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RELATIONS BETWEEN THE SMALL FUNCTIONS AND THE SOLUTIONS OF CERTAIN SECOND-ORDER DIFFERENTIAL EQUATIONS

HUIFANG LIU, ZHIQIANG MAO

Abstract. In this paper, we investigate the relations between the small functions and the solutions, first, second derivatives, and differential polynomial of the solutions to the differential equation

$$f'' + A_1 e^{P(z)} f' + A_0 e^{Q(z)} f = 0,$$

where $P(z) = a_n z^n + \dots + a_0$, $Q(z) = b_n z^n + \dots + b_0$ are polynomials with degree n $(n \geq 1)$, a_i, b_i $(i = 0, 1, \dots, n)$, $a_n b_n \neq 0$ are complex constants, $A_j(z) \not\equiv 0$ $(j = 0, 1)$ are entire functions with $\sigma(A_j) < n$.

1. Main Results

In this paper, we use the standard notation of Nevanlinna's value distribution theory [7]. In addition, we use notations $\sigma(f)$, $\lambda(f)$, $\overline{\lambda}(f)$ to denote the order of growth, the exponent of convergence of the zero-sequence and the sequence of distinct zeros of $f(z)$ respectively. A meromorphic function $g(z)$ is called a small function of a meromorphic function $f(z)$ if $T(r, g) = o(T(r, f))$, as $r \to +\infty$.

Consider the differential equation

$$f'' + A_1 e^{P(z)} f' + A_0 e^{Q(z)} f = 0, \tag{1.1}$$

where $P(z)$, $Q(z)$ are polynomials with degree n $(n \geq 1)$, $A_j(z) \not\equiv 0$ $(j = 0, 1)$ are entire functions with $\sigma(A_1) < \deg P$, $\sigma(A_0) < \deg Q$. If $\deg P \neq \deg Q$, then every solution of (1.1) has infinite order [5, p. 419]. If $\deg P = \deg Q$, then equation (1.1) may have a solution of finite order. Indeed $f(z) = z$ satisfies $f'' + ze^z f' - e^z f = 0$. Kwon [8] studied the growth of solutions of equation (1.1) with $\deg P = \deg Q$, and obtained the following result.

Theorem 1.1. *Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ be entire functions with $\sigma(A_j) < n$, $P(z) = a_n z^n + \dots + a_0$, $Q(z) = b_n z^n + \dots + b_0$ be polynomials with degree n $(n \geq 1)$, where a_i, b_i $(i = 0, 1, \dots, n)$, $a_n b_n \neq 0$ are complex constants such that $\arg a_n \neq \arg b_n$ or $a_n = c b_n (0 < c < 1)$. Then every solution $f \not\equiv 0$ of equation (1.1) has infinite order.*

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Chen and Shon [4] studied the differential equation

$$f'' + A_1 e^{az} f' + A_0 e^{bz} f = 0 \tag{1.2}$$

and obtained

Theorem 1.2. *Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ be entire functions with $\sigma(A_j) < 1$, a, b be complex constants such that $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb (0 < c < 1)$. If $\varphi(z) \not\equiv 0$ is an entire function with finite order, then every solution $f \not\equiv 0$ of equation (1.2) satisfies $\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \overline{\lambda}(f'' - \varphi) = \infty$.*

Theorem 1.3. *Let $A_j(z), a, b$ satisfy the hypotheses of Theorem 1.2, $d_0(z)$, $d_1(z)$, $d_2(z)$ be polynomials not all equal to zero, $\varphi(z) \not\equiv 0$ is an entire function of order less than 1. If $f \not\equiv 0$ is a solution of equation (1.2), then the differential polynomials $g(z) = d_2 f'' + d_1 f' + d_0 f$ satisfy $\overline{\lambda}(g - \varphi) = \infty$.*

In this paper we go deeply into the study of the relations of the small functions and solutions of the differential equation (1.1) and obtain the following theorem.

Theorem 1.4. *Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$, $P(z)$, $Q(z)$ satisfy the hypotheses of Theorem 1.1. If $\varphi(z) \not\equiv 0$ is an entire function with finite order, then every solution $f \not\equiv 0$ of equation (1.1) satisfies $\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \overline{\lambda}(f'' - \varphi) = \infty$.*

Theorem 1.5. *Let $A_j(z) \not\equiv 0)$ $(j = 0, 1)$, $P(z)$, $Q(z)$ satisfy the hypotheses of Theorem 1.1, $d_0(z)$, $d_1(z)$, $d_2(z)$ be polynomials that are not all equal to zero, $\varphi(z) \not\equiv 0$ is an entire function of order that is less than n . If $f \not\equiv 0$ is a solution of equation (1.1), then the differential polynomials $g(z) = d_2 f'' + d_1 f' + d_0 f$ satisfy $\overline{\lambda}(g - \varphi) = \infty$.*

2. Auxiliary Lemmas

Lemma 2.1 ([3]). *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < +\infty$. Then for any given $\varepsilon > 0$, there is a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\varphi \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = R_0(\varphi) > 1$, such that for all z satisfying $\arg z = \varphi$ and $|z| \geq R_0$, we have $\exp\{-r^{\sigma+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\sigma+\varepsilon}\}$.*

Lemma 2.2. *Let $P(z) = (\alpha + i\beta)z^n + \dots$ $(\alpha, \beta$ are real, $|\alpha| + |\beta| \neq 0)$ be a polynomial with degree $n \geq 1$, $A(z) \not\equiv 0$ be a meromorphic function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$, then for any given $\varepsilon > 0$, there is a set $H_1 \subset [0, 2\pi)$ that has linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (H_1 \bigcup H_2)$ and a sufficiently large r, we have*

(i) *If $\delta(P, \theta) > 0$, then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta)r^n\} \leq |g(re^{i\theta})| \leq \exp\{(1 + \varepsilon)\delta(P, \theta)r^n\};$$

(ii) *If $\delta(P, \theta) < 0$, then*

$$\exp\{(1 + \varepsilon)\delta(P, \theta)r^n\} \leq |g(re^{i\theta})| \leq \exp\{(1 - \varepsilon)\delta(P, \theta)r^n\},$$

where $H_2 = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0\}$ is a finite set.

Proof. Rewrite $g(z)$ as $g(z) = we^{(\alpha+i\beta)z^n}$, where $w(z) = A(z)e^{P(z)-(\alpha+i\beta)z^n}$ is a meromorphic function with $\sigma(w) = s < n$. By lemma 2.1, for any given $\varepsilon (0 < \varepsilon < n - s)$, there is a set $H_1 \subset [0, 2\pi)$ that has linear measure zero, such that, if $\theta \in [0, 2\pi) \setminus H_1$, then there exists a constant $R = R(\theta) > 1$, for all z

satisfying $\arg z = \theta$ and $|z| \ge R$, we have $\exp\{-r^{s+\varepsilon}\} \le |w(z)| \le \exp\{r^{s+\varepsilon}\}$. By $|e^{(\alpha+i\beta)z^n}| = e^{Re(\alpha+i\beta)z^n = e^{\delta(P,\theta)r^n}}$, when $\theta \in [0,2\pi)\backslash(H_1 \bigcup H_2)$ and $|z| = r > R$, we have $\exp\{-r^{s+\varepsilon} + \delta(P,\theta)r^n\} \le |g(z)| \le \exp\{r^{s+\varepsilon} + \delta(P,\theta)r^n\}$. So by the above inequality and $\delta(P,\theta) > 0$ or $\delta(P,\theta) < 0$, we complete the proof. $\square$

Lemma 2.3 ([4]). *Let $f(z)$ be an entire function with infinite order, $d_j(z)$ ($j = 0,1,2$) be polynomials that are not all equal to zero. Then*

$$w(z) = d_2 f'' + d_1 f' + d_0 f$$

has infinite order.

Lemma 2.4. *Let $a_i, b_i (i = 0,1\dots n)$ be complex constants such that $a_n b_n \ne 0$ and $\arg a_n \ne \arg b_n$ or $a_n = cb_n (0 < c < 1)$, $P(z) = a_n z^n + \dots + a_0$, $Q(z) = b_n z^n + \dots + b_0$. We denote index sets by*

$$\begin{gathered}
\Lambda_1 = \{0, P\};\\
\Lambda_2 = \{0, P, Q, 2P, P+Q\};\\
\Lambda_3 = \{0, P, Q, 2P, P+Q, 2Q, 3P, 2P+Q, P+2Q\};\\
\Lambda_4 = \big\{0, P, Q, 2P, P+Q, 2Q, 3P, 2P+Q, P+2Q,\\
3Q, 4P, 3P+Q, 2P+2Q, P+3Q\big\}.
\end{gathered}$$

Then

- (i) *If H_j ($j \in \Lambda_1$) and H_Q are all meromorphic functions of orders that are less than n, $H_Q \not\equiv 0$, setting $\psi_1(z) = \Sigma_{j\in\Lambda_1} H_j(z) e^j$, then $\psi_1(z) + H_Q e^Q \not\equiv 0$.*
- (ii) *If H_j ($j \in \Lambda_2$) and H_{2Q} are all meromorphic functions of orders that are less than n, $H_{2Q} \not\equiv 0$, setting $\psi_2(z) = \Sigma_{j\in\Lambda_2} H_j(z) e^j$, then $\psi_2(z) + H_{2Q} e^{2Q} \not\equiv 0$.*
- (iii) *If $H_j (j \in \Lambda_3)$ and H_{3Q} are all meromorphic functions of orders that are less than n, $H_{3Q} \not\equiv 0$, setting $\psi_3(z) = \Sigma_{j\in\Lambda_3} H_j(z) e^j$, then $\psi_3(z) + H_{3Q} e^{3Q} \not\equiv 0$.*
- (iv) *If $H_j (j \in \Lambda_4)$ and H_{4Q} are all meromorphic functions of orders that are less than n, $H_{4Q} \not\equiv 0$, setting $\psi_4(z) = \Sigma_{j\in\Lambda_4} H_j(z) e^j$, then $\psi_4(z) + H_{4Q} e^{4Q} \not\equiv 0$.*
- (v) *The derived function of $\psi_j(z)$ ($j = 1,\dots,4$) keep the above properties of $\psi_j(z)$, and also it can be expressed by $\psi_j(z)$. $\psi_j(z)$ may be different at different places, but preserve the above properties. $\psi_2(z)\psi_2(z)$(it denotes the product of two $\psi_2(z)$, and two $\psi_2(z)$ may be different.) is of properties of $\psi_4(z)$, we write $\psi_2(z)\psi_2(z) = \psi_4(z)$. Similarly we have*
$$\psi_1(z)\psi_1(z) = \psi_2(z), \psi_1(z)\psi_2(z) = \psi_3(z), \psi_1(z)\psi_3(z) = \psi_4(z).$$
- (vi) *let $\psi_{20}(z), \psi_{21}(z), \psi_{22}(z)$ have the form of $\psi_2(z)$ which is defined as in (ii), $\varphi(z) \not\equiv 0$ is a meromorphic function with finite order and $H_{2Q} \not\equiv 0$ are all meromorphic functions of orders that are less than n. Then*
$$\frac{\varphi''(z)}{\varphi(z)}\psi_{22}(z) + \frac{\varphi'(z)}{\varphi(z)}\psi_{21}(z) + \psi_{20}(z) + H_{2Q} e^{2Q} \not\equiv 0.$$

Proof. Properties (i)–(iv) are similar, and the properties of $\psi_j(z)$ ($j = 1,\dots,4$) in (v) are clear, so we only prove (ii) and (vi). For the proof of (ii). We consider two cases:

Case 1: $\arg a_n \ne \arg b_n$. Then $\arg(a_n + b_n)$, $\arg a_n$, $\arg b_n$ are three distinct arguments. Set $\sigma(H_0) = \beta < n$, by Lemma 2.1, for any given $\varepsilon (0 < \varepsilon < \min\{\frac{1}{5}, n - \beta\})$, there exists a set $E_0 \subset [0,2\pi)$ that has linear measure zero, such that if

$\theta \in [0, 2\pi) \setminus E_0$, then there is a constant $R = R(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r \geq R$, we have

$$|H_0(z)| \leq \exp\{r^{\beta+\varepsilon}\}. \tag{2.1}$$

By lemma 2.2, there exists a ray $\arg z = \theta \in [0, 2\pi) \setminus (E_0 \cup E_1 \cup E_2)$, where $E_1 \subset [0, 2\pi)$ has linear measure zero, $E_2 = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0$ or $\delta(Q, \theta) = 0$ or $\delta(P+Q, \theta) = 0\}$ is a finite set, such that $\delta(P, \theta) < 0$, $\delta(P+Q, \theta) < 0$, $\delta(Q, \theta) > 0$, and for the above given ε, we have, when r is sufficiently large,

$$|H_{2Q}e^{2Q}| \geq \exp\{(1-\varepsilon)2\delta(Q, \theta)r^n\}, \tag{2.2}$$

$$|H_Q e^Q| \leq \exp\{(1+\varepsilon)\delta(Q, \theta)r^n\}, \tag{2.3}$$

$$|H_{P+Q}e^{P+Q}| \leq \exp\{(1-\varepsilon)\delta(P+Q, \theta)r^n\} < 1, \tag{2.4}$$

$$|H_{2P}e^{2P}| \leq \exp\{(1-\varepsilon)2\delta(P, \theta)r^n\} < 1, \tag{2.5}$$

$$|H_P e^P| \leq \exp\{(1-\varepsilon)\delta(P, \theta)r^n\} < 1. \tag{2.6}$$

If $\psi_2(z) + H_{2Q}e^{2Q} \equiv 0$, then by (2.1)-(2.6), we have

$$\begin{aligned} \exp\{(1-\varepsilon)2\delta(Q, \theta)r^n\} &\leq |H_{2Q}e^{2Q}| \\ &\leq \exp\{r^{\beta+\varepsilon}\} + \exp\{(1+\varepsilon)\delta(Q, \theta)r^n\} + 3 \\ &\leq 3\exp\{r^{\beta+\varepsilon}\}\exp\{(1+\varepsilon)\delta(Q, \theta)r^n\}. \end{aligned}$$

Because $2(1-\varepsilon) - (1+\varepsilon) = 1 - 3\varepsilon > \frac{2}{5}$, we have

$$\exp\{\frac{2}{5}\delta(Q, \theta)r^n\} \leq 3\exp\{r^{\beta+\varepsilon}\}.$$

This is a contradiction to $\beta + \varepsilon < n$. Hence $\psi_2(z) + H_{2Q}e^{2Q} \not\equiv 0$.

Case 2: $a_n = cb_n (0 < c < 1)$. Set $\sigma(H_0) = \beta < n$. By Lemmas 2.1 and 2.2, for any given $\varepsilon (0 < \varepsilon < \min\{\frac{1-c}{5}, n - \beta\})$, there exist set $E_j \subset [0, 2\pi) (j = 0, 1, 2)$ that have linear measure zero, E_j are defined as in the case (1) respectively. We take the ray $\theta \in [0, 2\pi) \setminus (E_0 \cup E_1 \cup E_2)$, such that $\delta(Q, \theta) > 0$, and when $|z| = r$ is sufficiently large, we have (2.1)-(2.3) and

$$|H_{P+Q}e^{P+Q}| \leq \exp\{(1+\varepsilon)(1+c)\delta(Q, \theta)r^n\}, \tag{2.7}$$

$$|H_{2P}e^{2P}| \leq \exp\{(1+\varepsilon)2c\delta(Q, \theta)r^n\}, \tag{2.8}$$

$$|H_P e^P| \leq \exp\{(1+\varepsilon)c\delta(Q, \theta)r^n\}. \tag{2.9}$$

If $\psi_2(z) + H_{2Q}e^{2Q} \equiv 0$, then by (2.1)-(2.3), and (2.7)-(2.9), we have

$$\begin{aligned} \exp\{(1-\varepsilon)2\delta(Q, \theta)r^n\} &\leq |H_{2Q}e^{2Q}| \\ &\leq \exp\{r^{\beta+\varepsilon}\} + 2\exp\{(1+\varepsilon)(1+c)\delta(Q, \theta)r^n\} \\ &\quad + \exp\{(1+\varepsilon)2c\delta(Q, \theta)r^n\} + \exp\{(1+\varepsilon)c\delta(Q, \theta)r^n\}. \end{aligned} \tag{2.10}$$

Because $0 < \varepsilon < \min\{\frac{1-c}{5}, n - \beta\}$, when $r \to +\infty$, we have

$$\frac{\exp\{r^{\beta+\varepsilon}\}}{\exp\{(1-\varepsilon)2\delta(Q, \theta)r^n\}} \to 0, \tag{2.11}$$

$$\frac{\exp\{(1+\varepsilon)(1+c)\delta(Q, \theta)r^n\}}{\exp\{(1-\varepsilon)2\delta(Q, \theta)r^n\}} \to 0, \tag{2.12}$$

$$\frac{\exp\{(1+\varepsilon)2c\delta(Q,\theta)r^n\}}{\exp\{(1-\varepsilon)2\delta(Q,\theta)r^n\}} \to 0, \tag{2.13}$$

$$\frac{\exp\{(1+\varepsilon)c\delta(Q,\theta)r^n\}}{\exp\{(1-\varepsilon)2\delta(Q,\theta)r^n\}} \to 0. \tag{2.14}$$

By (2.10)-(2.14), we get a contradiction. Hence $\psi_2(z) + H_{2Q}e^{2Q} \not\equiv 0$.

Proof of (vi). By $\sigma(\varphi) < \infty$ and [6, p. 89] we know, for any given $\varepsilon > 0$, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, if $\theta \in [0, 2\pi) \setminus E$, then there exists a constant $R = R(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| \geq R$, we have

$$|\frac{\varphi^{(k)}(z)}{\varphi(z)}| \leq |z|^{k(\sigma(\varphi)-1+\varepsilon)} \quad (k = 1, 2).$$

So on the ray $\arg z = \theta \in [0, 2\pi) \setminus E$, $\frac{\varphi^{(k)}(z)}{\varphi(z)} H_j(z)e^j$ $(k = 1, 2, j \in \Lambda_2)$ keep the properties of $H_j e^j$ which are defined as in (2.1), (2.3)–(2.6) or (2.1), (2.3), (2.7)–(2.9). Using a similar reasoning to that in the proof of (ii), we can prove (vi). $\square$

Lemma 2.5 ([2]). *Suppose that $A_0, \dots, A_{k-1}, F \not\equiv 0$ are finite-order meromorphic functions. If f is an infinite-order meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_0 f = F,$$

then f satisfies $\overline{\lambda}(f) = \lambda(f) = \sigma(f) = \infty$.

3. Proofs of Theorems

Proof of Theorem 1.4. Suppose that $f(z) \not\equiv 0$ is a solution of (1.1). First of all we prove that $\overline{\lambda}(f-\varphi) = \infty$. Set $g_0 = f - \varphi$, then $\sigma(g_0) = \sigma(f) = \infty$,$\overline{\lambda}(g_0) = \overline{\lambda}(f-\varphi)$. Substituting $f = g_0 + \varphi$,$f' = g_0' + \varphi'$,$f'' = g_0'' + \varphi''$ into equation (1.1), we have

$$g_0'' + A_1 e^{P(z)} g_0' + A_0 e^{Q(z)} g_0 = -(\varphi'' + A_1 e^{P(z)}\varphi' + A_0 e^{Q(z)}\varphi). \tag{3.1}$$

We remark that (3.1) may have finite-order solution (For example when $\varphi(z) = z$, $g_0 = -z$ solves the equation (3.1)). But here we discuss only the case $\sigma(g_0) = \infty$.

By $\varphi(z)$ being a finite-order entire function and Theorem 1.1, we know $\varphi'' + A_1 e^{P(z)}\varphi' + A_0 e^{Q(z)}\varphi \not\equiv 0$. Hence by lemma 2.5, we have $\overline{\lambda}(g_0) = \sigma(g_0) = \infty$, i.e. $\overline{\lambda}(f-\varphi) = \infty$.

Secondly we prove $\overline{\lambda}(f'-\varphi) = \infty$. Set $g_1 = f' - \varphi$, then $\sigma(g_1) = \sigma(f') = \sigma(f) = \infty$,$\overline{\lambda}(g_1) = \overline{\lambda}(f'-\varphi)$. Differentiating both sides of equation (1.1), we get

$$f''' + A_1 e^{P(z)} f'' + [(A_1 e^{P(z)})' + A_0 e^{Q(z)}]f' + (A_0 e^{Q(z)})' f = 0. \tag{3.2}$$

Substituting $f = -\frac{1}{A_0 e^{Q(z)}}[f'' + A_1 e^{P(z)} f']$ into (3.2), we get

$$f''' + [A_1 e^{P(z)} - \frac{(A_0 e^{Q(z)})'}{A_0 e^{Q(z)}}]f'' + [(A_1 e^{P(z)})' + A_0 e^{Q(z)} - \frac{(A_0 e^{Q(z)})'}{A_0 e^{Q(z)}} A_1 e^{P(z)}]f' = 0. \tag{3.3}$$

Substituting $f' = g_1 + \varphi$, $f'' = g_1' + \varphi'$, $f''' = g_1'' + \varphi''$ into equation (3.3), we get

$$g_1'' + h_1 g_1' + h_0 g_1 = h, \tag{3.4}$$

where $h_1 = A_1 e^{P(z)} - \frac{(A_0 e^{Q(z)})'}{A_0 e^{Q(z)}}$,

$$h_0 = (A_1 e^{P(z)})' + A_0 e^{Q(z)} - \frac{(A_0 e^{Q(z)})'}{A_0 e^{Q(z)}} A_1 e^{P(z)},$$

$$-h=\varphi''-(\frac{A_0'}{A_0}+Q')\varphi'+[A_1\varphi'+A_1'\varphi+P'A_1\varphi-\frac{A_0'}{A_0}A_1\varphi-Q'A_1\varphi]e^P+A_0\varphi e^Q.$$

Now we prove $h\not\equiv 0$. If $h\equiv 0$, then

$$\frac{\varphi''}{\varphi}-(\frac{A_0'}{A_0}+Q')\frac{\varphi'}{\varphi}+[\frac{\varphi'}{\varphi}+\frac{A_1'}{A_1}+P'-\frac{A_0'}{A_0}-Q']A_1e^P+A_0e^Q=0. \tag{3.5}$$

By $\sigma(\varphi)<\infty$, $\sigma(A_j)<n$ $(j=0,1)$ and [6, p. 89], for any given $0<\varepsilon<\frac{1-c}{1+2c}$ (c is defined as in Theorem 1.4), there exists a set $E_0\subset[0,2\pi)$ that has linear measure zero, if $\theta\in[0,2\pi)\setminus E_0$, then there exists a constant $R=R(\theta)>1$, such that for all z satisfying $\arg z=\theta$ and $|z|\geq R$, we have

$$|\frac{\varphi^{(k)}(z)}{\varphi(z)}|\leq|z|^{k(\sigma(\varphi)-1+\varepsilon)}\quad(k=1,2), \tag{3.6}$$

$$|\frac{A_j'(z)}{A_j(z)}|\leq|z|^{\sigma(A_j)-1+\varepsilon}\quad(j=0,1). \tag{3.7}$$

Since $P(z)$, $Q(z)$ are polynomials with degree n, when $|z|=r$ is sufficiently large, we have

$$|P'(z)|\leq r^n\quad\text{and}\quad|Q'(z)|\leq r^n. \tag{3.8}$$

So by (3.6)-(3.8), there exists a positive constant M, such that for all z satisfying $\arg z=\theta\in[0,2\pi)\setminus E_0$, we have, when $|z|=r$ is sufficiently large,

$$\big|(\frac{A_0'}{A_0}+Q')\frac{\varphi'}{\varphi}\big|\leq r^M, \tag{3.9}$$

$$\big|\frac{\varphi'}{\varphi}+\frac{A_1'}{A_1}+P'-\frac{A_0'}{A_0}-Q'\big|\leq r^M. \tag{3.10}$$

If $\arg a_n\neq\arg b_n$, then by lemma 2.2, there exists a ray $\arg z=\theta\in[0,2\pi)\setminus(E_0\cup E_1\cup E_2)$, $E_1\subset[0,2\pi)$ having linear measure zero, $E_2=\{\theta\in[0,2\pi);\delta(P,\theta)=0$ or $\delta(Q,\theta)=0\}$ being a finite set, such that $\delta(P,\theta)<0,\delta(Q,\theta)>0$, and for the above given ε, we have, when r is sufficiently large,

$$|A_0e^Q|\geq\exp\{(1-\varepsilon)\delta(Q,\theta)r^n\}, \tag{3.11}$$

$$|A_1e^P|\leq\exp\{(1-\varepsilon)\delta(P,\theta)r^n\}<1. \tag{3.12}$$

So by (3.5),(3.6) and (3.9)-(3.12), we get

$$\exp\{(1-\varepsilon)\delta(Q,\theta)r^n\}\leq|A_0e^Q|\leq r^{2(\sigma(\varphi)-1+\varepsilon)}+r^M+r^M.$$

This is absurd.

If $a_n=cb_n(0<c<1)$, then by lemma 2.2, there exists a ray $\arg z=\theta\in[0,2\pi)\setminus(E_0\cup E_1\cup E_2)$, where E_0,E_1 and E_2 are defined as the above, such that $\delta(Q,\theta)>0$, and for the above given ε, when r is sufficiently large, we have (3.11) and

$$|A_1e^P|\leq\exp\{(1+\varepsilon)c\delta(Q,\theta)r^n\}. \tag{3.13}$$

So by (3.5), (3.6), (3.9)-(3.11) and (3.13), we get

$$\begin{aligned}\exp\{(1-\varepsilon)\delta(Q,\theta)r^n\}&\leq|A_0e^Q|\\&\leq r^{2(\sigma(\varphi)-1+\varepsilon)}+r^M+r^M\exp\{(1+\varepsilon)c\delta(Q,\theta)r^n\}\\&\leq 3\exp\{(1+2\varepsilon)c\delta(Q,\theta)r^n\}.\end{aligned}$$

This is a contradiction to $0 < \varepsilon < \frac{1-c}{1+2c}$. From the above proof, we get $h \not\equiv 0$. From $h \not\equiv 0$ and lemma 2.5 we get $\overline{\lambda}(g_1) = \sigma(g_1) = \infty$. Hence $\overline{\lambda}(f' - \varphi) = \infty$.

Finally we prove that $\overline{\lambda}(f'' - \varphi) = \infty$. Set $g_2 = f'' - \varphi$, then $\sigma(g_2) = \sigma(f'') = \sigma(f) = \infty$, $\overline{\lambda}(g_2) = \overline{\lambda}(f'' - \varphi)$. Differentiating both sides of equation (3.2), we get

$$f^{(4)} + A_1e^P f''' + [2(A_1e^P)' + A_0e^Q]f'' + [(A_1e^P)'' + 2(A_0e^Q)']f' + (A_0e^Q)''f = 0. \tag{3.14}$$

Substituting $f = -\frac{1}{A_0e^Q}[f'' + A_1e^P f']$ into (3.14), we get

$$\begin{aligned} &f^{(4)} + A_1e^P f''' + [2(A_1e^P)' + A_0e^Q - \frac{(A_0e^Q)''}{A_0e^Q}]f'' \\ &+[(A_1e^P)'' + 2(A_0e^Q)' - \frac{(A_0e^Q)''}{A_0e^Q}A_1e^P]f' = 0. \end{aligned} \tag{3.15}$$

By (3.3) and (3.15), we have

$$f^{(4)} + H_3f''' + H_2f'' = 0, \tag{3.16}$$

where

$$H_3 = A_1e^P - \frac{\varphi_1(z)}{\varphi_2(z)}, \tag{3.17}$$

$$H_2 = 2(A_1e^P)' + A_0e^Q - \frac{(A_0e^Q)''}{A_0e^Q} - \frac{\varphi_1(z)}{\varphi_2(z)}[A_1e^P - \frac{(A_0e^Q)'}{A_0e^Q}], \tag{3.18}$$

$$\varphi_1(z) = (A_1e^P)'' + 2(A_0e^Q)' - \frac{(A_0e^Q)''}{A_0e^Q}A_1e^P, \tag{3.19}$$

$$\varphi_2(z) = (A_1e^P)' + A_0e^Q - \frac{(A_0e^Q)'}{A_0e^Q}A_1e^P, \tag{3.20}$$

and $\varphi_2(z) \not\equiv 0$ by Lemma 2.4 (i). Clearly, $H_3, H_2, \varphi_1(z), \varphi_2(z)$ are meromorphic functions with $\sigma(\varphi_k) \le n(k = 1, 2)$, $\sigma(H_j) \le n(j = 2, 3)$.

Substituting $f'' = g_2 + \varphi, f''' = g_2' + \varphi', f^{(4)} = g_2'' + \varphi''$ into (3.16),

$$g_2'' + H_3g_2' + H_2g_2 = -(\varphi'' + H_3\varphi' + H_2\varphi).$$

If we can prove that $-(\varphi'' + H_3\varphi' + H_2\varphi) \not\equiv 0$, then by lemma 2.5, we get $\overline{\lambda}(g_2) = \sigma(g_2) = \infty$. Hence $\overline{\lambda}(f'' - \varphi) = \infty$. Now we prove $-(\varphi'' + H_3\varphi' + H_2\varphi) \not\equiv 0$. Notice that

$$(A_1e^P)' = (A_1' + A_1P')e^P, \quad (A_1e^P)'' = (A_1'' + 2A_1'P' + A_1(P')^2 + A_1P'')e^P,$$
$$\frac{(A_0e^Q)'}{A_0e^Q} = \frac{A_0'}{A_0} + Q', \quad \frac{(A_0e^Q)''}{A_0e^Q} = \frac{A_0''}{A_0} + 2\frac{A_0'}{A_0}Q' + (Q')^2 + Q''.$$

So by (3.17)-(3.20), we have

$$\varphi_1(z) = B_1e^P + 2(A_0' + A_0Q')e^Q, \tag{3.21}$$

$$\varphi_2(z) = B_2e^P + A_0e^Q, \tag{3.22}$$

$$H_3 = \frac{1}{\varphi_2(z)}H_4, \tag{3.23}$$

$$H_2 = \frac{1}{\varphi_2(z)}[A_0^2e^{2Q} + H_5], \tag{3.24}$$

where

$$\begin{aligned} H_5 &= [2A_0(A_1' + A_1P') + A_0B_2 - 2A_1(A_0' + A_0Q')]e^{P+Q} \\ &\quad + [2B_2(A_1' + A_1P') - A_1B_1]e^{2P} - [A_0'' + 2A_0'Q' + A_0(Q')^2 \\ &\quad + A_0Q'' - 2(\frac{A_0'}{A_0} + Q')(A_0' + A_0Q')]e^Q \\ &\quad - [B_2(\frac{A_0''}{A_0} + 2\frac{A_0'}{A_0}Q' + (Q')^2 + Q'') - B_1(\frac{A_0'}{A_0} + Q')]e^P, \\ H_4 &= A_1A_0e^{P+Q} + A_1B_2e^{2P} - 2(A_0' + A_0Q')e^Q - B_1e^P, \end{aligned}$$

$$B_1 = A_1'' + 2A_1'P' + A_1(P')^2 + A_1P'' - \frac{A_1}{A_0}(A_0'' + 2A_0'Q' + A_0(Q')^2 + A_0Q''),$$

$$B_2 = A_1' + A_1P' - A_1(\frac{A_0'}{A_0} + Q').$$

Clearly, B_1, B_2 are meromorphic functions with $\sigma(B_j) < n$ $(j = 1, 2)$. By (3.22)-(3.24), we see that

$$-(\frac{\varphi''}{\varphi} + H_3\frac{\varphi'}{\varphi} + H_2) = -\frac{1}{\varphi_2(z)}\{\frac{\varphi''}{\varphi}\varphi_2(z) + \frac{\varphi'}{\varphi}H_4 + H_5 + A_0^2e^{2Q}\}.$$

As $\varphi_2(z)$, H_4, H_5 have the form of $\psi_2(z)$ which is defined as in lemma 2.4 (ii), so by lemma 2.4 (i) and (vi), we get $\frac{\varphi''}{\varphi}\varphi_2(z) + \frac{\varphi'}{\varphi}H_4 + H_5 + A_0^2e^{2Q} \not\equiv 0$, $\varphi_2(z) \not\equiv 0$. Hence $-(\varphi'' + H_3\varphi' + H_2\varphi) \not\equiv 0$. $\square$

Proof of Theorem 1.5. First, we suppose $d_2 \not\equiv 0$. Suppose that $f \not\equiv 0$ is a solution of equation (1.1), by Theorem 1.1 we have $\sigma(f) = \infty$. Set $w = d_2f'' + d_1f' + d_0f - \varphi$, then $\sigma(w) = \sigma(g) = \sigma(f) = \infty$ by lemma 2.3.

To prove that $\overline{\lambda}(g - \varphi) = \infty$, we need to prove only that $\overline{\lambda}(w) = \infty$. Substituting $f'' = -A_1e^Pf' - A_0e^Qf$ into w, we get

$$w = (d_1 - d_2A_1e^P)f' + (d_0 - d_2A_0e^Q)f - \varphi. \tag{3.25}$$

Differentiating both sides of equation (3.25), and replacing f'' with $f'' = -A_1e^Pf' - A_0e^Qf$, we obtain

$$\begin{aligned} w' &= [d_2A_1^2e^{2P} - ((d_2A_1)' + P'd_2A_1 + d_1A_1)e^P - d_2A_0e^Q + d_0 + d_1']f' \\ &\quad + [d_2A_0A_1e^{P+Q} - ((d_2A_0)' + Q'd_2A_0 + d_1A_0)e^Q + d_0']f - \varphi'. \end{aligned} \tag{3.26}$$

Set

$$\begin{gathered} \alpha_1 = d_1 - d_2A_1e^P, \quad \alpha_0 = d_0 - d_2A_0e^Q, \\ \beta_1 = d_2A_1^2e^{2P} - ((d_2A_1)' + P'd_2A_1 + d_1A_1)e^P - d_2A_0e^Q + d_0 + d_1', \\ \beta_0 = d_2A_0A_1e^{P+Q} - ((d_2A_0)' + Q'd_2A_0 + d_1A_0)e^Q + d_0'. \end{gathered}$$

Then we have

$$\begin{gathered} \alpha_1f' + \alpha_0f = w + \varphi \\ \beta_1f' + \beta_0f = w' + \varphi'. \end{gathered}$$

Set

$$
\begin{aligned}
h &= \alpha_1\beta_0 - \alpha_0\beta_1 \\
&= [d_1 - d_2A_1e^P][d_2A_0A_1e^{P+Q} - ((d_2A_0)' + Q'd_2A_0 + d_1A_0)e^Q + d_0'] \\
&\quad - [d_0 - d_2A_0e^Q][d_2A_1^2e^{2P} - ((d_2A_1)' + P'd_2A_1 + d_1A_1)e^P \\
&\quad - d_2A_0e^Q + d_0 + d_1'].
\end{aligned} \tag{3.27}
$$

Now check all terms of h. Since the term $\pm d_2^2A_1^2A_0e^{2P+Q}$ is eliminated, by (3.27) we can write

$$
h = \psi_2(z) - d_2^2A_0^2e^{2Q}, \tag{3.28}
$$

where $\psi_2(z)$ is defined as in lemma 2.4 (ii). By $d_2 \not\equiv 0$, $A_0 \not\equiv 0$ and lemma 2.4 (ii), we see that $h \not\equiv 0$. By (3.25) and (3.26), we obtain

$$
\begin{aligned}
f' &= \frac{1}{h}\{-(d_0 - d_2A_0e^Q)(w' + \varphi') \\
&\quad + [d_2A_0A_1e^{P+Q} - ((d_2A_0)' + Q'd_2A_0 + d_1A_0)e^Q + d_0'](w + \varphi)\} \\
&= \frac{1}{h}\{-(d_0 - d_2A_0e^Q)w' + \Phi_{10}w + \varphi d_2A_0A_1e^{P+Q} \\
&\quad + [d_2A_0\varphi' - ((d_2A_0)' + Q'd_2A_0 + d_1A_0)\varphi]e^Q + \psi_1\},
\end{aligned} \tag{3.29}
$$

where Φ_{10} is an entire function with $\sigma(\Phi_{10}) \le n$, ψ_1 is defined as in lemma 2.4 (i).

$$
\begin{aligned}
f &= \frac{1}{h}\{(d_1 - d_2A_1e^P)(w' + \varphi') \\
&\quad - [d_2A_1^2e^{2P} - ((d_2A_1)' + P'd_2A_1 + d_1A_1)e^P - d_2A_0e^Q + d_0 + d_1'](w + \varphi)\} \\
&= \frac{1}{h}\{(d_1 - d_2A_1e^P)w' + \Phi_{00}w - \varphi d_2A_1^2e^{2P} + \varphi d_2A_0e^Q + \psi_1\},
\end{aligned} \tag{3.30}
$$

where Φ_{00} is an entire function with $\sigma(\Phi_{00}) \le n$, ψ_1 is defined as in lemma 2.4 (i). Differentiating both sides of equation (3.29), and by (3.28), we get

$$
f'' = \frac{1}{h^2}\{(-d_2^3A_0^3e^{3Q} + \psi_3)w'' + \Phi_{21}w' + \Phi_{20}w + \psi_4\}, \tag{3.31}
$$

where Φ_{21} and Φ_{20} are entire functions with $\sigma(\Phi_{21}) \le n$, $\sigma(\Phi_{20}) \le n$, ψ_3, ψ_4 are defined as in lemma 2.4 (iii)-(iv). Substituting (3.28)-(3.31) into (1.1), we obtain

$$
\begin{aligned}
&(-d_2^3A_0^3e^{3Q} + \psi_3)w'' + \Phi_{21}w' + \Phi_{20}w + \psi_4 \\
&+ A_1e^{P(z)}(\psi_2(z) - d_2^2A_0^2e^{2Q})\{-(d_0 - d_2A_0e^Q)w' + \Phi_{10}w + \varphi d_2A_0A_1e^{P+Q} \\
&+ [d_2A_0\varphi' - ((d_2A_0)' + Q'd_2A_0 + d_1A_0)\varphi]e^Q + \psi_1\} \\
&+ A_0e^{Q(z)}(\psi_2(z) - d_2^2A_0^2e^{2Q})\{(d_1 - d_2A_1e^P)w' \\
&+ \Phi_{00}w - \varphi d_2A_1^2e^{2P} + \varphi d_2A_0e^Q + \psi_1\} = 0,
\end{aligned}
$$

namely

$$
(-d_2^3A_0^3e^{3Q} + \psi_3)w'' + \Phi_1w' + \Phi_0w = F, \tag{3.32}
$$

where Φ_1 and Φ_0 are entire functions with $\sigma(\Phi_1) \le n$, $\sigma(\Phi_0) \le n$, and

$$\begin{aligned}
-F &= \psi_4 + (A_1 e^P \psi_2 - d_2^2 A_1 A_0^2 e^{(P+2Q)})(\varphi d_2 A_0 A_1 e^{P+Q} \\
&\quad + [d_2 A_0 \varphi' - ((d_2 A_0)' + Q' d_2 A_0 + d_1 A_0)\varphi] e^Q + \psi_1)) \\
&\quad + (A_0 e^Q \psi_2 - d_2^2 A_0^3 e^{3Q})(-\varphi d_2 A_1^2 e^{2P} + \varphi d_2 A_0 e^Q + \psi_1) \\
&= \psi_4 + A_1^2 A_0 \psi_2 \varphi d_2 e^{2P+Q} + A_1 \psi_2 [d_2 A_0 \varphi' - (d_2 A_0)'\varphi - Q' d_2 A_0 \varphi \\
&\quad - d_1 A_0 \varphi] e^{P+Q} + A_1 e^P \psi_1 \psi_2 - d_2^2 A_1 A_0^2 e^{(P+2Q)} \psi_1 - d_2^2 A_1 A_0^2 [d_2 A_0 \varphi' \\
&\quad - (d_2 A_0)'\varphi - Q' d_2 A_0 \varphi - d_1 A_0 \varphi] e^{P+3Q} - d_2^3 A_0^3 A_1^2 \varphi e^{2P+3Q} \\
&\quad - \varphi \psi_2 d_2 A_0 A_1^2 e^{2P+Q} + \varphi d_2^3 A_1^2 A_0^3 e^{2P+3Q} + \psi_2 \varphi d_2 A_0^2 e^{2Q} \\
&\quad - \varphi d_2^3 A_0^4 e^{4Q} + A_0 e^Q \psi_1 \psi_2 - d_2^2 A_0^3 e^{3Q} \psi_1.
\end{aligned} \tag{3.33}$$

Since every ψ_2 in (3.33) is equal to that in (3.28), so the terms $\pm A_1^2 A_0 \psi_2 \varphi d_2 e^{2P+Q}$ and $\pm \varphi d_2^3 A_1^2 A_0^3 e^{2P+3Q}$ are eliminated. By lemma 2.4 (iv), we know that

$$\begin{gathered}
A_1 \psi_2 [d_2 A_0 \varphi' - (d_2 A_0)'\varphi - Q' d_2 A_0 \varphi - d_1 A_0 \varphi] e^{P+Q}, \\
A_1 e^P \psi_1 \psi_2, \quad -d_2^2 A_1 A_0^2 e^{(P+2Q)} \psi_1, \\
-d_2^2 A_1 A_0^2 [d_2 A_0 \varphi' - (d_2 A_0)'\varphi - Q' d_2 A_0 \varphi - d_1 A_0 \varphi] e^{P+3Q}, \\
\psi_2 \varphi d_2 A_0^2 e^{2Q}, \quad A_0 e^Q \psi_1 \psi_2, \quad -d_2^2 A_0^3 e^{3Q} \psi_1
\end{gathered}$$

having all forms of ψ_4, by (3.33), we obtain

$$-F = -\varphi d_2^3 A_0^4 e^{4Q} + \psi_4. \tag{3.34}$$

By lemma 2.4 (iii)-(iv) and $d_2 \not\equiv 0$, $\varphi \not\equiv 0$, $A_0 \not\equiv 0$ and $\sigma(\varphi) < n$, we see that

$$F \not\equiv 0, \quad -d_2^3 A_0^3 e^{3Q} + \psi_3 \not\equiv 0. \tag{3.35}$$

By equation (3.32), lemma 2.5, $\sigma(w) = \infty$ and (3.35), we obtain $\overline{\lambda}(w) = \sigma(w) = \infty$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$, $d_0 \not\equiv 0$. Using a similar reasoning to that above, we get $\overline{\lambda}(w) = \sigma(w) = \infty$. Finally, if $d_2 \equiv 0$, $d_1 \not\equiv 0$, $d_0 \equiv 0$ or $d_2 \equiv 0$, $d_1 \equiv 0$, $d_0 \not\equiv 0$, then for $w = d_j f^{(j)} - \varphi$ ($j = 1$ or 0), we can consider $\frac{w}{d_j} = f^{(j)} - \frac{\varphi}{d_j}$. Since $\overline{\lambda}(w) = \overline{\lambda}(\frac{w}{d_j})$ (d_j being polynomials), using a similar reasoning as in Theorem 1.4 and $\sigma(w) = \infty$, we get $\overline{\lambda}(w) = \sigma(w) = \infty$. $\square$

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