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EXISTENCE OF SOLUTIONS TO A THIRD-ORDER MULTI-POINT PROBLEM ON TIME SCALES

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Abstract. We are concerned with the existence and form of solutions to nonlinear third-order three-point and multi-point boundary-value problems on general time scales. Using the corresponding Green function, we prove the existence of at least one positive solution using the Guo-Krasnosel'skii fixed point theorem. Moreover, a third-order multi-point eigenvalue problem is formulated, and eigenvalue intervals for the existence of a positive solution are found.

1. Introduction

We will establish the corresponding Green function whereby conditions can be given such that a positive solution exists for the following nonlinear third-order three-point boundary value problem on arbitrary time scales

$$(px^{\Delta\Delta})^{\nabla}(t) + a(t)f(x(t)) = 0, \quad t \in [t_1, t_3]_{\mathbb{T}}, \tag{1.1}$$

$$x(\rho(t_1)) = x^{\Delta}(\rho(t_1)) = 0, \quad x^{\Delta}(\sigma(t_3)) - \alpha x^{\Delta}(t_2) = 0, \tag{1.2}$$

where: p is a right-dense continuous, real-valued function with $0 < p(t) \le 1$ on $\mathbb{T}$; the boundary points from $\mathbb{T}$ satisfy $t_1 < t_2 < t_3$, with $t_2/\alpha \in \mathbb{T}$ such that

- (H1) the constants d and α satisfy
$$d := \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha \int_{\rho(t_1)}^{t_2} \frac{\Delta\tau}{p(\tau)} > 0 \quad \text{and} \quad 1 < \alpha < \frac{\int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}}{\int_{\rho(t_1)}^{t_2} \frac{\Delta\tau}{p(\tau)}};$$
- (H2) the continuous function $f : [0,\infty) \to [0,\infty)$ is such that the following exist:
$$f_0 := \lim_{x\to 0^+} \frac{f(x)}{x} \quad f_\infty := \lim_{x\to\infty} \frac{f(x)}{x}.$$
- (H3) the left-dense continuous function $a : [\rho(t_1), \sigma(t_3)]_{\mathbb{T}} \to [0,\infty)$ is such that a is not identically zero on $[t_2/\alpha, t_2]_{\mathbb{T}}$.

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If $\mathbb{T}=\mathbb{R}$, then (1.1), (1.2) is the ordinary third-order three-point boundary value problem

$$(px'')'(t)+a(t)f(x(t))=0, \quad t\in[t_1,t_3]_{\mathbb{R}},$$
$$x(t_1)=x'(t_1)=0, \quad x'(t_3)-\alpha x'(t_2)=0.$$

If $\mathbb{T}=\mathbb{Z}$, then (1.1), (1.2) is the discrete third-order three-point boundary value problem

$$\nabla\big(p\Delta^2 x)(t)+a(t)f(x(t)\big)=0, \quad t\in[t_1,t_3]_{\mathbb{Z}},$$
$$x(t_1-1)=\Delta x(t_1-1)=0, \quad \Delta x(t_3+1)-\alpha\Delta x(t_2)=0,$$

where $\Delta y(t)=y(t+1)-y(t)$ and $\nabla y(t)=y(t)-y(t-1)$. As a final illustration, if $\mathbb{T}$ is a quantum time scale for some real $q>1$, then (1.1), (1.2) is the third-order three-point quantum boundary value problem

$$D^q\left(pD_q(D_q x)\right)(t)+a(t)f(x(t))=0, \quad t\in[t_1,t_3]_{\mathbb{T}},$$
$$x(t_1/q)=D_q x(t_1/q)=0, \quad D_q x(qt_3)-\alpha D_q x(t_2)=0,$$

where the quantum derivatives are given by the difference quotients

$$D_q y(t)=\frac{y(qt)-y(t)}{(q-1)t} \quad\text{and}\quad D^q y(t)=\frac{y(t)-y(t/q)}{(1-1/q)t}.$$

Third-order differential equations, though less common in applications than even-order problems, nevertheless do appear, for example in the study of quantum fluids; see Gamba and Jüngel [5]. Here we approach a third-order three-point problem on general time scales, namely on any nonempty closed subset of the real line, to include the discrete, continuous, and quantum calculus as special cases. Of late there have been several papers on third-order boundary value problems. Hopkins and Kosmatov [9]; Li [10]; Liu, Ume, and Kang [11, 12]; and Minghe and Chang [13] have all recently considered third-order problems. All of these papers, however, were two-point problems with $\mathbb{T}=\mathbb{R}$. Graef and Yang [6], Sun [16], and Wong [17] consider three-point focal problems, while Palamides and Smyrlis consider the three-point boundary conditions

$$x(0)=x''(\eta)=x(1)=0, \quad \mathbb{T}=[0,1]_{\mathbb{R}}.$$

On general time scales there are also a few papers on third-order problems. Sun [15] considers a third-order two-point boundary value problem; a couple of papers on third-order three-point boundary value problems considered on general time scales are [1, 2] in the right-focal case. Note that boundary value problems on time scales that utilize both delta and nabla derivatives, such as the one here, were first introduced by Atici and Guseinov [3]. For more on existence of solutions to boundary value problems, see [4, Chapters 4 and 6-9], the text by Guo and Lakshmikantham [7], and Zhang and Liu [18].

Problem (1.1), (1.2) is an extension of the unit interval boundary value problem

$$x'''(t)+a(t)f(x(t))=0, \quad t\in(0,1)_{\mathbb{R}},$$
$$x(0)=x'(0)=0, \quad \alpha x'(\eta)=x'(1),$$

to arbitrary time scales [8]; in other words, take $\mathbb{T}=\mathbb{R}$, $p\equiv 1$, $t_1=0$, $t_2=\eta$ and $t_3=1$ in (1.1), (1.2) to get the results in [8]. One could also consider a third-order problem with derivatives in the order of nabla, nabla, delta, but the results would be similar; other permutations of nablas and/or deltas lead to a Green function that is less easy to calculate.

2. Preliminary lemmas

Underlying our technique will be the Green function for the homogeneous, third-order, three-point boundary-value problem

$$-(px^{\Delta\Delta})^{\nabla}(t) = 0, \quad t \in [t_1, t_3]_{\mathbb{T}}, \tag{2.1}$$

$$x(\rho(t_1)) = x^{\Delta}(\rho(t_1)) = 0, \quad \alpha x^{\Delta}(t_2) = x^{\Delta}(\sigma(t_3)). \tag{2.2}$$

The Green function for (2.1), (2.2) will be defined, nonnegative, and bounded above on $[\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}} \times [t_1, \sigma(t_3)]_{\mathbb{T}}$, as will be shown in the following lemmas.

Lemma 2.1. *For $y \in C_{ld}[\rho(t_1), \sigma(t_3)]_{\mathbb{T}}$, the boundary value problem*

$$(px^{\Delta\Delta})^{\nabla}(t) + y(t) = 0, \quad t \in [t_1, t_3]_{\mathbb{T}}, \tag{2.3}$$

$$x(\rho(t_1)) = x^{\Delta}(\rho(t_1)) = 0, \quad \alpha x^{\Delta}(t_2) = x^{\Delta}(\sigma(t_3)) \tag{2.4}$$

has a unique solution $x(t) = \int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)y(s)\nabla s$, where the Green function corresponding to the problem (2.1), (2.2) *is given by*

$$G(t,s) = \begin{cases} \frac{1}{d}\Big(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha\int_s^{t_2} \frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u & \\ -\int_s^t\int_s^u \frac{\Delta\tau}{p(\tau)}\Delta u & : s \le \min\{t_2, t\} \\ \frac{1}{d}\Big(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha\int_s^{t_2} \frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u & : t \le s \le t_2 \\ \frac{1}{d}\Big(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u - \int_s^t\int_s^u \frac{\Delta\tau}{p(\tau)}\Delta u & : t_2 \le s \le t \\ \frac{1}{d}\Big(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u & : \max\{t_2, t\} \le s. \end{cases} \tag{2.5}$$

Proof. We follow the approach given in the case $\mathbb{T} = \mathbb{R}$ in [8]. If $\rho(t_1) \le t \le t_2$, then

$$\begin{aligned} x(t) &= \int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)y(s)\nabla s \\ &= \int_{\rho(t_1)}^{t}\Big[\frac{1}{d}\Big(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha\int_s^{t_2} \frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u \\ &\quad - \int_s^t\int_s^u \frac{\Delta\tau}{p(\tau)}\Delta u\Big]y(s)\nabla s \\ &\quad + \int_t^{t_2}\Big[\frac{1}{d}\Big(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha\int_s^{t_2} \frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u\Big]y(s)\nabla s \\ &\quad + \int_{t_2}^{\sigma(t_3)}\Big[\frac{1}{d}\Big(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u\Big]y(s)\nabla s \\ &= \frac{1}{d}\Big[\int_{\rho(t_1)}^{t_2}\Big(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha\int_s^{t_2} \frac{\Delta\tau}{p(\tau)}\Big)y(s)\nabla s + \int_{t_2}^{\sigma(t_3)}\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}y(s)\nabla s\Big] \\ &\quad \times \Big[\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u\Big] - \int_{\rho(t_1)}^t\Big(\int_s^t\int_s^u \frac{\Delta\tau}{p(\tau)}\Delta u\Big)y(s)\nabla s. \end{aligned}$$

If $\sigma^2(t_3) \ge t \ge t_2$, then

$$\begin{aligned}
x(t) &= \int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)y(s)\nabla s \\
&= \int_{\rho(t_1)}^{t_2} \Big[\frac{1}{d}\Big(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha\int_s^{t_2}\frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^{t}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u \\
&\quad - \int_s^t\int_s^u \frac{\Delta\tau}{p(\tau)}\Delta u\Big] y(s)\nabla s \\
&\quad + \int_{t_2}^{t}\Big[\frac{1}{d}\Big(\int_s^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^{t}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u - \int_s^t\int_s^u\frac{\Delta\tau}{p(\tau)}\Delta u\Big]y(s)\nabla s \\
&\quad + \int_t^{\sigma(t_3)}\Big[\frac{1}{d}\Big(\int_s^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^{t}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big]y(s)\nabla s \\
&= \frac{1}{d}\Big[\int_{\rho(t_1)}^{t_2}\Big(\int_s^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)} - \alpha\int_s^{t_2}\frac{\Delta\tau}{p(\tau)}\Big)y(s)\nabla s + \int_{t_2}^{\sigma(t_3)}\int_s^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}y(s)\nabla s\Big] \\
&\quad \times \Big[\int_{\rho(t_1)}^{t}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big] - \int_{\rho(t_1)}^{t}\Big(\int_s^t\int_s^u\frac{\Delta\tau}{p(\tau)}\Delta u\Big)y(s)\nabla s.
\end{aligned}$$

For the remainder of the proof let

$$k(s) := \frac{1}{d}\Big[\int_{\rho(t_1)}^{t_2}\Big(\int_s^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)} - \alpha\int_s^{t_2}\frac{\Delta\tau}{p(\tau)}\Big)y(s)\nabla s + \int_{t_2}^{\sigma(t_3)}\int_s^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}y(s)\nabla s\Big].$$

Thus for all $t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$,

$$x(t) = k(s)\int_{\rho(t_1)}^{t}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u - \int_{\rho(t_1)}^{t}\left(\int_s^t\int_s^u\frac{\Delta\tau}{p(\tau)}\Delta u\right)y(s)\nabla s.$$

Note that $x(\rho(t_1)) = 0$. Taking a delta derivative,

$$x^{\Delta}(t) = k(s)\int_{\rho(t_1)}^{t}\frac{\Delta\tau}{p(\tau)} - \int_{\rho(t_1)}^{t}\left(\int_s^t\frac{\Delta\tau}{p(\tau)}\right)y(s)\nabla s.$$

Again it is easy to see that $x^{\Delta}(\rho(t_1)) = 0$. To verify the third boundary condition, check that

$$\begin{aligned}
&x^{\Delta}(\sigma(t_3)) - \alpha x^{\Delta}(t_2) \\
&= dk(s) - \int_{\rho(t_1)}^{\sigma(t_3)}\Big(\int_s^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Big)y(s)\nabla s + \alpha\int_{\rho(t_1)}^{t_2}\Big(\int_s^{t_2}\frac{\Delta\tau}{p(\tau)}\Big)y(s)\nabla s \\
&= 0.
\end{aligned}$$

It follows that the boundary conditions (2.4) are satisfied. To finish the proof, another delta derivative yields

$$x^{\Delta\Delta}(t) = \frac{k(s)}{p(t)} - \int_{\rho(t_1)}^{t}\frac{y(s)}{p(t)}\nabla s,$$

which results in $(px^{\Delta\Delta})^{\nabla}(t) = -y(t)$, so that (2.3) is satisfied as well. $\square$

We now seek bounds on the Green function given in (2.5). For later reference, set

$$g(s) := \frac{1}{d}(\alpha+1)\big(\sigma^2(t_3)-\rho(t_1)\big)\Big(\int_{\rho(t_1)}^{s}\frac{\Delta\tau}{p(\tau)}\Big)\Big(\int_{s}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Big). \tag{2.6}$$

Lemma 2.2. *Assume* (H1). *The Green function* (2.5) *corresponding to the problem* (2.1)*,* (2.2) *satisfies*

$$0 \le G(t,s) \le g(s)$$

for $(t,s)\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}\times[t_1,\sigma(t_3)]_{\mathbb{T}}$.

Proof. First we show that $G(t,s)$ is nonnegative. For $t\le s\le t_2$, consider the coefficient

$$c_1(s) := \frac{1}{d}\Big(\int_{s}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)} - \alpha\int_{s}^{t_2}\frac{\Delta\tau}{p(\tau)}\Big).$$

Since $c_1(\rho(t_1))=1$ and $c_1^{\Delta}(s)=\frac{\alpha-1}{dp(s)}>0$, $c_1(s)\ge 1$ for all $s\ge t_1$. It follows that branches 1,2, and 4 of $G(t,s)$ in (2.5) are nonnegative, so we consider the third branch of $G(t,s)$:

$$\frac{1}{d}\Big[\Big(\int_{s}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^{t}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u - d\int_{s}^{t}\int_{s}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big].$$

For $t_2\le s\le t$ let

$$v(t,s) := \Big(\int_{s}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^{t}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u - d\int_{s}^{t}\int_{s}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u.$$

Then

$$\begin{aligned}
v(t,s) &= \Big(\int_{s}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Big)\Big[\int_{\rho(t_1)}^{s}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u + \int_{s}^{t}\int_{\rho(t_1)}^{s}\frac{\Delta\tau}{p(\tau)}\Delta u\Big]\\
&\quad + \Big(\int_{s}^{t}\int_{s}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big)\Big(\int_{s}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)} - d\Big)\\
&= \Big(\int_{s}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Big)\Big[\int_{\rho(t_1)}^{s}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u + \int_{s}^{t}\int_{\rho(t_1)}^{s}\frac{\Delta\tau}{p(\tau)}\Delta u\Big]\\
&\quad + \alpha\Big(\int_{\rho(t_1)}^{t_2}\frac{\Delta\tau}{p(\tau)}\Big)\Big(\int_{s}^{t}\int_{s}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big) - \Big(\int_{\rho(t_1)}^{s}\frac{\Delta\tau}{p(\tau)}\Big)\Big(\int_{s}^{t}\int_{s}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big)\\
&= \Big(\int_{s}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Big)\Big(\int_{\rho(t_1)}^{s}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big) + \alpha\Big(\int_{\rho(t_1)}^{t_2}\frac{\Delta\tau}{p(\tau)}\Big)\Big(\int_{s}^{t}\int_{s}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big)\\
&\quad + \Big(\int_{\rho(t_1)}^{s}\frac{\Delta\tau}{p(\tau)}\Big)\Big[\Big(\int_{s}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Big)\Big(\int_{s}^{t}\frac{\Delta\tau}{p(\tau)}\Big) - \int_{s}^{t}\int_{s}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big].
\end{aligned}$$

Clearly this is nonnegative if the last term in brackets is nonnegative. For the remainder of the proof set

$$j(t) := \Big(\int_{s}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Big)\Big(\int_{s}^{t}\frac{\Delta\tau}{p(\tau)}\Big) - \int_{s}^{t}\int_{s}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u$$

for $t_2\le s\le t$. Then $j(s)=0$, and

$$j^{\Delta}(t) = \frac{1}{p(t)}\int_{s}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)} - \int_{s}^{t}\frac{\Delta\tau}{p(\tau)} \ge 0$$

since $0 < p(t) \le 1$ on $\mathbb{T}$ by assumption. Thus $j(t)$ is nonnegative, guaranteeing overall that $v(t,s)$ is nonnegative, so that ultimately $G(t,s)$ is nonnegative as claimed.

Now we show that $G(t,s) \le g(s)$ on $[\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}} \times [t_1, \sigma(t_3)]_{\mathbb{T}}$, where g is given in (2.6). For any fixed $s \in [t_1, \sigma(t_3)]_{\mathbb{T}}$, a delta derivative of $G(t,s)$ with respect to t yields

$$G^{\Delta_t}(t,s) = \begin{cases} \frac{1}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha \int_s^{t_2} \frac{\Delta\tau}{p(\tau)}\right) \int_{\rho(t_1)}^t \frac{\Delta\tau}{p(\tau)} - \int_s^t \frac{\Delta\tau}{p(\tau)} & : s \le \min\{t_2, t\} \\ \frac{1}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha \int_s^{t_2} \frac{\Delta\tau}{p(\tau)}\right) \int_{\rho(t_1)}^t \frac{\Delta\tau}{p(\tau)} & : t \le s \le t_2 \\ \frac{1}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\right) \int_{\rho(t_1)}^t \frac{\Delta\tau}{p(\tau)} - \int_s^t \frac{\Delta\tau}{p(\tau)} & : t_2 \le s \le t \\ \frac{1}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\right) \int_{\rho(t_1)}^t \frac{\Delta\tau}{p(\tau)} & : \max\{t_2, t\} \le s. \end{cases}$$

Then rewriting we see that

$$\begin{aligned} 0 &\le G^{\Delta_t}(t,s) \\ &= \frac{1}{d}\begin{cases} \left(\int_{\rho(t_1)}^s \frac{\Delta\tau}{p(\tau)}\right)\left[d + (\alpha-1)\int_{\rho(t_1)}^t \frac{\Delta\tau}{p(\tau)}\right] & : s \le \min\{t_2, t\} \\ \left(\int_{\rho(t_1)}^t \frac{\Delta\tau}{p(\tau)}\right)\left[d + (\alpha-1)\int_{\rho(t_1)}^s \frac{\Delta\tau}{p(\tau)}\right] & : t \le s \le t_2 \\ \left(\int_t^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\right)\int_{\rho(t_1)}^s \frac{\Delta\tau}{p(\tau)} + \alpha\left(\int_s^t \frac{\Delta\tau}{p(\tau)}\right)\int_{\rho(t_1)}^{t_2} \frac{\Delta\tau}{p(\tau)} & : t_2 \le s \le t \\ \left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\right)\int_{\rho(t_1)}^t \frac{\Delta\tau}{p(\tau)} & : \max\{t_2, t\} \le s, \end{cases} \end{aligned}$$

and rewriting again we obtain

$$\begin{aligned} 0 \le G^{\Delta_t}(t,s) &\le \begin{cases} \frac{\alpha}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\right)\int_{\rho(t_1)}^s \frac{\Delta\tau}{p(\tau)} & : s \le \min\{t_2, t\} \\ \frac{\alpha}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\right)\int_{\rho(t_1)}^s \frac{\Delta\tau}{p(\tau)} & : t \le s \le t_2 \\ \frac{\alpha+1}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\right)\int_{\rho(t_1)}^s \frac{\Delta\tau}{p(\tau)} & : t_2 \le s \le t \\ \frac{\alpha}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\right)\int_{\rho(t_1)}^s \frac{\Delta\tau}{p(\tau)} & : \max\{t_2, t\} \le s \end{cases} \\ &\le \frac{g(s)}{\big(\sigma^2(t_3) - \rho(t_1)\big)}. \end{aligned} \tag{2.7}$$

Delta integration from $\rho(t_1)$ to t yields $G(t,s) \le g(s)$ for $g(s)$ given in (2.6). ☐

Lemma 2.3. *Assume* (H1). *The Green function* (2.5) *corresponding to the problem* (2.1), (2.2) *satisfies*

$$G(t,s) \ge \gamma g(s), \quad \gamma := \frac{\min\{\alpha-1, \alpha\}\int_{\rho(t_1)}^{t_2/\alpha}\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u}{(\alpha+1)\big(\sigma^2(t_3) - \rho(t_1)\big)\int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}} \in (0,1) \tag{2.8}$$

for $(t,s) \in [t_2/\alpha, t_2]_{\mathbb{T}} \times [t_1, \sigma(t_3)]_{\mathbb{T}}$, *where* $g(s)$ *is given in* (2.6).

Proof. If $s = \sigma(t_3)$, or if t_1 is a left-dense point and $s = t_1$, then the result follows from (2.6). Thus consider the cases where $(t,s) \in [t_2/\alpha, t_2]_{\mathbb{T}} \times (\rho(t_1), \sigma(t_3))_{\mathbb{T}}$. For

$s \le t \le t_2$,

$$\begin{aligned}
\frac{G(t,s)}{g(s)} &= \frac{\Big(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha\int_s^{t_2} \frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u - d\int_s^t\int_s^u \frac{\Delta\tau}{p(\tau)}\Delta u}{dg(s)} \\
&= \frac{d\Big(\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u - \int_s^t\int_s^u \frac{\Delta\tau}{p(\tau)}\Delta u\Big)}{dg(s)} \\
&\quad + \frac{(\alpha-1)\Big(\int_{\rho(t_1)}^s \frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u}{dg(s)} \\
&\ge \frac{(\alpha-1)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u}{(\alpha+1)\big(\sigma^2(t_3)-\rho(t_1)\big)\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}} \ge \gamma.
\end{aligned}$$

For $t \le s \le t_2$,

$$\begin{aligned}
\frac{G(t,s)}{g(s)} &= \frac{\Big(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha\int_s^{t_2} \frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u}{dg(s)} \\
&= \frac{d\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u + (\alpha-1)\Big(\int_{\rho(t_1)}^s \frac{\Delta\tau}{p(\tau)}\Big)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u}{dg(s)} \\
&\ge \frac{(\alpha-1)\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u}{(\alpha+1)\big(\sigma^2(t_3)-\rho(t_1)\big)\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}} \ge \gamma.
\end{aligned}$$

For $t \le t_2 \le s$,

$$\frac{G(t,s)}{g(s)} = \frac{\int_{\rho(t_1)}^t\int_{\rho(t_1)}^u \frac{\Delta\tau}{p(\tau)}\Delta u}{(\alpha+1)\big(\sigma^2(t_3)-\rho(t_1)\big)\int_{\rho(t_1)}^s \frac{\Delta\tau}{p(\tau)}} \ge \gamma.$$

In all cases the statement holds. □

3. An existence result on cones

Let $\mathcal{B}$ denote the Banach space $C[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$ with the norm

$$\|x\| = \sup_{t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}} |x(t)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{x \in \mathcal{B} : x(t) \ge 0 \text{ for } t \in [\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}, x(t) \ge \gamma\|x\| \text{ on } [t_2/\alpha, t_2]_{\mathbb{T}}\}.$$

For $x \in \mathcal{P}$, define

$$Ax(t) := \int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)a(s)f(x(s))\nabla s, \quad t \in [\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}.$$

From Lemma 2.2, for $t \in [\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$,

$$0 \le Ax(t) = \int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)a(s)f(x(s))\nabla s \le \int_{\rho(t_1)}^{\sigma(t_3)} g(s)a(s)f(x(s))\nabla s,$$

so that

$$\|Ax\| \le \int_{\rho(t_1)}^{\sigma(t_3)} g(s)a(s)f(x(s))\nabla s. \tag{3.1}$$

By Lemma 2.3 and (3.1), for $t \in [t_2/\alpha, t_2]_{\mathbb{T}}$,

$$Ax(t) = \int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)a(s)f(x(s))\nabla s \geq \gamma \int_{\rho(t_1)}^{\sigma(t_3)} g(s)a(s)f(x(s))\nabla s \geq \gamma \|Ax\|,$$

giving us

$$Ax(t) \geq \gamma\|Ax\|, \quad [t_2/\alpha, t_2]_{\mathbb{T}},$$

and $A\mathcal{P} \subset \mathcal{P}$. Furthermore, it is straightforward to verify that $A : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator whose fixed points are solutions of (1.1), (1.2).

To establish an existence result we will employ the following fixed point theorem due to Guo and Krasnosel'skii [7], and seek a fixed point of T in $\mathcal{P}$.

Theorem 3.1. *Let E be a Banach space, $P \subseteq E$ be a cone, and suppose that $\mathcal{S}_1$, $\mathcal{S}_2$ are bounded open balls of E centered at the origin with $\overline{\mathcal{S}}_1 \subset \mathcal{S}_2$. Suppose further that $L : P \cap (\overline{\mathcal{S}}_2 \setminus \mathcal{S}_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ly\| \leq \|y\|$, $y \in P \cap \partial\mathcal{S}_1$ *and* $\|Ly\| \geq \|y\|$, $y \in P \cap \partial\mathcal{S}_2$, *or*
(ii) $\|Ly\| \geq \|y\|$, $y \in P \cap \partial\mathcal{S}_1$ *and* $\|Ly\| \leq \|y\|$, $y \in P \cap \partial\mathcal{S}_2$

holds. Then L has a fixed point in $P \cap (\overline{\mathcal{S}}_2 \setminus \mathcal{S}_1)$.

Theorem 3.2. *Assume* (H1), (H2), *and* (H3) *hold. Then the boundary value problem* (1.1), (1.2) *has at least one positive solution if*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(f is superlinear), or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(f is sublinear).*

Proof. The proof in the general time-scale setting is similar to that given in the case $\mathbb{T} = \mathbb{R}$ in [8] and is omitted. □

4. Nonlinear multi-point problem

In the next two sections we consider the related multi-point boundary value problem

$$(px^{\Delta\Delta})^{\nabla}(t) + \lambda a(t)f(x(t)) = 0, \quad t \in [t_1, t_3]_{\mathbb{T}}, \tag{4.1}$$

$$x(\rho(t_1)) = x^{\Delta}(\rho(t_1)) = 0, \quad x^{\Delta}(\sigma(t_3)) - \alpha x^{\Delta}(t_2) = \sum_{i=1}^{n} \alpha_i x^{\Delta}(\xi_i), \tag{4.2}$$

where: p is a right-dense continuous, real-valued function with $0 < p(t) \leq 1$ on $\mathbb{T}$; $\lambda > 0$ is a real scalar; the boundary points from $\mathbb{T}$ satisfy $t_1 < t_2 < t_3$, with $t_2/\alpha \in \mathbb{T}$ such that

(K1) the constants d and α satisfy

$$d := \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha \int_{\rho(t_1)}^{t_2} \frac{\Delta\tau}{p(\tau)} > 0 \quad \text{and} \quad 1 < \alpha < \frac{\int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}}{\int_{\rho(t_1)}^{t_2} \frac{\Delta\tau}{p(\tau)}};$$

(K2) the coefficients satisfy $\alpha_i \geq 0$ for $i = 1, 2, \cdots, n$ and the points $\xi_i \in (\rho(t_1), \sigma^2(t_3))_{\mathbb{T}}$ are such that

$$\xi_1 < \xi_2 < \cdots < \xi_n \quad \text{and} \quad d - \sum_{i=1}^{n} \alpha_i \int_{\rho(t_1)}^{\xi_i} \frac{\Delta\tau}{p(\tau)} > 0;$$

(K3) the continuous function $f : [0,\infty) \to [0,\infty)$ is such that the following exist:

$$f_0 := \lim_{x\to 0^+} \frac{f(x)}{x}, \quad f_\infty := \lim_{x\to\infty} \frac{f(x)}{x};$$

(K4) the left-dense continuous function $a : [\rho(t_1), \sigma(t_3)]_{\mathbb{T}} \to [0,\infty)$ is such that

$$\exists t_* \in [t_2/\alpha, t_2]_{\mathbb{T}} \ni a(t_*) > 0. \tag{4.3}$$

By the novelty of the multi-point boundary conditions, problem (4.1), (4.2) is introduced for the first time on any time scale, including $\mathbb{R}$, $\mathbb{Z}$, and the quantum time scale.

We now turn our attention to the problem

$$(px^{\Delta\Delta})^{\nabla}(t) + \lambda y(t) = 0, \quad t \in [t_1, t_3]_{\mathbb{T}}, \tag{4.4}$$

with multi-point boundary conditions (4.2), where $y : [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}} \to (0,\infty)$ is a left-dense continuous function, and $\lambda > 0$.

Lemma 4.1. *Assume* (K1) *and* (K2). *If $y \in C_{ld}[\rho(t_1), \sigma^2(t_3)]$ with $y \geq 0$, then the nonhomogeneous dynamic equation* (4.4) *with boundary conditions* (4.2) *has a unique solution x^* on $t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$ given by*

$$x^*(t) = \lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)y(s)\nabla s + B(y) \int_{\rho(t_1)}^{t} \int_{\rho(t_1)}^{u} \frac{\Delta\tau}{p(\tau)} \Delta u \Big), \tag{4.5}$$

where: $G(t,s)$ is the Green function (2.5) *of the boundary value problem* (2.1), (2.2) *and the functional B is defined by*

$$B(y) := \Big(d - \sum_{i=1}^{n} \alpha_i \int_{\rho(t_1)}^{\xi_i} \frac{\Delta\tau}{p(\tau)}\Big)^{-1} \sum_{i=1}^{n} \alpha_i \int_{\rho(t_1)}^{\sigma(t_3)} G^{\Delta_t}(\xi_i, s)y(s)\nabla s. \tag{4.6}$$

Proof. Let $y \in C_{ld}[\rho(t_1), \sigma^2(t_3)]$ with $y \geq 0$; we show that the function x^* given in (4.5) is a solution of (4.4) with conditions (4.2) only if $B(y)$ is given by (4.6). If x^* is a solution of (4.4), (4.2), then

$$x^*(t) = \lambda \int_{\rho(t_1)}^{t} G(t,s)y(s)\nabla s + \lambda \int_{t}^{\sigma(t_3)} G(t,s)y(s)\nabla s + B \int_{\rho(t_1)}^{t} \int_{\rho(t_1)}^{u} \frac{\Delta\tau}{p(\tau)} \Delta u$$

for some constant B. Taking the delta derivative with respect to t yields

$$x^{*\Delta}(t) = \lambda \int_{\rho(t_1)}^{t} G^{\Delta}(t,s)y(s)\nabla s + \lambda \int_{t}^{\sigma(t_3)} G^{\Delta}(t,s)y(s)\nabla s + B \int_{\rho(t_1)}^{t} \frac{\Delta\tau}{p(\tau)};$$

clearly $x^*(\rho(t_1)) = x^{*\Delta}(\rho(t_1)) = 0$ as $G(t,s)$ satisfies (1.2). Since p times the delta derivative of this expression is

$$\begin{aligned}
(px^{*\Delta\Delta})(t) &= \lambda \int_{\rho(t_1)}^{t} \Big[\frac{1}{d}\Big(\int_{s}^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha \int_{s}^{t_2} \frac{\Delta\tau}{p(\tau)}\Big) - 1\Big] y(s)\nabla s \\
&\quad + \lambda \int_{t}^{t_2} \Big[\frac{1}{d}\Big(\int_{s}^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha \int_{s}^{t_2} \frac{\Delta\tau}{p(\tau)}\Big)\Big] y(s)\nabla s + B \\
&\quad + \frac{\lambda}{d} \int_{t_2}^{\sigma(t_3)} \Big(\int_{s}^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\Big) y(s)\nabla s,
\end{aligned}$$

we see that

$$\begin{aligned}(px^{*\Delta\Delta})^{\nabla}(t) &= \lambda\Big[\frac{1}{d}\Big(\int_t^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)} - \alpha\int_t^{t_2}\frac{\Delta\tau}{p(\tau)}\Big) - 1\Big]y(t)\\ &\quad - \lambda\Big[\frac{1}{d}\Big(\int_t^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)} - \alpha\int_t^{t_2}\frac{\Delta\tau}{p(\tau)}\Big)\Big]y(t) = -\lambda y(t)\end{aligned}$$

and (4.4) holds; this works regardless of the placement of t in $[t_1, t_3]_{\mathbb{T}}$. To meet the other boundary condition in (4.2), we must have that

$$Bd = \sum_{i=1}^{n}\alpha_i\Big[\int_{\rho(t_1)}^{\sigma(t_3)} G^{\Delta}(\xi_i, s)y(s)\nabla s + B\int_{\rho(t_1)}^{\xi_i}\frac{\Delta\tau}{p(\tau)}\Big],$$

from which (4.6) follows. □

Corollary 4.2. *Assume* (K1) *and* (K2). *If $y \in C_{ld}[\rho(t_1), \sigma^2(t_3)]$ with $y \geq 0$, then the unique solution x^* as in* (4.5) *of the problem* (4.4), (4.2) *satisfies $x^*(t) \geq 0$ for $t \in [\rho(t_1), \sigma^2(t_3)]$.*

Proof. From Lemma 2.2 we know that on $[\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}} \times [t_1, \sigma(t_3)]_{\mathbb{T}}$ the Green function (2.5) satisfies $G(t, s) \geq 0$. ¿From equation (2.7) and (K2) we have that $B(y) \geq 0$ for $y \geq 0$. □

Lemma 4.3. *Assume* (K1) *and* (K2). *If $y \in C_{ld}[\rho(t_1), \sigma^2(t_3)]$ with $y \geq 0$, then the unique solution x^* as in* (4.5) *of the problem* (4.4), (4.2) *satisfies*

$$\gamma\|x^*\| \leq x^*(t), \quad t \in [t_2/\alpha, t_2]_{\mathbb{T}}$$

for γ given in (2.8).

Proof. Let $y \in C_{ld}[\rho(t_1), \sigma^2(t_3)]$ with $y \geq 0$. From previous work in Lemma 2.2, it is clear that for all $t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$ and $B(y)$ given in (4.6),

$$x^*(t) \leq \lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} g(s)y(s)\nabla s + B(y)\int_{\rho(t_1)}^{\sigma^2(t_3)}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big). \tag{4.7}$$

For $t \in [t_2/\alpha, t_2]_{\mathbb{T}}$, from Lemma 2.3 and the definition of γ in (2.8) we have

$$\begin{aligned} x^*(t) &= \lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)y(s)\nabla s + B(y)\int_{\rho(t_1)}^{t}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big)\\ &\geq \lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} \gamma g(s)y(s)\nabla s + B(y)\int_{\rho(t_1)}^{t_2/\alpha}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big)\\ &= \gamma\lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} g(s)y(s)\nabla s + B(y)\frac{(\alpha+1)\int_{\rho(t_1)}^{\sigma^2(t_3)}\int_{\rho(t_1)}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Delta u}{\min\{\alpha-1, \alpha\}}\Big)\\ &\geq \gamma\|x^*\|. \end{aligned} \tag{4.8}$$

The proof is complete. □

5. Eigenvalue intervals

To establish eigenvalue intervals for the eigenvalue problem (4.1), (4.2) we will employ Theorem 3.1. To that end, let $\mathcal{B}$ denote the Banach space $C[\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$ with the norm $\|x\| = \sup_{t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}} |x(t)|$. Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{x \in \mathcal{B} : x(t) \geq 0 \text{ for } t \in [\rho(t_1),\ \sigma^2(t_3)]_{\mathbb{T}},\ x(t) \geq \gamma\|x\| \text{ on } [t_2/\alpha, t_2]_{\mathbb{T}}\},$$

where γ is given in (2.8). When $x \in \mathcal{P}$ define the operator $T : \mathcal{P} \to \mathcal{B}$ for $t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$ by

$$(Tx)(t) := \lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)a(s)f(x(s))\nabla s + B(af(x))\int_{\rho(t_1)}^{t}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big), \tag{5.1}$$

using (4.6). We seek a fixed point of T in $\mathcal{P}$ by establishing the hypotheses of Theorem 3.1.

Lemma 5.1. *Assume* (K1) *through* (K4)*. Then $T : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. Consider the integral operator T in (5.1). If $x \in \mathcal{P}$, then by Lemma 2.2 we have, as in (4.7) and (4.8),

$$(Tx)(t) \leq \lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} g(s)a(s)f(x(s))\nabla s + B(af(x))\int_{\rho(t_1)}^{\sigma^2(t_3)}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big),$$

so that for $t \in [t_2/\alpha, t_2]_{\mathbb{T}}$,

$$\begin{aligned}
(Tx)(t) &\geq \lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} \gamma g(s)a(s)f(x(s))\nabla s + B(af(x))\int_{\rho(t_1)}^{t_2/\alpha}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big)\\
&= \gamma\lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} g(s)a(s)f(x(s))\nabla s + B(af(x))\frac{(\alpha+1)\int_{\rho(t_1)}^{\sigma^2(t_3)}\int_{\rho(t_1)}^{\sigma(t_3)}\frac{\Delta\tau}{p(\tau)}\Delta u}{\min\{\alpha-1,\alpha\}}\Big)\\
&\geq \gamma\|Tx\|.
\end{aligned}$$

Therefore, $T : \mathcal{P} \to \mathcal{P}$. Moreover, T is completely continuous by a typical application of the Ascoli-Arzela Theorem. $\square$

For $G(t,s)$ in (2.5) and B (4.6), define the constant

$$J := \int_{\rho(t_1)}^{\sigma(t_3)} g(s)a(s)\nabla s + B(a)\int_{\rho(t_1)}^{\sigma^2(t_3)}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u. \tag{5.2}$$

Theorem 5.2. *Assume* (K1) *through* (K4)*. Then for each λ satisfying*

$$\frac{1}{f_\infty\gamma\int_{t_2/\alpha}^{t_2} G(\sigma^2(t_3),s)a(s)\nabla s} < \lambda < \frac{1}{f_0 J} \tag{5.3}$$

there exists at least one positive solution of (4.1), (4.2) *in $\mathcal{P}$.*

Proof. Let J be as in (5.2), λ as in (5.3), and let $\epsilon > 0$ be such that

$$\frac{1}{(f_\infty-\epsilon)\gamma\int_{t_2/\alpha}^{t_2} G(\tau,s)a(s)\nabla s} \leq \lambda \leq \frac{1}{(f_0+\epsilon)J}. \tag{5.4}$$

First consider f_0. There exists an $R_1 > 0$ such that $f(x) \le (f_0+\epsilon)x$ for $0 < x \le R_1$ by the definition of f_0. Pick $x \in \mathcal{P}$ with $\|x\| = R_1$. ¿From (4.6) we have $|B(af(x))| \le B(a)\|f(x)\|$. Using Lemma 2.2 we have

$$\begin{aligned}
(Tx)(t) &= \lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)a(s)f(x(s))\nabla s + B(af(x))\int_{\rho(t_1)}^{t}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big)\\
&\le \lambda\|f(x)\|\Big(\int_{\rho(t_1)}^{\sigma(t_3)} g(s)a(s)\nabla s + B(a)\int_{\rho(t_1)}^{\sigma^2(t_3)}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big)\\
&\le \lambda(f_0+\epsilon)\|x\|J \le \|x\|
\end{aligned}$$

from the right side of (5.4). As a result, $\|Tx\| \le \|x\|$. Thus, take

$$\Omega_1 := \{x \in \mathcal{B} : \|x\| < R_1\}$$

so that $\|Tx\| \le \|x\|$ for $x \in \mathcal{P} \cap \partial\Omega_1$.

Next consider f_∞. Again by definition there exists an $R_2' > R_1$ such that $f(x) \ge (f_\infty - \epsilon)x$ for $x \ge R_2'$; take $R_2 = \max\{2R_1, R_2'/\Gamma\}$. If $x \in \mathcal{P}$ with $\|x\| = R_2$, then for $s \in [t_2/\alpha, t_2]_{\mathbb{T}}$ we have

$$x(s) \ge \gamma\|x\| = \gamma R_2. \tag{5.5}$$

Define $\Omega_2 := \{x \in \mathcal{B} : \|x\| < R_2\}$; using (5.5) for $s \in [t_2/\alpha, t_2]$ we get

$$\begin{aligned}
&(Tx)(\sigma^2(t_3))\\
&= \lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} G(\sigma^2(t_3),s)a(s)f(x(s))\nabla s + B(af(x))\int_{\rho(t_1)}^{\sigma^2(t_3)}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big)\\
&\ge \lambda\int_{t_2/\alpha}^{t_2} G(\sigma^2(t_3),s)a(s)f(x(s))\nabla s \ge \lambda(f_\infty-\epsilon)\int_{t_2/\alpha}^{t_2} G(\sigma^2(t_3),s)a(s)x(s)\nabla s\\
&\ge \lambda(f_\infty-\epsilon)\gamma R_2\int_{t_2/\alpha}^{t_2} G(\sigma^2(t_3),s)a(s)\nabla s \ge R_2 = \|x\|,
\end{aligned}$$

where we have used the left side of (5.4). Hence we have shown that

$$\|Tx\| \ge \|x\|, \quad x \in \mathcal{P} \cap \partial\Omega_2.$$

An application of Theorem 3.1 yields the conclusion of the theorem; in other words, T has a fixed point x in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $R_1 \le \|x\| \le R_2$. $\square$

Theorem 5.3. *Assume* (K1) *through* (K4)*. Then for each λ satisfying*

$$\frac{1}{f_0\gamma\int_{t_2/\alpha}^{t_2} G(\sigma^2(t_3),s)a(s)\nabla s} < \lambda < \frac{1}{f_\infty J} \tag{5.6}$$

there exists at least one positive solution of (4.1), (4.2) *in $\mathcal{P}$.*

Proof. Let J be as in (5.2), λ as in (5.6), and let $\epsilon > 0$ be such that

$$\frac{1}{(f_0-\epsilon)\gamma\int_{t_2/\alpha}^{t_2} G(\tau,s)a(s)\nabla s} \le \lambda \le \frac{1}{(f_\infty+\epsilon)J}. \tag{5.7}$$

Again let T be the operator defined in (5.1). We once more seek a fixed point of T in $\mathcal{P}$ by establishing the hypotheses of Theorem 3.1.

First consider f_0. There exists an $R_1 > 0$ such that $f(x) \geq (f_0 - \epsilon)x$ for $0 < x \leq R_1$ by the definition of f_0. Pick $x \in \mathcal{P}$ with $\|x\| = R_1$. For $s \in [t_2/\alpha, t_2]_{\mathbb{T}}$ we have

$$x(s) \geq \gamma\|x\| = \gamma R_1. \tag{5.8}$$

Using the left side of (5.7) and (5.8) we get, for $s \in [t_2/\alpha, t_2]_{\mathbb{T}}$,

$$\begin{aligned}
&(Tx)(\sigma^2(t_3)) \\
&= \lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} G(\sigma^2(t_3), s)a(s)f(x(s))\nabla s + B(af(x))\int_{\rho(t_1)}^{\sigma^2(t_3)}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big) \\
&\geq \lambda(f_0 - \epsilon)\int_{t_2/\alpha}^{t_2} G(\sigma^2(t_3), s)a(s)x(s)\nabla s \\
&\geq \lambda(f_0 - \epsilon)R_1\gamma\int_{t_2/\alpha}^{t_2} G(\sigma^2(t_3), s)a(s)\nabla s \geq R_1 = \|x\|,
\end{aligned}$$

Therefore $\|Tx\| \geq \|x\|$. This motivates us to define

$$\Omega_1 := \{x \in \mathcal{B} : \|x\| < R_1\},$$

whereby our work above confirms

$$\|Tx\| \geq \|x\|, \quad x \in \mathcal{P} \cap \partial\Omega_1.$$

Next consider f_∞. Again by definition there exists an $R_2' > R_1$ such that $f(x) \leq (f_\infty + \epsilon)x$ for $x \geq R_2'$; take $R_2 = \max\{2R_1, R_2'/\Gamma\}$. If f is bounded, there exists $M > 0$ with $f(x) \leq M$ for all $x \in (0, \infty)$. Let

$$R_2 := \max\Big\{2R_2',\ \lambda M\Big(\int_{\rho(t_1)}^{\sigma(t_3)} g(s)a(s)\nabla s + B(a)\int_{\rho(t_1)}^{\sigma^2(t_3)}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big)\Big\}.$$

If $x \in \mathcal{P}$ with $\|x\| = R_2$, then we have

$$(Tx)(t) \leq \lambda M\Big(\int_{\rho(t_1)}^{\sigma(t_3)} g(s)a(s)\nabla s + B(a)\int_{\rho(t_1)}^{\sigma^2(t_3)}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big) \leq R_2 = \|x\|.$$

As a result, $\|Tx\| \leq \|x\|$. Thus, take

$$\Omega_2 := \{x \in \mathcal{B} : \|x\| < R_2\}$$

so that $\|Tx\| \leq \|x\|$ for $x \in \mathcal{P} \cap \partial\Omega_2$. If f is unbounded, take $R_2 := \max\{2R_1, R_2'\}$ such that $f(x) \leq f(R_2)$ for $0 < x \leq R_2$. If $x \in \mathcal{P}$ with $\|x\| = R_2$, then we have

$$\begin{aligned}
(Tx)(t) &= \lambda\Big(\int_{\rho(t_1)}^{\sigma(t_3)} G(t, s)a(s)f(x(s))\nabla s + B(af(x))\int_{\rho(t_1)}^{t}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big) \\
&\leq \lambda f(R_2)\Big(\int_{\rho(t_1)}^{\sigma(t_3)} g(s)a(s)\nabla s + B(a)\int_{\rho(t_1)}^{\sigma^2(t_3)}\int_{\rho(t_1)}^{u}\frac{\Delta\tau}{p(\tau)}\Delta u\Big) \\
&\leq \lambda(f_\infty + \epsilon)R_2 J \leq R_2 = \|x\|,
\end{aligned}$$

where we have used the left side of (5.7). Hence we have shown that

$$\|Tx\| \leq \|x\|, \quad x \in \mathcal{P} \cap \partial\Omega_2$$

if we take

$$\Omega_2 := \{x \in \mathcal{B} : \|x\| < R_2\}.$$

As before an application of Theorem 3.1 yields the conclusion that T has a fixed point x in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $R_1 \leq \|x\| \leq R_2$. $\square$

Corollary 5.4. *Assume* (K1) *through* (K4)*. If f is sublinear (i.e., $f_0 = \infty$ and $f_\infty = 0$), or if f is superlinear (i.e., $f_0 = 0$ and $f_\infty = \infty$), then for any $\lambda > 0$ the boundary value problem* (4.1)*,* (4.2) *has at least one positive solution in $\mathcal{P}$.*

Proof. For the superlinear claim, use (5.3) of Theorem 5.2; for the sublinear claim, use (5.6) of Theorem 5.3. □

As remarked earlier, the results in this section are new for ordinary differential equations (when $\mathbb{T} = \mathbb{R}$) and for difference equations (when $\mathbb{T} = \mathbb{Z}$).

We now provide an example to illustrate that conditions (K1) through (K4) are naturally satisfied.

Example 5.5. Consider for $\mathbb{T} = \mathbb{R}$ and the following choices: $t_1 = 0$, $t_2 = 1/2$, $t_3 = 1$; $p \equiv 1$; continuous $f(x)$ such that f_0 and f_∞ exist; $\alpha = 3/2$; $\alpha_1 = 1/2$; $\xi_1 = 1/4$; $a(t) = t$. Then the boundary value problem (4.1), (4.2) has at least one positive solution in $\mathcal{P}$ for any

$$\frac{1718.45}{f_\infty} < \lambda < \frac{0.824}{f_0}.$$

With these choices, (4.1), (4.2) reduces to a third-order four-point boundary value problem

$$\begin{gathered} x'''(t) + \lambda t f(x(t)) = 0, \quad t \in [0,1]_{\mathbb{R}}, \\ x(0) = x'(0) = 0, \\ x'(1) - 3x'(1/2)/2 = x'(1/4)/2. \end{gathered}$$

It is not difficult to verify that conditions (K1) through (K4) are satisfied. Some calculations lead to $\gamma = 1/90$, $g(s) = 10s(1-s)$, $B(s) = 73/96$, $J = 233/192$, and $\int_{1/3}^{1/2} sG(1,s)ds = 181/3456$, so that we can find the interval in (5.3) to be

$$\frac{311040}{181 f_\infty} < \lambda < \frac{192}{233 f_0},$$

which is approximately

$$\frac{1718.45}{f_\infty} < \lambda < \frac{0.824}{f_0}.$$

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