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A diffusion equation for composite materials *

Mohamed El Hajji

Abstract

In this article, we study the asymptotic behavior of solutions to the diffusion equation with non-homogeneous Neumann boundary conditions. This equation models a composite material that occupies a perforated domain, in $\mathbb{R}^N$, with small holes whose sizes are measured by a number r_ε. We examine the case when $r_\varepsilon < \varepsilon^{N/(N-2)}$ with zero-average data around the holes, and the case when $\lim_{\varepsilon\to 0} r_\varepsilon/\varepsilon = 0$ with nonzero-average data.

1 Introduction

As a model for a composite material occupying a perforated domain in $\mathbb{R}^N$, the diffusion equation with non-homogeneous boundary conditions has been the object of many studies. In particular, we are interested in the properties of the solution to the equation

$$\begin{gathered} -\operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega_\varepsilon, \\ (A^\varepsilon(x)\nabla u_\varepsilon)\cdot n = h_\varepsilon \ \text{ on } \partial S_\varepsilon, \\ u_\varepsilon = 0 \ \text{ on } \partial\Omega, \end{gathered} \tag{1}$$

where Ω_ε is the perforated domain obtained by extracting a set of holes S_ε from Ω, f and h_ε are given functions, and A^ε is an operator in the space

$$\begin{aligned} M(\alpha,\beta;\Omega) \quad = \quad & \big\{A \in [L^\infty(\mathbb{R}^N)]^{N^2} : (A(x)\lambda,\lambda) \geq \alpha|\lambda|^2, \\ & |A(x)\lambda| \leq \beta|\lambda| \,\forall\lambda\in\mathbb{R}^N, p.p\cdot x\in\Omega\big\} \end{aligned}$$

which is defined for all real numbers α and β.

When $h_\varepsilon \in L^2(\partial S_\varepsilon)$ and the domain has holes of size r_ε, solutions to (1) have been studied by D. Cioranescu and P. Donato [3] for $r_\varepsilon = \varepsilon$ and $A^\varepsilon(x) = A(\frac{x}{\varepsilon})$ with $A \in M(\alpha,\beta;\Omega)$, and by C. Conca and P. Donato [5] for $A = I$ and $r_\varepsilon \ll \varepsilon$. Using the concept of H^0-convergence introduced by M. Briane et al. [2], P. Donato and M. El Hajji [6] showed convergence of solutions in not-necessarily

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periodic domains. The H^0-convergence is proven by showing strong convergence in $H^{-1}(\Omega)$ of the distribution, concentrated on the boundary of S_ε, given by

$$\langle \nu_h^\varepsilon, \varphi\rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)} \;=\; \langle h_\varepsilon, \varphi\rangle_{H^{-1/2}(\partial S_\varepsilon),\, H^{1/2}(\partial S_\varepsilon)}, \quad \forall \varphi \in H_0^1(\Omega). \tag{2}$$

This method allows the study the asymptotic behavior of solutions to (1) for $h_\varepsilon \in L^2(\partial S_\varepsilon)$ with $r_\varepsilon > \varepsilon^{N/N-2}$ (see [6]), and for perforated domains with double periodicity with $h_\varepsilon \in H^{-1/2}(\partial S_\varepsilon)$ and $r_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$ (see T. Levy [8]).

In this article, we study the perforated domains with $r_\varepsilon < \varepsilon^{N/N-2}$, and perforated domains such that $r_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$. In these situation the distribution given by (2) does not converge strongly in $H^{-1}(\Omega)$, and so the method described above can not be applied. In spite of this, we describe the asymptotic behavior of solutions to (1) using oscillating test functions. This method was introduced by L. Tartar [10] and has been used by many authors.

2 Statement of the main result

Let Ω be a bounded open set of $\mathbb{R}^N$, $Y = [0, l_1[\times .. \times [0, l_N[$ be a representative cell, and S be an open set of Y with smooth boundary ∂S such that $\overline{S} \subset Y$. Let ε and r_ε be terms of positive sequences such that $r_\varepsilon \le \varepsilon$. Let c denote positive constants independent of ε. We denote by $\tau(r_\varepsilon \overline{S})$ the set of translations of $r_\varepsilon \overline{S}$ of the form $\varepsilon k_1 + r_\varepsilon \overline{S}$ with $k \in \mathbb{Z}^N$. Let $k_l = (k_1 l_1, .., k_N l_N)$ represent the holes in $\mathbb{R}^N$.

We assume that the holes $\tau(r_\varepsilon \overline{S})$ do not intersect the boundary $\partial\Omega$. If S_ε is the set of the holes enclosed in Ω, it follows that there exists a finite set $\mathcal{K} \in \mathbb{Z}^N$ such that

$$S_\varepsilon = \bigcup_{k\in\mathcal{K}} r_\varepsilon(k_l + \overline{S}).$$

We set

$$\Omega_\varepsilon = \Omega \setminus \overline{S_\varepsilon}, \tag{3}$$

and denote by $\chi_{\Omega_\varepsilon}$ the characteristic function of Ω_ε. Let V_ε denote the Hilbert space

$$V_\varepsilon = \left\{ v \in H^1(\Omega_\varepsilon), v_{|\partial\Omega} = 0 \right\}$$

equipped with the H^1-norm. Let $A(y) = (a_{ij}(y))_{ij}$ be a matrix such that

$$A \in \left(L^\infty\left(\mathbb{R}^N\right)\right)^{N^2},$$

A is Y-periodic, and there there exist $\alpha > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(y)\,\lambda_i \lambda_j \ge \alpha\,|\lambda|^2, \quad \text{a.e. } y \text{ in } \mathbb{R}^N, \quad \forall\, \lambda \in \mathbb{R}^N. \tag{4}$$

We note that for every $\varepsilon > 0$,

$$A^\varepsilon(x) = A(\frac{x}{\varepsilon}) \quad \text{a.e. } x \text{ in } \mathbb{R}^N. \tag{5}$$

In this paper, we study the system

$$\begin{aligned} -\operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon, \\ (A^\varepsilon \nabla u_\varepsilon).n = h_\varepsilon \quad \text{on } \partial S_\varepsilon, \\ u_\varepsilon = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{6}$$

where Ω_ε is given by (3), A^ε is given by (5), and h_ε is given by

$$h^\varepsilon(x) = h(\frac{x}{r_\varepsilon}), \tag{7}$$

where $h \in L^2(\partial S)$ is Y-periodic function. Set

$$I_h = \frac{1}{|Y|} \int_{\partial S} h \, d\sigma \, . \tag{8}$$

We examine the case where $r_\varepsilon < \varepsilon^{N/(N-2)}$ with $I_h \neq 0$, and the case where $\lim_{\varepsilon\to o} r_\varepsilon/\varepsilon = 0$ with $I_h = 0$. The following result describes the asymptotic behavior of the solution to (6) in the two cases.

Theorem 1 *Let u_ε be the solution of (6). Suppose that one of the following hypotheses is satisfied*

$$I_h \neq 0 \quad \textit{and} \quad \begin{cases} \lim\limits_{\varepsilon\to 0} \dfrac{r_\varepsilon}{\varepsilon^{N/(N-2)}} = 0 & \textit{if } N > 2, \\ \lim\limits_{\varepsilon\to 0} \varepsilon^{-2}(\ln(\varepsilon/r_\varepsilon))^{-1} = 0 & \textit{if } N = 2, \end{cases} \tag{9}$$

or

$$I_h = 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon} = 0\, . \tag{10}$$

Then, for every $\varepsilon > 0$, there exists an extension operator P_ε defined from V_ε to $H^1_0(\Omega)$ satisfying

$$P_\varepsilon \in \mathcal{L}\left(V_\varepsilon, H^1_0(\Omega)\right), \tag{11}$$

$$(P_\varepsilon v)_{|\Omega_\varepsilon} = v \ \forall v \in V_\varepsilon, \tag{12}$$

$$\|\nabla\left(P_\varepsilon v\right)\|_{(L^2(\Omega))^N} \leq C\, \|\nabla v\|_{(L^2(\Omega_\varepsilon))^N}\, , \ \forall v \in V_\varepsilon\, . \tag{13}$$

such that

$$(\frac{r_\varepsilon}{\varepsilon})^{-N/2} P_\varepsilon u_\varepsilon \rightharpoonup u \quad \textit{weakly in } H^1_0(\Omega), \quad \textit{for } N > 2$$

and

$$P_\varepsilon[(\frac{r_\varepsilon}{\varepsilon})^{-1/2}(\log\frac{\varepsilon}{r_\varepsilon})^{-1/2} u_\varepsilon] \rightharpoonup u \quad \textit{weakly in } H^1_0(\Omega), \quad \textit{for } N = 2,$$

where u is the solution of the problem

$$\begin{aligned} -\operatorname{div}(A^0.\nabla u) = 0 \quad \textit{in } \Omega, \\ u = 0 \quad \textit{on } \partial\Omega, \end{aligned}$$

and the matrix $A^0 = (a^0_{ij})_{ij}$ *has entries*

$$a^0_{ij} = \frac{1}{|Y|} \int_Y (a_{ji} - \sum_{k=1}^{N} a_{ki} \frac{\partial \chi_j}{\partial y_k} dy), \tag{14}$$

and χ_j *is a* Y*-periodic function that satisfies*

$$-\operatorname{div}(A^t \nabla (y_j - \chi_j)) = 0 \quad \text{in } Y \tag{15}$$

Remark. One can replace the first equation of system (6) by

$$-\operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f_\varepsilon \quad \text{in } \Omega_\varepsilon,$$

with

$$\begin{gathered} (\tfrac{r_\varepsilon}{\varepsilon})^{-N/2} f_\varepsilon \rightharpoonup f \quad \text{weakly in } L^2(\Omega) \quad \text{if } N > 2, \\ (\tfrac{r_\varepsilon}{\varepsilon})^{-1} (\ln(\varepsilon / r_\varepsilon))^{-1/2} f_\varepsilon \rightharpoonup f \quad \text{weakly in } L^2(\Omega) \quad \text{if } N = 2. \end{gathered}$$

Then u will be the solution of

$$\begin{gathered} -\operatorname{div}(A^0 . \nabla u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

This approach has been used in [5] for the case $A = I$, in [3] when $A = I$ and $r_\varepsilon = \varepsilon$, and in [6] for the case where $r_\varepsilon > \varepsilon^{N/(N-2)}$ using the H^0-convergence and some arguments given by S. Kaizu in [7].

3 Proof of the main result

Observe first that S_ε is admissible in Ω, in the sense of the H^0-convergence ([6, 4, 5, 10]). Then there exists an extension operator P_ε satisfying (14).

On the other hand, the matrix A^0 can be defined by

$$A^0 \lambda = \mathcal{M}_Y ({}^t A \nabla w_\lambda) = \frac{1}{|Y|} \int_Y {}^t A \nabla w - \lambda \, dy, \quad \forall \lambda \in \mathbb{R}^N,$$

where for every $\lambda \in \mathbb{R}$, w_λ is the solution of the problem

$$\begin{gathered} -\operatorname{div}({}^t A \nabla w_\lambda) = 0 \quad \text{in } Y, \\ \text{with } w_\lambda - \lambda y \quad Y\text{-periodic}. \end{gathered} \tag{16}$$

For $x \in \mathbb{R}^N$, let

$$w^\varepsilon_\lambda(x) = \varepsilon w_\lambda(\frac{x}{\varepsilon}). \tag{17}$$

To simplify notation, let

$$\delta^\varepsilon = \begin{cases} (r_\varepsilon/\varepsilon)^{-N/2} & \text{if } N > 2, \\ (r_\varepsilon/\varepsilon)^{-1} (\ln(\varepsilon/r_\varepsilon))^{-1/2} & \text{if } N = 2. \end{cases} \tag{18}$$

Taking u_ε as a test function in the variational formulation of (6), and using the classical techniques of *a priori* estimates, one can easily show the existence of two constants c and c' independent of ε such that

$$c' \leq \|\nabla(\delta^\varepsilon u_\varepsilon)\|_{L^2(\Omega_\varepsilon)} \leq c.$$

Hence, from (13), up to a subsequence,

$$P_\varepsilon(\delta^\varepsilon u_\varepsilon) \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega). \tag{19}$$

Set now $\xi^\varepsilon = A^\varepsilon \nabla[P_\varepsilon(\delta^\varepsilon u_\varepsilon)]$. Then using (19), (11)-(13) and (4)-(5), one shows that ξ^ε is bounded in $L^2(\Omega)$, and so up to a subsequence

$$\xi^\varepsilon \rightharpoonup \xi \quad \text{weakly in } L^2(\Omega). \tag{20}$$

Case where (9) is satisfied. Let $\phi \in D(\Omega)$. Then from the variational formulation of (6), one has

$$\int_\Omega \chi_{\Omega_\varepsilon} \xi^\varepsilon.\nabla\phi\,dx = \delta^\varepsilon \int_{\partial S_\varepsilon} h_\varepsilon \phi\,d\sigma\,. \tag{21}$$

If $N(\varepsilon)$ denotes the number of the holes included in Ω, one has then

$$\begin{aligned} |\delta^\varepsilon \int_{\partial S_\varepsilon} h_\varepsilon\phi\,d\sigma| &\leq \|\phi\|_{L^\infty(\Omega)}\delta^\varepsilon \sum_{k\in\mathcal{K}} \int_{\partial(r_\varepsilon(S+k))} |h(\frac{x}{r_\varepsilon})|\,d\sigma(x) \\ &\leq c\delta^\varepsilon N(\varepsilon) r_\varepsilon^{N-1} \int_{\partial S} |h|\,d\sigma \\ &\leq c\delta^\varepsilon \frac{r_\varepsilon^{N-1}}{\varepsilon^N} |\partial S|^{1/2} \|h\|_{L^2(\partial S)}. \end{aligned} \tag{22}$$

From (18), one can write

$$\delta^\varepsilon \frac{r_\varepsilon^{N-1}}{\varepsilon^N} = \begin{cases} (\frac{r_\varepsilon{}^{N-2}}{\varepsilon^N})^{1/2} & \text{if } N > 2, \\ [\varepsilon^{-2}(\ln(\varepsilon/r_\varepsilon))^{-1}]^{1/2} & \text{if } N = 2, \end{cases}$$

and so in virtue of (9),

$$\lim_{\varepsilon\to 0} \delta^\varepsilon \frac{r_\varepsilon{}^{N-1}}{\varepsilon^N} = 0, \tag{23}$$

hence

$$\lim_{\varepsilon\to 0} \delta^\varepsilon \int_{\partial S_\varepsilon} h_\varepsilon\phi\,d\sigma = 0\,.$$

On the other hand, it is easy to show that

$$\chi_{\Omega_\varepsilon} \to 1 \quad \text{strongly in } L^p(\Omega) \quad \forall p \in [1,\infty[,$$

hence

$$\int_\Omega \chi_\varepsilon \xi^\varepsilon.\nabla\phi\,dx \to \int_\Omega \xi.\nabla\phi\,dx. \tag{24}$$

One can deduce that, as $\varepsilon \to 0$ in (21),

$$\int_\Omega \xi \nabla \phi \, dx = 0, \quad \forall \phi \in D(\Omega).$$

Consequently

$$-\operatorname{div} \xi = 0 \quad \text{in } \Omega. \tag{25}$$

It remains to identify the function ξ. Let w_λ^ε be the function defined by (16)-(17). Then

$$w_\lambda^\varepsilon \rightharpoonup \lambda x \quad \text{weakly in } H^1(\Omega), \text{ and so } L^p(\Omega) \text{ strong } \forall p < 2^*$$

where $1/2^* = 1/2 - 1/N$, with $N \geq 2$. Let $\phi \in D(\Omega)$, by choosing $\phi w_\lambda^\varepsilon$ as a test function in the variational formulation of (6), one has

$$\int_{\Omega_\varepsilon} \xi^\varepsilon \nabla(\phi w_\lambda^\varepsilon) \, dx = \delta^\varepsilon \int_{\partial S_\varepsilon} h_\varepsilon \phi w_\lambda^\varepsilon \, dx \, . \tag{26}$$

To pass to the limit as $\varepsilon \to 0$ in (26), we set

$$\int_{\Omega_\varepsilon} \xi^\varepsilon \nabla(\phi w_\lambda^\varepsilon) \, dx = J_1^\varepsilon + J_2^\varepsilon \, , \tag{27}$$

where

$$J_1^\varepsilon = \int_{\Omega_\varepsilon} \xi^\varepsilon \nabla \phi . w_\lambda^\varepsilon \, dx \quad \text{and} \quad J_2^\varepsilon = \int_{\Omega_\varepsilon} \xi^\varepsilon \nabla w_\lambda^\varepsilon . \phi \, dx.$$

Using the results given by G. Stampacchia in [9] (see also [1]), one can deduce that $w_\lambda \in L^\infty(Y)$, so

$$\chi_{\Omega_\varepsilon} w_\lambda^\varepsilon \to \lambda x \quad \text{strongly in } L^2(\Omega). \tag{28}$$

This with convergence (20), gives

$$J_1^\varepsilon = \int_\Omega \chi_{\Omega_\varepsilon} w_\lambda^\varepsilon \xi^\varepsilon \nabla \phi \, dx \to \int_\Omega \lambda x \xi \nabla \phi \, dx \quad \text{as } \varepsilon \to 0 \, . \tag{29}$$

Now, we may write

$$J_2^\varepsilon = \int_\Omega \xi^\varepsilon \nabla w_\lambda^\varepsilon \phi \, dx - \int_{S_\varepsilon} \xi^\varepsilon \nabla w_\lambda^\varepsilon \phi \, dx \, . \tag{30}$$

One the one hand,

$$\begin{aligned} \int_\Omega \xi^\varepsilon \nabla w_\lambda^\varepsilon \phi \, dx &= \int_\Omega {}^t A^\varepsilon \nabla w_\lambda^\varepsilon \nabla [P_\varepsilon(\delta^\varepsilon u_\varepsilon)\phi] \, dx - \int_\Omega {}^t A^\varepsilon \nabla w_\lambda^\varepsilon \nabla \phi P_\varepsilon[(\delta^\varepsilon u_\varepsilon] \, dx \\ &= -\int_\Omega {}^t A^\varepsilon \nabla w_\lambda^\varepsilon \nabla \phi P_\varepsilon[(\delta^\varepsilon u_\varepsilon] \, dx \end{aligned}$$

because, from the definition of w^ε_λ,

$$\int_\Omega {}^t A^\varepsilon \nabla w^\varepsilon_\lambda \nabla [P_\varepsilon(\delta^\varepsilon u_\varepsilon)\phi]\, dx = 0\,.$$

From the definition of A^0, ${}^t A^\varepsilon \nabla w^\varepsilon_\lambda \rightharpoonup A^0\lambda$ weakly in $L^2(\Omega)$. From (19), up to a subsequence,

$$P_\varepsilon(\delta^\varepsilon u_\varepsilon) \to u \quad \text{strongly in} L^2(\Omega),$$

which implies

$$\int_\Omega {}^t A^\varepsilon \nabla w^\varepsilon_\lambda \nabla\phi P_\varepsilon(\delta^\varepsilon u_\varepsilon)\, dx \to \int_\Omega A^0\lambda u \nabla\phi\, dx\,.$$

Hence

$$\int_\Omega \xi^\varepsilon \nabla w^\varepsilon_\lambda \phi\, dx \to -\int_\Omega A^0\lambda\nabla\phi u\, dx\,. \tag{31}$$

On the other hand,

$$|\int_{S_\varepsilon} \xi^\varepsilon \nabla w^\varepsilon_\lambda \phi\, dx| \le c\|\xi^\varepsilon\|_{L^2(\Omega)}\|\nabla w^\varepsilon_\lambda\|_{L^2(S_\varepsilon)}. \tag{32}$$

Since $\|\xi^\varepsilon\|_{L^2(\Omega)}$ is bounded,

$$|\int_{S_\varepsilon} \xi^\varepsilon \nabla w^\varepsilon_\lambda \phi\, dx| \le c\|\nabla w^\varepsilon_\lambda\|_{L^2(S_\varepsilon)}. \tag{33}$$

Note that

$$\begin{aligned}
\|\nabla w^\varepsilon_\lambda\|^2_{L^2(S_\varepsilon)} &= \int_{S_\varepsilon} |(\nabla w^\varepsilon_\lambda)(x)|^2\, dx \\
&= \sum_{k\in\mathcal{K}} \int_{r_\varepsilon(S+k)} |(\nabla w^\varepsilon_\lambda)(x)|^2\, dx \\
&= \sum_{k\in\mathcal{K}} \int_{r_\varepsilon(S+k)} |(\nabla_y w_\lambda)(\frac{x}{\varepsilon})|^2\, dy \\
&= N(\varepsilon)\varepsilon^N \int_{\frac{r_\varepsilon}{\varepsilon}S} |(\nabla_y w_\lambda)(y)|^2\, dy \\
&\le c\int_{\frac{r_\varepsilon}{\varepsilon}S} |\nabla w_\lambda|^2\, dy\,.
\end{aligned}$$

Since $r_\varepsilon/\varepsilon \to 0$ and $w_\lambda \in H^1(Y)$, it follows that

$$\int_{r_\varepsilon S/\varepsilon} |\nabla w_\lambda|^2\, dy \to 0\,.$$

Using (33), one has

$$\int_{S_\varepsilon} \xi^\varepsilon \nabla w^\varepsilon_\lambda \phi\, dx \to 0 \quad \text{as } \varepsilon \to 0.$$

This, with (30) and convergence (31) imply that

$$J_2^\varepsilon \to -\int_\Omega A^0\lambda\nabla\phi u\,dx. \tag{34}$$

Next we pass to the limit in the right hand of (26). With the same argument as in (22),

$$\begin{aligned}|\delta^\varepsilon\int_{\partial S_\varepsilon} h_\varepsilon\phi w_\lambda^\varepsilon\,d\sigma| &\le c\delta^\varepsilon N(\varepsilon)r_\varepsilon^{N-1}\int_{\partial S}|hw_\lambda|\,d\sigma\\ &\le c\delta^\varepsilon\frac{r_\varepsilon^{N-1}}{\varepsilon^N}\|w_\lambda\|_{L^2(\partial S)}|\partial S|^{1/2}\|h\|_{L^2(\partial S)}.\end{aligned}$$

Since we have shown that

$$\lim_{\varepsilon\to 0}\delta^\varepsilon\frac{r_\varepsilon^{N-1}}{\varepsilon^N}=0,$$

from (9) one deduces that

$$\delta^\varepsilon\int_{\partial S_\varepsilon} h_\varepsilon\phi w_\lambda^\varepsilon\,d\sigma\to 0.$$

Finally, by passing to the limit as $\varepsilon\to 0$ in (26), and using (32) and (34) one obtains

$$\int_\Omega \lambda x\xi\nabla\phi\,dx-\int_\Omega A^0\lambda\nabla\phi u\,dx=0,$$

hence, from (25) it follows that

$$\int_\Omega \xi\lambda\phi\,dx=\int_\Omega A^0\lambda\nabla u\phi\,dx\forall\phi\in D(\Omega),\forall\lambda\in\mathbb{R}^N,$$

i.e., $\xi=A^0\nabla u$.

Case where (10) is satisfied. Let $\phi\in D(\Omega)$. Then from the variational formulation of (6),

$$\int_\Omega \chi_{\Omega_\varepsilon}\xi^\varepsilon.\nabla\phi\,dx=\delta^\varepsilon\int_{\partial S_\varepsilon} h_\varepsilon\phi\,d\sigma. \tag{35}$$

The arguments used the proof of (24) can be applied here to obtain

$$\int_\Omega \chi_{\Omega_\varepsilon}\xi^\varepsilon.\nabla\phi\,dx\to\int_\Omega \xi.\nabla\phi\,dx\,.$$

To pass to the limit in the right-hand side of (35), we introduce N as the solution to

$$\begin{gathered}-\operatorname{div}N=0\quad\text{in } S,\\ N.n=-h\quad\text{on }\partial S\,.\end{gathered}$$

The existence of N is assured by the hypothesis $I_h = 0$. Set $N_\varepsilon(x) = N(\dfrac{x-\varepsilon k}{r_\varepsilon})$, for x in $(\varepsilon Y \setminus r_\varepsilon S)_k$. Then

$$\int_{\partial S_\varepsilon} h_\varepsilon \phi \, d\sigma = \int_{S_\varepsilon} \nabla\phi . N_\varepsilon \, dx,$$

hence

$$\delta^\varepsilon | \int_{\partial S_\varepsilon} h_\varepsilon \phi \, d\sigma| \le \delta^\varepsilon \|\nabla\phi\|_{L^2(S_\varepsilon)} \|N_\varepsilon\|_{L^2(S_\varepsilon)}.$$

Note that $\|N_\varepsilon\|_{L^2(S_\varepsilon)} \le c(\frac{r_\varepsilon}{\varepsilon})^{N/2} \|N\|_{L^2(S)}$, so

$$\delta^\varepsilon | \int_{\partial S_\varepsilon} h_\varepsilon \phi \, d\sigma| \le c\delta^\varepsilon (\frac{r_\varepsilon}{\varepsilon})^{N/2} \|\nabla\phi\|_{L^1(S_\varepsilon)}. \tag{36}$$

Since

$$\delta^\varepsilon (\frac{r_\varepsilon}{\varepsilon})^{N/2} = \begin{cases} 1 & \text{if } N > 2 \\ (\ln(\varepsilon/r_\varepsilon))^{-1/2} & \text{if } N = 2, \end{cases}$$

it follows from (36), when $N = 2$, that

$$\lim_{\varepsilon\to 0} \delta^\varepsilon | \int_{\partial S_\varepsilon} h_\varepsilon \phi \, d\sigma| = 0\,.$$

For $N > 2$, one has

$$\delta^\varepsilon | \int_{\partial S_\varepsilon} h_\varepsilon \phi \, d\sigma| \le c\|\nabla\phi\|_{L^2(S_\varepsilon)}.$$

Since $\chi_{\Omega_\varepsilon} \to 1$ strongly in $L^p(\Omega)$, for all $p \in [1, \infty[$ and $\phi \in D(\Omega)$, one deduces that

$$\int_\Omega (1 - \chi_\varepsilon)|\nabla\phi|^2 \, dx \to 0\,.$$

Hence, by passing to the limit as $\varepsilon \to 0$ in (35), one obtains

$$\int_\Omega \xi . \nabla\phi \, dx = 0\,,$$

then $-\operatorname{div} \xi = 0 \quad \text{in } \Omega$.

Let w_λ^ε be the function defined by (16)-(17) and $\phi \in D(\Omega)$. As in the previous case, by using $\phi w_\lambda^\varepsilon$ as a test function in the variational formulation of (6), one has

$$\int_{\Omega_\varepsilon} \xi^\varepsilon . \nabla(\phi w_\lambda^\varepsilon) \, dx = \delta^\varepsilon \int_{\partial S_\varepsilon} h_\varepsilon \phi w_\lambda^\varepsilon \, d\sigma\,.$$

From (27), (29) and (34), one has

$$\int_{\Omega_\varepsilon} \xi^\varepsilon . \nabla(\phi w_\lambda^\varepsilon) \, dx \to \int_\Omega \lambda x . \nabla\phi \, dx - \int_\Omega A^0 \lambda . \nabla\phi u \, dx. \tag{37}$$

Now we show that

$$\delta^\varepsilon \Big| \int_{\partial S_\varepsilon} h_\varepsilon \phi w_\lambda^\varepsilon \, d\sigma \Big| \to 0. \tag{38}$$

One has

$$\begin{aligned}
&\delta^\varepsilon \Big| \int_{\partial S_\varepsilon} h_\varepsilon \phi w_\lambda^\varepsilon \, d\sigma \Big| \\
&= \delta^\varepsilon \Big| \int_{S_\varepsilon} \nabla(\phi w_\lambda^\varepsilon).N_\varepsilon \, dx \Big| \\
&\leq \delta^\varepsilon \Big| \int_{S_\varepsilon} \nabla\phi . w_\lambda^\varepsilon . N_\varepsilon \, d\sigma \Big| + \delta^\varepsilon \Big| \int_{S_\varepsilon} \phi . \nabla w_\lambda^\varepsilon . N_\varepsilon \, d\sigma \Big| \\
&\leq \delta^\varepsilon \|N_\varepsilon\|_{L^2(S_\varepsilon)} \|\nabla\phi w_\lambda^\varepsilon\|_{L^2(S_\varepsilon)} + \delta^\varepsilon \|N_\varepsilon\|_{L^2(S_\varepsilon)} \|\phi \nabla w_\lambda^\varepsilon\|_{L^2(S_\varepsilon)} \\
&\leq c\delta^\varepsilon \big(\frac{r_\varepsilon}{\varepsilon}\big)^{N/2} \left\{ \|\nabla\phi w_\lambda^\varepsilon\|_{L^2(S_\varepsilon)} + \|\phi \nabla w_\lambda^\varepsilon\|_{L^2(S_\varepsilon)} \right\} \\
&\leq c\delta^\varepsilon \big(\frac{r_\varepsilon}{\varepsilon}\big)^{N/2} \left\{ \|\nabla\phi\|_{L^\infty(\Omega)} \|w_\lambda^\varepsilon\|_{L^2(S_\varepsilon)} + \|\phi\|_{L^\infty(\Omega)} \|\nabla w_\lambda^\varepsilon\|_{L^2(S_\varepsilon)} \right\} . \\
&\leq c\delta^\varepsilon \big(\frac{r_\varepsilon}{\varepsilon}\big)^{N/2} \left\{ \|w_\lambda^\varepsilon\|_{L^2(S_\varepsilon)} + \|\nabla w_\lambda^\varepsilon\|_{L^2(S_\varepsilon)} \right\} .
\end{aligned}$$

Note that

$$\begin{aligned}
\|\nabla w_\lambda^\varepsilon\|^2_{L^2(S_\varepsilon)} &= \int_{S_\varepsilon} |\nabla w_\lambda^\varepsilon|^2 \, dx = \sum_{k\in\mathcal{K}} \int_{r_\varepsilon(S+k)} |\nabla w_\lambda^\varepsilon|^2 \, dx \\
&\leq N(\varepsilon)\varepsilon^N \int_{\frac{r_\varepsilon}{\varepsilon} S} |\nabla w_\lambda|^2 \, dx \leq c \int_{\frac{r-\varepsilon}{\varepsilon} S} |\nabla w_\lambda|^2 \, dx \, .
\end{aligned}$$

Since $w_\lambda \in H^1(S)$ and $r_\varepsilon/\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0} \int_{\frac{r_\varepsilon}{\varepsilon} S} |\nabla w_\lambda|^2 \, dx = 0 \, .$$

Hence $\|\nabla w_\lambda^\varepsilon\|_{L^2(S_\varepsilon)} \to 0$. On the other hand, one has $\|w_\lambda^\varepsilon\|_{L^2(S_\varepsilon)} \leq c$. Finally, as

$$\lim_{\varepsilon\to 0} \delta^\varepsilon \big(\frac{r_\varepsilon}{\varepsilon}\big)^{N/2} = 0 \, ,$$

one deduces (38). This and (37) completes the proof, using the same arguments as in the previous case.

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MOHAMED EL HAJJI
Université de Rouen, UFR des Sciences
UPRES-A 60 85 (Labo de Math.)
76821 Mont Saint Aignan, France
e-mail: Mohamed.Elhajji@univ-rouen.fr