Electronic Journal of Differential Equations, Vol. 2005(2005), No. 70, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

MULTIPLE POSITIVE SOLUTIONS FOR SINGULAR SEMI-POSITONE DELAY DIFFERENTIAL EQUATION

XIAN XU

Abstract. In this paper, we obtain new existence results for multiple positive solutions of a delay singular differential boundary-value problem. Our main tool is the fixed point index method.

1. Introduction

Singular differential boundary-value problems arise from many branches of basic mathematics and applied mathematics. Many techniques have been developed to establish the existence of positive solutions of various classes of singular differential boundary-value problems; [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13] and the references therein. In particular, the authors of [1, 5] obtained some existence results for positive solutions of some singular functional differential equations. Motivated by [1, 5], in this paper we will consider the following singular delay differential equation

$$\begin{gathered} y'' + f(t, y(t-a)) = 0, \quad t \in (0,1]\backslash\{a\}, \\ y(t) = \mu(t), \quad t \in [-a, 0], \\ y(1) = 0, \end{gathered} \tag{1.1}$$

where $0 < a < 1$,

$$\mu(t) \in C[-a, 0], \quad \mu(0) = 0, \quad \mu(t) > 0, \quad \forall t \in [-a, 0), \tag{1.2}$$

and the nonlinear term $f(t, y)$ satisfies

$$\phi_0(t)h_0(y) - p(t) \le f(t, y) \le \phi(t)(g(y) + h(y)), \tag{1.3}$$

ϕ, ϕ_0, p are in $C((0,1], R^+)$, g is in $C((0,+\infty), R^+)$, h_0, h are in $C(R^+, R^+)$, and $R^+ = [0, +\infty)$.

Problem (1.1) is a singular semi-positone boundary-value problem because p is allowed to be nonnegative on interval $(0,1)$ and f may have singularity at $t = 0$ and $y = 0$. Recently, there have been many papers considered the singular semi-positone boundary-value problems; see [2, 8, 13, 11, 7] and the references therein. But most of these papers paid attention to the existence of positive solutions of the

2000 *Mathematics Subject Classification.* 34B15, 34B25.
Key words and phrases. Multiple positive solutions; fixed point index; semi-positone delay differential equation.

Submitted May 7, 2005. Published June 30, 2005.
Supported by grant 04KJB110138 from the Natural Science Foundation of the Jiangsu Education Committee, China.

singular semi-positone boundary-value problems and there were fewer papers that discuss the existence of multiple positive solutions of the singular semi-positone problems. To cover up this gap, in this paper we will establish some existence results for multiple positive solutions of singular delay differential semi-positone boundary value problem (1.1). It is difficult to show the existence and multiplicity results for positive solutions of semi-positone problems. For our purpose, a special cone will be needed to establish the multiplicity results for positive solutions of semi-positone problems.

Let $P = \{x \in C[-a,1] : x(t) \geq 0 \text{ for all } t \in [-a,1]\}$. It is well known that $C[-a,1]$ is a real Banach space with the maximum norm $\|x\| = \max_{t\in[-a,1]} |x(t)|$, and P a normal cone of $C[-a,1]$. By a positive solution of (1.1) we mean a function $x \in P$ satisfying (1.1) and $x(t) \not\equiv 0$.

Throughout this paper, we will assume that (1.2) and (1.3) hold.

2. Preliminary Lemmas

Let us list some assumptions to be used.

(H1) $g : (0,+\infty) \to R^+$ is continuous and decreasing, $h,\ h_0 : R^+ \to R^+$ are continuous and increasing.

(H2) For any constant $k_0 > 0$,

$$\int_0^a [\phi(s)g(\mu(s-a)) + g(k_0 s) + p(s)]ds < +\infty.$$

(H3) There exists $R_0 \geq 2\int_0^1 p(s)ds$ such that

$$R_0 - 2\sqrt{B(R_0)} > A, \tag{2.1}$$

where

$$A = \int_0^a [\phi(s)(g(\mu(s-a)) + h(\mu(s-a)+1)) + p(s)]ds + \int_a^1 \phi(s)g(\frac{1}{2}R_0 a(s-a))ds,$$

$$B(R_0) = 2\sup_{t\in[a,1]}[\phi(t)+p(t)]\int_0^{R_0}[g(\frac{1}{2}s) + h(s+1) + 1]ds.$$

(H4) There exist $u_1 > R_0$ and $[\alpha,\beta] \subset (a,1)$ such that

$$\alpha(1-\beta-a)h_0(\frac{1}{2}u_1)\int_{\alpha+a}^{\beta+a}\phi_0(s)ds > u_1.$$

Remark 2.1. The nonlinear term f of the form (1.3) in the case $\phi_0(t) = p(t) = 0$ for all $t \in [0,1]$ has been studied by many authors [1, 10, 5, 11].

Let $Q = \{x \in P : x(t) \geq \|x\|t(1-t) \text{ for } t \in [0,1]\}$. It is easy to see that Q is a cone of $C[-a,1]$. For each $x \in P$, let

$$[x(t-a)]^* = \max\{x(t-a) + x_0(t-a) - w(t-a), \widetilde{x}_0(t-a)\}, \quad \forall t \in [0,1],$$

where

$$x_0(t)=\begin{cases}\mu(t), & t\in[-a,0],\\ 0, & t\in(0,1],\end{cases}$$

$$\widetilde{x}_0(t)=\begin{cases}0, & t\in[-a,0],\\ \frac{1}{2}R_0t(1-t), & t\in(0,1],\end{cases}$$

$$w(t)=\begin{cases}0, & t\in[-a,0],\\ \int_0^1 G(t,s)p(s)ds, & t\in(0,1],\end{cases}$$

$$G(t,s)=\begin{cases}s(1-t), & s\le t,\\ t(1-s), & s>t.\end{cases}$$

For each positive integer n, let us define an operator $T_n:P\to P$ by

$$(T_nx)(t)=\begin{cases}0, & t\in[-a,0];\\ \int_0^1 G(t,s)[f(s,[x(s-a)]^*+n^{-1})+p(s)]ds, & t\in(0,1].\end{cases}\tag{2.2}$$

Lemma 2.2 ([6]). *Let X be a retract of the real Banach space E and X_1 be a bounded convex retract of X. Let U be a nonempty open set of X and $U\subset X_1$. suppose that $A:X_1\to X$ is completely continuous, $A(X_1)\subset X_1$ and A has no fixed points on $X_1\backslash U$. Then $i(A,U,X)=1$.*

Lemma 2.3. *Suppose that (H1) and (H2) hold. Then $T_n:P\to Q$ is a completely continuous operator for each positive integer n.*

Proof. Let n be a fixed positive integer, and $y=T_nx$ for some $x\in P$. Suppose that $\|y\|_{[0,1]}=y(t_0)$ for some $t_0\in(0,1)$, where $\|y\|_{[0,1]}=\max_{t\in[0,1]}|y(t)|$. Since

$$y(t)\ge y(s)=0,\quad t\in[-a,1],\ s\in[-a,0],$$

it follows that $\|y\|=\|y\|_{[0,1]}$. It is easy to see that

$$y''(t)=-f(t,[x(t-a)]^*+n^{-1})-p(t)\le 0,\ \forall t\in(0,1].$$

Therefore, the graph of $y(t)$ is concave down on $(0,1)$. For any $0\le t\le t_0$, we have

$$y(t)=y((1-\frac{t}{t_0})\cdot 0+\frac{t}{t_0}t_0)\ge\|y\|t(1-t).$$

Similarly,

$$y(t)=y(\frac{1-t}{1-t_0}t_0+(1-\frac{1-t}{1-t_0})\cdot 1)\ge\|y\|t(1-t),\quad \forall t_0\le t\le 1.$$

Hence, $T_n:P\to Q$.

Now, we show that T_n is a completely continuous operator for every positive integer n. It is easy to see that T_n is a continuous and bounded operator for every positive integer n. Let $B\subset P$ be a bounded set such that $\|x\|\le L$ for all $x\in B$ and some $L>0$. Then, we can easily see that

$$x_0(t-a)+\widetilde{x}_0(t-a)\le[x(t-a)]^*\le\|x\|+\|w\|+\|\mu\|+R_0,\quad \forall x\in B,\ t\in[0,1].\tag{2.3}$$

Thus, for any $t_1,\ t_2 \in [0,1]$, we have

$$\begin{aligned} &|(T_nx)(t_1) - (T_nx)(t_2)| \\ &\le \int_0^1 |G(t_1,s) - G(t_2,s)|[\phi(s)(g(x_0(s-a) + \widetilde{x}_0(s-a)) \\ &\quad + h(L + \|w\| + \|\mu\| + R_0 + 1)) + p(s)]ds. \end{aligned} \tag{2.4}$$

Then the uniform continuity of $G(t,s)$ on $[0,1] \times [0,1]$, (2.4) and the assumption (H2) imply that $T_n(B)$ is an equicontinuous set on [0,1]. Obviously, $T_n(B)$ is equicontinuous on $[-a,0]$. Thus, $T_n : P \to Q$ is a completely continuous operator. The proof is complete. ☐

Lemma 2.4. *Let $\Omega_0 = \{x \in Q : \|x\| < R_0\}$. Suppose that (H1)-(H3) hold. Then for every positive integer n,*

$$i(T_n, \Omega_0, Q) = 1\,.$$

Proof. We claim that

$$z \neq \lambda T_n z, \quad \lambda \in [0,1],\ z \in \partial\Omega_0. \tag{2.5}$$

where $\partial\Omega_0$ denotes the boundary of Ω_0 in Q. In fact, if (2.5) is not true, then there exist $\lambda_0 \in [0,1]$, $z_0 \in \partial\Omega_0$, and positive integer n_0 such that $z_0 = \lambda_0 T_{n_0} z_0$. From $z_0 \in Q$, we have

$$z_0(t) \ge \|z_0\| t(1-t) = R_0 t(1-t),\ t \in [0,1]. \tag{2.6}$$

On the other hand, using the fact that $G(t,s) \le t(1-t)$ for $(t,s) \in [0,1] \times [0,1]$, we have

$$w(t) = \int_0^1 G(t,s)p(s)ds \le \Big(\int_0^1 p(s)ds\Big)t(1-t), \quad \forall t \in [0,1]. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$z_0(t) - w(t) \ge \frac{1}{2} z_0(t) \ge \frac{1}{2} R_0 t(1-t), \forall t \in [0,1]. \tag{2.8}$$

From $z_0 = \lambda_0 T_{n_0} z_0$, by direct computation, we have

$$\begin{gathered} z_0''(t) + \lambda_0[f(t, [z_0(t-a)]^* + n^{-1}) + p(t)] = 0, \quad t \in (0,1], \\ z_0(t) = 0, \quad t \in [-a, 0), \\ z_0(1) = 0. \end{gathered} \tag{2.9}$$

By (2.8) and (2.9), we get that

$$[z_0(t-a)]^* = \begin{cases} \mu(t-a), & t \in [0,a], \\ z_0(t-a) - w(t-a), & t \in (a,1]. \end{cases} \tag{2.10}$$

It follows from (2.9) that $z_0''(t) \le 0$ for $t \in (0,1]$. Thus, the graph of $z_0(t)$ is concave down on $(0,1)$, and so, there exists $t_0 \in (0,1)$ such that

$$\begin{gathered} \|z_0\| = z_0(t_0), \quad z'(t_0) = 0, \quad z_0'(t) \ge 0 \quad \text{on } (0, t_0), \\ and z'(t) \le 0 \quad \text{on } (t_0, 1). \end{gathered}$$

Therefore, we have the following two cases:
Case (a): $t_0 \le a$. By (2.9) and (2.10), we have

$$-z_0''(t) \le \phi(t)(g(\mu(t-a)) + h(\mu(t-a) + 1)) + p(t), \quad \forall t \in (0, t_0).$$

Integrating from $t (t \in (0, t_0))$ to t_0, we have

$$z_0'(t) \le \int_0^{t_0} [\phi(s)(g(\mu(s-a) + h(\mu(s-a)+1)) + p(s)]ds, \quad t \in (0, t_0].$$

Then integrating from 0 to t_0, we have

$$z_0(t_0) \le \int_0^{t_0} s[\phi(s)(g(\mu(s-a)) + h(\mu(s-a)+1)) + p(s)]ds \le A. \tag{2.11}$$

Case (b): $t_0 > a$. By (2.8), (2.9) and (2.10), we have

$$\begin{aligned} -z_0''(t) &\le \phi(t)(g(z_0(t-a) - w(t-a)) + h(z_0(t-a)+1)) + p(t) \\ &\le \phi(t)(g(\frac{1}{2} z_0(t-a)) + h(z_0(t-a)+1)) + p(t), \quad \forall t \in [a, t_0]. \end{aligned}$$

Since $z_0'(t-a) \ge z_0'(t)$ for $t \in [a, t_0]$, we have

$$-z_0''(t) z_0'(t) \le [\phi(t)(g(\frac{1}{2} z_0(t-a)) + h(z_0(t-a)+1)) + p(t)] z_0'(t-a),$$

for all $t \in [a, t_0]$. Integrating from $t (t \in [a, t_0])$ to t_0, we have

$$\begin{aligned} &[z_0'(t)]^2 \\ &\le 2 \sup_{t \in [a,1]} [\phi(t) + p(t)] \int_t^{t_0} [g(\frac{1}{2} z_0(s-a)) + h(z_0(s-a)+1) + 1] z_0'(s-a) ds \\ &\le 2 \sup_{t \in [a,1]} [\phi(t) + p(t)] \int_{z_0(t-a)}^{z_0(t_0-a)} [g(\frac{1}{2} s) + h(s+1) + 1] ds \\ &\le 2 \sup_{t \in [a,1]} [\phi(t) + p(t)] \int_0^{z_0(t_0)} [g(\frac{1}{2} s) + h(s+1) + 1] ds \\ &= B(R_0) \end{aligned}$$

and so

$$z_0'(t) \le \sqrt{B(R_0)}, \quad \forall t \in [a, t_0]. \tag{2.12}$$

Then integrating from a to t_0, we have

$$z_0(t_0) \le z_0(a) + \sqrt{B(R_0)}. \tag{2.13}$$

On the other hand, by (2.9) and (2.10), we have

$$-z_0''(t) \le \phi(t)[g(\mu(t-a)) + h(\mu(t-a)+1)] + p(t), \quad \forall t \in (0, a].$$

Integrating from $t (t \in (0, a))$ to a, by (2.12), we have

$$z_0'(t) \le z_0'(a) + \int_0^a [\phi(s)(g(\mu(s-a)) + h(\mu(s-a)+1) + p(s)]ds \le \sqrt{B(R_0)} + A.$$

Then integrating from 0 to a, we have

$$z_0(a) \le \sqrt{B(R_0)} + A. \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$z_0(t_0) \le 2\sqrt{B(R_0)} + A \tag{2.15}$$

Since $z_0(t_0) = R_0$, from (2.11) and (2.15), we have

$$R_0 \le 2\sqrt{B(R_0)} + A,$$

which is a contradiction to (H3). Thus (2.5) holds. By the properties of fixed point index, we have

$$i(T_n, \Omega_0, Q) = i(\theta, \Omega_0, Q) = 1.$$

The proof is completed. $\square$

Remark 2.5. The inequality (2.1) played an important role in the proof of Lemma 2.4. This type of inequality has been employed extensively in the literature [1, 5, 9]. The main idea of our proof of Lemma 2.4 is derived from [1].

3. Main Results

Theorem 3.1. *Suppose that (H1)-(H4) hold. Assume that*

$$\lim_{x\to+\infty} \frac{h(x)}{x} = 0. \tag{3.1}$$

Then (1.1) *has at least two positive solutions.*

Proof. For each positive integer n, let us define the operator T_n by (2.2). It follows from Lemma 2.3 that $T_n : Q \to Q$ is a completely continuous operator for every n. Let δ be a positive number such that

$$0 < \delta < \min\big\{1, (\int_0^1 \phi(s)ds)^{-1}\big\}.$$

It follows from (3.1) that there exists $R > u_1$ such that $h(x) \le \delta x$ for all $x \ge R$. Since $h : R^+ \to R^+$ is increasing, then

$$h(x) \le \delta x + h(R), \quad \forall x \in R^+.$$

By (H4), there exists $\widetilde{u}_1 > u_1$ such that

$$\alpha(1-\beta-a)h(\frac{1}{2}\widetilde{u}_1)\int_{\alpha+a}^{\beta+a} \phi_0(s)ds > \widetilde{u}_1. \tag{3.2}$$

Put

$$R_1 = \max\big\{2\widetilde{u}_1, \frac{2[A + \|w\| + (\|w\| + \|\mu\| + R_0 + 1 + h(R))\int_0^1 \phi(s)ds]}{1 - \delta\int_0^1 \phi(s)ds}\big\}, \tag{3.3}$$

$$\begin{gathered}
\Omega_0 = \{x \in Q : \|x\| < R_0\},\\
\Omega_1 = \{x \in Q : \|x\| < R_1\},\\
\Omega_{01} = \{x \in Q : \|x\| < R_1, \quad \inf_{t\in[\alpha,\beta]} x(t) > u_1\},\\
U_{01} = \{x \in Q : \|x\| < R_1 \quad \inf_{t\in[\alpha,\beta]} x(t) > \widetilde{u}_1\}.
\end{gathered}$$

It is easy to see that Ω_0, Ω_1, Ω_{01} and U_{01} are bounded open convex sets of Q, and that

$$\Omega_0 \subset \Omega_1, \quad \Omega_{01} \subset \Omega_1, \quad U_{01} \subset \Omega_1, \quad \Omega_0 \cap \Omega_{01} = \emptyset, \quad U_{01} \subset \Omega_{01}.$$

For each positive integer n and $x \in \overline{\Omega}_1$, by (2.3) and (3.3), we have

$$\begin{aligned}
&(T_nx)(t)\\
&\le \int_0^1 G(t,s)[\phi(s)(g([x(s-a)]^*+n^{-1})+h([x(s-a)]^*+n^{-1}))+p(s)]ds\\
&\le \int_0^a \phi(s)(g(\mu(s-a))ds+\int_a^1 \phi(s)(g(\frac{1}{2}R_0a(s-a))ds\\
&+h(\|x\|+\|w\|+\|\mu\|+R_0+1))\int_0^1 \phi(s)ds+\|w\| \qquad (3.4)\\
&\le A+\|w\|+[\delta(\|x\|+\|w\|+\|\mu\|+R_0+1)+h(R)]\int_0^1 \phi(s)ds\\
&\le A+\|w\|+[\delta R_1+\|w\|+\|\mu\|+R_0+1+h(R)]\int_0^1 \phi(s)ds\\
&< R_1, \quad \forall t\in[0,1].
\end{aligned}$$

Since $(T_nx)(t)=0$ for $t\in[-a,0]$, it follows that $\|T_nx\|<R_1$ for all $x\in\overline{\Omega}_1$. Hence, $T_n(\overline{\Omega}_1)\subset\Omega_1$ for all positive integer n. By Lemma 2.2, we have for each positive integer n

$$i(T_n,\Omega_1,Q)=1. \qquad (3.5)$$

For any $x\in\overline{\Omega}_{01}$, by (3.4), we have $\|T_nx\|<R_1$. It is easy to see that for $x\in\overline{\Omega}_{01}$

$$[x(t-a)]^*=x(t-a)-w(t-a)\ge\frac{1}{2}x(t-a)\ge\frac{1}{2}u_1, \quad t\in[\alpha+a,\beta+a].$$

Then the assumption (H4) implies

$$\begin{aligned}
(T_nx)(t)&\ge\int_{\alpha+a}^{\beta+a} G(t,s)\phi_0(s)h_0([x(s-a)]^*)ds\\
&\ge\alpha(1-\beta-a)h_0(\frac{1}{2}u_1)\int_{\alpha+a}^{\beta+a}\phi_0(s)ds\\
&> u_1, \quad \forall t\in[\alpha,\beta],
\end{aligned}$$

and so, $T_n(\overline{\Omega}_{01})\subset\Omega_{01}$ for every positive integer n. Also by Lemma 2.2, we have

$$i(T_n,\Omega_{01},Q)=1 \qquad (3.6)$$

for every positive integer n. Similarly, by (3.2) we can show that

$$i(T_n,U_{01},Q)=1 \qquad (3.7)$$

for every positive integer n. It follows from (3.7) that for every positive integer n, T_n has at least one fixed point $\widetilde{x}_n\in\overline{U}_{01}$. It is easy to see that

$$\begin{aligned}
[\widetilde{x}_n(t-a)]^*&=\begin{cases}\mu(t-a), & t\in[0,a]\\ \widetilde{x}_n(t-a)-w(t-a), & t\in(a,1]\end{cases}\\
&\ge\begin{cases}\mu(t-a), & t\in(0,a]\\ \frac{1}{2}\widetilde{x}_n(t-a), & t\in(a,1]\end{cases}\\
&\ge x_0(t-a)+\widetilde{x}_0(t-a), \quad t\in[0,1],
\end{aligned}$$

and so

$$g([\widetilde{x}_n(t-a)]^*+n^{-1})\le g(x_0(t-a)+\widetilde{x}_0(t-a)), t\in(0,1]\backslash\{a\}. \qquad (3.8)$$

Using essentially the same argument as in Lemma 2.4, we see that there exists $t_n \in (0,1)$ such that $\widetilde{x}_n'(t_n) = 0$, and

$$-\widetilde{x}_n''(t) \le \phi(t)[g(x_0(t-a) + \widetilde{x}_0(t-a) + n^{-1}) + h(R_1+1)] + p(t),$$

for all $t \in (0,1]$. Integration yields

$$|\widetilde{x}_n'(t)| \le \int_0^1 [\phi(s)(g(x_0(s-a) + \widetilde{x}_0(s-a)) + h(R_1+1)) + p(s)]ds,$$

for all $t \in (0,1]$. This means that $\{\widetilde{x}_n\}$ is equicontinuous on $[0,1]$. Since $\widetilde{x}_n(t) = 0$ for $t \in [-a,0]$, $\{\widetilde{x}_n\}$ is equicontinuous on $[-a,0]$. Therefore, the Arzela-Ascoli Theorem guarantees the existence of a subsequence $\{\widetilde{x}_{n_i}\}$ of $\{\widetilde{x}_n\}$ and a function $x_{01} \in \overline{U}_{01}$ with $\widetilde{x}_{n_i}$ converging uniformly on $[-a,1]$ to x_{01} as $i \to \infty$. From $\widetilde{x}_n = T_n\widetilde{x}_n$, by (3.8) and using the Lebesgue dominated convergence Theorem, we have

$$\begin{aligned} x_{01}(t) &= \begin{cases} 0, & t \in [-a,0]; \\ \int_0^1 G(t,s)[f(s,[x_{01}(s-a)]^*) + p(s)]ds, & t \in (0,1]. \end{cases} \\ &= \begin{cases} 0, & t \in [-a,0]; \\ \int_0^1 G(t,s)[f(s, x_{01}(s-a) + x_0(s-a) & \\ -w(s-a)) + p(s)]ds, & t \in (0,1]. \end{cases} \end{aligned}$$

Let $y_{01}(t) = x_{01}(t) + x_0(t) - w(t)$ for $t \in [-a,1]$. Then, we have

$$y_{01}(t) = \begin{cases} \mu(t), & t \in [-a,0]; \\ \int_0^1 G(t,s)f(s, y_{01}(s-a))ds, & t \in (0,1]. \end{cases}$$

It is easily verfied that y_{01} is a positive solution of (1.1). It follows from (3.5), (3.6) and Lemma 2.4 that for every positive integer n

$$i(T_n, \Omega_1\backslash(\bar{\Omega}_{01} \cup \bar{\Omega}_0), Q) = i(T_n, \Omega_1, Q) - i(T_n, \Omega_{01}, Q) - i(T_n, \Omega_0, Q) = -1.$$

Therefore, T_n has at least one fixed point $\bar{x}_n \in \Omega_1\backslash(\bar{\Omega}_{01} \cup \bar{\Omega}_0)$ for every positive integer n. For every positive integer n, there is at least one point $t_n \in [\alpha,\beta]$ such that $\bar{x}_n(t_n) \le u_1$. In a similar way as above, we can show that there exist a subsequence $\{\bar{x}_{n_i}\}$ of $\{\bar{x}_n\}$, $x_1 \in \overline{\Omega_1\backslash(\bar{\Omega}_{01} \cup \bar{\Omega}_0)}$ and a point $t_0 \in [\alpha,\beta]$ such that $\bar{x}_{n_i}$ convergence uniformly on [-a,1] to x_1 as $i \to \infty$, and $x_1(t_0) \le u_1$. Let $y_1(t) = x_1(t) + x_0(t) - w(t)$ for $t \in [-a,1]$. Then y_1 is a positive solution of (1.1). Since

$$x_1(t_0) \le u_1 < \widetilde{u}_1 \le x_{01}(t_0),$$

y_{01} and y_1 are two distinct positive solutions of (1.1). □

Theorem 3.2. *Suppose that (H1)-(H4) hold, and that there exists $R_1 > u_1$ such that*

$$A + \|w\| + h(R_1 + \|w\| + \|\mu\| + 1)\int_0^1 \phi(s)ds < R_1,$$
$$\lim_{x\to+\infty} \frac{h_0(x)}{x} = +\infty.$$

Then (1.1) *has at least three positive solutions.*

Proof. For every positive integer n, let us define an operator T_n by (2.2). By (3.2), there exists a positive number $\bar{R}_1 > R_1$ such that

$$A + \|w\| + h(\bar{R}_1 + \|w\| + \|\mu\| + 1) \int_0^1 \phi(s)ds < \bar{R}_1. \tag{3.9}$$

Let us define the open sets Ω_0, Ω_{01}, Ω_1 and U_{01} as in Theorem 3.1. Let $U_1 = \{x \in Q : \|x\| < \bar{R}_1\}$. It is easy to see that for any $x \in \bar{\Omega}_1$ and $t \in [0,1]$

$$R_1 + \|w\| + \|\mu\| \geq [x(t-a)]^* \geq \begin{cases} \mu(t-a), & t \in [0,a]; \\ \frac{1}{2}R_0 a(t-a), & t \in [a,1]. \end{cases}$$

Since $h : R^+ \to R^+$ is increasing, we have

$$h([x(t-a)]^* + n^{-1}) \leq h(R_1 + \|w\| + \|\mu\| + 1), \forall t \in [0,1].$$

Therefore, by (3.2), we have

$$\begin{aligned} |(T_n x)(t)| &\leq A + \|w\| + h(R_1 + \|w\| + \|\mu\| + 1) \int_0^1 \phi(s)ds \\ &< R_1, \quad \forall x \in \bar{\Omega}_1,\ t \in [0,1] \end{aligned}$$

for any positive integer n. This means that $T_n(\bar{\Omega}_1) \subset \Omega_1$. Similarly, by (3.9), we can show $T_n(\bar{U}_1) \subset U_1$ for every positive integer n. By Lemma 2.2, we have

$$i(T_n, U_1, Q) = 1. \tag{3.10}$$

In a similar way as Theorem 3.1, we can show that (1.1) has at least two positive solutions y_1 and y_{01} such that

$$y_1(t) = x_1(t) + x_0(t) - w(t), \quad y_{01}(t) = x_{01}(t) + x_0(t) - w(t), \quad \forall t \in [-a,1],$$

where $x_1 \in \overline{\Omega_1 \backslash (\bar{\Omega}_{01} \cup \bar{\Omega}_0)}$ and $x_{01} \in \bar{U}_{01}$.

Now, we shall show the existence of the third positive solution of (1.1). Let

$$M > 2(\alpha(1-\beta) \sup_{t\in[0,1]} \int_{\alpha+a}^{\beta+a} G(t,s)\phi_0(s)ds)^{-1}. \tag{3.11}$$

By (3.2), there exists $\widetilde{R} > \bar{R}_1$ such that $h_0(y) \geq My$ for any $y \geq \widetilde{R}$. Set $R_2 = 2\widetilde{R}\alpha^{-1}(1-\beta)^{-1}$, $\Omega_2 = \{x \in Q : \|x\| < R_2\}$. Let $\psi_0 \in Q\backslash\{\theta\}$. We claim that for every positive integer n

$$y \neq T_n y + \lambda\psi_0, \quad \lambda \geq 0, \quad y \in \partial\Omega_2. \tag{3.12}$$

If not, then there exist $n_0 \in \mathbf{N}$, $y_0 \in \partial\Omega_2$ and $\lambda_0 \geq 0$ such that

$$y_0 = T_{n_0} y_0 + \lambda_0 \psi_0$$

It is easy to see that

$$\begin{aligned} [y_0(t-a)]^* &= y_0(t-a) - w(t-a) \\ &\geq \frac{1}{2}\|y_0\|(t-a)(1-t+a) \\ &\geq \frac{1}{2}R_2\alpha(1-\beta) > \widetilde{R}, \quad t \in [\alpha+a, \beta+a]. \end{aligned}$$

Therefore,

$$\begin{aligned} R_2 \ge y_0(t) &\ge \int_0^1 G(t,s)\phi_0(s)h_0([y_0(s-a)]^*)ds \\ &\ge \int_{\alpha+a}^{\beta+a} G(t,s)\phi_0(s)h_0(y_0(s-a)-w(s-a))ds \\ &\ge \frac{1}{2}MR_2\alpha(1-\beta)\int_{\alpha+a}^{\beta+a} G(t,s)\phi_0(s)ds, \quad t\in[0,1]. \end{aligned}$$

Hence

$$M \le 2(\alpha(1-\beta)\sup_{t\in[0,1]}\int_{\alpha+a}^{\beta+a} G(t,s)\phi_0(s)ds)^{-1},$$

which is a contradiction to (3.11). Hence, (3.12) holds. From the properties of the fixed point index, we have

$$i(T_n,\Omega_2,Q)=0.$$

It follows from (3.10) and (3) that

$$i(T_n,\Omega_2\backslash\bar{U}_1,Q)=i(T_n,\Omega_2,Q)-i(T_n,U_1,Q)=-1$$

for every positive integer n. Hence, T_n has at least one fixed point $\widetilde{x}_n \in \Omega_2\backslash\bar{U}_1$. Using essentially the same argument as in Theorem 3.1, we can show that there exist a subsequence $\{\widetilde{x}_{n_i}\}$ of $\{\widetilde{x}_n\}$, and $x_3 \in \overline{\Omega_2\backslash\bar{U}_1}$ such that $\widetilde{x}_{n_i} \to x_3 (i \to +\infty)$, and $y_3 = x_3 + x_0 - w$ is a positive solution of (1.1). The proof is completed. □

Corollary 3.3. *Suppose that (H1)-(H3) hold. Moreover, there exist R_i, $u_i (i = 1,2,\dots,n)$ with $R_0 < u_1 < R_1 < u_2 < R_2 < \dots < u_n < R_n$ such that*

$$A+\|w\|+h(R_i+\|w\|+\|\mu\|+1)\int_0^1\phi(s)ds < R_i, \quad i=1,2,\dots,n,$$

$$\alpha(1-\beta-a)h_0(\frac{1}{2}u_i)\int_{\alpha+a}^{\beta+a}\phi_0(s)ds > u_i, \quad i=1,2,\dots,n,$$

$$\lim_{x\to+\infty}\frac{h_0(x)}{x}=+\infty.$$

Then (1.1) *has at least* $2n+1$ *positive solutions.*

Corollary 3.4. *Suppose that (H1)-(H3) hold, and* $\lim_{x\to+\infty}\frac{h_0(x)}{x} = +\infty$. *Then* (1.1) *has at least one positive solution.*

We remark that the multiplicity results for positive solutions of singular semi-positone delay differential equations are new. Obviously, we can use the ideas of this paper to establish multiplicity results for positive solutions of the more general delay equation.

Example. Consider the delay differential boundary-value problem

$$\begin{gathered} y'' + 40\Big[\frac{1}{\sqrt{y(t-\frac{1}{4})}} + h(y(t-\frac{1}{4}))\Big] = \frac{1}{t^{1/4}}, \quad t\in(0,1]\backslash\{\frac{1}{4}\}, \\ y(t) = -t, \quad t\in[-\frac{1}{4},0], \\ y(1)=0, \end{gathered} \tag{3.13}$$

where $h : R^+ \to R^+$ is increasing, $h(y) = y^{1/4}$ for $y \in [0, 2 \times 10^4]$, and

$$\lim_{x\to+\infty} \frac{h(x)}{x} = 0\,.$$

Claim: If there exits $u_1 > 2\times10^4$ such that $h(\frac{1}{2}u_1) > 2u_1$, then the boundary-value problem (3.13) has at least two positive solutions.

Proof. Let $\phi(t) = \phi_0(t) = 40$ for $t \in [0, 1]$, $a = 1/4$, $g(y) = 1/\sqrt{y}$, $p(t) = 1/t^{1/4}$, $\mu(t) = -t$ for $t \in [-\frac{1}{4}, 0]$. It is easy to see (H1) and (H2) hold. Set $R_0 = 10^4$. Then, we have

$$\begin{aligned}
A &\le \int_0^{1/4} \big[40(\frac{1}{\sqrt{\frac{1}{4}-s}} + h(\frac{1}{4} - s + 1)) + \frac{1}{s^{1/4}}\big]ds + \int_{1/4}^1 \frac{40}{\sqrt{\frac{R_0}{8}(s-\frac{1}{4})}}\\
&\le 80\sqrt{\frac{1}{4}-s}\Big|_{1/4}^{0} + 10h(\frac{5}{4}) + \frac{4}{3} + \frac{320}{\sqrt{R_0}}\\
&= 40 + 10h(\frac{5}{4}) + \frac{4}{3} + \frac{320}{\sqrt{R_0}} \le 80,
\end{aligned}$$

$$\begin{aligned}
B(R_0) &= 2 \times (40 + \sqrt{2}) \int_0^{R_0} \big(\frac{1}{\sqrt{s/2}} + (s+1)^{\frac{1}{4}} + 1\big)ds\\
&< 84(2\sqrt{2R_0} + \frac{4}{5}(R_0 + 1)^{5/4} + R_0)\\
&< 340\sqrt{R_0} + 80(R_0 + 1)^{5/4} + 84R_0 < 8875040,
\end{aligned}$$

and

$$R_0 - 2\sqrt{B(R_0)} > R_0 - 2\sqrt{8875040} > 80,$$

which implies (H3). Put $[\alpha, \beta] = [\frac{1}{3}, \frac{1}{2}]$, $h(y) = h_0(y)$ for $y \in R^+$. It is easy to check that (H4) holds. Thus, by Theorem 3.1, the conclusion holds. The proof is completed. $\square$

References

[1] R. P. Agarwal and D. O'Regan; *Singular boundary value problems for superlinear second order ordinary and delay differential equations*, J. Differential Equations, 130, (1996), 333-355.

[2] Ravi P. Agarwal and D. O'Regan; *A note on existence of nonnegative solutions to singular semi-positone problems*, Nonlinear Anal., 36(1999), 615-622.

[3] Y. S. Choi; *A singular boundary value problem arising from near-ignition analysis of flame structure*, Diff. Integral Eq., 4(4), 1991, 891-895.

[4] R. Dalmasso; *Positive solutions of singular boundary value problems*, Nonlinear Anal., 27(6), 1996, 645-652.

[5] L. H. Erbe, Qingkai Kong; *Boundary value problems for singular second-order functional differential equations*, J. Comput. & Appl. Math., 53(1994), 377-388.

[6] Guo Dajun, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic press, Inc, New York, 1988.

[7] J. Janus, J. Myjak; *A generalized Emden-Fowler equation with a negative exponent*, Nonlinear Anal., 23(1994), 953-970.

[8] Liu Xiyu, Yan Baoqiang; *Boundary Irregular solutions to singular boundary value problems*, Nonlinear Anal., 32(5), 1998, 633-646.

[9] Donal O'Regan; *Singular second order boundary value problems*, Nonlinear Anal.,15(12), 1990, 1097-1109.

[10] D. O'Regan; *Existence principles and theory for singular Dirichlet boundary value problems*, Differential Equations Dynam. Systems, 3(1995), 289-304.

[11] Xu Xian; *Positive solutions for singular semi-positone boundary value problems*, J. Math. Anal. Appl., 273(2002), 480-491.
[12] Xian Xu; *Positive solutions of generalized Emden-Fowler equation*, Nonlinear Anal., 53(1), 2003, 23-44.
[13] Junyu Wang, Wenjie Gao; *A note on singular nonlinear two-point boundary value problems*, Nonlinear Anal., 39(2000), 281-287.

Xu Xian
Department of Mathematics, Xuzhou Normal University, Xuzhou, Jiangsu, 221116, China
E-mail address: xuxian68@163.com