents the degree of the regularity of the solution, which relates in the concrete case to the suitable Sobolev spaces, in correspondence with the required classical regularity (i.e. twice continuously differentiability in t and x) for the solution via Sobolev theorems. For clearness, in the concrete case: $D(\mathbf{A}(0)) = D(\mathbf{A}(t)) = H_0^1(\Omega) \cap H^2(\Omega)$, $D(\mathbf{A}^{1/2}(0)) = H_0^1(\Omega)$, and

$$D(\mathbf{A}^{(k+1)/2}(0)) = D(\mathbf{A}^{(k+1)/2}(t)) \subset H_0^1(\Omega) \cap H^{k+1}(\Omega).$$

In the concrete case, the above relation (2.2) is a consequence of the differentiability of $\mathbf{A}^{\nu/2}(t)$.

We suppose also that $\mathbf{A}(t)$ is strongly continuously differentiable on $D(\mathbf{A}(0))$ and set:

$$\mathbf{A}'(t)x := \frac{d}{dt}(\mathbf{A}(t)x), \quad t \in \mathbb{R}^+,\ x \in D(\mathbf{A}(0)).$$

Theorem 2.1 (v. Wahl [10, Satz 8, page 275]). *Let H be an abstract Hilbert space. Let $\mathbf{A}(t)$, $t \in \mathbb{R}^+$, be a family of self-adjoint, positive definite operators in H with a constant domain of definition (i.e. $D(\mathbf{A}(t)) = D(\mathbf{A}(0))$). Suppose that $\mathbf{A}(t)x$, $x \in D(\mathbf{A}(t))$, is continuously differentiable. Then the domain of definition of $\mathbf{A}^{1/2}(t)$ is also constant and $\mathbf{A}^{1/2}(t)x$, $x \in D(\mathbf{A}^{1/2}(0))$, is continuously differentiable with respect to t.*

We set

$$(\mathbf{A}^{1/2}(t))'x = \frac{d}{dt}(\mathbf{A}^{1/2}(t)x), \quad t \in \mathbb{R}^+,\ x \in D(\mathbf{A}^{1/2}(0)).$$

We need also the following 2 results.

Proposition 2.2 (v. Wahl [10, Hilfssatz 3, page 251]). *It is for all $t \in \mathbb{R}^+$, $\nu \in \mathbb{N}$: $\mathbf{A}^{\nu/2}(t)(\mathbf{A}^{1/2}(t))'\mathbf{A}^{-\frac{\nu+1}{2}}(t) \in L(H)$, with*

$$\|\mathbf{A}^{\nu/2}(t)(\mathbf{A}^{1/2}(t))'\mathbf{A}^{-(\nu+1)/2}(t)\| \leq L(\nu) + L(\nu+1), \quad t \in \mathbb{R}^+,\ \nu \in \mathbb{N}.$$

Proposition 2.3 (v. Wahl [10, Hilfssatz 3a, page 252]). *For every $\nu \in \mathbb{N}$ and $x \in H$, the function $\mathbf{A}^{\nu/2}(t)(\mathbf{A}^{1/2}(t))'\mathbf{A}^{-\frac{\nu+1}{2}}(t)x$ is continuous in t.*

Regarding the nonlinearity, let the mapping

$$\mathbf{M} : D(\mathbf{A}^{(k+1)/2}(0)) \to D(\mathbf{A}^{k/2}(0)) \tag{2.3}$$

satisfy the following conditions:

(I') Continuity: Let $\{u_\nu\}$ be a sequence in $D(\mathbf{A}^{(k+1)/2}(0))$ which converges to u in the graph norm of $\mathbf{A}^{(k+1)/2}(0)$. Then $\{\mathbf{M}u_\nu\}$ converges to $\mathbf{M}u$ in the graph norm of $D(\mathbf{A}^{k/2}(0))$.

(II') Local Lipschitz Condition: Let $u, v \in D(\mathbf{A}^{(k+1)/2}(0))$. If for $C \geq 0$, $t \geq 0$:
$$\sup_{\tau\in[t,t+1]} \big(\|\mathbf{A}^{(k+1)/2}(\tau)u\| + \|\mathbf{A}^{(k+1)/2}(\tau)v\|\big) \leq C$$
then it is for all $\tau \in [t, t+1]$,
$$\|\mathbf{A}^{k/2}(\tau)\big(\mathbf{M}u - \mathbf{M}v\big)\| \leq k(C,t)\, \|\mathbf{A}^{(k+1)/2}(\tau)(u-v)\|.$$
Here $k(\cdot,\cdot) \in C^0_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^{++})$ is a fixed function to be found.

(III') Null Point Property: $\mathbf{M}(0) = \mathbf{M}0 = 0$.

For the transition to the concrete case in $L^2(\Omega)$, the following definition is suitable.

Definition 2.4 (v. Wahl [10, page 253])**.** Let $\varphi \in D(\mathbf{A}^{(k+1)/2}(0))$, $\psi \in D(\mathbf{A}^{k/2}(0))$, $k \in \mathbb{N}$, $k \geq 1$. The function $u \in C^2([a,b], H)$ is called a k-regular solution of the differential equation (2.1) on $[a,b] \subset \mathbb{R}^+$ if $u(a) = \varphi$, $u'(a) = \psi$ and for every $t \in [a,b]$:
$$u(t) \in D(\mathbf{A}^{(k+1)/2}(0)), \quad u'(t) \in D(\mathbf{A}^{k/2}(0)), \quad u''(t) \in D(\mathbf{A}^{\frac{k-1}{2}}(0))$$
such that
$$u(\cdot) \in C^i\big([a,b], D(\mathbf{A}^{\frac{k-i+1}{2}}(0))\big), \quad i = 0,1,2.$$
Finally $u'' + \mathbf{A}(t)u + \mathbf{M}u = 0$ in $[a,b]$.

The above details are not directly applicable for our purpose because the directly treatment of (2.1) is not easy job. Therefore, we use a detour via the first order differential equation in abstract Hilbert space $\hat{H}$ (later we set $\hat{H} = H \times H$):
$$\begin{gathered} \hat{u}'(t) + \hat{\mathbf{A}}(t)\hat{u}(t) + \hat{\mathbf{M}}(t, \hat{u}(t)) = 0\,, \quad t \in [t_0, t_0 + T_0] \\ \hat{u}(t_0) = \hat{\varphi} \in D\big(\hat{\mathbf{A}}^k(0)\big). \end{gathered} \tag{2.4}$$
Here, $i\hat{\mathbf{A}}(t)$, $t \geq 0$, are a family of self-adjoint operators in Hilbert space $\hat{H}$ with $D(\hat{\mathbf{A}}^\nu(t)) = D(\hat{\mathbf{A}}^\nu(0))$, $\nu \in \mathbb{N}$. Further, every $\hat{\mathbf{A}}(t)$ possesses a bounded inverse in $\hat{H}$. Moreover, for every $\nu \in \mathbb{N}$, there exists a constant $\hat{L}(\nu)$ such that
$$\|\big(\hat{\mathbf{A}}^\nu(t) - \hat{\mathbf{A}}^\nu(s)\big)\hat{\mathbf{A}}^{-\nu}(r)\| \leq \hat{L}(\nu)|t-s|\,, \quad t,s,r \geq 0.$$
Because $i\hat{\mathbf{A}}(t)$ is self-adjoint,
$$\|\big(I + \alpha\hat{\mathbf{A}}(t)\big)^{-1}\| \leq 1\,, \quad \alpha > 0$$
and because
$$\begin{aligned} \|\big(\hat{\mathbf{A}}(t) - \hat{\mathbf{A}}(s)\big)\big(I + \hat{\mathbf{A}}(r)\big)^{-1}\| &\leq \|\big(\hat{\mathbf{A}}(t) - \hat{\mathbf{A}}(s)\big)\hat{\mathbf{A}}^{-1}(r)\|\, \|\hat{\mathbf{A}}(r)\big(I + \hat{\mathbf{A}}(r)\big)^{-1}\| \\ &\leq 2\hat{L}(1)|t-s|, \quad t,s,r \in \mathbb{R}^+, \end{aligned}$$
it follows by Kato [3, Theorem 3, page 210], (see also [10, page 244]) the existence of the evolution operator $\hat{U}(t,s) \in L(\hat{H})$, $t,s \in \mathbb{R}^+$, $t \geq s$, such that
$$\begin{gathered} \|\hat{\mathbf{U}}(t,s)\| \leq 1, \quad \hat{\mathbf{U}}(t,t) = I, \\ \hat{\mathbf{U}}(t,s)\hat{\mathbf{U}}(s,r) = \hat{\mathbf{U}}(t,r), \quad \hat{\mathbf{U}}(t,s)D\big(\hat{\mathbf{A}}(0)\big) \subset D\big(\hat{\mathbf{A}}(0)\big), \end{gathered} \tag{2.5}$$
as well as for every $x \in D\big(\hat{\mathbf{A}}(0)\big)$ and every $s \in \mathbb{R}^+$, $\hat{\mathbf{U}}(t,s)x$ is continuously differentiable in t with
$$\frac{\partial}{\partial t}\big(\hat{\mathbf{U}}(t,s)x\big) + \hat{\mathbf{A}}(t)\hat{\mathbf{U}}(t,s)x = 0. \tag{2.6}$$

Moreover, for every $x \in \hat{H}$, $\hat{\mathbf{U}}(t,s)x$ is simultaneously continuous in t,s. The evolution operator $\hat{\mathbf{U}}(t,s)$ is uniquely determined by those properties. The following 2 statements are a consequence of the above discussion.

Theorem 2.5 (v. Wahl [10, Satz 1, page 245])**.** *Let $\nu \in \mathbb{N}$. Then*

$$\hat{\mathbf{A}}(t)\hat{\mathbf{U}}(t,s)\hat{\mathbf{A}}^{-1}(s) \in L(\hat{H}),$$
$$\|\hat{\mathbf{A}}^{\nu}(t)\hat{\mathbf{U}}(t,s)\hat{\mathbf{A}}^{-\nu}(s)\| \le e^{\hat{L}(\nu)(t-s)}\,.$$

Proposition 2.6 (v. Wahl [10, Hilfssatz 2, page 246])**.** *Let $x \in \hat{H}$, $\nu \in \mathbb{N}$. Then the function $\hat{\mathbf{A}}^{\nu}(t)\hat{\mathbf{U}}(t,s)\hat{\mathbf{A}}^{-\nu}(s)x$ is continuous in t for a fixed s and continuous in s for a fixed t.*

Regarding the nonlinearity $\hat{\mathbf{M}}(t,\hat{u})$ in (2.4), we have the following conditions in conformity with our differential equation (2.1). Let

$$\hat{\mathbf{M}} : \mathbb{R}^+ \times D\big(\hat{\mathbf{A}}^k(0)\big) \to D\big(\hat{\mathbf{A}}^k(0)\big) \tag{2.7}$$

be a mapping with the following properties:

(I) Continuity: Let $\{t_\nu\}$, $t_\nu \in \mathbb{R}^+$, be a sequence which converges to t, and let $\{\hat{u}_\nu\}$ be a sequence in $D\big(\hat{\mathbf{A}}^k(0)\big)$ which converges in the graph norm of $D\big(\hat{\mathbf{A}}^k(0)\big)$ to $\hat{u}$. Then the sequence $\{\hat{\mathbf{M}}(t_\nu,\hat{u}_\nu)\}$ converges in the graph norm of $D(\hat{\mathbf{A}}^k(0))$ to $\hat{\mathbf{M}}(t,\hat{u})$.

(II) Local Lipschitz Condition: Let $\hat{u},\hat{v} \in D\big(\mathbf{A}^k(0)\big)$, $t \in \mathbb{R}^+$, $C \in \mathbb{R}^+$, and
$$\sup_{\tau\in[t,t+1]}\big(\|\hat{\mathbf{A}}^k(\tau)\hat{u}\| + \|\hat{\mathbf{A}}^k(\tau)\hat{v}\|\big) \le C.$$
Then for $\tau \in [t,t+1]$ it holds
$$\|\hat{\mathbf{A}}^k(\tau)\hat{\mathbf{M}}(\tau,\hat{u}) - \hat{\mathbf{A}}^k(\tau)\hat{\mathbf{M}}(\tau,\hat{v})\| \le \hat{k}(C,t)\|\hat{\mathbf{A}}^k(\tau)(\hat{u}-\hat{v})\|.$$
Here $\hat{k}(\cdot,\cdot) \in C^0_{\rm loc}\big(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^{++}\big)$ is a fixed function to be found.

(III) Null Point Property: $\hat{\mathbf{M}}(t,0) = 0$, $t \in \mathbb{R}^+$.

Now, we conform definition for the solution of (2.4).

Definition 2.7 (v. Wahl [10, page 247])**.** Let $\hat{\varphi} \in D\big(\hat{\mathbf{A}}^k(0)\big)$, $k \in \mathbb{N}$, $k \ge 1$. The function $\hat{u} \in C^1\big([a,b],\hat{H}\big)$ is called a k-regular solution of (2.4) on $[a,b] \subset \mathbb{R}^+$ if the following conditions are satisfied

(1) $\hat{u}(a) = \hat{\varphi}$;
(2) For every $t \in [a,b] : \hat{u}(t) \in D\big(\hat{\mathbf{A}}^k(0)\big)$, $\hat{u}'(t) \in D\big(\hat{\mathbf{A}}^{k-1}(0)\big)$;
(3) $\hat{\mathbf{A}}^{k-j}(0)\big(\frac{d^j}{dt^j}\hat{u}\big)(\cdot) \in C^0\big([a,b],\hat{H}\big)$, $j = 0,1$;
(4) $\hat{u}' + \hat{\mathbf{A}}(t)\hat{u} + \hat{\mathbf{M}}(t,\hat{u}) = 0$ in $[a,b]$.

In contrast to (2.1), there exists a suitable integral equation for (2.4), namely

$$\hat{u}(t) = \hat{U}(t,a)\hat{\varphi} - \int_a^t \hat{\mathbf{U}}(t,s)\hat{\mathbf{M}}(s,\hat{u}(s))ds\,, \quad t \in [a,b]. \tag{2.8}$$

According to v. Wahl [10, Satz 2, page 248], the use of the successive approximation proceeding on (2.8) yields the existence of a unique k-regular solution for the associated differential equation (2.4) on $[t_0,t_0+T_0]$ (i.e. with $t_0 = a$) for some $T_0 > 0$. Of course, a further existence interval $[t_0+T_0,t_0+T_0+T_1]$ is also available and with that the existence of a unique k-regular solution for (2.4) on a maximal interval $[t_0, T)$ with $\lim_{t\to T} \|\hat{\mathbf{A}}^k(t)\hat{u}(t)\| = \infty$, otherwise we can extend further. So, if there is an a priori estimate for $\|\hat{\mathbf{A}}^k(t)\hat{u}(t)\|$ on $[0,\infty)$ then the k-regular solution of (2.4) is extendable on the whole $[t_0,\infty)$.

We suppose now that the mapping property (2.3) and their conditions (I')–(III') are satisfied. We set $\hat{H} := H\times H$, $\hat{u} := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, and $\hat{\mathbf{A}}(t) := \begin{pmatrix} -i\mathbf{A}^{1/2}(t) & 0 \\ 0 & i\mathbf{A}^{1/2}(t) \end{pmatrix}$. Then for $t \in \mathbb{R}^+$, $\hat{u} := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in D\big(\hat{\mathbf{A}}^k(0)\big)$ and by Propositions 2.2 and 2.3, the mapping $\hat{\mathbf{M}}(t,\hat{u}) : \mathbb{R}^+ \times D\big(\hat{\mathbf{A}}^k(0)\big) \to D\big(\hat{\mathbf{A}}^k(0)\big)$ defined by

$$\hat{\mathbf{M}}(t,\hat{u}) := \frac{1}{2}\begin{pmatrix} \mathbf{M}(-i\mathbf{A}^{-1/2}(t)(u_1-u_2)) \\ \mathbf{M}(-i\mathbf{A}^{-1/2}(t)(u_1-u_2)) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} (\mathbf{A}^{1/2}(t))'\mathbf{A}^{-1/2}(t)(u_1-u_2) \\ -(\mathbf{A}^{1/2}(t))'\mathbf{A}^{-1/2}(t)(u_1-u_2) \end{pmatrix} \tag{2.9}$$

holds and fulfils obviously the mapping property (2.7) and their conditions (I)–(III) (see also v. Wahl [10, page 253]).

We set now the transition theorem from $\hat{H}$ to H.

Theorem 2.8 (v. Wahl [10, Satz 3, page 254])**.** *Let $\varphi \in D\big(\mathbf{A}^{(k+1)/2}(0)\big)$, $\psi \in D\big(\mathbf{A}^{k/2}(0)\big)$, $t_0 \in \mathbb{R}^+$, and set*

$$\hat{\varphi} := \frac{1}{2}\begin{pmatrix} i\mathbf{A}^{1/2}(t_0)\varphi + \psi \\ -i\mathbf{A}^{1/2}(t_0)\varphi + \psi \end{pmatrix}.$$

Moreover, let $\hat{u}(\cdot) \in C^1\big([t_0,t_0+T_0],\hat{H}]\big)$ be the unique k-regular solution of the differential equation (2.4). Then there exists a unique k-regular solution of the differential equation (2.1) on the interval $[t_0,t_0+T_0]$. It is

$$u = -i\mathbf{A}^{-1/2}(t)(u_1-u_2),$$

where $\hat{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

Corollary 2.9. *We have $u_1 - u_2 = i\mathbf{A}^{1/2}(t)u$. On the other hand, in the proof of* [10, Theorem 3 (Satz 3), page 255] *is shown that $u_1 + u_2 = u'$. It follows that*

$$u_1 = \frac{1}{2}\big(u' + i\mathbf{A}^{1/2}(t)u\big), \quad u_2 = \frac{1}{2}\big(u' - i\mathbf{A}^{1/2}(t)u\big).$$

Regarding the nonlinearity, we have

$$\hat{\mathbf{M}}(t,\hat{u}) = \frac{1}{2}\begin{pmatrix} \mathbf{M}u \\ \mathbf{M}u \end{pmatrix} + \frac{1}{2}\begin{pmatrix} i(\mathbf{A}^{1/2}(t))'u \\ -i(\mathbf{A}^{1/2}(t))'u \end{pmatrix}.$$

For the transition to the concrete case (i.e. $H = L^2(\Omega)$), we need the following well-known Sobolev Embedding Theorem (see v. Wahl [10, page 259]).

Theorem 2.10. *(1) Let $m \in \mathbb{N}\cup\{0\}$, $2 \le p < \infty$ with $\frac{1}{p} \ge \frac{1}{2} - \frac{m}{n}$. Then $H^m(\Omega) = H^{m,2}(\Omega) \subset L^p(\Omega)$ with a continuous embedding.*

(2) If $m > \frac{n}{2} + k$, $k \in \mathbb{N}\cup\{0\}$ then $H^m(\Omega) \subset C^k(\overline{\Omega})$, with a continuous embedding.

Based on this fact, in the k_0''-regular context (i.e. $5 = k_0'' + 1 > \frac{n}{2} + 2$, see the Abstract), for the solution of (2.1) we have the following result.

Proposition 2.11 (v. Wahl [10, Hilfssatz 8, page 263]). *In dimension $n = 5$, the 4-regular solution of Theorem 2.8 is as a function of $t, x_1, \dots, x_5$ twice classical continuously differentiable with respect to all these variables, and*

$$\Big(\frac{\partial^\nu}{\partial t^\nu} u\Big)(t, x_1, \dots, x_5) = \Big(\frac{d^\nu}{dt^\nu} u\Big)(t, x_1, \dots, x_5), \quad \nu = 1, 2.$$

Proof. By the Embedding Theorem 2.10 and Theorem 3.1 (set $x = (x_1, \dots, x_5)$),

$$\begin{aligned}
&\Big|\frac{u(t+h,x) - u(t,x)}{h} - \Big(\frac{d}{dt}u\Big)(t,x)\Big| \\
&\le c_1 \Big\|\frac{u(t+h,x) - u(t,x)}{h} - \Big(\frac{d}{dt}u\Big)(t,x\Big\|_{H^3(\Omega)} \\
&\le c_2 \Big\|\mathbf{A}^{\frac{3}{2}}(0)\Big(\frac{u(t+h) - u(t)}{h} - u'(t)\Big)\Big\| \to 0
\end{aligned}$$

as $h \to 0$. The other derivatives and continuities follow by a similar process. $\square$

In context with the Embedding Theorem above, we take from [10] the following 2 statements; the proofs depend merely on the Hölder inequality and the above Embedding Theorem 2.10 as in [10].

Proposition 2.12 (v. Wahl [10, Hilfssatz 6, page 260]). *Let $m, N \in \mathbb{N}$, $m > \frac{n}{2}$, and $\alpha_1, \dots, \alpha_N$ are multi-indexes belong to $\mathbb{R}^n$ with $\alpha := \sum_{\nu=1}^N \alpha_\nu$, $|\alpha| \le m$. If $u_1, \dots, u_N \in H^m(\Omega)$ then $\prod_{\nu=1}^N u_\nu \in H^m(\Omega)$ and*

$$\Big\|\prod_{\nu=1}^N D^{\alpha_\nu} u_\nu\Big\| \le c_1 \prod_{\nu=1}^N \|u_\nu\|_{H^m(\Omega)}\,.$$

Proposition 2.13 (v. Wahl[10, Hilfssatz 7, page 261]). *Let $m, N \in \mathbb{N}$, $m \ge \frac{n}{2}$, and let $\alpha_1, \dots, \alpha_N$ be as in Proposition 2.11. If $u_1, \dots, u_N \in H^{m+1}(\Omega)$, then there exists $\nu_0 \in \mathbb{N}$, $1 \le \nu_0 \le N$ such that*

$$\Big\|\prod_{\nu=1}^N D^{\alpha_\nu} u_\nu\Big\| \le c_2 \|u_{\nu_0}\|_{H^{m+1}(\Omega)} \prod_{\nu \ne \nu_0} \|u_\nu\|_{H^m(\Omega)}$$

holds. Especially, $\prod_{\nu=1}^N u_\nu \in H^{m+1}(\Omega)$.

3. Differential operators $\mathbf{A}(t)$

In the following let $n = 5$ and $\Omega \subset \mathbb{R}^5$ be bounded or unbounded domain with $\partial\Omega \in C^\infty$. Define the operators

$$\mathbf{A}(t) := -\sum_{i,j=1}^5 \frac{\partial}{\partial x_i}\big\{a_{ij}(t,x)\frac{\partial}{\partial x_j}\big\} + c(t,x) + \lambda, \quad t \in \mathbb{R}^+$$

with the following properties:

1. $a_{ij}(t,x), c(t,x) \in C^\infty(\mathbb{R}^+ \times \bar{\Omega}, \mathbb{R})$ $(i,j = 1, \dots, 5)$. Moreover these functions and their derivatives are uniformly bounded;
2. Symmetry: $a_{ij}(t,x) = a_{ji}(t,x)$, $(i,j = 1, \dots, 5)$,
3. Ellipticity: The operators $\mathbf{A}(t)$ are uniformly elliptic on Ω, i.e. there is a $\mu_0 > 0$ such that for all $\xi = (\xi_1, \dots, \xi_5) \in \mathbb{C}^5$, $x \in \Omega$ the inequality:

$$\sum_{i,j=1}^5 a_{ij}(t,x)\xi_i\bar{\xi}_j \ge \mu_0|\xi|^2$$

holds, where μ_0 is independent from t;

(4) Positive definiteness: λ is sufficiently large. This condition is necessary to guarantee the positive definiteness of $\mathbf{A}(t)$ and it has only formally meaning because this λ can be subtracted at the end from the nonlinearity. The positive definiteness of $\mathbf{A}(t)$ is necessary because we will operate with the root of $\mathbf{A}(t)$.

Of course, under the above assumptions, $\mathbf{A}(t)u$, $u \in D(\mathbf{A}(0))$, is infinitely differentiable with respect to t with:

$$\mathbf{A}^{(\nu)}(t)u = -\sum_{i,j=1}^{5} \frac{\partial}{\partial x_i}\Big\{\Big(\frac{\partial^\nu}{\partial t^\nu} a_{ij}(t,x)\Big)\frac{\partial u}{\partial x_j}\Big\} + \frac{\partial^\nu}{\partial t^\nu} c(t,x)u.$$

It is well-known that under the above conditions the operators $\mathbf{A}(t)$ are self-adjoint and positive definite on the constant domain of definition $D(\mathbf{A}(t)) = D(\mathbf{A}(0)) = H_0^1(\Omega) \cap H^2(\Omega)$ with $D(\mathbf{A}^{1/2}(t)) = H_0^1(\Omega)$ (see Tanabe [8, page 113]). Furthermore for $m \in \mathbb{N}$,

$$D(\mathbf{A}^{\frac{m}{2}}(t)) \subset H_0^1(\Omega) \cap H^m(\Omega).$$

Theorem 3.1. *For $m \in \mathbb{N}$ and $u \in D(\mathbf{A}^{\frac{m}{2}}(0))$ it is:*

$$c_1(m)\|u\|_{H^m(\Omega)} \le \|\mathbf{A}^{\frac{m}{2}}(0)u\| \le c_2(m)\|u\|_{H^m(\Omega)}. \tag{3.1}$$

Proof. By induction. We prove first that

$$c_1(1)\|u\|_{H^1(\Omega)} \le \|\mathbf{A}^{1/2}(0)u\| \le c_2(1)\|u\|_{H^1(\Omega)}, \quad u \in D(\mathbf{A}^{1/2}(0)) = H_0^1(\Omega).$$

For $u \in D(\mathbf{A}(0)) = H_0^1(\Omega) \cap H^2(\Omega)$,

$$\begin{aligned}
(\mathbf{A}(0)u,u) &= -\sum_{i,j=1}^{5}\int_\Omega \big(a_{ij}(0,x)u_{x_i}\big)_{x_j}\bar{u}\,\mathrm{d}x + \int_\Omega \big(c(0,x)+\lambda\big)u\bar{u}\,\mathrm{d}x\\
&= \sum_{i,j=1}^{5}\int_\Omega a_{ij}(0,x)u_{x_i}\bar{u}_{x_j}\,\mathrm{d}x + \int_\Omega \big(c(0,x)+\lambda\big)|u|^2\,\mathrm{d}x\\
&\ge \mu_0\sum_{i=1}^{5}\int_\Omega |u_{x_i}|^2\,\mathrm{d}x + \int_\Omega \big(c(0,x)+\lambda\big)|u|^2\,\mathrm{d}x \quad \text{(ellipticity condition)}\\
&\ge c_1(1)^2\|u\|_{H^1(\Omega)}^2. \quad (\lambda \text{ sufficiently large})
\end{aligned}$$

On the other hand it is obvious via the Hölder inequality that

$$\begin{aligned}
(\mathbf{A}(0)u,u) &\le \sup_{\bar\Omega}\big|a_{ij}(0,x)\big|\sum_{i,j=1}^{5}\Big(\int_\Omega |u_{x_i}|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega |u_{x_j}|^2\,\mathrm{d}x\Big)^{1/2} + \lambda_0\int_\Omega |u|^2\,\mathrm{d}x\\
&\le c_2(1)^2\|u\|_{H^1(\Omega)}^2,
\end{aligned}$$

i.e. for $u \in D(\mathbf{A}(0))$ we have

$$c_1(1)\|u\|_{H^1(\Omega)} \le \big\|\mathbf{A}^{1/2}(0)u\big\| \le c_2(1)\|u\|_{H^1(\Omega)}.$$

Let now $u \in D(\mathbf{A}^{1/2}(0)) = H_0^1(\Omega)$. There exists a sequence $u_n \in C_0^\infty(\Omega)$ with $\|u_n - u\|_{H^1(\Omega)} \to 0$ as $n \to \infty$. From the above, we have

$$\|\mathbf{A}^{1/2}(0)(u_n - u_m)\| \le c_2(1)\|u_n - u_m\|_{H^1(\Omega)} \to 0 \quad (n,m \to \infty),$$

i.e. $\mathbf{A}^{1/2}(0)u_n \to v$, $u_n \to u \quad (n \to \infty)$. Thus $v = \mathbf{A}^{1/2}(0)u$. The assertion follows via limits for

$$c_1(1)\|u_n\|_{H^1(\Omega)} \le \|\mathbf{A}^{1/2}(0)u_n\| \le c_2(1)\|u_n\|_{H^1(\Omega)}.$$

Suppose now that (3.1) is true for $m \le k$, i.e.,

$$c_1(k)\|u\|_{H^k(\Omega)} \le \|\mathbf{A}^{k/2}(0)u\| \le c_2(k)\|u\|_{H^k(\Omega)}, \quad u \in D(\mathbf{A}^{k/2}(0)).$$

Let now $u \in D(\mathbf{A}^{\frac{k+1}{2}}(0))$, i.e. $\mathbf{A}(0)u \in D(\mathbf{A}^{\frac{k-1}{2}}(0))$. According to Agmon, Douglis and Nirenberg [1, Theorem 15.2, page 704],

$$\begin{aligned} \|u\|_{H^{k+1}(\Omega)} &\le c_1\big(\|\mathbf{A}(0)u\|_{H^{k-1}(\Omega)} + \|u\|\big) \le c_2\|\mathbf{A}(0)u\|_{H^{k-1}(\Omega)} \\ &\le \frac{1}{c_1(k+1)}\|\mathbf{A}^{\frac{k+1}{2}}(0)u\|, \end{aligned}$$

i.e. $c_1(k+1)\|u\|_{H^{k+1}(\Omega)} \le \|\mathbf{A}^{\frac{k+1}{2}}(0)u\|$, $u \in D(\mathbf{A}^{\frac{k+1}{2}}(0))$. The another direction of the proof is obvious. □

4. Nonlinearity

Let $k, k', k'' \in \mathbb{N}$ with

$$k > \frac{n}{2}, \quad k' \ge \frac{n}{2}, \quad k'' > \frac{n}{2} + 1, \quad k''_0 := \min\{k''\}.$$

That means with $n = 5$: $k \ge 3$, $k' \ge 3$, $k'' \ge 4$, $k''_0 = 4$.

Now, let $\mathbf{F} \in C^{k''+3}_{\rm loc}(\mathbb{R}^+, \mathbb{R})$. For $\mathbf{M}u := \mathbf{F}'\big(|u|^2\big)u + \zeta u$, it is obvious that $\mathbf{M} : H^k(\Omega) \to L^2(\Omega)$. Further the following result holds.

Proposition 4.1. *For $\frac{n}{2} < k < k'' + 1$,*

$$\mathbf{M} : H^k(\Omega) \to H^k(\Omega).$$

Proof. Take $u \in H^k(\Omega)$. Then

$$\|\mathbf{F}'\big(|u|^2\big)u\|^2_{H^k(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha \mathbf{F}'\big(|u|^2\big)u\|^2,$$

where $D^\alpha \mathbf{F}'\big(|u|^2\big)u$, $|\alpha| \le k$, is a finite sum of terms of the form

$$\mathbf{F}^{(\nu)}\big(|u|^2\big) \prod_{\rho=1}^{N} D^{\alpha_\rho} u_\rho$$

and $1 \le \nu \le |\alpha| + 1$, $\sum_{\rho=1}^N |\alpha_\rho| = |\alpha|$, $u_\rho \in \{u, \bar{u}\}$. The use of the Embedding Theorem 2.10 and Propostion 2.12 yield

$$\Big\|\mathbf{F}^{(\nu)}\big(|u|^2\big) \prod_{\rho=1}^{N} D^{\alpha_\rho} u_\rho\Big\| \le c\Big(\sup_{0 \le \sigma \le c'\|u\|_{H^k}} \big|\mathbf{F}^{(\nu)}(\sigma^2)\big|^2\Big)^{1/2} \|u\|^N_{H^k} < \infty.$$

□

For the proof of the desired mapping property, we need the following 2 well-known facts.

Proposition 4.2. *Let $u \in H^1(\Omega)$ and $u|_{\partial\Omega} = 0$ a.e. Then it is $u \in H^1_0(\Omega)$.*

Proposition 4.3. *Let $u \in H^1_0(\Omega) \cap H^k(\Omega)$, $k > \frac{n}{2}$, then $u|_{\partial\Omega} = 0$ a.e.*

Theorem 4.4. *Suppose $n = 5$ and $\mathbf{F} \in C^7_{\rm loc}(\mathbb{R}^+, \mathbb{R})$, $u \in D(\mathbf{A}^{5/2}(0))$. Then*
$$\mathbf{M}u = \mathbf{F}'(|u|^2)u + \zeta u \in D(\mathbf{A}^2(0)), \quad \zeta \in \mathbb{C},$$
i.e. the mapping property
$$\mathbf{M} : D(\mathbf{A}^{5/2}(0)) \to \ D(\mathbf{A}^2(0)) \tag{4.1}$$
holds. Moreover let $u, v \in D(\mathbf{A}^{5/2}(0))$, $C \geq 0$, $t \geq 0$. Then it holds the associated Lipschitz condition
$$\|\mathbf{A}^2(\tau)(\mathbf{M}u - \mathbf{M}v)\| \leq k(C,t)\|\mathbf{A}^{5/2}(\tau)(u - v)\|, \tag{4.2}$$
for $\tau \in [t, t+1]$ and
$$\sup_{\tau\in[t,t+1]} \Big(\|\mathbf{A}^{5/2}(\tau)u\| + \|\mathbf{A}^{5/2}(\tau)v\|\Big) \leq C$$
and where $k(\cdot,\cdot) \in C^0_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^{++})$ (i.e. the function $k(\cdot,\cdot)$ must be found here).

Proof. Since $u \in D(\mathbf{A}^{5/2}(0)) \subset H^1_0(\Omega) \cap H^5(\Omega)$, by Proposition 4.3: $u|_{\partial\Omega} = 0$ a.e.. According to Proposition 4.1, $\mathbf{F}'\big(|u|^2\big)u \in H^4(\Omega)$ and with that continuous on $\bar{\Omega}$ (Embedding Theorem 2.10), i.e. $\mathbf{F}'\big(|u|^2\big)u|_{\partial\Omega} = 0$ a.e., so that via Proposition 4.2: $\mathbf{F}'\big(|u|^2\big)u \in H^1_0(\Omega)$. Thus we have
$$\mathbf{F}'\big(|u|^2\big)u \in H^1_0(\Omega) \cap H^4(\Omega) \subset H^1_0(\Omega) \cap H^2(\Omega) = D(\mathbf{A}(0)).$$
Therefore,
$$\begin{aligned}
&\mathbf{A}(t)\mathbf{F}'\big(|u|^2\big)u \\
&= -\sum_{i,j=1}^5 a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}\mathbf{F}'\big(|u|^2\big)u - \sum_{i,j=1}^5 \big(a_{ij}(t,x)\big)_{x_i}\frac{\partial}{\partial x_j}\mathbf{F}'\big(|u|^2\big)u \\
&\quad + c(t,x)\mathbf{F}'\big(|u|^2\big)u + \lambda\mathbf{F}'\big(|u|^2\big)u \\
&= -\Big\{\mathbf{F}'\big(|u|^2\big)\sum_{i,j=1}^5 a_{ij}(t,x)u_{x_ix_j} + \sum_{i,j=1}^5 a_{ij}(t,x)\mathbf{F}''\big(|u|^2\big)u_{x_i}2\,\mathrm{Re}(\bar{u}u_{x_j}) \\
&\quad + \sum_{i,j=1}^5 a_{ij}(t,x)\mathbf{F}''\big(|u|^2\big)u_{x_j}2\,\mathrm{Re}(\bar{u}u_{x_i}) \\
&\quad + \sum_{i,j=1}^5 a_{ij}(t,x)\mathbf{F}''\big(|u|^2\big)u2\,\mathrm{Re}(\bar{u}u_{x_ix_j}) \\
&\quad + \sum_{i,j=1}^5 a_{ij}(t,x)\mathbf{F}''\big(|u|^2\big)u2\,\mathrm{Re}(\bar{u}_{x_j}u_{x_i}) \\
&\quad + \sum_{i,j=1}^5 a_{ij}(t,x)\mathbf{F}'''\big(|u|^2\big)u2\,\mathrm{Re}(\bar{u}u_{x_i})2\,\mathrm{Re}(\bar{u}u_{x_j})\Big\} \\
&\quad - \Big\{\mathbf{F}'\big(|u|^2\big)\sum_{i,j=1}^5 \big(a_{ij}(t,x)\big)_{x_i}u_{x_j} + \sum_{i,j=1}^5 \big(a_{ij}(t,x)\big)_{x_i}\mathbf{F}''\big(|u|^2\big)u2\,\mathrm{Re}(\bar{u}u_{x_j})\Big\} \\
&\quad + c(t,x)\mathbf{F}'\big(|u|^2\big)u + \lambda\mathbf{F}'\big(|u|^2\big)u.
\end{aligned}$$

In this sum, the four terms containing $\mathbf{F}'\big(|u|^2\big)$ represent

$$\mathbf{F}'\big(|u|^2\big)\mathbf{A}(t)u$$

with $\mathbf{A}(t)u \in D(\mathbf{A}^{3/2}(0)) \subset H_0^1(\Omega) \cap H^3(\Omega)$, i.e. $\mathbf{A}(t)u|_{\partial\Omega} = 0$ a.e. by Proposition 4.3, so that $\mathbf{F}'\big(|u|^2\big)\mathbf{A}(t)u|_{\partial\Omega} = 0$ a.e.. Furthermore it is obvious that $\mathbf{F}'\big(|u|^2\big)\mathbf{A}(t)u$ as well as the other terms admit 2 further derivatives, belong so to $H^2(\Omega)$. Since all the remaining terms contain u and are continuous, all the terms vanish on $\partial\Omega$, belong so to $H_0^1(\Omega)$ by Proposition 4.2. So, it is summarized:

$$\mathbf{A}(t)\mathbf{F}'\big(|u|^2\big)u \in H_0^1(\Omega) \cap H^2(\Omega) = D(\mathbf{A}(t)),$$

i.e. $\mathbf{F}'\big(|u|^2\big)u \in D(\mathbf{A}^2(0))$. The mapping property is proved.

Concerning the associated Lipschitz condition, we prove that first is of the form

$$\|\mathbf{A}^2(0)\{\mathbf{F}'\big(|u|^2\big)u - \mathbf{F}'\big(|v|^2\big)v\}\| \le \tilde{k}(c)\big\|\mathbf{A}^{5/2}(0)(u-v)\|$$

for all $u, v \in D(\mathbf{A}^{5/2}(0))$ with

$$\|\mathbf{A}^{5/2}(0)u\| + \|\mathbf{A}^{\frac{5}{5}}(0)v\| \le c,$$

where $\tilde{k}(\cdot) \in C^0_{\rm loc}(\mathbb{R}^+, \mathbb{R}^{++})$ is monotonically increasing.

Let now $u, v \in D(\mathbf{A}^{5/2}(0))$. By (3.1) we have

$$\begin{aligned}
\|\mathbf{A}^2(0)\{\mathbf{F}'\big(|u|^2\big)u - \mathbf{F}'\big(|v|^2\big)v\}\|^2 &\le c_2(4)^2\|\mathbf{F}'\big(|u|^2\big)u - \mathbf{F}'\big(|v|^2\big)v\|^2_{H^4(\Omega)} \\
&= c_2(4)^2 \sum_{|\alpha|\le 4} \|D^\alpha\{\mathbf{F}'\big(|u|^2\big)u - \mathbf{F}'\big(|v|^2\big)v\}\|^2.
\end{aligned}$$

Here $D^\alpha\big\{\mathbf{F}'\big(|u|^2\big)u - \mathbf{F}'\big(|v|^2\big)v\big\}$ consists of terms of the form

$$\mathbf{F}^{(\nu)}\big(|u|^2\big)\prod_{\rho=1}^N D^{\alpha_\rho}u_\rho - \mathbf{F}^{(\nu)}\big(|v|^2\big)\prod_{\rho=1}^N D^{\alpha_\rho}v_\rho,$$

where $1 \le \nu \le |\alpha|+1$, $\sum_{\rho=1}^N |\alpha_\rho| = |\alpha|$, $u_\rho \in \{u, \bar{u}\}$, $v_\rho \in \{v, \bar{v}\}$. For such term, we have

$$\begin{aligned}
&\|\mathbf{F}^{(\nu)}\big(|u|^2\big)\prod_{\rho=1}^N D^{\alpha_\rho}u_\rho - \mathbf{F}^{(\nu)}\big(|v|^2\big)\prod_{\rho=1}^N D^{\alpha_\rho}v_\rho\| \\
&\le \|\big\{\mathbf{F}^{(\nu)}\big(|u|^2\big) - \mathbf{F}^{(\nu)}\big(|v|^2\big)\big\}\prod_{\rho=1}^N D^{\alpha_\rho}v_\rho\| \\
&+ \|\mathbf{F}^{(\nu)}\big(|u|^2\big)\big\{\prod_{\rho=1}^N D^{\alpha_\rho}u_\rho - \prod_{\rho=1}^N D^{\alpha_\rho}v_\rho\big\}\| =: T_1 + T_2.
\end{aligned}$$

Since $\mathbf{F} \in C^7_{\rm loc}(\mathbb{R}^+, \mathbb{R})$, via the Embedding Theorem 2.10, Theorem 3.1, and Propositions 2.12 and 2.13, it follows that

$$\begin{aligned}
T_1 &= \|\int_0^1 \frac{\partial}{\partial\tau}\mathbf{F}^{(\nu)}\big(\big\{\tau|u| + (1-\tau)|v|\big\}^2\big)\,\mathrm{d}\tau \prod_{\rho=1}^N D^{\alpha_\rho}v_\rho\big\| \\
&\le 2\int_0^1 \Big\|\mathbf{F}^{(\nu+1)}\big(\big\{\tau|u| + (1-\tau)|v|\big\}^2\big)\big\{\tau|u| + (1-\tau)|v|\big\}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \big(|u|-|v|\big)\prod_{\rho=1}^{N} D^{\alpha_\rho}v_\rho\Big\|\,\mathrm{d}\tau\\
&\le c_1\Big(\sup_{0\le\sigma\le c_2\left(\|u\|_{H^5}+\|v\|_{H^5}\right)}\big|\mathbf{F}^{(\nu+1)}(\sigma^2)\big|^2\Big)^{1/2}\big(\|u\|_{H^5}+\|v\|_{H^5}\big)^{N+1}\|u-v\|_{H^5}\\
&\le c_3\Big(\sup_{0\le\sigma\le c_4\left(\|\mathbf{A}^{5/2}(0)u\|+\|\mathbf{A}^{5/2}(0)v\|\right)}\big|\mathbf{F}^{(\nu+1)}(\sigma^2)\big|^2\Big)^{1/2}\\
&\quad \times \big(\|\mathbf{A}^{5/2}(0)u\|+\|\mathbf{A}^{5/2}(0)v\|\big)^{N+1}\|\mathbf{A}^{5/2}(0)(u-v)\|\\
&\le \tilde{k}_1(c)\|\mathbf{A}^{5/2}(0)(u-v)\|,
\end{aligned}
$$

where $\|\mathbf{A}^{5/2}(0)u\|+\|\mathbf{A}^{5/2}(0)v\|\le c$ and $\tilde{k}_1(\cdot)\in C^0_{\rm loc}(\mathbb{R}^+,\mathbb{R}^{++})$ is the monotonically increasing multiplier.

$$
T_2\le\Big(\sup_{0\le\sigma\le c_5\|\mathbf{A}^{5/2}(0)u\|}\big|\mathbf{F}^{(\nu)}(\sigma^2)\big|^2\Big)^{1/2}\|\prod_{\rho=1}^{N}D^{\alpha_\rho}u_\rho-\prod_{\rho=1}^{N}D^{\alpha_\rho}v_\rho\|.
$$

Then

$$
\begin{aligned}
&\|\prod_{\rho=1}^{N}D^{\alpha_\rho}u_\rho-\prod_{\rho=1}^{N}D^{\alpha_\rho}v_\rho\|\\
&\le \|D^{\alpha_1}(u_1-v_1)D^{\alpha_2}u_2\cdots D^{\alpha_N}u_N\|\\
&+\|D^{\alpha_1}v_1(D^{\alpha_2}u_2\cdots D^{\alpha_N}u_N-D^{\alpha_2}v_2\cdots D^{\alpha_N}v_N)\|\\
&\le c_6\|u-v\|_{H^5}\|u\|^{N-1}_{H^5}+\|D^{\alpha_1}v_1D^{\alpha_2}(u_2-v_2)D^{\alpha_3}u_3\cdots D^{\alpha_N}u_N\|\\
&+\|D^{\alpha_1}v_1D^{\alpha_2}v_2(D^{\alpha_3}u_3\cdots D^{\alpha_N}u_N-D^{\alpha_3}v_3\cdots D^{\alpha_N}v_N)\|\\
&\le c_7\|\mathbf{A}^{5/2}(0)u\|^{N-1}\|\mathbf{A}^{5/2}(0)(u-v)\|\\
&\quad + c_8\|\mathbf{A}^{5/2}(0)v\|\,\|\mathbf{A}^{5/2}(0)u\|^{N-2}\|\mathbf{A}^{5/2}(0)(u-v)\|\\
&\quad +\|D^{\alpha_1}v_1D^{\alpha_2}v_2(D^{\alpha_3}u_3\cdots D^{\alpha_N}u_N-D^{\alpha_3}v_3\cdots D^{\alpha_N}v_N)\|\\
&\dots\\
&\le c_9\big(\|\mathbf{A}^{5/2}(0)u\|+\|\mathbf{A}^{5/2}(0)v\|\big)^{N-1}\|\mathbf{A}^{5/2}(0)(u-v)\|,
\end{aligned}
$$

i.e.

$$
T_2\le\tilde{k}_2(c)\|\mathbf{A}^{5/2}(0)(u-v)\|,
$$

where $\tilde{k}_2(\cdot)\in C^0_{\rm loc}(\mathbb{R}^+,\mathbb{R}^{++})$ is also monotonically increasing. Set $\tilde{k}(\cdot):=\tilde{k}_1(\cdot)+\tilde{k}_2(\cdot)$, and summarize the above as

$$
\|\mathbf{A}^2(0)\big\{\mathbf{F}'\big(|u|\big)^2u-\mathbf{F}'\big(|v|\big)^2v\big\}\|\le\tilde{k}(c)\|\mathbf{A}^{5/2}(0)(u-v)\|
$$

with $\|\mathbf{A}^{5/2}(0)u\|+\|\mathbf{A}^{5/2}(0)v\|\le c$.

The operators $\mathbf{A}(t)$ fulfill condition (2.2):

$$
\|\big(\mathbf{A}^{\nu/2}(t)-\mathbf{A}^{\nu/2}(s)\big)\mathbf{A}^{-\nu/2}(r)\|\le L(\nu)|t-s|,\quad t,s,r,\in\mathbb{R}^+,\quad \nu\in\mathbb{N}.
$$

This implies

$$
\|\mathbf{A}^{\nu/2}(t)\mathbf{A}^{-\nu/2}(0)\|,\,\|\mathbf{A}^{\nu/2}(0)\mathbf{A}^{-\nu/2}(t)\|\le 1+L(\nu)t,\quad t\in\mathbb{R}^+,
$$

and for $u \in D\big(\mathbf{A}^{\nu/2}(0)\big)$:

$$\big\|\mathbf{A}^{\nu/2}(0)u\| = \|\mathbf{A}^{\nu/2}(0)\mathbf{A}^{-\nu/2}(t)\mathbf{A}^{\nu/2}(t)u\| \le \big(1+L(\nu)t\big)\|\mathbf{A}^{\nu/2}(t)u\big\|, \quad t \in \mathbb{R}^+.$$

Take now $t \in \mathbb{R}^+$, $\tau \in [t, t+1]$. Then

$$\begin{aligned}
&\|\mathbf{A}^2(\tau)\big\{\mathbf{F}'\big(|u|^2\big)u - \mathbf{F}'\big(|v|^2\big)v\big\}\| \\
&= \|\mathbf{A}^2(\tau)\mathbf{A}^{-2}(0)\mathbf{A}^2(0)\big\{\mathbf{F}'\big(|u|^2\big)u - \mathbf{F}'\big(|v|^2\big)v\big\}\| \\
&\le \|\mathbf{A}^2(\tau)\mathbf{A}^{-2}(0)\|\|\mathbf{A}^2(0)\big\{\mathbf{F}'\big(|u|^2\big)u - \mathbf{F}'\big(|v|^2\big)v\big\}\| \\
&\le (1+L(4)\tau)\tilde{k}\big(\|\mathbf{A}^{5/2}(0)u\| + \|\mathbf{A}^{5/2}(0)v\|\big)\|\mathbf{A}^{5/2}(0)(u-v)\| \\
&\le (1+L(4)(t+1))(1+L(5)(t+1))\tilde{k}\big((1+L(5)(t+1)) \\
&\quad \times \sup_{\tau\in[t,t+1]}\big(\|\mathbf{A}^{5/2}(\tau)u\| + \|\mathbf{A}^{5/2}(\tau)v\|\big)\big)\|\mathbf{A}^{5/2}(\tau)(u-v)\| \\
&\le \big(1+L(4)(t+1)\big)\big(1+L(5)(t+1)\big)\tilde{k}\big((1+L(5)(t+1))C\big)\|\mathbf{A}^{5/2}(\tau)(u-v)\|
\end{aligned}$$

with

$$\sup_{\tau\in[t,t+1]}\big(\|\mathbf{A}^{5/2}(\tau)u\| + \|\mathbf{A}^{5/2}(\tau)v\|\big) \le C.$$

Thus, with

$$k(C,t) := \big(1+L(4)(t+1)\big)\big(1+L(5)(t+1)\big)\tilde{k}\big((1+L(5)(t+1))C\big) + |\zeta|,$$

the associated Lipschitz condition (4.2) is satisfied. □

Proposition 4.5. *Let $u \in D(\mathbf{A}^{5/2}(0))$ and $\nu = 3$ or 4 as well as $\mathbf{F} \in C^7(\mathbb{R}^+, \mathbb{R})$, $\zeta \in \mathbb{C}$. Then*

$$\|\mathbf{A}^{\nu/2}(0)\big(\mathbf{F}'\big(|u|^2\big)u + \zeta u\big)\| \le f_\nu\big(\|\mathbf{A}^{\nu/2}(0)u\|\big)\|\mathbf{A}^{\frac{\nu+1}{2}}(0)u\|$$

with $f_\nu \in C^0_{\rm loc}(\mathbb{R}^+, \mathbb{R}^{++})$ monotonically increasing.

Proof. Since $u \in D(\mathbf{A}^{5/2}(0))$, by Theorem 4.4,

$$\mathbf{F}'\big(|u|^2\big)u + \zeta u \in D(\mathbf{A}^2(0)) \subseteq D(\mathbf{A}^{\nu/2}(0)) \subset H^1_0(\Omega) \cap H^\nu(\Omega).$$

Moreover, by Theorem 3.1,

$$\begin{aligned}
\|\mathbf{A}^{\nu/2}(0)\big(\mathbf{F}'\big(|u|^2\big)u + \zeta u\big)\|^2 &\le c_2(\nu)\|\mathbf{F}'\big(|u|^2\big)u + \zeta u\|^2_{H^\nu(\Omega)} \\
&= c_2(\nu)\sum_{|\alpha|\le\nu}\|D^\alpha\big(\mathbf{F}'\big(|u|^2\big)u + \zeta u\big)\|^2,
\end{aligned}$$

where $D^\alpha\mathbf{F}'\big(|u|^2\big)u$ consists of a finite sum of terms of the form

$$\mathbf{F}^{(\mu)}\big(|u|^2\big)\prod_{\rho=1}^N D^{\alpha_\rho}u_\rho$$

and $1 \le \mu \le |\alpha| + 1$, $\sum_{\rho=1}^N |\alpha_\rho| = |\alpha| \le \nu$, $u_\rho \in \{u, \bar{u}\}$.

Since $u \in H^5(\Omega) \subseteq H^{\nu+1}(\Omega) \subset H^\nu(\Omega)$, $\nu = 3$ or $4 > \frac{n}{2}$, it follows via the Embedding Theorem 2.10 and Proposition 2.12 that

$$\|\mathbf{F}^{(\mu)}\big(|u|^2\big)\prod_{\rho=1}^N D^{\alpha_\rho}u_\rho\|^2$$

$$
\begin{aligned}
&= \int_\Omega \big|\mathbf{F}^{(\mu)}\big(|u|^2\big)\big|^2 \big|\prod_{\rho=1}^N D^{\alpha_\rho} u_\rho\big|^2 \,\mathrm{d}x \\
&\le \Big(\sup_{0\le\sigma\le c_1\|u\|_{H^\nu}} \big|\mathbf{F}^{(\mu)}(\sigma^2)\big|\Big)^2 \|\prod_{\rho=1}^N D^{\alpha_\rho} u_\rho\|^2 \\
&\le c_2\Big(\sup_{0\le\sigma\le c_1\|u\|_{H^\nu}} \big|\mathbf{F}^{(\mu)}(\sigma^2)\big|\Big)^2 \|u\|^{2N}_{H^\nu(\Omega)} \\
&\le c_2\Big(\sup_{0\le\sigma\le c_1\|u\|_{H^\nu}} \big|\mathbf{F}^{(\mu)}(\sigma^2)\big|\Big)^2 \|u\|^{2(N-1)}_{H^\nu(\Omega)}\|u\|^2_{H^{\nu+1}(\Omega)} \\
&\le c_3\Big(\sup_{0\le\sigma\le c_3\|\mathbf{A}^{\nu/2}(0)u\|} \big|\mathbf{F}^{(\mu)}(\sigma^2)\big|^2\Big)^2 \|\mathbf{A}^{\nu/2}(0)u\|^{2(N-1)}\|\mathbf{A}^{\frac{\nu+1}{2}}(0)u\|^2,
\end{aligned}
$$

i.e. we have the assertion. $\square$

Proposition 4.5 is necessary to increase the regularity farther than $D(\mathbf{A}^{3/2}(0))$. The next Proposition is necessary to ensure the a priori estimates for $\|\mathbf{A}^{1/2}(t)u(t)\|$, $\|\mathbf{A}(t)u(t)\|$ and $\|\mathbf{A}^{3/2}(t)u(t)\|$, where $u(t)\in D(\mathbf{A}^{5/2}(0))$ is the 4-regular solution in the maximal interval $[0,T)$.

Proposition 4.6. *Let $u\in D(\mathbf{A}^{5/2}(0))$ and as previously $\mathbf{F}\in C^7_{\rm loc}(\mathbb{R}^+,\mathbb{R})$.*

(a) *If $|\mathbf{F}'(\sigma)|\le c\sigma^{1/3}$, then*
$$\|\mathbf{F}'\big(|u|^2\big)u\| \le c_1\|\mathbf{A}^{1/2}(0)u\|^{5/3}.$$

(b) *Moreover, if*
$$|\mathbf{F}''(\sigma)| \le \begin{cases} c\sigma^{\frac13-1} & \sigma\ge 1\\ c & \sigma\le 1,\end{cases}$$
i.e. $|\mathbf{F}''(\sigma)\sigma|\le c\sigma^{1/3}$, then
$$\|\mathbf{A}^{1/2}(0)\mathbf{F}'\big(|u|^2\big)u\| \le c_2\|\mathbf{A}^{1/2}(0)u\|^{\frac23}\|\mathbf{A}(0)u\|.$$

(c) *In addition if*
$$|\mathbf{F}'''(\sigma)| \le \begin{cases} c\sigma^{\frac13-2} & \sigma\ge 1\\ c & \sigma\le 1,\end{cases}$$
then
$$\|\mathbf{A}(0)\mathbf{F}'\big(|u|^2\big)u\| \le f_2\big(\|\mathbf{A}(0)u\|\big)\|\mathbf{A}^{3/2}(0)u\|,$$
with monotonically increasing $f_2\in C^0_{\rm loc}(\mathbb{R}^+,\mathbb{R}^{++})$.

The proof of the above proposition is exactly the same proof of Proposition 4.5 under the consideration of the growth restrictions.

5. A priori estimates and the classical solvability

The classical solvability of (1.1) corresponds with the 4-regular solution for:
$$
\begin{gathered}
u'' + \mathbf{A}(t)u + \mathbf{M}u = 0\\
u(0)=\varphi\in D(\mathbf{A}^{5/2}(0)),\quad u'(0)=\psi\in D(\mathbf{A}^2(0))\,.
\end{gathered}
\tag{5.1}
$$
On the other hand, the 4-regular solution of (2.4) corresponds to this 4-regular soluton of (5.1) by Theorem 2.8. By Theorem 4.4, the mapping property (2.3) and their conditions (I')–(I"') are satisfied for $k=4$ which implies the mapping

property (2.7) and their conditions (I)–(III) with $k = 4$. That means a unique 4-regular solution for (5.1) first on a maximal interval $[0, T)$. For extending this solution on the whole $[0, \infty)$, we need still a priori estimates for $\|\hat{u}(t)\|$, $\|\hat{\mathbf{A}}(t)\hat{u}(t)\|$, $\dots$, $\|\hat{\mathbf{A}}^4(t)\hat{u}(t)\|$. Each of these a priori estimates depends on the one before. By Corollary 2.9,

$$\hat{u}(t) = \frac{1}{2}\begin{pmatrix} u'(t) + i\mathbf{A}^{1/2}(t)u(t) \\ u'(t) - i\mathbf{A}^{1/2}(t)u(t) \end{pmatrix}, \quad \hat{\mathbf{A}}(t) = \begin{pmatrix} -i\mathbf{A}^{1/2}(t) & 0 \\ 0 & i\mathbf{A}^{1/2}(t) \end{pmatrix},$$
$$\hat{\mathbf{M}}(t, \hat{u}(t)) = \frac{1}{2}\begin{pmatrix} \mathbf{M}u(t) \\ \mathbf{M}u(t) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} i(\mathbf{A}^{1/2}(t))'u(t) \\ -i(\mathbf{A}^{1/2}(t))'u(t) \end{pmatrix}.$$

Originally $\hat{u}(t)$ is the solution of

$$\hat{u}(t) + \hat{\mathbf{A}}(t)\hat{u}(t) + \hat{\mathbf{M}}(t, \hat{u}(t)) = 0, \quad \hat{u}(0) = \hat{\varphi}$$

as well as of the associated integral equation

$$\hat{u}(t) = \hat{\mathbf{U}}(t, 0)\hat{\varphi} - \int_0^t \hat{\mathbf{U}}(t, s)\hat{\mathbf{M}}(s, \hat{u}(s))ds\,.$$

Theorem 5.1. *Let* $\varphi \in D(\mathbf{A}^{5/2}(0))$, $\psi \in D(\mathbf{A}^2(0))$, $\mathbf{F} \in C^7_{\rm loc}(\mathbb{R}^+, \mathbb{R})$ *with* $\mathbf{F}(\sigma) \geq 0$. *Furthermore let the conditions*

$$|\mathbf{F}'(\sigma)| \leq k_1\sigma^{1/3}, \quad |\mathbf{F}''(\sigma)| \leq k_1\sigma^{-2/3}, \quad |\mathbf{F}'''(\sigma)| \leq k_1\sigma^{-3/2}$$

be satisfied for $\sigma \geq 1$. *Then there exists a unique 4-regular solution for* (5.1) *on* $[0, \infty)$.

Proof. We show first the a priori estimate for $\|\hat{u}(t)\|$. We have

$$\|\hat{u}(t)\| = \left\|\begin{pmatrix} u' + i\mathbf{A}^{1/2}(t)u \\ u' - i\mathbf{A}^{1/2}(t)u \end{pmatrix}\right\| = \Big(\|u'\|^2 + \|\mathbf{A}^{1/2}(t)u\|^2\Big)^{1/2},$$

where

$$\frac{1}{\sqrt{2}}\Big(\|u'\| + \|\mathbf{A}^{1/2}(t)u\|\Big) \leq \Big(\|u'\|^2 + \|\mathbf{A}^{1/2}(t)u\|^2\Big)^{1/2} \leq \|u'\| + \|\mathbf{A}^{1/2}(t)u\|.$$

Scalar multiplication of (5.1) with u' yields

$$(u'', u') + (\mathbf{A}(t)u, u') + (\mathbf{M}u, u') = 0.$$

Then (see von Wahl [10, page 270])

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}(t)u, u) &= (\mathbf{A}(t)u, u') + (\mathbf{A}'(t)u, u) + (u', \mathbf{A}(t)u) \\ &= (\mathbf{A}'(t)u, u) + 2\,\mathrm{Re}(\mathbf{A}(t)u, u'), \end{aligned}$$

where

$$\begin{aligned} (\mathbf{A}'(t)u, u) &= \big((\mathbf{A}^{1/2}(t)\mathbf{A}^{1/2}(t))'u, u\big) \\ &= \big((\mathbf{A}^{1/2}(t))'\mathbf{A}^{1/2}(t)u, u\big) + \big(\mathbf{A}^{1/2}(t)(\mathbf{A}^{1/2}(t))'u, u\big) \\ &= \big(\mathbf{A}^{1/2}(t)u, (\mathbf{A}^{1/2}(t))'u\big) + \big((\mathbf{A}^{1/2}(t))'u, \mathbf{A}^{1/2}(t)u\big) \\ &= 2\,\mathrm{Re}\big(\mathbf{A}^{1/2}(t)u, (\mathbf{A}^{1/2}(t))'u\big) \\ &\leq 2\|\mathbf{A}^{1/2}(t)u\|\,\|(\mathbf{A}^{1/2}(t))'u\| \\ &\leq 2c\|\mathbf{A}^{1/2}(t)u\|^2, \end{aligned}$$

(see von Wahl [10, page 279]). So, the above scalar multiplication with $\tilde{\mathbf{M}}u := \mathbf{M}u - \zeta u$ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u'\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{A}^{1/2}(t)u\|^2 &= \big(\mathbf{A}'(t)u, u\big) - 2\operatorname{Re}(\tilde{\mathbf{M}}u, u') + 2\operatorname{Re}\zeta(u, u')\\ &\le 2c\|\mathbf{A}^{1/2}(t)u\|^2 - 2\operatorname{Re}(\tilde{\mathbf{M}}u, u') + 2\operatorname{Re}\zeta(u, u').\end{aligned}$$

Integrating from 0 to t yields

$$\begin{aligned}&\|u'\|^2 + \|\mathbf{A}^{1/2}(t)u\|^2\\ &\le \|\mathbf{A}^{1/2}(0)\varphi\|^2 + \|\psi\|^2 + 2c\int_0^t \|\mathbf{A}^{1/2}(s)u(s)\|^2\,\mathrm{d}s\\ &\quad - \int_0^t 2\operatorname{Re}\big(\tilde{\mathbf{M}}u(s), u'(s)\big)\,\mathrm{d}s + |\zeta|\int_0^t \big(\|u'(s)\|^2 + \|\mathbf{A}^{1/2}(s)u(s)\|^2\big)\,\mathrm{d}s\end{aligned}$$

where (see [10, page 271])

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{F}\big(|u(t,x)|^2\big)\,\mathrm{d}x = 2\operatorname{Re}\big(\tilde{\mathbf{M}}u, u'\big).$$

It follows that

$$\begin{aligned}\|u'\|^2 + \|\mathbf{A}^{1/2}(t)u\|^2 &\le \|\mathbf{A}^{1/2}(0)\varphi\|^2 + \|\psi\|^2 + \int_\Omega \mathbf{F}\big(|\varphi|^2\big)\,\mathrm{d}x\\ &\quad - \int_\Omega \mathbf{F}\big(|u(t,x)|^2\big)\,\mathrm{d}x + c_1\int_0^t \big(\|u'(s)\|^2 + \|\mathbf{A}^{1/2}(s)u(s)\|^2\big)\,\mathrm{d}s,\end{aligned}$$

so that via Gronwall's Lemma the a priori estimate for $\|u'\|^2 + \|\mathbf{A}^{1/2}(t)u\|^2$ and with that for $\|\hat{u}(t)\|$ follows.

To obtain the a priori estimate for $\|\hat{\mathbf{A}}(t)\hat{u}(t)\|$, we operate with

$$\hat{u}(t) = \hat{\mathbf{U}}(t,0)\hat{\varphi} - \int_0^t \hat{\mathbf{U}}(t,s)\hat{\mathbf{M}}(s,\hat{u}(s))\,\mathrm{d}s,$$

which by Theorem 2.5 implies

$$\begin{aligned}\|\hat{\mathbf{A}}(t)\hat{u}(t)\| &\le \|\hat{\mathbf{A}}(t)\hat{\mathbf{U}}(t,0)\hat{A}^{-1}(0)\|\,\|\hat{\mathbf{A}}(0)\hat{\varphi}\|\\ &\quad + \int_0^t \|\hat{\mathbf{A}}(t)\hat{\mathbf{U}}(t,s)\hat{\mathbf{A}}^{-1}(s)\|\,\|\hat{\mathbf{A}}(s)\hat{\mathbf{M}}(s,\hat{u}(s))\|\,\mathrm{d}s\\ &\le e^{\hat{L}(1)t}\|\hat{\mathbf{A}}(0)\hat{\varphi}\| + \int_0^t e^{\hat{L}(1)(t-s)}\|\hat{\mathbf{A}}(s)\hat{\mathbf{M}}(s,\hat{u}(s))\|\,\mathrm{d}s\\ &\le e^{\hat{L}(1)t}\|\hat{\mathbf{A}}(0)\hat{\varphi}\| + 2e^{\hat{L}(1)t}\int_0^t \|\mathbf{A}^{1/2}(s)\mathbf{M}u(s)\,\mathrm{d}s\|\\ &\quad + 2e^{\hat{L}(1)t}\int_0^t \|\mathbf{A}^{1/2}(s)(\mathbf{A}^{1/2}(s))'u(s)\|\,\mathrm{d}s.\end{aligned}$$

By Proposition 2.2,

$$\begin{aligned}\|\mathbf{A}^{1/2}(s)(\mathbf{A}^{1/2}(s))'u(s)\| &\le \|\mathbf{A}^{1/2}(s)(\mathbf{A}^{1/2}(s))'\mathbf{A}^{-1}(s)\|\,\|\mathbf{A}(s)u(s)\|\\ &\le c_2\|\mathbf{A}(s)u(s)\|.\end{aligned}$$

and via Proposition 4.6(b) and the above a priori estimate for $\|\mathbf{A}(t)u(t)\|$, we continue the above calculation on $[0,T)$:

$$\begin{aligned}\|\hat{\mathbf{A}}(t)\hat{u}(t)\| &\le e^{\hat{L}(1)t}\|\hat{\mathbf{A}}(0)\hat{\varphi}\| + c_3 e^{\hat{L}(1)t}\int_0^t \big(\|\mathbf{A}^{1/2}(s)u(s)\|^{\frac{2}{3}}+1\big)\|\mathbf{A}(s)u(s)\|\,\mathrm{d}s\\ &\le c_1(T)+c_2(T)\int_0^t \Big(\|\mathbf{A}^{1/2}(s)u'(s)\|+\|\mathbf{A}(s)u(s)\|\Big)\,\mathrm{d}s\\ &\le c_1(T)+\sqrt{2}c_2(T)\int_0^t \|\hat{\mathbf{A}}(s)\hat{u}(s)\|\,\mathrm{d}s,\end{aligned}$$

so that the desired a priori estimate for $\|\hat{\mathbf{A}}(t)\hat{u}(t)\|$ follows by Gronwall's Lemma.

We estimate now $\|\hat{\mathbf{A}}^2(t)\hat{u}(t)\|$. On $[0,T)$ we have

$$\begin{aligned}\|\hat{\mathbf{A}}^2(t)\hat{u}(t)\| &\le \|\hat{\mathbf{A}}^2(t)\hat{\mathbf{U}}(t,0)\hat{\mathbf{A}}^{-2}(0)\|\,\|\hat{\mathbf{A}}^2(0)\hat{\varphi}\|\\ &\quad + \int_0^t \|\hat{\mathbf{A}}^2(t)\hat{\mathbf{U}}(t,s)\hat{\mathbf{A}}^{-2}(s)\|\,\|\hat{\mathbf{A}}^2(s)\hat{\mathbf{M}}(s,\hat{u}(s))\|\,\mathrm{d}s\\ &\le e^{\hat{L}(2)T}\|\hat{\mathbf{A}}^2(0)\hat{\varphi}\| + e^{\hat{L}(2)T}\int_0^t \|\hat{\mathbf{A}}^2(s)\hat{\mathbf{M}}(s,\hat{u}(s))\|\,\mathrm{d}s.\end{aligned}$$

By Propositions 4.6(c) and 2.2, under the consideration of the previous estimate, we have

$$\begin{aligned}\|\hat{\mathbf{A}}^2(s)\hat{\mathbf{M}}(s,\hat{u}(s))\| &\le c_3\Big(\|\mathbf{A}(s)\mathbf{M}u(s)\|+\|\mathbf{A}(s)(\mathbf{A}^{1/2}(s))'u(s)\|\Big)\\ &\le c_3'\Big(f_2\big(\|\mathbf{A}(s)u(s)\|\big)+1\Big)\|\mathbf{A}^{3/2}(s)u(s)\|\\ &\le c_3''(T)\|\mathbf{A}^{3/2}(s)u(s)\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|\hat{\mathbf{A}}^2(t)\hat{u}(t)\| &\le c_3(T)+c_4(T)\int_0^t \Big(\|\mathbf{A}(s)u'(s)\|+\|\mathbf{A}^{3/2}(s)u(s)\|\Big)\,\mathrm{d}s\\ &\le c_3(T)+\sqrt{2}c_4(T)\int_0^t \|\hat{\mathbf{A}}^2(s)\hat{u}(s)\|\,\mathrm{d}s.\end{aligned}$$

So the a priori estimate for $\|\hat{\mathbf{A}}^2(t)\hat{u}(t)\|$ follows via Gronwall's Lemma.

This power 2 of $\hat{\mathbf{A}}^2(t)\hat{u}(t)$ is the critical power, so that the remaining a priori estimates for $\|\hat{\mathbf{A}}^\nu(t)\hat{u}(t)\|$, $\nu=3,4$, follow immediately via Proposition 4.5 applied on the integral equation directly since a sufficient regularity is available meanwhile (see [10, pages 269, 270]). $\square$

A consequence of Proposition 2.11 is the following result.

Corollary 5.2. *The 4-regular solution of the previous Theorem 5.1 for* (5.1) *and with that for* (1.1) *is as a function of* $t, x_1, \dots, x_5$ *twice classical continuously differentiable. Moreover, because of* $u(t,x)$, $u'(t,x)$, $u''(t,x)$ *belong to* $H_0^1(\Omega)\cap H_0^3(\Omega)$, *it follows by Proposition 4.3 that for all* $t\in[0,\infty)$: $u(t,x)$, $u'(t,x)$, $u''(t,x)\big|_{\partial\Omega}=0$ *a. e.*

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