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TWO-SCALE CONVERGENCE OF A MODEL FOR FLOW IN A PARTIALLY FISSURED MEDIUM

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Abstract. The distributed-microstructure model for the flow of single phase fluid in a partially fissured composite medium due to Douglas-Peszyńska-Showalter [12] is extended to a quasi-linear version. This model contains the geometry of the local cells distributed throughout the medium, the flux exchange across their intricate interface with the imbedded fissure system, and the secondary flux resulting from diffusion paths within the matrix. Both the exact but highly singular micro-model and the macro-model are shown to be well-posed, and it is proved that the solution of the micro-model is two-scale convergent to that of the macro-model as the spatial parameter goes to zero. In the linear case, the effective coefficients are obtained by a partial decoupling of the homogenized system.

1. Introduction

A fissured medium is a structure consisting of a porous and permeable matrix which is interlaced on a fine scale by a system of highly permeable fissures. The majority of fluid transport will occur along flow paths through the fissure system, and the relative volume and storage capacity of the porous matrix is much larger than that of the fissure system. When the system of fissures is so well developed that the matrix is broken into individual blocks or cells that are isolated from each other, there is consequently no flow directly from cell to cell, but only an exchange of fluid between each cell and the surrounding fissure system. This is the *totally fissured* case that arises in the modeling of granular materials. In the more general *partially fissured* case of composite media, not only the fissure system but also the matrix of cells may be connected, so there is some flow directly within the cell matrix. The developments below concern this more general model with the additional component of a global flow through the matrix.

An exact microscopic model of flow in a fissured medium treats the regions occupied by the fissure system and by the porous matrix as two Darcy media with different physical parameters. The resulting discontinuities in the parameter values

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across the matrix-fissure interface are severe, and the characteristic width of the fissures is very small in comparison with the size of the matrix blocks. Consequently, any such exact microscopic model, written as a classical interface problem, is numerically and analytically intractable. For the case of a totally fissured medium, these difficulties were overcome by constructing models which describe the flow on two scales, macroscopic and microscopic; see [2, 4, 5, 13, 23]. A macro-model for flow in a totally fissured medium was obtained as the limit of an exact micro-model with properly chosen scaling of permeability in the porous matrix. It is an example of a distributed microstructure model. Derivations of these two–scale models have been based on averaging over the exact geometry of the region (see [2, 3]) or by the construction of a continuous distribution of blocks over the region as in [23] or by assuming some periodic structure for the domain that permits the use of homogenization methods [8, 9]. (See [15] or [16] for a review, and for more information on homogenization see [7, 21].) This model was extended in [12] to the partially fissured case. The novelty in this construction was to represent the flow in the matrix by a parallel construction in the style of [6, 24]. Thus, two flows are introduced in the exact micro-model for the matrix, one is the slow scale flow of [5] which leads to local storage, and the additional one is the global flow within the matrix. A formal asymptotic expansion was used in [12] to derive the corresponding distributed microstructure model. See [10, 11] for another approach to modeling flow in a partially fissured medium and [15] for further discussion and related works. Here we extend the considerations to a quasi-linear version, and we use two-scale convergence to prove the convergence of the micro-model to the corresponding macro-model.

Our plan for this project is as follows. In the remainder of this section, we briefly recall the partial differential equations that describe the flow through a homogeneous medium in order to introduce some notation. Then we describe in turn various function spaces of L^p or of Sobolev type, the two-scale convergence procedure, and basic results for weak and strong formulations of the Cauchy problem in Banach space. In Section 2 we describe a nonlinear version of the *micro-model* from [12] for flow through a partially fissured medium and show that this system leads to a well-posed initial-boundary-value problem. In Section 3 we show that this micro-model has a two-scale limit as the parameter $\varepsilon \to 0$, and this limit satisfies a variational identity. The point of Section 4 is to establish that this limit satisfies additional properties which collectively comprise the homogenized *macro-model*. These results on the well-posedness of the macro-model are sumarized and completed in Section 5. There we relate the weak and strong formulations of the macro-model problem to the corresponding realizations as a Cauchy problem for a nonlinear evolution equation in Banach space. We also develop a simpler and useful reduced system to describe this limit, and we show that it agrees with the usual homogenized model from [12] in the linear case.

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We begin with a review of notation in the context of the flow of a single phase slightly compressible liquid through a *homogeneous medium*. Thus the density $\rho(x,t)$ and pressure $p(x,t)$ are related by the *state equation* $\rho = \rho_0 e^{\kappa p}$, and the equation for conservation of mass is given by

$$c(x)\frac{\partial\rho}{\partial t} - \nabla\cdot\sum_{j=1}^{N}(\rho k_j(\rho\frac{\partial p}{\partial x_j})\frac{\partial p}{\partial x_j}) = f(x,t).$$

The state equation yields the relationship $\frac{\partial\rho}{\partial x_j} = \kappa\rho\frac{\partial p}{\partial x_j}$, so the conservation equation can be written

$$c(x)\frac{\partial\rho}{\partial t} - \nabla\cdot\sum_{j=1}^{N}(k_j(\frac{1}{\kappa}\frac{\partial\rho}{\partial x_j})\frac{1}{\kappa}\frac{\partial\rho}{\partial x_j}) = f(x,t).$$

Finally, by introducing the *flow potential* $u(w) = \int_0^w \rho\,dp$, we have

$$c(x)\frac{\partial u}{\partial t} - \nabla\cdot\mu(\nabla u) = f(x,t),$$

where the *flux* is given componentwise by the negative of the function $\mu(\nabla u) \equiv \frac{1}{\kappa}\sum_{j=1}^{N} k_j(\frac{\partial u}{\partial x_j})\frac{\partial u}{\partial x_j}$. We shall assume below that this is a monotone function of the gradient. The classical Forchheimer-type corrections to the Darcy law for fluids lead to such functions with growth of order $p = \frac{3}{2}$.

Various spaces of functions on a bounded (for simplicity) domain Ω in $\mathbb{R}^N$ with smooth boundary $\partial\Omega \equiv \Gamma$ will be used. For each $1 < p < \infty$, $L^p(\Omega)$ is the usual *Lebesgue space* of (equivalence classes of) p-th power summable functions, and $W^{1,p}(\Omega)$ is the *Sobolev space* of functions which belong to $L^p(\Omega)$ together with their first order derivatives. The *trace* map $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ is the restriction to boundary values.

Let $Y = [0,1]^N$ denote the unit cube. Corresponding spaces of Y *-periodic* functions will be denoted by a subscript #. For example, $C_\#(Y)$ is the Banach space of functions which are defined on all of $\mathbb{R}^N$ and which are continuous and Y-periodic. Similarly, $L^p_\#(Y)$ is the Banach space of functions in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ which are Y-periodic. For this space we take the norm of $L^p(Y)$ and note that $L^p_\#(Y)$ is equivalent to the space of Y-periodic extensions to $\mathbb{R}^N$ of the functions in $L^p(Y)$. Similarly, we define $W^{1,p}_\#(Y)$ to be the Banach space of Y-periodic extensions to $\mathbb{R}^N$ of those functions in $W^{1,p}(Y)$ for which the trace (or boundary values) agree on opposite sides of the boundary, ∂Y, and its norm is the usual norm of $W^{1,p}(Y)$. The linear space $C^\infty_\#(Y) \equiv C_\#(Y) \cap C^\infty(\mathbb{R}^N)$ is dense in both of $L^p_\#(Y)$ and $W^{1,p}_\#(Y)$.

Various spaces of *vector-valued* functions will arise in the developments below. If $\mathbb{B}$ is a Banach space and X is a topological space, then $C(X;\mathbb{B})$ denotes the space of continuous $\mathbb{B}$-valued functions on X with the corresponding supremum norm, and for any measure space Ω we let $L^p(\Omega;\mathbb{B})$ denote the space of p-th power norm-summable (equivalence classes of) functions on Ω with values in $\mathbb{B}$. When $X = [0,T]$ or $\Omega = (0,T)$ is the indicated time interval, we denote the corresponding *evolution spaces* by $C(0,T;\mathbb{B})$ and $L^p(0,T;\mathbb{B})$, respectively.

Next we quote some definitions and results on *two-scale convergence* from [1] slightly modified to allow for homogenization with a parameter (which we denote by t). These changes do not affect the proofs from [1] in any essential way.

Definition 1.1. A function, $\psi(x,t,y)\in L^{p'}(\Omega\times(0,T),C_\#(Y))$, which is Y-periodic in y and which satisfies

$$\lim_{\varepsilon\to 0}\int_{\Omega\times(0,T)}\psi\left(x,t,\frac{x}{\varepsilon}\right)^{p'}\,dx\,dt=\int_{\Omega\times(0,T)}\int_Y\psi(x,t,y)^{p'}\,dy\,dx\,dt,$$

is called an *admissible* test function. Here p' is the conjugate of p, that is, $\frac{1}{p}+\frac{1}{p'}=1$.

Definition 1.2. A sequence u^ε in $L^p((0,T)\times\Omega)$ *two-scale* converges to $u_0(x,t,y)\in L^p((0,T)\times\Omega\times Y)$ if for any admissible test function $\psi(x,t,y)$,

$$\lim_{\varepsilon\to 0}\int_\Omega\int_{(0,T)}u^\varepsilon(x,t)\psi\left(x,t,\frac{x}{\varepsilon}\right)\,dt\,dx=\int_\Omega\int_{(0,T)}\int_Y u_0(x,t,y)\psi(x,t,y)\,dy\,dt\,dx. \tag{1.1}$$

Theorem 1.1. If u^ε is a bounded sequence in $L^p((0,T)\times\Omega)$, then there exists a function $u_0(x,t,y)$ in $L^p((0,T)\times\Omega\times Y)$ and a subsequence of u^ε which two-scale converges to u_0. Moreover, the subsequence u^ε converges weakly in $L^p((0,T)\times\Omega)$ to $u(x,t)=\int_Y u_0(x,t,y)\,dy$.

When the sequence, u^ε, is $W^{1,p}$-bounded, we get more information.

Theorem 1.2. Let u^ε be a bounded sequence in $L^p(0,T;W^{1,p}(\Omega))$ that converges weakly to u in $L^p((0,T);W^{1,p}(\Omega))$. Then u^ε two-scale converges to u, and there is a function $U(x,t,y)$ in $L^p((0,T)\times\Omega;W^{1,p}_\#(Y)/\mathbb{R})$ such that, up to a subsequence, $\nabla_x u^\varepsilon$ two-scale converges to $\nabla_x u(x,t)+\nabla_y U(x,t,y)$.

Theorem 1.3. Let u^ε and $\varepsilon\nabla_x u^\varepsilon$ be two bounded sequences in $L^p((0,T)\times\Omega))$. Then there exists a function $U\,(x,t,y)$ in $L^p((0,T)\times\Omega;W^{1,p}_\#(Y)/\mathbb{R})$ such that, up to a subsequence, u^ε and $\varepsilon\nabla_x u^\varepsilon$ two-scale converge to $U\,(x,t,y)$ and $\nabla_y U(x,t,y)$, respectively.

Finally, we formulate the *Cauchy problem* or *initial-value problem* for an evolution equation in Banach space in a form that will be convenient for our applications below. Let V be a reflexive Banach space with dual V'; we shall set $\mathcal{V}=L^p(0,T;V)$ for $1<p<\infty$, and its dual is $\mathcal{V}'\cong L^{p'}(0,T;V')$. Let V be dense and continuously embedded in a Hilbert space H, so that $V\hookrightarrow H$ and we can identify $H'\hookrightarrow V'$ by restriction.

Proposition 1.4. The Banach space $W_p(0,T)\equiv\{u\in\mathcal{V}:u'\in\mathcal{V}'\}$ is contained in $C([0,T],H)$. Moreover, if $u\in W_p(0,T)$ then $|u(\cdot)|_H^2$ is absolutely continuous on $[0,T]$,

$$\frac{d}{dt}|u(t)|_H^2=2u'\big(t\big)\big(u(t)\big)\quad\text{a.e.}\quad t\in[0,T]\ ,$$

and there is a constant C for which

$$\|u\|_{C([0,T],H)}\le C\|u\|_{W_p(0,T)}\ ,\quad u\in W_p\ .$$

Corollary 1.5. If $u,v\in W_p(0,T)$ then $(u(\cdot),v(\cdot))_H$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\big(u(t),v(t)\big)_H=u'\big(t\big)\big(v(t)\big)+v'\big(t\big)\big(u(t)\big)\ ,\qquad\text{a.e.}\quad t\in[0,T]\ .$$

Suppose we are given a (not necessarily linear) function $\mathcal{A} : V \to V'$ and $u_0 \in H$, $f \in \mathcal{V}'$. Then consider the *Cauchy Problem* to find

$$u \in \mathcal{V} : u'(t) + \mathcal{A}\big(u(t)\big) = f(t) \quad \text{in } \mathcal{V}' , \qquad u(0) = u_0 \text{ in } H . \tag{1.2}$$

It is understood that $u' \in \mathcal{V}'$ in (1.2), so it follows from Proposition 1.4 that u is continuous into H and the condition on $u(0)$ is meaningful. If $\mathcal{A}$ is known to map $\mathcal{V}$ into $\mathcal{V}'$, i.e., the realization of $\mathcal{A} : V \to V'$ as an operator on $\mathcal{V}$ has values in $\mathcal{V}'$, then (1.2) is equivalent to the *variational formulation*

$$\begin{gathered} u \in \mathcal{V} : \text{ for every } v \in \mathcal{V} \text{ with } v' \in \mathcal{V}' \text{ and } v(T) = 0 \\ - \int_0^T \big(u(t), v'(t)\big)_H \, dt + \int_0^T \mathcal{A}\big(u(t)\big) v(t) \, dt = \int_0^T f(t) v(t) \, dt + \big(u_0, v(0)\big)_H . \end{gathered} \tag{1.3}$$

The equivalence of the strong and variational formulations of the Cauchy problem will be used freely in all of our applications below. See Chapter III of [22] for the above and related results on the Cauchy problem.

2. The Micro-Model

We consider a structure consisting of fissures and matrix periodically distributed in a domain Ω in $\mathbb{R}^N$ with period εY, where $\varepsilon > 0$. Let the unit cube $Y = [0,1]^N$ be given in complementary parts, Y_1 and Y_2, which represent the local structure of the fissure and matrix, respectively. Denote by $\chi_j(y)$ the characteristic function of Y_j for $j = 1, 2$, extended Y-periodically to all of $\mathbb{R}^N$. Thus, $\chi_1(y) + \chi_2(y) = 1$. We shall assume that both of the sets $\{y \in \mathbb{R}^N : \chi_j(y) = 1\}$, $j = 1, 2$ are smooth. With the assumptions that we make on the coefficients below to obtain coercivity estimates, it is not necessary to assume further that these sets are also connected. The domain Ω is thus divided into the two subdomains, Ω_1^ε and Ω_2^ε, representing the *fissures* and *matrix* respectively, and given by

$$\Omega_j^\varepsilon \equiv \{x \in \Omega : \chi_j\left(\frac{x}{\varepsilon}\right) = 1\}, \quad j = 1, 2.$$

Let $\Gamma_{1,2}^\varepsilon \equiv \partial\Omega_1^\varepsilon \cap \partial\Omega_2^\varepsilon \cap \Omega$ be that part of the interface of Ω_1^ε with Ω_2^ε that is interior to Ω, and let $\Gamma_{1,2} \equiv \partial Y_1 \cap \partial Y_2 \cap Y$ be the corresponding interface in the local cell Y. Likewise, let $\Gamma_{2,2} \equiv \bar{Y}_2 \cap \partial Y$ and denote by $\Gamma_{2,2}^\varepsilon$ its periodic extension which forms the interface between those parts of the matrix Ω_2^ε which lie within neighboring εY-cells.

The flow potential of the fluid in the fissures Ω_1^ε is denoted by $u_1^\varepsilon(x,t)$ and the corresponding flux there is given by $-\mu_1\left(\frac{x}{\varepsilon}, \nabla u_1^\varepsilon\right)$. The flow potential in the matrix Ω_2^ε is represented as the sum of two parts, one component $u_2^\varepsilon(x,t)$ with flux $-\mu_2\left(\frac{x}{\varepsilon}, \nabla u_2^\varepsilon\right)$ which accounts for the global diffusion through the pore system of the matrix , and the second component $u_3^\varepsilon(x,t)$ with flux $-\varepsilon\mu_3\left(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon\right)$ and corresponding very high frequency spatial variations which lead to local storage in the matrix. The *total flow potential* in the matrix Ω_2^ε is then $\alpha u_2^\varepsilon + \beta u_3^\varepsilon$. (Here $\alpha + \beta = 1$ with $\alpha \geq 0$ and $\beta > 0$.)

In the following, we shall set $Y_3 = Y_2$ and likewise set $\chi_3 = \chi_2$ in order to simplify notation. For $j = 1, 2, 3$, let $\mu_j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and assume that for every $\vec{\xi} \in \mathbb{R}^N$, $\mu_j\left(\cdot, \vec{\xi}\right)$ is measurable and Y-periodic and for a.e. $y \in Y$, $\mu_j(y, \cdot)$ is continuous. In addition, assume that we have positive constants k, C, c_0 and

$1 < p < \infty$ such that for every $\vec{\xi}$, $\vec{\eta} \in \mathbb{R}^N$ and a.e. $y \in Y$

$$\left|\mu_j\left(y, \vec{\xi}\right)\right| \leq C\left|\vec{\xi}\right|^{p-1} + k \tag{2.1}$$

$$\left(\mu_j(y, \vec{\xi}) - \mu_j(y, \vec{\eta})\right) \cdot \left(\vec{\xi} - \vec{\eta}\right) \geq 0 \tag{2.2}$$

$$\mu_j\left(y, \vec{\xi}\right) \cdot \vec{\xi} \geq c_0\left|\vec{\xi}\right|^p - k. \tag{2.3}$$

Let $c_j \in C_\#(Y)$ be given such that

$$0 < c_0 \leq c_j(y) \leq C, \quad 1 \leq j \leq 3. \tag{2.4}$$

Since these are given on $\mathbb{R}^N$, we can define for $j = 1, 2, 3$ the corresponding scaled coefficients at $x \in \Omega_j^\varepsilon$, $\vec{\xi} \in \mathbb{R}^N$ by

$$c_j^\varepsilon(x) \equiv c_j\left(\frac{x}{\varepsilon}\right), \ \mu_j^\varepsilon\left(x, \vec{\xi}\right) \equiv \mu_j\left(\frac{x}{\varepsilon}, \vec{\xi}\right).$$

The exact micro-model introduced in [12] for diffusion in a partially fissured medium is given by the system

$$\frac{\partial}{\partial t}\left(c_1^\varepsilon(x)\, u_1^\varepsilon(x,t)\right) - \vec{\nabla} \cdot \mu_1^\varepsilon\left(x, \vec{\nabla} u_1^\varepsilon(x,t)\right) = 0 \qquad \text{in } \Omega_1^\varepsilon \tag{2.5}$$

$$\frac{\partial}{\partial t}\left(c_2^\varepsilon(x)\, u_2^\varepsilon(x,t)\right) - \vec{\nabla} \cdot \mu_2^\varepsilon\left(x, \vec{\nabla} u_2^\varepsilon(x,t)\right) = 0 \qquad \text{in } \Omega_2^\varepsilon \tag{2.6}$$

$$\frac{\partial}{\partial t}\left(c_3^\varepsilon(x)\, u_3^\varepsilon(x,t)\right) - \varepsilon\vec{\nabla} \cdot \mu_3^\varepsilon\left(x, \varepsilon\vec{\nabla} u_3^\varepsilon(x,t)\right) = 0 \qquad \text{in } \Omega_2^\varepsilon \tag{2.7}$$

$$u_1^\varepsilon = \alpha u_2^\varepsilon + \beta u_3^\varepsilon \qquad \text{on } \Gamma_{1,2}^\varepsilon \tag{2.8}$$

$$\alpha\mu_1^\varepsilon\left(x, \vec{\nabla} u_1^\varepsilon(x,t)\right) \cdot \vec{\nu}_1 = \mu_2^\varepsilon\left(x, \vec{\nabla} u_2^\varepsilon(x,t)\right) \cdot \vec{\nu}_1 \qquad \text{on } \Gamma_{1,2}^\varepsilon \tag{2.9}$$

$$\beta\mu_1^\varepsilon\left(x, \vec{\nabla} u_1^\varepsilon(x,t)\right) \cdot \vec{\nu}_1 = \varepsilon\mu_3^\varepsilon\left(x, \varepsilon\vec{\nabla} u_3^\varepsilon(x,t)\right) \cdot \vec{\nu}_1 \qquad \text{on } \Gamma_{1,2}^\varepsilon \tag{2.10}$$

where $\vec{\nu}_1$ is the unit outward normal on $\partial\Omega_1^\varepsilon$. We shall similarly let $\vec{\nu}_2$ denote the unit outward normal on $\partial\Omega_2^\varepsilon$, so $\vec{\nu}_1 = -\vec{\nu}_2$ on $\Gamma_{1,2}^\varepsilon$. The first equation is the conservation of mass in the fissure system. In the matrix, Ω_2^ε, we have two components of the flow potential. The first is the usual flow through the matrix, and the second component is scaled by ε^p to represent the very high frequency variations in flow that result from the relatively very low permeability of the matrix. Each of these is assumed to satisfy a corresponding conservation equation. The total flow potential in the matrix is given by the convex combination $\alpha u_2^\varepsilon + \beta u_3^\varepsilon$ where $\alpha \geq 0$, $\beta > 0$ denote the corresponding fractions of each, so $\alpha + \beta = 1$. Thus, the first interface condition is the continuity of flow potential, and the remaining conditions determine the corresponding partition of flux across the interface. Since the boundary conditions will play no essential role in the development, we shall assume homogeneous Neumann boundary conditions

$$\mu_1^\varepsilon\left(x, \vec{\nabla} u_1^\varepsilon(x,t)\right) \cdot \vec{\nu}_1 = 0 \qquad \text{on } \partial\Omega_1^\varepsilon \cap \partial\Omega\,, \tag{2.11}$$

$$\mu_2^\varepsilon\left(x, \vec{\nabla} u_2^\varepsilon(x,t)\right) \cdot \vec{\nu}_2 = 0 \qquad \text{and} \tag{2.12}$$

$$\mu_3^\varepsilon\left(x, \vec{\nabla} u_3^\varepsilon(x,t)\right) \cdot \vec{\nu}_2 = 0 \qquad \text{on } \partial\Omega_2^\varepsilon \cap \partial\Omega\,. \tag{2.13}$$

The system is completed by the *initial conditions*

$$u_1^\varepsilon(\cdot,0) = u_1^0(\cdot), \quad u_2^\varepsilon(\cdot,0) = u_2^0(\cdot), \quad u_3^\varepsilon(\cdot,0) = u_3^0(\cdot) \tag{2.14}$$

in H^ε.

Next we develop the *variational formulation* for the initial-boundary-value problem (2.5)-(2.14) and show that the resulting *Cauchy problem* is well posed in the appropriate function space. Define the *state space*

$$H^\varepsilon \equiv L^2(\Omega_1^\varepsilon) \times L^2(\Omega_2^\varepsilon) \times L^2(\Omega_2^\varepsilon),$$

a Hilbert space with the inner product

$$([u_1,u_2,u_3],[\varphi_1,\varphi_2,\varphi_3])_{H^\varepsilon} \equiv \int_{\Omega_1^\varepsilon} c_1^\varepsilon(x)\, u_1(x)\, \varphi_1(x)\, dx + \int_{\Omega_2^\varepsilon} \left[c_2^\varepsilon(x)\, u_2(x)\, \varphi_2(x) + c_3^\varepsilon(x)\, u_3(x)\, \varphi_3(x)\right] dx\,.$$

Let $\gamma_j^\varepsilon : W^{1,p}(\Omega_j^\varepsilon) \to L^p\left(\partial\Omega_j^\varepsilon\right)$ be the usual trace maps on the respective spaces for $j = 1,2,\ \varepsilon > 0$, and define the *energy space*

$$V^\varepsilon \equiv H^\varepsilon \cap \{[u_1,u_2,u_3] \in W^{1,p}(\Omega_1^\varepsilon) \times W^{1,p}(\Omega_2^\varepsilon) \times W^{1,p}(\Omega_2^\varepsilon) : \gamma_1^\varepsilon u_1 = \alpha\gamma_2^\varepsilon u_2 + \beta\gamma_2^\varepsilon u_3 \text{ on } \Gamma_{1,2}^\varepsilon\}.$$

Note that V^ε is a Banach space when equipped with the norm

$$\|[u_1,u_2,u_3]\|_{V^\varepsilon} \equiv \|\chi_1^\varepsilon u_1\|_{L^2(\Omega)} + \|\chi_2^\varepsilon u_2\|_{L^2(\Omega)} + \|\chi_2^\varepsilon u_3\|_{L^2(\Omega)} + \left\|\chi_1^\varepsilon \vec\nabla u_1\right\|_{L^p(\Omega)} + \left\|\chi_2^\varepsilon \vec\nabla u_2\right\|_{L^p(\Omega)} + \left\|\chi_2^\varepsilon \vec\nabla u_3\right\|_{L^p(\Omega)}.$$

If we multiply each of (2.5), (2.6), (2.7) by the corresponding $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$ for which $[\varphi_1,\varphi_2,\varphi_3] \in V^\varepsilon$, integrate over the corresponding domains, and make use of (2.9)-(2.13), we find that the triple of functions $\vec u^\varepsilon(\cdot) \equiv [u_1^\varepsilon(\cdot), u_2^\varepsilon(\cdot), u_3^\varepsilon(\cdot)]$ in $L^p(0,T;V^\varepsilon)$ satisfies

$$\left(\frac{\partial}{\partial t}[u_1^\varepsilon(t),u_2^\varepsilon(t),u_3^\varepsilon(t)],[\varphi_1,\varphi_2,\varphi_3]\right)_{H^\varepsilon} + \mathcal{A}^\varepsilon([u_1^\varepsilon(t),u_2^\varepsilon(t),u_3^\varepsilon(t)])([\varphi_1,\varphi_2,\varphi_3]) = 0$$

for all $[\varphi_1,\varphi_2,\varphi_3] \in V^\varepsilon$, where we define the operator $\mathcal{A}^\varepsilon : V^\varepsilon \to (V^\varepsilon)'$ by

$$\mathcal{A}^\varepsilon([u_1,u_2,u_3])([\varphi_1,\varphi_2,\varphi_3]) \equiv \int_{\Omega_1^\varepsilon} \mu_1^\varepsilon\left(x, \vec\nabla u_1(x)\right) \cdot \vec\nabla\varphi_1(x)\, dx + \int_{\Omega_2^\varepsilon} \left\{\mu_2^\varepsilon\left(x, \vec\nabla u_2(x)\right) \cdot \vec\nabla\varphi_2(x) + \mu_3^\varepsilon\left(x, \varepsilon\vec\nabla u_3(x)\right) \cdot \varepsilon\vec\nabla\varphi_3(x)\right\} dx$$

for $[u_1,u_2,u_3]$, $[\varphi_1,\varphi_2,\varphi_3] \in V^\varepsilon$. Thus, the variational form of this problem is to find, for each $\varepsilon > 0$ and $[u_1^0,u_2^0,u_3^0] \in H^\varepsilon$ a triple of functions $\vec u^\varepsilon(\cdot) \equiv [u_1^\varepsilon(\cdot), u_2^\varepsilon(\cdot), u_3^\varepsilon(\cdot)]$ in $L^p(0,T;V^\varepsilon)$ such that

$$\frac{d}{dt}\vec u^\varepsilon(\cdot) + \mathcal{A}^\varepsilon \vec u^\varepsilon(\cdot) = 0 \text{ in } L^{p'}(0,T;(V^\varepsilon)') \tag{2.15}$$

and

$$\vec u^\varepsilon(0) = \vec u^0 \text{ in } H^\varepsilon. \tag{2.16}$$

Conversely, a solution of (2.15) will satisfy (2.5)-(2.8), and if that solution is sufficiently smooth, then it will also satisfy (2.5)-(2.13).

The assumptions (2.1), (2.2) and (2.3) guarantee that $\mathcal{A}^{\varepsilon}$ satisfies the hypotheses of [22, Proposition III.4.1], so there is a unique solution $\vec{u}^{\varepsilon} \equiv [u_1^{\varepsilon}, u_2^{\varepsilon}, u_3^{\varepsilon}]$ in $L^p(0,T;V^{\varepsilon})$ of (2.15) and (2.16). Note that since $\frac{d}{dt}\vec{u}^{\varepsilon} \in L^{p'}\left(0,T;(V^{\varepsilon})'\right)$ that $\vec{u}^{\varepsilon} \in C([0,T];H^{\varepsilon})$ and so (2.16) is meaningful by Proposition 1.4 above.

3. Two-scale limits

We introduce the scaled characteristic functions

$$\chi_j^{\varepsilon}(x) \equiv \chi_j\left(\frac{x}{\varepsilon}\right), \quad j = 1, 2.$$

These will be used to denote the *zero-extension* of various functions. In particular, for any function w defined on Ω_j^{ε} the product $\chi_j^{\varepsilon} w$ is understood to be defined on all of Ω as the zero extension of w. Similarly, if w is given on Y_j, then $\chi_j w$ is the corresponding zero extension to all of Y.

Our starting point is a preliminary convergence result for the solutions described above.

Lemma 3.1. There exist a pair of functions u_j in $L^p\left(0,T;W^{1,p}(\Omega)\right)$, $j = 1, 2$, and triples of functions U_j in $L^p((0,T)\times\Omega;W_{\#}^{1,p}(Y)/\mathbb{R})$, g_j in $L^{p'}((0,T)\times\Omega\times Y^N))$, $u_j^* \in L^2(\Omega\times Y)$ for $j = 1, 2, 3$, and a subsequence taken from the sequence of solutions of (2.15)-(2.16) above, hereafter denoted by $\vec{u}^{\varepsilon} = [u_1^{\varepsilon}, u_2^{\varepsilon}, u_3^{\varepsilon}]$, which two-scale converges as follows:

$$\chi_1^{\varepsilon} u_1^{\varepsilon} \stackrel{2}{\to} \chi_1(y)\, u_1(x,t) \tag{3.1}$$

$$\chi_1^{\varepsilon} \vec{\nabla} u_1^{\varepsilon} \stackrel{2}{\to} \chi_1(y)\left[\vec{\nabla} u_1(x,t) + \vec{\nabla}_y U_1(x,y,t)\right] \tag{3.2}$$

$$\chi_2^{\varepsilon} u_2^{\varepsilon} \stackrel{2}{\to} \chi_2(y)\, u_2(x,t) \tag{3.3}$$

$$\chi_2^{\varepsilon} \vec{\nabla} u_2^{\varepsilon} \stackrel{2}{\to} \chi_2(y)\left[\vec{\nabla} u_2(x,t) + \vec{\nabla}_y U_2(x,y,t)\right] \tag{3.4}$$

$$\chi_2^{\varepsilon} u_3^{\varepsilon} \stackrel{2}{\to} \chi_2(y)\, U_3(x,y,t) \tag{3.5}$$

$$\varepsilon \chi_2^{\varepsilon} \vec{\nabla} u_3^{\varepsilon} \stackrel{2}{\to} \chi_2(y)\, \vec{\nabla}_y U_3(x,y,t) \tag{3.6}$$

$$\chi_1^{\varepsilon} \mu_1^{\varepsilon}\left(\vec{\nabla} u_1^{\varepsilon}\right) \stackrel{2}{\to} \chi_1(y)\, \vec{g}_1(x,y,t) \tag{3.7}$$

$$\chi_2^{\varepsilon} \mu_2^{\varepsilon}\left(\vec{\nabla} u_2^{\varepsilon}\right) \stackrel{2}{\to} \chi_2(y)\, \vec{g}_2(x,y,t) \tag{3.8}$$

$$\chi_2^{\varepsilon} \mu_3^{\varepsilon}\left(\varepsilon\vec{\nabla} u_3^{\varepsilon}\right) \stackrel{2}{\to} \chi_2(y)\, \vec{g}_3(x,y,t) \tag{3.9}$$

$$\chi_j^{\varepsilon} u_j^{\varepsilon}(\cdot, T) \stackrel{2}{\to} \chi_j(y) u_j^*(x), \quad j = 1,\ 2,\ 3. \tag{3.10}$$

Proof. Using Proposition 1.4 and (2.15) we can write

$$\frac{1}{2}\frac{d}{dt}\left([u_1^{\varepsilon}, u_2^{\varepsilon}, u_3^{\varepsilon}], [u_1^{\varepsilon}, u_2^{\varepsilon}, u_3^{\varepsilon}]\right)_{H^{\varepsilon}} + \ \mathcal{A}^{\varepsilon}\left([u_1^{\varepsilon}, u_2^{\varepsilon}, u_3^{\varepsilon}]\right)\left([u_1^{\varepsilon}, u_2^{\varepsilon}, u_3^{\varepsilon}]\right) = 0.$$

Integrating in t gives

$$\frac{1}{2}\left\|\vec{u}^{\varepsilon}(t)\right\|_{H^{\varepsilon}}^2 - \frac{1}{2}\left\|\vec{u}^{\varepsilon}(0)\right\|_{H^{\varepsilon}}^2 + \int_0^t \mathcal{A}^{\varepsilon}\left([u_1^{\varepsilon}, u_2^{\varepsilon}, u_3^{\varepsilon}]\right)\left([u_1^{\varepsilon}, u_2^{\varepsilon}, u_3^{\varepsilon}]\right) dt = 0 \tag{3.11}$$

which, with the assumption (2.3) yields

(3.12)
$$\frac{1}{2}\left\|\vec{u}^{\varepsilon}(t)\right\|_{H^{\varepsilon}}^{2}+c_0\int_0^t\left(\left\|\chi_1^{\varepsilon}\vec{\nabla}u_1^{\varepsilon}\right\|_{L^p(\Omega)}^{p}+\left\|\chi_2^{\varepsilon}\vec{\nabla}u_2^{\varepsilon}\right\|_{L^p(\Omega)}^{p}+\left\|\varepsilon\chi_2^{\varepsilon}\vec{\nabla}u_3^{\varepsilon}\right\|_{L^p(\Omega)}^{p}\right)dt$$
$$\leq\frac{1}{2}\left\|\left[\chi_1u_1^0,\chi_2u_2^0,\chi_2u_3^0\right]\right\|_{H^{\varepsilon}}^{2}+t\left|k\right|,\quad 0\leq t\leq T.$$

Thus, $\vec{u}^{\varepsilon}(\cdot)$ is bounded in $L^{\infty}(0,T;H^{\varepsilon})$, and so $\chi_1^{\varepsilon}u_1^{\varepsilon}$, $\chi_2^{\varepsilon}u_2^{\varepsilon}$, and $\chi_2^{\varepsilon}u_3^{\varepsilon}$ are bounded in $L^{\infty}(0,T;L^2(\Omega))$. Also, $\chi_1^{\varepsilon}\vec{\nabla}u_1^{\varepsilon}$, $\chi_2^{\varepsilon}\vec{\nabla}u_2^{\varepsilon}$ and $\varepsilon\chi_2^{\varepsilon}\vec{\nabla}u_3^{\varepsilon}$ are bounded in $L^p(0,T;L^p(\Omega)^N)$. We obtain (3.1) through (3.4) exactly as in [1, Theorem 2.9] by Theorem 1.2. Statements (3.5) and (3.6) follow from Theorem 1.3. Finally, from (2.1) and the bounds already established, we have that $\chi_j^{\varepsilon}\mu_j^{\varepsilon}\left(x,\vec{\nabla}u_j^{\varepsilon}(x,t)\right)$ (for $j=1,2$) and $\chi_2^{\varepsilon}\mu_3^{\varepsilon}\left(x,\varepsilon\vec{\nabla}u_3^{\varepsilon}(x,t)\right)$ are bounded in $L^{p'}\left([0,T],L^{p'}(\Omega)\right)$ due to (2.3), (3.12) and

$$\begin{aligned}\int_0^T\int_{\Omega}\chi_j^{\varepsilon}\left|\mu_j\left(\frac{x}{\varepsilon},\vec{\xi}(x)\right)\right|^{p'}dxdt &\leq \int_0^T\int_{\Omega}\chi_j^{\varepsilon}\left|\vec{\xi}(x)\right|^{(p-1)p'}dxdt\\ &= \int_0^T\int_{\Omega}\chi_j^{\varepsilon}\left|\vec{\xi}(x)\right|^{p}dxdt.\end{aligned}$$

Thus $\chi_j^{\varepsilon}\mu_j^{\varepsilon}\left(x,\vec{\nabla}u_j^{\varepsilon}(x,t)\right)$ and $\chi_2^{\varepsilon}\mu_3^{\varepsilon}\left(x,\varepsilon\vec{\nabla}u_3^{\varepsilon}(x,t)\right)$ converge as stated. □

Define the *flow potential* $u^{\varepsilon}\equiv\chi_1^{\varepsilon}u_1^{\varepsilon}+\chi_2^{\varepsilon}\left(\alpha u_2^{\varepsilon}+\beta u_3^{\varepsilon}\right)\in L^p\left(0,T;W^{1,p}(\Omega)\right)$ for each $\varepsilon>0$, and note that on $\Gamma_{1,2}^{\varepsilon}$

$$\gamma_1^{\varepsilon}u^{\varepsilon}=\gamma_1^{\varepsilon}u_1^{\varepsilon}=\alpha\gamma_2^{\varepsilon}u_2^{\varepsilon}+\beta\gamma_2^{\varepsilon}u_3^{\varepsilon}=\gamma_2^{\varepsilon}u^{\varepsilon}.$$

Thus

$$\varepsilon\vec{\nabla}u^{\varepsilon}=\varepsilon\chi_1^{\varepsilon}\vec{\nabla}u_1^{\varepsilon}+\chi_2^{\varepsilon}\left(\alpha\varepsilon\vec{\nabla}u_2^{\varepsilon}+\beta\varepsilon\vec{\nabla}u_3^{\varepsilon}\right)\in L^p\left([0,T]\times\Omega\right)$$

and from Lemma (3.1) we see that

$$u^{\varepsilon}\xrightarrow{2}\chi_1(y)\,u_1(x)+\chi_2(y)\left(\alpha u_2(x,t)+\beta U_3(x,y,t)\right)$$

and

$$\varepsilon\vec{\nabla}u^{\varepsilon}\xrightarrow{2}\chi_2(y)\,\beta\vec{\nabla}_yU_3(x,y,t)\,.$$

Now let $\vec{\varphi}\in C_0^{\infty}\left(\Omega,C_{\#}^{\infty}\left(Y^N\right)\right)$ and note that

$$\int_{\Omega}\varepsilon\vec{\nabla}u^{\varepsilon}(x,t)\cdot\vec{\varphi}\left(x,\frac{x}{\varepsilon}\right)dx=$$
$$-\int_{\Omega}u^{\varepsilon}(x,t)\left[\varepsilon\vec{\nabla}\cdot\vec{\varphi}\left(x,\frac{x}{\varepsilon}\right)+\vec{\nabla}_y\cdot\vec{\varphi}\left(x,\frac{x}{\varepsilon}\right)\right]dx.$$

Taking two-scale limits on both sides yields

(3.13)
$$\int_{\Omega}\int_Y\beta\chi_2(y)\,\vec{\nabla}_yU_3(x,y,t)\cdot\vec{\varphi}(x,y)\,dxdy=$$
$$-\int_{\Omega}\int_Y\left(\chi_1(y)\,u_1(x,t)+\chi_2(y)\left(\alpha u_2(x,t)+\beta U_3(x,y,t)\right)\right)\vec{\nabla}_y\cdot\vec{\varphi}(x,y)\,dxdy.$$

The divergence theorem shows that the left hand side of (3.13) is simply

$$\int_\Omega \int_{Y_2} \beta \vec{\nabla}_y U_3(x,y,t) \cdot \vec{\varphi}(x,y)\,dxdy = -\int_\Omega \int_{Y_2} \beta U_3(x,y,t)\,\vec{\nabla}_y \cdot \vec{\varphi}(x,y)\,dxdy + \int_\Omega \int_{\partial Y_2} \beta U_3(x,s,t)\,\vec{\varphi}(x,s)\cdot \vec{\nu}_2 dxds$$

while the right hand side of (3.13) can be written

$$-\int_\Omega \int_{Y_1} u_1(x,t)\,\vec{\nabla}_y \cdot \vec{\varphi}(x,y)\,dxdy - \int_\Omega \int_{Y_2} (\alpha u_2(x,t) + \beta U_3(x,y,t))\,\vec{\nabla}_y \cdot \vec{\varphi}(x,y)\,dxdy.$$

We see that (3.13) yields

$$\begin{aligned}
&\int_\Omega \int_{\partial Y_2} \beta U_3(x,s,t)\,\vec{\varphi}(x,s)\cdot \vec{\nu}_2 dxds = \\
&\quad -\int_\Omega \int_{Y_1} u_1(x,t)\,\vec{\nabla}_y \cdot \vec{\varphi}(x,y)\,dxdy - \int_\Omega \int_{Y_2} \alpha u_2(x,t)\,\vec{\nabla}_y \cdot \vec{\varphi}(x,y)\,dxdy \\
&\quad = -\int_\Omega \int_{\partial Y_1} u_1(x,t)\,\vec{\varphi}(x,s)\cdot \vec{\nu}_1 dxds - \int_\Omega \int_{\partial Y_2} \alpha u_2(x,t)\,\vec{\varphi}(x,s)\cdot \vec{\nu}_2 dxds.
\end{aligned}$$

Since U_3 and $\vec{\varphi}$ are periodic on $\Gamma_{2,2}$, this shows that

$$\beta U_3 + \alpha u_2 = u_1 \quad \text{on } \partial Y_1 \cap \partial Y_2 \equiv \Gamma_{1,2}\,. \tag{3.14}$$

Next we seek a variational statement which is satisfied by the limits obtained in Lemma 3.1. Choose smooth functions

$$\varphi_j \in L^p\left(0,T; W^{1,p}(\Omega)\right),\ j=1,2, \quad \Phi_j \in L^p\left((0,T)\times\Omega; W^{1,p}_{\#}(Y)\right),\ j=1,2,3,$$

such that

$$\frac{\partial \varphi_j}{\partial t} \in L^{p'}\left(0,T; W^{1,p}(\Omega)'\right),\ j=1,2,\ \frac{\partial \Phi_3}{\partial t} \in L^{p'}\left((0,T)\times\Omega; W^{1,p}_{\#}(Y)'\right),$$

and $\beta\Phi_3(x,y,t) = \varphi_1(x,t) - \alpha\varphi_2(x,t)$ for $y \in \Gamma_{1,2}$. In the following we shall use the notation $(\cdot)_{,t}$ to represent the time derivative $\frac{\partial}{\partial t}(\cdot)$. Apply (2.15) to the triple $[\varphi_1(x,t)+\varepsilon\Phi_1\left(x,\frac{x}{\varepsilon},t\right),\ \varphi_2(x,t)+\varepsilon\Phi_2\left(x,\frac{x}{\varepsilon},t\right),\ \Phi_3^\varepsilon\left(x,\frac{x}{\varepsilon},t\right)]$ in $L^p(0,T;V^\varepsilon)$, where we define $\Phi_3^\varepsilon(x,y,t) \equiv \Phi_3(x,y,t) + \frac{\varepsilon}{\beta}\Phi_1(x,y,t) - \frac{\varepsilon\alpha}{\beta}\Phi_2(x,y,t)$. Then integrate by parts in t to obtain

$$
\begin{aligned}
(3.15)\quad & -\sum_{j=1}^{2}\int_0^T\int_{\Omega_j^\varepsilon} c_j^\varepsilon u_j^\varepsilon\left(\varphi_{j,t}+\varepsilon\Phi_{j,t}\right)dxdt-\int_0^T\int_{\Omega_2^\varepsilon} c_3^\varepsilon u_3^\varepsilon\Phi_{3,t}^\varepsilon\,dxdt\\
&+\sum_{j=1}^{2}\int_{\Omega_j^\varepsilon} c_j^\varepsilon u_j^\varepsilon(x,T)\left(\varphi_j(x,T)+\varepsilon\Phi_j\left(x,\frac{x}{\varepsilon},T\right)\right)dx+\int_{\Omega_2^\varepsilon} c_3^\varepsilon u_3^\varepsilon(x,T)\,\Phi_3^\varepsilon\left(x,\frac{x}{\varepsilon},T\right)dx\\
&-\sum_{j=1}^{2}\int_{\Omega_j^\varepsilon} c_j^\varepsilon u_j^0\left(\varphi_j(x,0)+\varepsilon\Phi_j\left(x,\frac{x}{\varepsilon},0\right)\right)dx-\int_{\Omega_2^\varepsilon} c_3^\varepsilon u_3^0\Phi_3^\varepsilon\left(x,\frac{x}{\varepsilon},0\right)dx\\
&+\sum_{j=1}^{2}\int_0^T\int_{\Omega_j^\varepsilon}\mu_j^\varepsilon\left(x,\vec\nabla u_j^\varepsilon(x,t)\right)\cdot\vec\nabla\left(\varphi_j(x,t)+\varepsilon\Phi_j\left(x,\frac{x}{\varepsilon},t\right)\right)dxdt\\
&+\int_0^T\int_{\Omega_2^\varepsilon}\mu_3^\varepsilon\left(x,\varepsilon\vec\nabla u_3^\varepsilon(x,t)\right)\cdot\varepsilon\left[\vec\nabla\Phi_3^\varepsilon\left(x,\frac{x}{\varepsilon},t\right)+\frac{1}{\varepsilon}\vec\nabla_y\Phi_3^\varepsilon\left(x,\frac{x}{\varepsilon},t\right)\right]dxdt=0.
\end{aligned}
$$

Letting $\varepsilon\to 0$ in (3.15) now yields

$$
\begin{aligned}
(3.16)\quad & -\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j} c_j(y)\,u_j(x,t)\,\varphi_{j,t}(x,t)\;dydxdt\\
&-\int_0^T\int_\Omega\int_{Y_2} c_3(y)\,U_3(x,y,t)\,\Phi_{3,t}(x,y,t)\,dydxdt\\
&+\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)\,u_j^*(x)\,\varphi_j(x,T)\;dydx+\int_\Omega\int_{Y_2} c_3(y)\,u_3^*(x)\,\Phi_3(x,y,T)\,dydx\\
&-\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)\,u_j^0(x)\,\varphi_j(x,0)\;dydx-\int_\Omega\int_{Y_2} c_3(y)\,u_3^0(x)\,\Phi_3(x,y,0)\,dydx\\
&+\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\vec g_j(x,y,t)\cdot\left[\vec\nabla\varphi_j(x,t)+\vec\nabla_y\Phi_j(x,y,t)\right]dydxdt\\
&+\int_0^T\int_\Omega\int_{Y_2}\vec g_3(x,y,t)\cdot\vec\nabla_y\Phi_3(x,y,t)\;dydxdt=0.
\end{aligned}
$$

We can summarize the preceding as follows. Define the *energy space*

$$
\begin{aligned}
W\equiv\{[u_1,\,u_2,\,U_1,\,U_2,\,U_3]\in W^{1,p}(\Omega)^2\times L^p\left(\Omega;W_{\#}^{1,p}(Y)\right)^3:\\
\beta U_3(x,y)=u_1(x)-\alpha u_2(x)\text{ for }y\in\Gamma_{1,2}\}.
\end{aligned}
$$

We have shown that the limit obtained in Lemma 3.1 satisfies

$$[u_1,\,u_2,\,U_1,\,U_2,\,U_3]\in L^p(0,T;W)$$

and by density, (3.16) holds for all $[\varphi_1,\,\varphi_2,\,\Phi_1,\,\Phi_2,\,\Phi_3]\in L^p(0,T;W)$ such that $\frac{d}{dt}[\varphi_1,\,\varphi_2,\,0,\,0,\,\Phi_3]\in L^{p'}(0,T;W')$. It remains to find the strong form of the problem and to identify the *flux* terms $\vec g_j$.

4. The Homogenized Problem

We shall decouple the variational identity (3.16) in order to obtain the strong form of our homogenized system. This will be accomplished by making special choices of the test functions $[\varphi_1, \varphi_2, \Phi_1, \Phi_2, \Phi_3]$ as above, and the strong form will be displayed below in Corollary 5.2. First we choose $\varphi_1, \varphi_2, \Phi_1, \Phi_2$ all equal to zero, and choose Φ_3 as above and to vanish at $t = 0$ and $t = T$ and on $\Gamma_{1,2}$. Together with the identity (3.14) from above, this gives at *a.e.* $x \in \Omega$ the *cell system*

$$c_3(y)\frac{\partial U_3(x,y,t)}{\partial t} - \vec{\nabla}_y \cdot \vec{g}_3(x,y,t) = 0\,, \quad y \in Y_2\,, \tag{4.1}$$

$$U_3 \text{ and } \vec{g}_3 \cdot \vec{\nu} \text{ are } Y\text{-periodic on } \Gamma_{2,2}\,, \tag{4.2}$$

$$\beta U_3 = u_1 - \alpha u_2 \text{ on } \Gamma_{1,2}\,. \tag{4.3}$$

Next let φ_1 be as above and vanish at $t = 0$ and $t = T$, and choose Φ_3 by the requirement that $\beta\Phi_3(x, y, t) = \varphi_1(x, t)$ for $y \in Y_2$. With the remaining test functions all zero, this yields the *macro-fissure equation*

$$\left(\int_{Y_1} c_1(y)\,dy\right)\frac{\partial u_1(x,t)}{\partial t} + \frac{1}{\beta}\frac{\partial}{\partial t}\int_{Y_2} c_3(y)U_3(x,y,t)\,dy = \vec{\nabla}\cdot\int_{Y_1}\vec{g}_1(x,y,t)\,dy\,. \tag{4.4}$$

Similarly we choose φ_2 as above and vanishing at $t = 0$ and $t = T$ and let Φ_3 be determined by $\beta\Phi_3(x, y, t) = -\alpha\varphi_2(x, t)$ for $y \in Y_1$ to obtain the *macro-matrix equation*

$$\left(\int_{Y_2} c_2(y)\,dy\right)\frac{\partial u_2(x,t)}{\partial t} - \frac{\alpha}{\beta}\frac{\partial}{\partial t}\int_{Y_2} c_3(y)U_3(x,y,t)\,dy = \vec{\nabla}\cdot\int_{Y_2}\vec{g}_2(x,y,t)\,dy\,. \tag{4.5}$$

Finally, by setting the test functions $\varphi_1, \varphi_2, \Phi_3$ all equal to zero and by choosing Φ_1, Φ_2 as above, we obtain the pair of systems

$$\vec{\nabla}_y \cdot \vec{g}_j(x,y,t) = 0 \quad y \in Y_j\,, \tag{4.6}$$

$$\vec{g}_j \cdot \vec{\nu} = 0 \text{ on } \Gamma_{1,2} \text{ and } \vec{g}_j \cdot \vec{\nu} \text{ is } Y\text{-periodic on } \partial Y_j \cap \partial Y \text{ for } j = 1,2. \tag{4.7}$$

Note that (4.1) and (4.4) and (4.5) hold in $L^{p'}((0,T)\times\Omega; W^{1,p}_{\#}(Y)')$ and $L^{p'}(0,T; W^{1,p}(\Omega)')$, respectively. Substituting (4.1)-(4.7) in (3.16) gives the boundary conditions

$$\int_{Y_1}\vec{g}_1(x,y,t)\,dy\cdot\vec{\nu_1} = 0 \quad \text{and} \tag{4.8}$$

$$\int_{Y_2}\vec{g}_2(x,y,t)\,dy\cdot\vec{\nu_2} = 0 \quad \text{on } \partial\Omega \tag{4.9}$$

and the initial and final conditions

$$U_3(x,y,0) = u_3^0(x)\,,\ U_3(x,y,T) = u_3^*(x,y) \tag{4.10}$$

and

$$u_j(x,0) = u_j^0(x)\,,\ u_j(x,T) = u_j^*(x) \qquad \text{for } j = 1,2 \tag{4.11}$$

in $L^2(\Omega\times Y_2)$ and $L^2(\Omega)$ respectively. The final conditions appearing above will be used only to identify the functions $\vec{g}_i(x,y,t)$ below; they are *not* part of the problem. Note also that using (4.10) and (4.11), integrating by parts in t in (3.16), and replacing the test functions φ_j (for $j=1,2$) and Φ_j (for $j=1,2,3$) with sequences converging to u_j and U_j gives the following "homogenized" version of (3.11),

$$
\begin{aligned}
(4.12)\quad & \frac{1}{2}\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)\left|u_j(x,T)\right|^2 dydx+\frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)\left|U_3(x,y,T)\right|^2 dydx\\
&-\left[\frac{1}{2}\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)\left|u_j^0(x)\right|^2 dydx+\frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)\left|u_3^0(x)\right|^2 dydx\right]\\
&+\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\vec{g}_j(x,y,t)\cdot\left[\vec{\nabla}u_j(x,t)+\vec{\nabla}_yU_j(x,y,t)\right]dydxdt\\
&+\int_0^T\int_\Omega\int_{Y_2}\vec{g}_3(x,y,t)\cdot\vec{\nabla}_yU_3(x,y,t)\,dydxdt=0.
\end{aligned}
$$

It remains to find $\vec{g}_1,\ \vec{g}_2$ and $\vec{g}_3$ in terms u_1, u_2, U_1, U_2 and U_3. To this end, let $\vec{\phi}$ and $\vec{\xi}$ be in $C_0^\infty\left([0,T]\times\Omega;C_\#^\infty(Y)\right)^N$ and $\Phi_1,\Phi_2,\Phi_3\in C_0^\infty\left([0,T]\times\Omega;C_\#^\infty(Y)\right)$ and for $\varepsilon>0$, define the triple of functions

$$\eta_j^\varepsilon(x,t)=\chi_j\left(\frac{x}{\varepsilon}\right)\vec{\nabla}u_j(x,t)+\varepsilon\chi_j\left(\frac{x}{\varepsilon}\right)\vec{\nabla}\Phi_j\left(x,\frac{x}{\varepsilon},t\right)+\lambda\vec{\phi}\left(x,\frac{x}{\varepsilon},t\right),\quad j=1,2,$$

and

$$\eta_3^\varepsilon(x,t)=\chi_2\left(\frac{x}{\varepsilon}\right)\left(\varepsilon\vec{\nabla}\Phi_3\left(x,\frac{x}{\varepsilon},t\right)+\lambda\vec{\xi}\left(x,\frac{x}{\varepsilon},t\right)\right).$$

Note that each $\eta_j^\varepsilon(x,t)$ and (because of the continuity assumption) $\mu_j\left(\frac{x}{\varepsilon},\eta_j^\varepsilon(x,t)\right)$ ($j=1,2,3$) arises from an admissible test function, and we have the two-scale convergence

$$\eta_j^\varepsilon\xrightarrow{2}\eta_j(x,y,t)\equiv\chi_j(y)\vec{\nabla}u_j(x,t)+\chi_j(y)\vec{\nabla}_y\Phi_j(x,y,t)+\lambda\vec{\phi}(x,y,t),\quad j=1,2,$$

$$\eta_3^\varepsilon\xrightarrow{2}\eta_3(x,y,t)\equiv\chi_2(y)\left(\vec{\nabla}_y\Phi_3(x,y,t)\right)+\lambda\vec{\xi}(x,y,t).$$

By (2.2) we have

$$
\begin{aligned}
(4.13)\quad & \sum_{j=1}^{2}\int_0^T\int_{\Omega_j^\varepsilon}\left(\mu_j^\varepsilon\left(x,\vec{\nabla}u_j^\varepsilon\right)-\mu_j^\varepsilon\left(x,\eta_j^\varepsilon\right)\right)\left(\vec{\nabla}u_j^\varepsilon-\eta_j^\varepsilon\right)dxdt\\
&+\int_0^T\int_{\Omega_2^\varepsilon}\left(\mu_3^\varepsilon\left(x,\varepsilon\vec{\nabla}u_3^\varepsilon\right)-\mu_3^\varepsilon\left(x,\eta_3^\varepsilon\right)\right)\left(\varepsilon\vec{\nabla}u_3^\varepsilon-\eta_3^\varepsilon\right)dxdt\geq 0.
\end{aligned}
$$

Expanding (4.13) and employing (3.11) at $t = T$ gives

$$\frac{1}{2}\sum_{j=1}^{2}\left\{\int_{\Omega_j^\varepsilon} c_j^\varepsilon\left(\left|u_j^0\right|^2 - \left|u_j^\varepsilon(x,T)\right|^2\right)dx\right\} + \frac{1}{2}\int_{\Omega_2^\varepsilon} c_3^\varepsilon\left(\left|u_3^0\right|^2 - \left|u_3^\varepsilon(x,T)\right|^2\right)dx$$
$$-\sum_{j=1}^{2}\int_0^T\int_{\Omega_j^\varepsilon}\left\{\mu_j^\varepsilon\left(x,\eta_j^\varepsilon\right)\cdot\left(\vec{\nabla}u_j^\varepsilon - \eta_j^\varepsilon\right) + \mu_j^\varepsilon\left(x,\vec{\nabla}u_j^\varepsilon\right)\cdot\eta_j^\varepsilon\right\}dxdt$$
$$-\int_0^T\int_{\Omega_2^\varepsilon}\left\{\mu_3^\varepsilon\left(x,\eta_3^\varepsilon\right)\cdot\left(\varepsilon\vec{\nabla}u_3^\varepsilon - \eta_3^\varepsilon\right) + \mu_3^\varepsilon\left(x,\varepsilon\vec{\nabla}u_3^\varepsilon\right)\cdot\eta_3^\varepsilon\right\}dxdt \geq 0.$$

Let $\varepsilon \to 0$ and apply the two-scale convergence results above to obtain

$$\begin{aligned}(4.14)\quad &\frac{1}{2}\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)\left|u_j^0(x)\right|^2 dydx + \frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)\left|u_3^0(x)\right|^2 dydx\\
&-\frac{1}{2}\lim_{\varepsilon\to 0}\left[\sum_{j=1}^{2}\int_{\Omega_j^\varepsilon} c_j^\varepsilon\left|u_j^\varepsilon(x,T)\right|^2 dx + \frac{1}{2}\int_{\Omega_2^\varepsilon} c_3^\varepsilon\left|u_3^\varepsilon(x,T)\right|^2 dx\right]\\
&-\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\mu_j\left(y,\eta_j(x,y,t)\right)\cdot\left(\vec{\nabla}_y U_j(x,y) - \vec{\nabla}_y\Phi_j(x,y,t) - \lambda\vec{\phi}(x,y,t)\right)dydxdt\\
&-\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\vec{g}_j(x,y,t)\cdot\left(\vec{\nabla}u_j(x,t) + \vec{\nabla}_y\Phi_j(x,y,t) + \lambda\vec{\phi}(x,y,t)\right)dydxdt\\
&-\int_0^T\int_\Omega\int_{Y_2}\mu_3\left(y,\eta_3(x,y,t)\right)\cdot\left(\vec{\nabla}_y U_3(x,y,t) - \vec{\nabla}_y\Phi_3(x,y,t) - \lambda\vec{\xi}(x,y,t)\right)dydxdt\\
&-\int_0^T\int_\Omega\int_{Y_2}\vec{g}_3(x,y,t)\cdot\left(\vec{\nabla}_y\Phi_3(x,y,t) + \lambda\vec{\xi}(x,y,t)\right)dydxdt \geq 0.\end{aligned}$$

Set $\vec{\phi} = \chi_1\vec{\theta}_1 + \chi_2\vec{\theta}_2$ where $\vec{\theta}_j \in C_0^\infty([0,T]\times\Omega, C^\infty(Y_j))$ and $\chi_j\vec{\theta}_j$ is Y-periodic. Following [1] we note that since each μ_j is continuous in the second variable, we may replace Φ_j by sequences converging strongly in $L^p((0,T)\times\Omega; W_\#^{1,p}(Y)/\mathbb{R})$ to $\chi_1(y)U_1(x,y,t)$, $\chi_2(y)U_2(x,y,t)$ and $\chi_2(y)U_3(x,y,t)$ for $j = 1, 2$ and 3, respectively. Thus (4.14) becomes

$$\begin{aligned}(4.15)\quad &\frac{1}{2}\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)\left|u_j^0(x)\right|^2 dydx + \frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)\left|u_3^0(x)\right|^2 dydx\\
&-\frac{1}{2}\lim_{\varepsilon\to 0}\left[\sum_{j=1}^{2}\int_{\Omega_j^\varepsilon} c_j^\varepsilon\left|u_j^\varepsilon(x,T)\right|^2 dx + \frac{1}{2}\int_{\Omega_2^\varepsilon} c_3^\varepsilon\left|u_3^\varepsilon(x,T)\right|^2 dx\right]\end{aligned}$$

$$+\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\mu_j\left(y,\vec{\nabla}u_j\left(x,t\right)+\vec{\nabla}_yU_j\left(x,y,t\right)+\lambda\vec{\theta}_j\left(x,y,t\right)\right)\cdot\lambda\vec{\theta}_j\left(x,y,t\right)dydxdt$$
$$-\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\vec{g}_j\left(x,y,t\right)\cdot\left(\vec{\nabla}u_j\left(x,t\right)+\vec{\nabla}_yU_j\left(x,y,t\right)+\lambda\vec{\theta}_j\left(x,y,t\right)\right)dydxdt$$
$$+\int_0^T\int_\Omega\int_{Y_2}\mu_3\left(y,\vec{\nabla}_yU_3\left(x,y,t\right)+\lambda\vec{\xi}\left(x,y,t\right)\right)\cdot\left(\lambda\vec{\xi}\left(x,y,t\right)\right)dydxdt$$
$$-\int_0^T\int_\Omega\int_{Y_2}\vec{g}_3\left(x,y,t\right)\cdot\left(\vec{\nabla}_yU_3\left(x,y,t\right)+\lambda\vec{\xi}\left(x,y,t\right)\right)dydxdt\geq 0.$$

We now employ (4.10), (4.11) and (4.12) in (4.15) to obtain

(4.16)
$$\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\mu_j\left(y,\vec{\nabla}u_j\left(x,t\right)+\vec{\nabla}_yU_j\left(x,y,t\right)+\lambda\vec{\theta}_j\left(x,y,t\right)\right)\cdot\lambda\vec{\theta}_j\left(x,y,t\right)dydxdt$$
$$+\int_0^T\int_\Omega\int_{Y_2}\mu_3\left(y,\vec{\nabla}_yU_3\left(x,y,t\right)+\lambda\vec{\xi}\left(x,y,t\right)\right)\cdot\lambda\vec{\xi}\left(x,y,t\right)dydxdt$$
$$-\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\vec{g}_j\left(x,y,t\right)\cdot\lambda\vec{\theta}_j\left(x,y,t\right)dydxdt-\int_0^T\int_\Omega\int_{Y_2}\vec{g}_3\left(x,y,t\right)\cdot\lambda\vec{\xi}\left(x,y,t\right)dydxdt$$
$$\geq\frac{1}{2}\lim_{\varepsilon\to 0}\left[\sum_{j=1}^{2}\int_{\Omega_j^\varepsilon}c_j^\varepsilon\left|u_j^\varepsilon\left(x,T\right)\right|^2dx+\frac{1}{2}\int_{\Omega_2^\varepsilon}c_3^\varepsilon\left|u_3^\varepsilon\left(x,T\right)\right|^2dx\right]$$
$$-\frac{1}{2}\sum_{j=1}^{2}\int_\Omega\int_{Y_j}c_j\left(y\right)\left|u_j\left(x,T\right)\right|^2\,dydx+\frac{1}{2}\int_\Omega\int_{Y_2}c_3\left(y\right)\left|U_3\left(x,y,T\right)\right|^2dydx.$$

The right hand side of (4.16) is non-negative by [1, Proposition 1.6], so dividing by λ and letting $\lambda\to 0$ gives

(4.17)
$$\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\left[\mu_j\left(y,\vec{\nabla}u_j\left(x,t\right)+\vec{\nabla}_yU_j\left(x,y,t\right)\right)-\vec{g}_j\left(x,y,t\right)\right]\cdot\vec{\theta}_j\left(x,y,t\right)dydxdt$$
$$+\int_0^T\int_\Omega\int_{Y_2}\left[\mu_3\left(y,\vec{\nabla}_yU_3\left(x,y,t\right)\right)-\vec{g}_3\left(x,y,t\right)\right]\cdot\vec{\xi}\left(x,y,t\right)dydxdt\geq 0.$$

This holds for all $\vec{\theta}_1$, $\vec{\theta}_2$, and $\vec{\xi}$, so

$$\mu_j\left(y,\vec{\nabla}u_j\left(x,t\right)+\vec{\nabla}_yU_j\left(x,y,t\right)\right)=\vec{g}_j\left(x,y,t\right)\ \text{ in } Y_j,\qquad j=1,2,$$

and

$$\mu_3\left(y,\vec{\nabla}_yU_3\left(x,y,t\right)\right)=\vec{g}_3\left(x,y,t\right)\ \text{ in } Y_2.$$

These identities complete the strong form of the homogenized problem. We shall summarize and complement these results in the following section.

5. The Main Result

Theorem 5.1. Assume that (2.1)-(2.4) hold, that $\beta > 0$, and that $u_1^0, u_2^0,$ and $u_3^0 \in L^2(\Omega)$ are given. Then the limits $[u_1,\ u_2,\ U_1,\ U_2,\ U_3]$ established above in Lemma 3.1 are the unique solution

$$u_j \in L^p(0,T;W^{1,p}(\Omega)),\ j=1,2, \quad U_j \in L^p((0,T)\times\Omega;W^{1,p}_{\#}(Y_j)/\mathbb{R}),\ j=1,2,3,$$

with $\beta U_3(x,y,t) = u_1(x,t) - \alpha u_2(x,t)$ for $y \in \Gamma_{1,2}$ of the **homogenized system**

$$\begin{aligned}
(5.1)\quad & -\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j} c_j(y)\, u_j(x,t)\, \varphi_{j,t}(x,t)\ dydxdt \\
& \qquad -\int_0^T\int_\Omega\int_{Y_2} c_3(y)\, U_3(x,y,t)\, \Phi_{3,t}(x,y,t)\, dydxdt \\
& -\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)\, u_j^0(x)\, \varphi_j(x,0)\ dydx - \int_\Omega\int_{Y_2} c_3(y)\, u_3^0(x)\, \Phi_3(x,y,0)\, dydx \\
& +\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\mu_j\left(y, \vec{\nabla}u_j(x,t) + \vec{\nabla}_y U_j(x,y,t)\right)\cdot\left[\vec{\nabla}\varphi_j(x,t) + \vec{\nabla}_y\Phi_j(x,y,t)\right] dydxdt \\
& \qquad + \int_0^T\int_\Omega\int_{Y_2}\mu_3\left(y, \vec{\nabla}_y U_3(x,y,t)\right)\cdot\vec{\nabla}_y\Phi_3(x,y,t)\ dydxdt = 0,
\end{aligned}$$

for all

$$\varphi_j \in L^p\left(0,T;W^{1,p}(\Omega)\right),\ j=1,2, \quad \Phi_j \in L^p\left((0,T)\times\Omega;W^{1,p}_{\#}(Y_j)\right),\quad j=1,2,3,$$

for which

$$\frac{\partial\varphi_j}{\partial t} \in L^{p'}\left(0,T;(W^{1,p}(\Omega))'\right),\ j=1,2, \quad \frac{\partial\Phi_3}{\partial t} \in L^{p'}\left((0,T)\times\Omega;(W^{1,p}_{\#}(Y_j))'\right),$$
$$\beta\Phi_3(x,y,t) = \varphi_1(x,t) - \alpha\varphi_2(x,t) \quad \text{for } y \in \Gamma_{1,2},$$

and

$$\varphi_1(x,T) = \varphi_2(x,T) = \Phi_3(x,y,T) = 0.$$

Only the uniqueness needs yet to be verified, and this will follow below. In particular, U_1 and U_2 are determined within a constant for each $t \in (0,T)$, so each of these is unique up to a corresponding function of t. We shall check that (5.1) is just the *variational form* of the Cauchy problem for an appropiate evolution equation in Banach space, and that the corresponding *strong problem* is described as follows. The state space is given by

$$H \equiv \{[\varphi_1,\ \varphi_2,\ \Phi_3] \in L^2(\Omega)\times L^2(\Omega)\times L^2\left(\Omega;L^2(Y_2)\right)$$

with the scalar product

$$(\mathbf{u},\varphi)_H \equiv \sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)\, dy\, u_j(x)\, \varphi_j(x)\, dx + \int_\Omega\int_{Y_2} c_3(y)\, U_3(x,y)\, \Phi_3(x,y)\, dy\, dx.$$

Define the *energy space*

$$V \equiv \{[\varphi_1,\, \varphi_2,\, \Phi_3] \in H \cap \left(W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times L^p\left(\Omega; W^{1,p}_{\#}(Y_2)\right)\right) : \\ \beta\Phi_3(x,y) = \varphi_1(x) - \alpha\varphi_2(x) \text{ for } y \in \Gamma_{1,2}\}$$

and the corresponding *evolution space* by $\mathcal{V} = L^p(0,T;V)$.

Corollary 5.2. The triple $\mathbf{u}(\cdot) \equiv [u_1(\cdot),\ u_2(\cdot),\ U_3(\cdot)]$ is the unique solution $\mathbf{u}(\cdot) \in \mathcal{V}$ with $\mathbf{u}'(\cdot) \in \mathcal{V}'$ of the **strong homogenized system**

$$\left(\int_{Y_1} c_1(y)\, dy\right) \frac{\partial u_1\,(x,t)}{\partial t} + \frac{1}{\beta}\frac{\partial}{\partial t}\int_{Y_2} c_3(y) U_3(x,y,t)\, dy \\ = \vec{\nabla}\cdot \int_{Y_1} \mu_1\left(y, \vec{\nabla} u_1\,(x,t) + \vec{\nabla}_y U_1\,(x,y,t)\right) dy\,,$$

$$\left(\int_{Y_2} c_2(y)\, dy\right) \frac{\partial u_2\,(x,t)}{\partial t} - \frac{\alpha}{\beta}\frac{\partial}{\partial t}\int_{Y_2} c_3(y) U_3(x,y,t)\, dy \\ = \vec{\nabla}\cdot \int_{Y_2} \mu_2\left(y, \vec{\nabla} u_2\,(x,t) + \vec{\nabla}_y U_2\,(x,y,t)\right) dy\,,$$

$$c_3\,(y)\, \frac{\partial U_3\,(x,y,t)}{\partial t} - \vec{\nabla}_y \cdot \mu_3\left(y, \vec{\nabla}_y U_3\,(x,y,t)\right) = 0\,, \quad y \in Y_2\,,$$
$$U_3\,(x,y,t) \text{ and } \mu_3\left(y, \vec{\nabla}_y U_3\,(x,y,t)\right)\cdot \vec{\nu} \text{ are } Y\text{-periodic on } \Gamma_{2,2}\,,$$
$$\beta U_3 = u_1 - \alpha u_2 \text{ on } \Gamma_{1,2}\,,$$

with the boundary conditions

$$\int_{Y_1} \mu_1\left(y, \vec{\nabla} u_1\,(x,t) + \vec{\nabla}_y U_1\,(x,y,t)\right) dy \cdot \vec{\nu_1} = 0 \text{ and}$$
$$\int_{Y_2} \mu_2\left(y, \vec{\nabla} u_2\,(x,t) + \vec{\nabla}_y U_2\,(x,y,t)\right) dy \cdot \vec{\nu_2} = 0 \text{ on } \partial\Omega,$$

and the initial conditions

$$u_j\,(x,0) = u_j^0\,(x) \text{ for } j = 1,2, \qquad U_3\,(x,y,0) = u_3^0\,(x)\,,$$

where $U_1 \in L^p((0,T)\times\Omega; W^{1,p}_{\#}(Y_1)/\mathbb{R})$, $U_2 \in L^p((0,T)\times\Omega; W^{1,p}_{\#}(Y_2)/\mathbb{R})$ are solutions of the *local problems*

$$\vec{\nabla}_y \cdot \mu_j\left(y, \vec{\nabla}_y U_j\,(x,y,t) + \vec{\nabla} u_j\,(x,t)\right) = 0\,, \quad y \in Y_j\,,$$
$$\mu_j\left(y, \vec{\nabla}_y U_j\,(x,y,t) + \vec{\nabla} u_j\,(x,t)\right)\cdot \nu = 0 \text{ on } \Gamma_{1,2}, \qquad Y\text{-periodic on } \Gamma_{2,2}\ j = 1,2.$$

Proof. Define an operator $\mathcal{A} : V \to V'$ by

$$\langle \mathcal{A}\mathbf{u}, \varphi\rangle \equiv \sum_{j=1}^{2} \int_\Omega \int_{Y_j} \{\mu_j(y, \vec{\nabla} u_j\,(x) + \vec{\nabla}_y U_j(x,y)\} \cdot (\vec{\nabla}\varphi_j\,(x))\, dydx \\ + \int_\Omega \int_{Y_3} \{\mu_3(y, \vec{\nabla}_y U_3(x,y)\} \cdot (\vec{\nabla}_y \Phi_3(x,y))\, dydx, \\ \mathbf{u} = [u_1, u_2, U_3],\ \varphi = [\varphi_1, \varphi_2, \Phi] \in V, \tag{5.2}$$

where $U_1(x,y)$ and $U_2(x,y)$ are determined by

$$(5.3)\quad U_j \in L^p\left(\Omega; W^{1,p}_{\#}(Y_j)\right):$$
$$\int_\Omega \int_{Y_j} \{\mu_j(y, \vec{\nabla}_y U_j(x,y) + \vec{\nabla} u_j\,(x)\} \cdot (\vec{\nabla}_y \Phi(x,y))\, dy dx = 0,$$
$$\Phi \in L^p\left(\Omega; W^{1,p}_{\#}(Y_j)\right),$$

for $j = 1$, 2. It has been already shown in Section 4 that $[u_1, u_2, U_1, U_2, U_3]$ satisfies the homogenized system of Theorem 5.1, and from this it follows that $\mathbf{u}(\cdot)$ satisfies the variational form of the Cauchy problem (1.3). It is easy to check that $\mathcal{A}$ is monotone and bounded $\mathcal{V} \to \mathcal{V}'$, so $\mathbf{u}(\cdot)$ is the unique solution as well of the strong problem (1.2), and this is realized as the strong homogenized system of Corollary 5.2. □

Remark 5.1. Theorem 5.1 describes the limiting form of the original micro-model from Section 2 as a system for the five unknowns u_1, u_2, U_1, U_2, and U_3. This system can also be realized from an evolution equation based on the variational identity (3.16) on the space W, but this would be of *degenerate* type: the time derivatives of U_1 and U_2 do not occur in the system. However, by following the suggestion implicit in Corollary 5.2 we were able to incorporate the local functions U_1 and U_2 in the definition of the operator $\mathcal{A}$ and thereby to write our limiting system as a non-degenerate evolution equation on the space V with three components.

In the linear case, one can carry this decoupling even further and represent each of the functions U_1 and U_2 in terms of the corresponding u_1 or u_2 in order to obtain a closed system for the remaining three unknowns. Suppose that we have symmetric Y-periodic coefficient functions $a^1_{ij}(y) \in C\,(Y_1)$ and $a^2_{ij}(y) \in C\,(Y_2)$ (which are zero off their respective domains). We assume that there is a $c_0 > 0$, independent of y, such that

$$\sum_{i,j=1}^N a^k_{ij}(y)\xi_i\xi_j \geq c_0|\xi|^2, \quad y \in Y_k \quad \text{for } k = 1, 2.$$

Extending each a^k_{ij} to all of $\mathbb{R}$ by periodicity, we define for $k = 1, 2$ and $\vec{\xi} \in \mathbb{R}^N$

$$\mu^\varepsilon_k\left(x, \vec{\xi}\right)_i = \sum_{j=1}^N a^k_{ij}\left(\frac{x}{\varepsilon}\right)\xi_j.$$

Then with $p = 2$, the results developed above apply. For each of $k = 1, 2$ we isolate from (5.1) the following problem for $U_k(x,y,t)$:

Find $U_k \in L^2\left((0,T) \times \Omega; W^{1,2}_{\#}\,(Y_k)\right)$ such that

$$(5.4)\quad \int_0^T \int_\Omega \int_{Y_k} \mu_k\left(y, \vec{\nabla} u_k\,(x,t) + \vec{\nabla}_y U_k\,(x,y,t)\right) \cdot \vec{\nabla}_y \Phi_k\,(x,y,t)\ dy dx dt = 0$$
$$\text{for all } \Phi_k \in L^2\left((0,T) \times \Omega; W^{1,2}_{\#}\,(Y_k)\right).$$

The “input” to this problem is $\vec{\nabla} u_k\,(x,t)$, independent of y, so this permits us to separate variables with the following construction:

For $1 \le i \le N$, define $W_i^k(y)$ to be the solution of

$$-\vec{\nabla}_y \cdot \left[\mu_k(y, \vec{\nabla}_y W_i^k(y) + \vec{e}_i)\right] = 0 \quad \text{in} \quad Y_k \ ,$$
$$\mu_k(y, \vec{\nabla}_y W_i^k(y) + \vec{e}_i) \cdot \vec{n} = 0 \ \text{ on } \ \partial Y_k \sim \partial Y$$
$$W_i^k(\cdot) \text{ is } Y\text{-periodic.}$$

Then by linearity we can write

$$U_k(x,y,t) = \sum_{j=1}^{N} \frac{\partial u_k}{\partial x_j}(x,t) W_j(y) \ .$$

If we substitute this into (5.1) with $\Phi_k(x,y,t) = \sum_{j=1}^{N} \frac{\partial \varphi_k(x,t)}{\partial x_j} W_j^k(y)$, we obtain the **decoupled homogenized system**

$$-\sum_{k=1}^{2} \int_0^T \int_\Omega \tilde{c}_k u_k\,(x,t)\, \varphi_{k,t}\,(x,t)\ dxdt - \sum_{k=1}^{2} \int_\Omega \tilde{c}_k u_k^0\,(x)\, \varphi_k\,(x,0)\ dx$$
$$+ \sum_{k=1}^{2} \int_0^T \int_\Omega \int_{Y_k} \sum_{i,j=1}^{N} A_{ij}^k \frac{\partial u_k}{\partial x_i}(x,t) \frac{\partial \varphi_k}{\partial x_j}\,(x,t)\ dydxdt = 0 \quad \text{for } k = 1,2$$

$$-\int_0^T \int_\Omega \int_{Y_2} c_3\,(y)\, U_3\,(x,y,t)\, \Phi_{3,t}\,(x,y,t)\, dydxdt$$
$$-\int_\Omega \int_{Y_2} c_3\,(y)\, u_3^0\,(x)\, \Phi_3\,(x,y,0)\, dydx$$
$$+\int_0^T \int_\Omega \int_{Y_2} \mu_3 \left(y, \vec{\nabla}_y U_3\,(x,y,t)\right) \cdot \vec{\nabla}_y \Phi_3\,(x,y,t)\ dydxdt = 0.$$

where the coefficients are given by

$$\tilde{c}_k = \int_{Y_k} c_k\, dy \quad , \quad A_{ij}^k = \int_{Y_k} \mu_k(y, \vec{\nabla}_y W_i^k\,(y) + \vec{e}_i) \cdot (\vec{\nabla}_y W_j^k\,(y) + \vec{e}_j)\, dy\,.$$

These are the usual *effective coefficients* which are constants that result from the "averaging" due to the homogenization procedure.

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