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MOVING-BOUNDARY PROBLEMS FOR THE TIME-FRACTIONAL DIFFUSION EQUATION

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Abstract. We consider a one-dimensional moving-boundary problem for the time-fractional diffusion equation. The time-fractional derivative of order $\alpha \in (0,1)$ is taken in the sense of Caputo. We study the asymptotic behavior, as t tends to infinity, of a general solution by using a fractional weak maximum principle. Also, we give some particular exact solutions in terms of Wright functions.

1. Introduction

The beginning of fractional calculus dates to the 19th century. Abel, Liouville, Riemann and Letnikov proposed several definitions of fractional derivatives motivated by the idea of providing a novel operator that includes the classical concept of derivative. However, these definitions were not used until a century later. It was in the 1950s when the study of fractional differential equations gained relevance. From that moment, many authors pointed out that derivatives and integrals of non-integer order were useful for describing properties of various real-world materials such as polymers and some types of non-homogeneous solids [1, 6, 23, 24].

Among the fractional partial differential equations, we have the fractional diffusion equation (FDE), obtained from the standard diffusion equation by replacing the first order time-derivative by a fractional derivative. This equation was studied in [9, 10, 21, 11, 26, 35]. Several applications have been considered in the past two decades: Mainardi studied the FDE in the context of the theory of linear viscoelasticity in [22]. Voller et al. [32] stated that the FDE can be derived if we consider a new kind of heat flux involving the memory of the material, instead of the classical local flux and we replace it in the balance heat equation (see [13] where a general theory of heat conduction for materials with memory is presented). FDEs ha been solved numerically by several authors; see a general discussion in [7]. Some numerical results for a fractional Stefan problem based on a finite difference method adapted to the FDE are presented in [4]. A finite difference scheme for an initial-boundary value problem is presented in [16].

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In this article, we consider a moving-boundary problem for the FDE in which a time-fractional derivative of order $\alpha \in (0,1)$ in the sense of Caputo [5]. More precisely, we consider the problem

$$\begin{gathered} {}_0D_t^\alpha u(x,t) = \lambda^2 u_{xx}(x,t), \quad s_1(t) < x < s_2(t),\ 0 < t \le T,\ 0 < \alpha < 1, \\ u(s_1(t),t) = g(t), \quad 0 < t \le T, \\ u(s_2(t),t) = h(t), \quad 0 < t \le T, \\ u(x,0) = f(x), \quad a \le x \le b, \quad s_1(0) = a, \quad s_2(0) = b, \end{gathered} \tag{1.1}$$

where

$$ {}_0D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u_t(x,\tau) d\tau, \tag{1.2}$$

and the given functions s_1, s_2, g, h and f are continuous functions is their respective domains.

Problem (1.1) has not yet been deeply studied (some works related to it are [3, 15, 18, 27, 30, 31]) and our purpose is to present some explicit solutions and then analyze the asymptotic behavior of a general solution as t tends to infinity. In partial differential equations of parabolic type, the asymptotic behavior of a solution is closely linked to the maximum principles (weak and strong) valid for this kind of problems. These statements are not yet valid for fractional parabolic operators, but we gather some results known at the moment for the FDE (which is a particular case of of a fractional differential operator of parabolic type) related to maximum principles [2, 20, 19]. Using these results in conjunction with the benefits of the Mittag-Leffler functions, the asymptotic behaviour is obtained.

2. Some exact solutions

Definition 2.1. Let $[a,b] \subset \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ be such that $n-1 < \alpha < n$, and let $f \in W^n(a,b) = \{f \in \mathcal{C}^n(a,b] : f^{(n)} \in L^1[a,b]\}$ be. The *fractional Caputo derivative of order* α is defined by

$$ {}_aD^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n \\ f^{(n)}(x), & \alpha = n. \end{cases} $$

Proposition 2.2. *Let $\alpha \in \mathbb{R}^+$. Then the fractional Caputo derivative of order α is a linear operator such that:*

(a) ${}_aD^\alpha(C) = 0$ *for every* $C \in \mathbb{R}$.
(b) ${}_aD^\alpha((t-a)^\beta) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}(t-a)^{\beta-\alpha}$.

Definition 2.3. For every $x \in \mathbb{R}$, $\rho > 0$ the *Mittag-Leffler* function is defined by

$$ E_\rho(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(\rho k + 1)}, \quad x \in \mathbb{R},\ \rho > 0. \tag{2.1}$$

Definition 2.4. For every $x \in \mathbb{R}$, $\rho > -1$ and $\beta \in \mathbb{R}$ the *Wright* function [33] is defined by

$$ W(x;\rho;\beta) = \sum_{k=0}^\infty \frac{x^k}{k!\Gamma(\rho k + \beta)}. \tag{2.2}$$

The Mainardi function [12] is a special case of the Wright function defined by

$$M_\rho(x) = W(-x, -\rho, 1-\rho) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma(-\rho n + 1 - \rho)}, \quad x \in \mathbb{C},\ 0 < \rho < 1. \tag{2.3}$$

Proposition 2.5. *The Mittag-Leffler function defined in* (2.1) *satisfies the following properties:*

(a) *E_ρ is an entire function if $\rho > 0$.*
(b) *$\lim_{t\to\infty} E_\rho(-t) = 0$ for every $\rho > 0$.*
(c) *${}_0D^\rho(E_\rho(\mu x^\rho)) = \mu E_\rho(\mu x^\rho)$ for every $x, \mu \in \mathbb{R}$, $\rho > 0$.*
(d) *E_ρ is completely monotonic on the negative real axis. In particular $E_\rho(-t)$ is a decreasing function in $\mathbb{R}^+$.*

See [14] for items (a), (b) and (c), and [21] for item (d).

Proposition 2.6. *The Wright function satisfies the following properties:*

(a) *The Wright function* (2.2) *is an entire function if $\rho > -1$.*
(b) *The derivative of the Wright function can be computed as $\frac{\partial}{\partial x} W(x, \rho, \beta) = W(x, \rho, \rho + \beta)$.*
(c) *For all $\alpha, c \in \mathbb{R}^+$, $\rho \in (0, 1)$, $\beta \in \mathbb{R}$ we have*
$${}_0D^\alpha(x^{\beta-1} W(-cx^{-\rho}, -\rho, \beta)) = x^{\beta-\alpha-1} W(-cx^{-\rho}, -\rho, \beta - \alpha).$$
(d) *The following limits involving the parameter α hold:*
$$\lim_{\alpha \nearrow 1} M_{\alpha/2}(x) = M_{1/2}(x) = \frac{e^{-\frac{x^2}{4}}}{\sqrt{\pi}}, \quad \lim_{\alpha \nearrow 1} [1 - W(-x, -\frac{\alpha}{2}, 1)] = \operatorname{erf}(\frac{x}{2})$$
where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$. This result allows us to call function $1 - W(-\cdot, -\frac{\alpha}{2}, 1)$ as the "fractional erf function".
(e) *The "fractional erf function" $1 - W(-\cdot, -\frac{\alpha}{2}, 1)$ is a positive and strictly increasing function in $\mathbb{R}^+$ such that $0 < 1 - W(-x, -\frac{\alpha}{2}, 1) < 1$, for all $x > 0$.*
(f) *For every $\alpha \in (0, 1)$, and $\beta > 0$,*
$$\lim_{x\to\infty} W(-x, -\frac{\alpha}{2}, \beta) = 0.$$

For items (a) and (b), see [34]. A proof of (c) is given in [25], and a proof of (d) and (e) can be founded in [28]. For (f), see [12]

We will consider the following two regions related to problem (1.1):

$$\Omega_0 = \{(x, t) : s_1(t) < x < s_2(t),\ 0 < t \le T\},$$
$$\partial_p\Omega_0 = \{(s_1(t), t) : 0 < t \le T\} \cup \{(s_2(t), t) : 0 < t \le T\} \cup \{(x, 0) : a \le x \le b\},$$

where the latter is called parabolic boundary.

Definition 2.7. A function $u = u(x, t)$ is a solution of problem (1.1) if

- u is defined in $[a_0, b_0] \times [0, T]$, where $a_0 := \min\{s_1(t) : t \in [0, T]\}$ and $b_0 := \max\{s_2(t) : t \in [0, T]\}$.
- $u \in CW_{\Omega_0} := C(\Omega_0) \cap W_t^1((0, T)) \cap C_x^2(\Omega_0)$, where

$$W_t^1((0, T)) := \{f(x, \cdot) \in C^1((0, T)) \cap L^1(0, T) \text{for every fixed } x \in [a_0, b_0]\}.$$

- u is continuous in $\Omega_0 \cup \partial_p \Omega_0$ except perhaps at $(a,0)$ and $(b,0)$ where

$$0 \le \liminf_{(x,t)\to(a,0)} u(x,t) \le \limsup_{(x,t)\to(a,0)} u(x,t) < +\infty,$$
$$0 \le \liminf_{(x,t)\to(b,0)} u(x,t) \le \limsup_{(x,t)\to(b,0)} u(x,t) < +\infty.$$

- u satisfies the conditions in (1.1).

Remark 2.8. We require u to be defined in $[a_0, b_0] \times [0, T]$ since the fractional derivative ${}_0D_t^\alpha u(x,t)$ involves values $u_t(x,\tau)$ for all τ in $[0,t]$.

Problem 1. Consider $s_1(t) = 0$, $s_2(t) = t^{\alpha/2}, g(t) = 1$ and $h(t) = 0$ in problem (1.1). Note that condition (1.1)(4) is not considered because $a = b = 0$. Taking $\beta = 1$ and $\rho = \frac{\alpha}{2}$ in Proposition 2.6(c), using Proposition 2.6(b) and the principle of superposition (valid due to Proposition 2.2), we can state that function

$$u(x,t) = a + b[1 - W(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1)]$$

is a solution to the following initial-boundary problem associated with the FDE,

$$\begin{gathered} {}_0D_t^\alpha u(x,t) = u_{xx}(x,t), \quad 0 < x,\ 0 < t \le T,\ 0 < \alpha < 1, \\ u(0,t) = a, \quad 0 < t \le T, \\ u(x,0) = a + b, \quad 0 < x. \end{gathered} \tag{2.4}$$

Clearly $a = 1$. Evaluating u at the curve $s_2(t) = t^{\alpha/2}$, $0 < t < T$, we obtain

$$u(t^{\alpha/2}, t) = 1 + b[1 - W(-1, -\frac{\alpha}{2}, 1)].$$

Since $1 - W(-1, -\frac{\alpha}{2}, 1) \neq 0$ by Proposition 2.6 item (d), we can take $b = -\frac{1}{1 - W(-1, -\frac{\alpha}{2}, 1)}$, and state that

$$u(x,t) = 1 - \frac{1}{1 - W(-1, -\frac{\alpha}{2}, 1)}[1 - W(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1)] \tag{2.5}$$

is a solution to the problem

$$\begin{gathered} {}_0D_t^\alpha u(x,t) = u_{xx}(x,t), \quad 0 < x < t^{\alpha/2},\ 0 < t \le T,\ 0 < \alpha < 1, \\ u(0,t) = 1, \quad 0 < t \le T, \\ u(t^{\alpha/2}, t) = 0, \quad 0 < t \le T. \end{gathered} \tag{2.6}$$

Problem 2. Consider $s_1(t) = -t^{\alpha/2}$, $s_2(t) = t^{\alpha/2}$, in problem (1.1). Functions g and h will be determined latter. It is reasonable to try to find a solution related to the solution (2.5). But now we have to deal with negative values of the variable x.

From [21] and [8], we can state that function

$$u_1(x,t) = \int_{-\infty}^{\infty} \frac{1}{2t^{\alpha/2}} M_{\alpha/2}(\frac{|x-\xi|}{t^{\alpha/2}}) f(\xi) d\xi \tag{2.7}$$

is a solution to the Cauchy problem

$$\begin{gathered} {}_0D_t^\alpha u(x,t) = u_{xx}(x,t), \quad x \in \mathbb{R},\ 0 < t \le T,\ 0 < \alpha < 1, \\ u(x,0) = f(x), \quad x \in \mathbb{R}. \end{gathered} \tag{2.8}$$

for any piecewise continuous and bounded function f. By considering

$$f(x)=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0,\end{cases}$$

(as it was done in [17] and [29]) it results that

$$u_1(x,t)=\int_0^\infty \frac{1}{2t^{\alpha/2}}\big[M_{\alpha/2}\big(\frac{|x-\xi|}{t^{\alpha/2}}\big)-M_{\alpha/2}\big(\frac{|x+\xi|}{t^{\alpha/2}}\big)\big]d\xi. \tag{2.9}$$

In particular, for $x<0$ we have that $|x-\xi|=\xi-x$ for all $\xi>0$, and using Proposition 2.6 items 2 and 5, it results that

$$\begin{aligned} u_1(x,t)&=\int_0^\infty \frac{1}{2t^{\alpha/2}}\big[M_{\alpha/2}\big(\frac{\xi-x}{t^{\alpha/2}}\big)-M_{\alpha/2}\big(\frac{|x+\xi|}{t^{\alpha/2}}\big)\big]d\xi\\ &=\int_0^\infty \frac{1}{2t^{\alpha/2}}M_{\alpha/2}\big(\frac{\xi-x}{t^{\alpha/2}}\big)d\xi-\int_0^{-x}\frac{1}{2t^{\alpha/2}}M_{\alpha/2}\big(\frac{-(x+\xi)}{t^{\alpha/2}}\big)d\xi\\ &\quad-\int_{-x}^\infty \frac{1}{2t^{\alpha/2}}M_{\alpha/2}\big(\frac{x+\xi}{t^{\alpha/2}}\big)d\xi\\ &=\int_0^\infty \frac{1}{2t^{\alpha/2}}M_{\alpha/2}\big(\frac{\xi-x}{t^{\alpha/2}}\big)d\xi-\int_0^{-x}\frac{1}{2t^{\alpha/2}}M_{\alpha/2}\big(\frac{-(x+\xi)}{t^{\alpha/2}}\big)d\xi\\ &\quad-\int_{-x}^\infty \frac{1}{2t^{\alpha/2}}M_{\alpha/2}\big(\frac{x+\xi}{t^{\alpha/2}}\big)d\xi\\ &=W\big(\frac{x}{t^{\alpha/2}},-\frac{\alpha}{2},1\big)-1. \end{aligned}$$

Then, the function

$$u(x,t)=\begin{cases}1-W(-\frac{x}{t^{\alpha/2}},-\frac{\alpha}{2},1), & 0<x<t^{\alpha/2}\\ W(\frac{x}{t^{\alpha/2}},-\frac{\alpha}{2},1)-1, & -t^{\alpha/2}<x<0\end{cases}$$

is a solution to (1.1)(1) for every $x\neq 0$, $-t^{\alpha/2}<x<t^{\alpha/2}$. Moreover, due to the linearity of the Caputo derivative we can state that

$$u(x,t)=\begin{cases}a[1-W(-\frac{x}{t^{\alpha/2}},-\frac{\alpha}{2},1)]+b, & 0<x<t^{\alpha/2}\\ a[W(\frac{x}{t^{\alpha/2}},-\frac{\alpha}{2},1)-1]+b, & -t^{\alpha/2}<x<0\end{cases}$$

is a solution to (1.1)(1) for every $x\neq 0$, $-t^{\alpha/2}<x<t^{\alpha/2}$.

We would like to extend this solution to the values of $x=0$, but if we extend u to this values, it results that

$$u(x,t)=\begin{cases}a[1-W(-\frac{x}{t^{\alpha/2}},-\frac{\alpha}{2},1)]+b, & 0\le x<t^{\alpha/2}\\ a[W(\frac{x}{t^{\alpha/2}},-\frac{\alpha}{2},1)-1]+b, & -t^{\alpha/2}<x<0,\end{cases} \tag{2.10}$$

is a $\mathcal{C}^1_x(D_T)$ function, but $u_{xx}(0^+,t)\neq u_{xx}(0^-,t)$.

To obtain a $\mathcal{C}^2_x(\Omega_0)$ solution according to definition 2.7 we apply the following Lemma proved in [11].

Lemma 2.9. *Let $v(x,t)$ be a solution of the FDE such that $F(x,t)=\int_x^\infty v(\xi,t)d\xi$ is well defined for every $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, $\lim_{x\to\infty}\frac{\partial v}{\partial x}(x,t)=0$, $|\frac{\partial}{\partial\tau}v(\xi,\tau)|\le g(\xi)$ in $L^1(x,\infty)$ and*

$$\frac{\frac{\partial}{\partial\tau}v(\xi,\tau)}{(t-\tau)^\alpha}\in L^1((x,\infty)\times(0,t)).$$

Then $\int_x^\infty v(\xi,t)d\xi$ *is a solution to the FDE.*

From Proposition 2.6 and using estimates made in [11], it yields that $v(x,t) = W(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1)$ is under the assumptions of Lemma 2.9 for every $x \geq 0$. Then

$$\int_x^\infty v(\xi,t)d\xi = t^{\alpha/2} W(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1+\frac{\alpha}{2})$$

satisfies (1.1)(1). By using the linearity of the Caputo derivative and the principle of superposition we can state that

$$w_{\rm pos}(x,t) = a[x + t^{\alpha/2} W(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1+\frac{\alpha}{2})] + bx \tag{2.11}$$

is a solution to the FDE such that $\frac{\partial}{\partial x} w_{\rm pos}(x,t) = u(x,t)$ for every $x \geq 0$, $t > 0$.

For negative values of x we enunciate an analogous lemma.

Lemma 2.10. *Let* $v(x,t)$ *be a solution of the FDE such that* $F(x,t) = \int_{-\infty}^x v(\xi,t)d\xi$ *is well defined for every* $(x,t) \in \mathbb{R}\times\mathbb{R}^+$, $\lim_{x\to-\infty} \frac{\partial v}{\partial x}(x,t) = 0$, $|\frac{\partial}{\partial \tau} v(\xi,\tau)| \leq g(\xi)$ *in* $L^1(-\infty,x)$ *and*

$$\frac{\frac{\partial}{\partial \tau} v(\xi,\tau)}{(t-\tau)^\alpha} \in L^1((-\infty,x)\times(0,t)).$$

Then $\int_{-\infty}^x v(\xi,t)d\xi$ *is a solution to the FDE.*

Proof. The required assumptions allows us to apply Fubini's theorem. Then

$$\begin{aligned}
{}_0D_t^\alpha F(x,t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial}{\partial\tau} F(x,\tau)}{(t-\tau)^\alpha} d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \Big(\frac{\partial}{\partial\tau} \int_{-\infty}^x v(\xi,\tau)d\xi\Big) d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \int_{-\infty}^x \frac{\partial}{\partial\tau} v(\xi,\tau)d\xi d\tau \\
&= \int_{-\infty}^x \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial}{\partial\tau} v(\xi,\tau)}{(t-\tau)^\alpha} d\tau \\
&= \int_{-\infty}^x {}_0D_t^\alpha v(\xi,t)d\xi \\
&= \int_{-\infty}^x \lambda^2 \frac{\partial^2 v}{\partial x^2}(\xi,t)d\xi \\
&= \lambda^2 \frac{\partial v}{\partial x}(\xi,t)\Big|_{-\infty}^x = \frac{\partial^2}{\partial x^2} F(x,t).
\end{aligned}$$

□

Applying Lemma 2.10 to function $v(x,t) = W(\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1)$ for $x < 0$, and reasoning as before, it yields that

$$w_{\rm neg}(x,t) = a[-x + t^{\alpha/2} W(\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1+\frac{\alpha}{2})] + bx \tag{2.12}$$

is a solution to the FDE such that $\frac{\partial}{\partial x} w_{\rm neg}(x,t) = u(x,t)$ for every $x < 0$, $t > 0$.

Combining (2.11) with (2.12) we have

$$w(x,t) = \begin{cases} a[x + t^{\alpha/2} W(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1+\frac{\alpha}{2})] + bx, & 0 \leq x < t^{\alpha/2} \\ a[-x + t^{\alpha/2} W(\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1+\frac{\alpha}{2})] + bx, & -t^{\alpha/2} < x < 0, \end{cases} \tag{2.13}$$

is a solution to (1.1)(1). Clearly w is a $\mathcal{C}^2_x(\Omega_0)$ function and by varying the parameters a and b we obtain different solutions associated with different boundary conditions. For example, if $b=\frac{1}{2}$ and $a=\frac{1}{2(1+W(-1,-\frac{\alpha}{2},1+\frac{\alpha}{2}))}$ it yields that (2.13) is a solution to the moving-boundary problem

$$\begin{gathered}
{}_0D^\alpha_t u(x,t)=\lambda^2\, u_{xx}(x,t),\quad -t^{\alpha/2}<x<t^{\alpha/2},\ 0<t\le T,\ 0<\alpha<1,\\
u(-t^{\alpha/2},t)=0,\quad 0<t\le T,\\
u(t^{\alpha/2},t)=t^{\alpha/2},\quad 0<t\le T.
\end{gathered}\tag{2.14}$$

3. Asymptotic behavior as t tends to infinity

Hereinafter we call L^α to the operator associated with the FDE, $L^\alpha:=\frac{\partial^2}{\partial x^2}-D^\alpha$. The following two results have been proved in [19] and [27] respectively.

Proposition 3.1. *Let $f\in W^1_t((0,T])\cup\mathcal{C}([0,T])$ be a function that attains its maximum at the point $t_0\in(0,T]$. Then for every $\alpha\in(0,1)$ it results that $D^\alpha f(t_0)\ge 0$.*

Remark 3.2. Note that this extremum principle is not valid either if $\alpha>1$ or if the fractional derivative is taken in the Riemann-Liouville sense.

Proposition 3.3. *If u is a function with $L^\alpha[u]>0$ (resp. $L^\alpha[u]<0$) in Ω_0, then u does not attain its maximum (resp. minimum) at Ω_0.*

Let us adapt here the next theorem obtained in [18] to the moving-boundary problem (1.1).

Theorem 3.4. *Let $u\in CW_{\Omega_0}$ be a solution of (1.1). Then either $u(x,t)\ge 0$ for all $(x,t)\in\overline{\Omega_0}$ or u attains its negative minimum on $\partial_p\Omega_0$.*

Proof. If $u\ge 0$ in $\overline{\Omega}_0$ the prove is finished. Now, suppose that there exists a point (x_0,t_0), such that $s_1(t_0)<x_0<s_2(t_0)$, $0<t_0\le T$ and $u(x_0,t_0)<\min_{\partial_p\Omega_0}u(x,t)=m\le 0$. Let $\epsilon=m-u(x_0,t_0)>0$ be and consider the auxiliary function

$$w(x,t)=u(x,t)-\frac{\epsilon}{2}\frac{T-t}{T},\quad (x,t)\in\overline{\Omega}_0.\tag{3.1}$$

Note that

$$w(x,t)\ge u(x,t)-\frac{\epsilon}{2}\quad \forall (x,t)\in\overline{\Omega}_0,\tag{3.2}$$

and that for every $(x,t)\in\partial_p\Omega_0$, it results that

$$\begin{aligned}
w(x_0,t_0)&=u(x_0,t_0)-\frac{\epsilon}{2}\frac{T-t_0}{T}\le u(x_0,t_0)\\
&=m-\epsilon\le u(x,t)-\epsilon\le w(x,t)+\frac{\epsilon}{2}-\epsilon\\
&=w(x,t)-\frac{\epsilon}{2}.
\end{aligned}$$

Consequently $w(x,t)>w(x_0,t_0)$ for all $(x,t)\in\partial_p\Omega_0$ and we can state that

$$w \text{ must attain its minimum at } \Omega_0.\tag{3.3}$$

On the other hand, from Proposition 2.2 it follows that

$$L^\alpha[w]=L^\alpha[u]-\frac{\epsilon}{2}\frac{\Gamma(2)}{\Gamma(2-\alpha)}\frac{t^{1-\alpha}}{T}<0\quad\forall (x,t)\in\Omega_0.$$

Now, applying Proposition 3.3 to w, it results that w can not attain its minimum in Ω_0, which contradicts (3.3). $\square$

Theorem 3.5. *Let u be a solution of the fractional moving-boundary problem* (1.1) *such that:*

(a) s_1 *is a decreasing continuous functions in* $\mathbb{R}_0^+$ *such that* $\lim_{t\to\infty} s_1(t) = a_0$,
(b) s_2 *is an increasing continuous functions in* $\mathbb{R}_0^+$ *such that* $\lim_{t\to\infty} s_2(t) = b_0$ *and* $s_1(t) < s_2(t)$ *for every* $t > 0$,
(c) f *is a non-negative continuous function defined in* $[a, b]$,
(d) g *and* h *are non-negative continuous functions defined in* $\mathbb{R}^+$ *such that* $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} h(t) = 0$.

Then $\lim_{t\to\infty} u(x, t) = 0$ *uniformly in* $[a_0, b_0]$.

Proof. Consider the function $\varphi(x) = \exp\{2b_0\} - \exp\{x\}$ defined in $[a_0, b_0]$. Clearly φ is a decreasing positive function in $[a_0, b_0]$, with $\varphi_{\min} = \varphi(b_0)$, $\varphi_{\max} = \varphi(a_0)$. Let $\Psi\colon [a_0, b_0] \times \mathbb{R}_0^+$ be the non-negative function defined by

$$\Psi(x,t) = \epsilon\varphi(x) + \frac{A}{\varphi_{\min}}\varphi(x)E_\alpha(-\gamma t^\alpha) \tag{3.4}$$

where ϵ, A and γ will be determined later, and E_α is the Mittag-Leffler function with parameter α. Applying the L^α operator to Ψ and using Proposition 2.5(c), it yields that

$$L^\alpha \Psi(x,t) = -\epsilon \exp\{x\} + \frac{A}{\varphi_{\min}}E_\alpha(-\gamma t^\alpha)[-\exp\{x\} + \gamma\varphi(x)]. \tag{3.5}$$

Applying Proposition 2.5(d) it results that

$$L^\alpha\Psi(x,t) < 0 \text{ for every } (x,t) \text{ in } [a_0, b_0] \times \mathbb{R}_0^+ \text{ if } \gamma < \frac{1}{\exp\{2b_0 - a_0\} - 1}. \tag{3.6}$$

Now, let $z = \Psi - u$ be. Taking into account the non negativity of the Mittag-Leffler function and that g and h are non–negative functions, such that $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} h(t) = 0$, there exist t_1 and t_2 such that the next inequalities hold

$$\begin{aligned} z(s_1(t),t) &= \epsilon\varphi(s_1(t)) + \frac{A\varphi(s_1(t))}{\varphi_{\min}}E_\alpha(-\gamma t^\alpha) - g(t) \\ &\geq \epsilon\varphi_{\min} - g(t) > 0 \quad \text{if } t > t_1. \end{aligned} \tag{3.7}$$

$$\begin{aligned} z(s_2(t),t) &= \epsilon\varphi(s_2(t)) + \frac{A\varphi(s_2(t))}{\varphi_{\min}}E_\alpha(-\gamma t^\alpha) - h(t) \\ &\geq \epsilon\varphi_{\min} - h(t) > 0 \quad \text{if } t > t_2. \end{aligned} \tag{3.8}$$

Consider $\tilde{t} = \max\{t_1, t_2\}$, $M_{\tilde{t}} = \max\{u(x,t); s_1(t) \leq x \leq s_2(t), 0 \leq t \leq \tilde{t}\}$ and take A such that

$$A \geq \frac{M_{\tilde{t}}}{E_\alpha(-\gamma \tilde{t}^\alpha)}. \tag{3.9}$$

Then

$$z(x,\tilde{t}) \geq 0, \quad s_1(\tilde{t}) < x < s_2(\tilde{t}). \tag{3.10}$$

Consider the region $\Omega_{\tilde{t}} = \{(x,t) : s_1(t) < x < s_2(t), \tilde{t} < t\}$. Note that inequalities (3.7), (3.8) and (3.10) imply that $z \geq 0$ in $\partial_p\Omega_{\tilde{t}}$. With the aim to prove that $z \geq 0$ in $\overline{\Omega_{\tilde{t}}}$, fix $T > \tilde{t}$ large enough and suppose that there exists a point $(x_0, t_0) \in \{(x,t) : s_1(t) < x < s_2(t), \tilde{t} < t \leq T\}$ where z attains its negative minimum. Clearly

$$z(x_0, t) \geq z(x_0, t_0) \quad \text{for all } \tilde{t} \leq t \leq T. \tag{3.11}$$

From Proposition 2.5(d) and (3.9) it results that

$$z(x_0, t) \geq 0 \quad \text{for all } t < \tilde{t}. \tag{3.12}$$

Inequalities (3.11) and (3.12) imply that the function $z(x_0, \cdot)$ attains its absolute minimum at t_0 in the interval $[0, T]$. Applying Proposition 3.1 it holds that $D^\alpha z(x_0, t_0) \leq 0$. Then

$$\begin{aligned} L^\alpha u(x_0, t_0) &= L^\alpha \Psi(x_0, t_0) - L^\alpha z(x_0, t_0) \\ &= L^\alpha \Psi(x_0, t_0) - \frac{\partial^2}{\partial x^2} z(x_0, t_0) + D^\alpha z(x_0, t_0) < 0 \end{aligned}$$

which is a contradiction. Therefore $z \geq 0$ in $\overline{\Omega_{\tilde{t}}}$, or equivalently

$$u(x, t) \leq \Psi(x, t) \quad \forall (x, t) \in \overline{\Omega_{\tilde{t}}}. \tag{3.13}$$

The non-negativity of functions f, g and h imply the non-negativity of u in $\partial\Omega_0$. Then Theorem 3.4 yields

$$0 \leq u(x, t) \quad \forall (x, t) \in \overline{\Omega_{\tilde{t}}}. \tag{3.14}$$

Extending u by 0 outside Ω_0, using (3.13) and (3.14) we can state that

$$0 \leq u(x, t) \leq \epsilon \varphi_{\max} + \frac{A\varphi_{\max}}{\varphi_{\min}} E_\alpha(-\gamma t^\alpha) \quad \text{for all } x \in [a_0, b_0], \text{ and all } t > \tilde{t}.$$

By Proposition 2.5(b), there exists $t_3 > 0$ (we can take $t_3 > \tilde{t}$) such that

$$0 \leq u(x, t) \leq 2\epsilon \varphi_{\max} \quad \text{for all } x \in [a_0, b_0], \text{ and all } t > t_3. \tag{3.15}$$

Taking into account that inequality (3.15) holds for every $\epsilon > 0$ it results that $\lim_{t\to\infty} u(x, t) = 0$ uniformly in $[a_0, b_0]$. □

Remark 3.6. The following initial-boundary-value problem was considered in [19]:

$$\begin{gathered} {}_0D_t^\alpha u(x, t) = \lambda^2 u_{xx}(x, t), \quad 0 < x < L,\ 0 < t,\ 0 < \alpha < 1, \\ u(0, t) = 0, \quad 0 < t, \\ u(L, t) = 0, \quad 0 < t, \\ u(x, 0) = f(x), \quad 0 \leq x \leq L. \end{gathered} \tag{3.16}$$

There a classical solution was obtained of the form

$$u(x, t) = \sum_{i=1}^{\infty} (f, X_i) E_\alpha(-\lambda_i t^\alpha) X_i(x), \tag{3.17}$$

where X_i, $i = 1, 2, dots$ are the eigenfunctions corresponding to the eigenvalues λ_i of the eigenvalue problem

$$\begin{gathered} X''(x) = \lambda X(x) \\ X(0) = X(L) = 0, \end{gathered}$$

and $f \in \mathcal{C}^1([0, L]) \cup \mathcal{C}^2(0, L)$, $f'' \in L^2(0, L)$ and $f(0) = f(L) = 0$. Note that if we ask f to be a non-negative function then Theorem 3.5 gives the uniformly convergence to zero in $[0, L]$ when $t \to \infty$ of (3.17).

Corollary 3.7. *Let u be a solution of the fractional initial-boundary-value problem*

$$\begin{gathered} {}_0D_t^\alpha u(x,t) = \lambda^2 u_{xx}(x,t), \quad 0 < x < L,\ 0 < t,\ 0 < \alpha < 1, \\ u(0,t) = g(t), \quad 0 < t, \\ u(L,t) = h(t), \quad 0 < t, \\ u(x,0) = f(x), \quad 0 \le x \le L, \end{gathered} \tag{3.18}$$

such that:

(a) *g is a continuous functions defined in $\mathbb{R}^+$ such that $\lim_{t\to\infty} g(t) = g_0$ and $g(t) \ge g_0$ for every $t \in \mathbb{R}^+$,*
(b) *h is a continuous functions defined in $\mathbb{R}^+$ such that $\lim_{t\to\infty} h(t) = h_0$ and $h(t) \ge h_0$ for every $t \in \mathbb{R}^+$,*
(c) *f is a continuous function defined in $[0,L]$ such that $f(x)-[\frac{h_0-g_0}{L}x+g_0] \ge 0$ for every $x \in [0,L]$.*

Then $\lim_{t\to\infty} u(x,t) = \frac{h_0-g_0}{L}x + g_0$ for every $x \in [0,L]$.

The proof of the above corollary, follows by applying Theorem 3.5 to the function $w(x,t) = u(x,t) - [\frac{h_0-g_0}{L}x + g_0]$.

Remark 3.8. The hypotheses $g(t) \ge m_1$ and $h(t) \ge m_2$ for every $t \in \mathbb{R}^+$ in Corollary 3.7 are essential because we need to guarantee the non-negativity of the solution in the hole parabolic boundary Ω_0 to apply Theorem 3.4, which is used in the proof of Theorem 3.5. This simple observation is interesting because it makes gains relevance to the memory of the operator that we are considering.

Note that if we consider the classical parabolic operator, the weak maximum principle is valid for every region $\Omega_{\tilde{t}}$ with $\tilde{t} \neq 0$, but for the "fractional weak maximum principle" (Theorem 3.4) this is not valid. This principle only holds in the hole region Ω_0, that is, the region including everything that happened from the initial time.

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