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EXISTENCE OF SOLUTIONS TO A PARATINGENT EQUATION WITH DELAYED ARGUMENT

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Abstract. In this work we prove the existence of solutions of a class of paratingent equations with delayed argument,

$$(Pt\,x)(t) \subset F([x]_t) \quad \text{for } t \geq 0$$

with the initial condition $x(t) = \xi(t)$ for $t \leq 0$. We use a fixed point theorem to obtain a solution and then provide an estimate for the solution.

1. Introduction

The first works on differential inclusions were published in 1934-35 by Marchaud [17] and Zaremba [26]. They used terms of contingent or paratingent equations. Later, Wasewski and his collaborators published a series of works and developed the elementary theory of differential inclusions [24, 25]. Within few years after the first publications, the differential inclusions resulted to be the basic tool in the optimal control theory. Starting from the pioneering work of Myshkis [18], there exists the whole series of papers devoted to paratingent and contingent differential inclusions with delay; see for example Campu [6, 7] and Kryzowa [15]. After this, many works appear on differential inclusions with delay, for example Deimling [8], Haddad [9, 10, 11, 12] Kamenskii et al. [14] and Zygmunt [27]. Recent results for differential inclusions with a finite delay $r > 0$ in spaces of Banach were obtained by Syam [23] and Castaing-Ibrahim [7]. Recently, Raczynski has successfully applied differential inclusions to simulation and modelling theory [19, 20, 21]. A more extended survey on differential inclusions can be found in the book of Aubin and Cellina [1], the book of K. Deimling [8], the book of M. Kamenskii [14] and the book of G. V. Smirnov [22].

In this work we study the existence of the solutions of the paratingent equation with delayed argument,

$$\begin{gathered} (Pt\,x)(t) \subset F([x]_t) \quad \text{for } t \geq 0, \\ x(t) = \xi(t) \quad \text{for } t \leq 0\,. \end{gathered}$$

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2. Preliminaries

Let (E,ρ) and (E',ρ') two metric spaces. By $\operatorname{Comp} E$, we denote the set of all the nonempty and compact subsets of E. When E is a vector space, $\operatorname{Conv} E$ denotes the set of all convex elements of $\operatorname{Comp} E$.

A set-valued map, $F : E \to \operatorname{Comp} E'$, is called upper semi-continuous in E, and denoted by u.s.c , if for any point $a \in E$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in B(a)_\delta \Rightarrow F(x) \subset B(F(a))_\varepsilon$ where $B(a)_\delta = B(a,\delta) = \{x \in E : \rho(a,x) < \delta\}$ and $B(F(a))_\varepsilon = B(F(a),\varepsilon) = \{y \in E' \text{ such as } z \in F(a) \text{ and } \rho'(y,z) < \varepsilon\}$. (see [2])

On the upper semi-continuity of a set-valued map, we have the following lemma (see [13]).

Lemma 2.1. *Let (E,ρ) and (E',ρ') be two metric spaces. A set-valued map, $F : E \to \operatorname{Comp} E'$, is u.s.c if and only if, for all sequences $\{x_i\} \in E$ and $\{y_i\} \in E'$ such that $\{x_i\} \to x_0$ and $\{y_i\} \in F(x_i)$, there exists a subsequence $\{y_{i_k}\}$ of $\{y_i\}$ which converges to $y_0 \in F(x_0)$.*

Let C the space of continuous functions $x : R \to R^n$ with the topology defined by an almost uniform convergence (i.e. a uniform convergence on each compact interval of $\mathbb{R}$). It is well know that the almost uniform convergence in C is equivalent to the convergence by the metric ρ defined as follows

$$\rho(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{(1, \sup|x(t) - y(t)|), -i \le t \le i\} \quad \text{for } x,y \in C.$$

Then C is a metric locally convex linear topological space. Let $\beta < 0$ be a fixed real number and let $I = [0,\infty[\subset \mathbb{R}$. If $x \in C$, the symbol $[x]_t$ denotes the restriction of x on the interval $[\beta,t]$ when $t \in I$ and $\|x\|_t = max\{|x(s)|, \beta \le s \le t\}$ with $|x| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$ for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

Let G denote the metric space whose elements are functions $[x]_t, [y]_u, \dots,$ where $t \in I$, $u \in I$, the distance between two functions $[x]_t$, $[y]_u$, being understood as a distance of their graphs in $R \times \mathbb{R}^n$ in the Hausdorff sense.

Paratingent of a function. Having a function $x \in C$ and $t \in I$, the set of limit points

$$\lim \frac{x(u_i) - x(s_i)}{u_i - s_i} = \alpha\,,$$

where $u_i \in I$, $s_i \in I$, $u_i \ne s_i$ $(i = 1,2,\dots)$, and $\lim u_i = \lim s_i = t$, is called the paratingent of x at the point t and denoted by $(Ptx)(t)$. It is easy to see that (Ptx) maps the interval I to the family of the nonempty and closed subsets of $\mathbb{R}^n$ (see [3]).

Paratingent equation with a delayed argument. Let a set-valued map $F : G \to \operatorname{Comp} \mathbb{R}^n$, be a relation of the form

$$(Pt\,x)(t) \subset F([x]_t) \quad \text{where } t \in I,\ x \in C. \tag{2.1}$$

is called paratingent equation with a delayed argument. Every function $x \in C$ satisfying (2.1) will be called the solution of these equation.

The generalized problem of Cauchy for (2.1) consists in the search for a solution of (2.1) which will be satisfy the initial condition

$$x(t) = \xi(t) \quad \text{for } t \in [\beta, 0] \tag{2.2}$$

where the function $\xi \in C$, called the initial function, is given in advance (i.e. the solution of (2.1) must contain a certain curve given in advance).

3. Existence of solutions

To show that the paratingent equation with delayed argument (2.1) with the initial condition (2.2) has at least one solution on interval $[0, T]$ ($T > 0$ an arbitrary real positive number), we assume the following hypothesis:

(H1) The set-valued mapping $F : G \to \operatorname{Conv} R^n$ is upper semi-continuous and satisfies the condition

$$F([x]_t) \subset \overline{B}(0, w(t, \|x\|_t)) \quad \text{for } t \geq 0 \tag{3.1}$$

where $\overline{B}(0, r)$ denotes the closed ball with center at 0 of $\mathbb{R}^n$ and radius r, $w(t, y)$ is a continuous function from $I \times I$ to I, increasing in y and such that the ordinary differential equation $y' = w(t, y)$, with the initial condition $y(0) = A$ (an arbitrary real positive number) has a maximal solution on all intervals I and for all A.

Theorem 3.1. *Under the hypothesis (H1), for each ζ, the paratingent equation with delayed argument* (2.1)–(2.2) *has a solution on* $[0, T]$*, with arbitrary* $T > 0$.

For the proof of this theorem we need some lemmas. First we will state Opial's theorem [16].

Lemma 3.2. *Let $w(t, y)$ a continuous function from $I \times I$ to I, increasing with respect to y and $M(t)$ a maximal solution of the ordinary differential equation $y' = w(t, y)$, with the initial condition $y(t_0) = y_0$, on the interval $[t_0, T]$, where $T > t_0$ (T an arbitrary positive real number). Let $m(t)$ be function which is continuous and increasing on $[t_0, T]$ and such that $m'(t) \leq w(t, m(t))$ almost everywhere on $[t_0, T]$. If $m(t_0) \leq y_0$, then $m(t) \leq M(t)$ for all $t \in [t_0, T]$.*

Lemma 3.3. *Let $x, y \in C$. If for all $t \geq 0$,*

$$(Pty)(t) \subset \overline{B}(0, w(s, \|x\|_t) \tag{3.2}$$

Then for all $t \geq 0$ and for all $h \geq 0$ we have

$$|y(t + h) - y(t)| \leq \int_t^{t+h} w(s, \|x\|_s) ds \tag{3.3}$$

Proof. Let T be fixed in I,

$$Q(h) = \int_t^{t+h} w(s,\ \|x\|_s) ds\ + 2\varepsilon(h + 1)\,,$$

and $R(h) = |y(t + h) - y(t)|$. It is suffices to prove that for each $\varepsilon > 0$ and each $h > 0$ we have

$$R(h) < Q(h)\,. \tag{3.4}$$

Suppose that there exist an $\varepsilon > 0$ such that (3.4) is not satisfied, and let h_0 the lower bound of the set $\{h > 0 : R(h) \geq Q(h)\}$. Since $R(0) = 0$ and $Q(0) = 2\epsilon$, we have $R(0) < Q(0)$, the number $h_0$is necessarily positive, i.e., $h_0 > 0$. If $R(h_0) > Q(h_0)$, there would be exist a real number $h' \in]0, h_0[$ such that $R(h') = Q(h_0)$, contrary to the definition of h_0. Therefore, we obtain

$$R(h_0) = Q(h_0) = |y(t + h_0) - y(t)|. \tag{3.5}$$

Let $\{h_i\}$, $i = 1, 2, \dots$, be an increasing sequence of positives numbers converging to h_0. We have $R(h_i) < Q(h_i)$ for $i = 1, 2, \dots$, from (3.5), we have

$$\begin{aligned}\frac{|y(t+h_0)-y(t+h_i)|}{h_0-h_i} &\geq \frac{|y(t+h_0)-y(t)|}{h_0-h_i} - \frac{|y(t+h_i)-y(t)|}{h_0-h_i}\\ &\geq \frac{|Q(t+h_0)-Q(t+h_i)|}{h_0-h_i}\\ &= 2\varepsilon + \frac{1}{h_0-h_i}\int_{t_0+h_i}^{t_0+h_0} w(s,\ \|x\|_s)ds\\ &= 2\varepsilon + w(u, \|x\|_u)\,,\end{aligned}$$

where $u \in [t_0+h_i, t_0+h_0]$. Therefore, starting at a certain integer N we have

$$\frac{|y(t+h_0)-y(t+h_i)|}{h_0-h_i} > \varepsilon + w(t+h_0, \|x\|_{t+h_o})\,.$$

Passing to limit, as $i \to \infty$, we have

$$\lim \frac{|y(t+h_0)-y(t+h_i)|}{h_0-h_i} \geq \varepsilon + w(t+h_0, \|x\|_{t+h_o}) > w(t+h_0, \|x\|_{t+h_o})\,.$$

However,

$$\lim \frac{|y(t+h_0)-y(t+h_i)|}{h_0-h_i} \in (Pt\,x)(t+h_0)\,;$$

thus we obtain a contradiction with hypothesis (3.2). Therefore, (3.3) must be true for all $t \in I$ and all $h > 0$. ☐

Lemma 3.4. *If $x \in C$ and $(Pt\,x)(t) \subset \overline{B}(0, w(t, \|x\|_t))$ for $t \in I$, then for all $t > 0$ we have $\|x\|_t \leq M(t)$ where $M(t)$ is the maximal solution of the ordinary differential equation $y' = w(t, y)$, with the initial condition $y(0) = \|x\|_0$.*

Proof. If $t \in I$ and $u \in [0, t]$, we have

$$|x(u)| = |x(u) - x(0) + x(0)| \leq |x(0)| + |x(u) - x(0)|.$$

However, $|x(0)| \leq \max\{|x(s)|, \beta \leq s \leq 0\}$, and according to Lemma 3.3 we obtain

$$|x(u) - x(0)| \leq \int_0^u w(s, \|x\|_s)ds\,.$$

Then

$$|x(u)| \leq \|x\|_0 + \int_0^u w(s, \|x\|_s)ds\,.$$

Letting $\|x\|_0 = \mu$, we obtain

$$\max\{|x(u)|,\, \beta \leq s \leq 0\} \leq \mu + \int_0^u w(s, \|x\|_s)ds;$$

however,

$$\|x\|_t \leq \mu + \int_0^u w(s, \|x\|_s)ds = \mu + \int_0^t w(s, \|x\|_s)ds\,.$$

If we assume $\lambda(t) = \|x\|_t$, we have

$$\lambda(t) \leq \mu + \int_0^t w(s, \|x\|_s)ds\,.$$

After derivation, we obtain $\lambda'(t) \leq w(t, \lambda(t))$. From this and using lemma 3.2, we obtain $\lambda(t) \leq M(t)$ for $t \geq 0$, where $M(t)$ is the maximal solution of the ordinary differential equation: $y' = w(t, y)$, with the initial condition $y(0) = \mu$. Finally we have $\|x\|_t \le M(t)$, for $t \ge 0$. □

Lemma 3.5. *Let $x, y \in C$ such that $\|x\|_t \le M(t)$ for $t \in I$, where $M(t)$ is the maximal solution of the ordinary differential equation: $z' = w(t, z)$, with the initial condition $z(0) = \|y\|_0$. If $(Pty)(t) \subset \overline{B}(0, w(t, \|x\|_t))$ for all $t \in I$; then $\|y\|_t \le M(t)$ for all $t \in I$.*

Proof. If $t \in I$ and $u \in [0, t]$, we have

$$|y(u)| = |y(u) - y(0) + y(0)| \le |y(0)| + |y(u) - y(0)|\,.$$

However, $|y(0)| \le max\{|y(s)|, \beta \le s \le 0\}$, and in view of Lemma 3.3 we have

$$|y(u) - y(0)| \le \int_0^u w(s, \|x\|_s) ds\,.$$

So that

$$|y(u)| \le \|y\|_0 + \int_0^u w(s, \|x\|_s) ds\,.$$

From the preceding inequality and hypothesis $\|x\|_t \le M(t)$, we obtain

$$|y(u)| \le \|y\|_0 + \int_0^u w(s,\, M(s)) ds\,.$$

Then

$$\max\{|y(s)|, \beta \le s \le 0\} \le \|y\|_0 + \int_0^u w(s,\, M(s)) ds\,;$$

in other words,

$$\|y\|_u \le \|y\|_0 + \int_0^u w(s,\, M(s)) ds\,.$$

If we pose $\lambda(u) = \|y\|_u$ and $\|y\|_0 = \eta$, we obtain

$$\lambda(u) \le \eta + \int_0^u w(s,\, M(s)) ds\,.$$

After derivation, we have $\lambda'(u) \le w(u, M(u)) = M'(u)$ for $u \ge 0$. Given that $\lambda(0) = M(0) = \eta$, and that the functions λ and M are positive on I, it follows that $\lambda(t) \le M(t)$ for $t \ge 0$; i.e.,

$$\|y\|_t \le M(t), \quad \text{for } t \ge 0.$$

□

Lemma 3.6. *Under the hypotheses of Lemma 3.5, the function y satisfies locally the Lipschitz condition*

$$|y(t) - y(t')| \le \Omega_T |t - t'|$$

where $\Omega_T = \max\{w(s, M(T)) : s \in [0, T]\}$, $t, t' \in [0, T]$, and T is an arbitrary positive number.

Proof. Let T an arbitrary positive number and $t', t \in [0, T]$. According to Lemma 3.3, we have

$$|y(t) - y(t')| \le \int_{t'}^t w(s, \|x\|_s) ds$$

However, in view of Lemma 3.5, we have $\|x\|s \le M(s)$ for $s \in [0, T]$. Therefore,

$$|y(t) - y(t') \le \int_{t'}^t w(s, \|x\|_s) ds \le \int_{t'}^t w(s, M(s)) ds \le \int_{t'}^t w(s, M(T)) ds$$

we obtain $|y(t)-y(t')| \le \Omega_T|t-t'|$ where $\Omega_T = \max\{w(s, M(T)), s \in [0,T]\}$. $\square$

Before proving the main theorem, we will still need some lemmas by Zygmunt [27].

Lemma 3.7. *Let x, y be functions in C and $\{x_i\}$, $\{y_i\}$, $i = 1, 2, \dots$ be subsequences of functions in C. If $x_i \to x$, $y_i \to y$, $(Pt\, y_i)(t) \subset F([x_i]_t)$ for $t > 0$, and $y_i(t) = \xi(t)$ for $t \le 0$, $i = 1, 2, \dots$. Then $(Pt\, y)(t) \subset F([x]_t)$ for $t \ge 0$, and $y(t) = \xi(t)$ for $t \le 0$.*

Lemma 3.8. *Let x, y be functions in C and $F : G \to \operatorname{Conv} R^n$ be an upper semi-continuous set-valued map. Define $G(t) = F([x]_i)$ for $t \ge 0$. Then the two following statements are equivalent.*

- (P1) $(Pt\, y)(t) \subset G(t)$
- (P2) *For all $t \in I$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\tau \in I$, all $\sigma \in I$, and $\tau \ne \sigma$, we have $\{|\tau - t| < \delta$ and $|\sigma - t| < \delta\} \Rightarrow \frac{y(\sigma)-y(\tau)}{\sigma-\tau} \in \overline{G(t)_\varepsilon}$, where $\overline{G(t)_\varepsilon}$ is the closure of the ϵ-neighborhood of $G(t)$.*

Lemma 3.9. *Let x, ξ be two functions in C and $F : G \to \operatorname{Conv} R^n$ be an upper semicontinuous set-valued map. Let us define $G(t) = F[x]_t$ for $t \ge 0$. Then there exist a function $y \in C$ such that $(Pt\, y)(t) \subset G(t)$ for $t \ge 0$ and $y(t) = \xi(t)$ for $t \le 0$.*

The proof of the three lemmas above can be found in [27]. Now we shall prove the main theorem.

Proof of Theorem 3.1. Let $T > 0$ be an arbitrary fixed real number. Let us consider the family Φ of functions $x \in C$ satisfying the following three conditions:

$$x(t) = \xi(t), \quad \text{for } t \in [\beta, 0] \tag{3.6}$$

$$\|x\|_t \le M(t), \quad \text{for } t \in [0,T] \tag{3.7}$$

$$|x(t) - x(t')| \le \Omega_T|t-t'|, \quad \text{for t} \in [0,T] \tag{3.8}$$

where $\Omega_T = \max\{w(s, M(T)), s \in [0,T]\}$ and $M(t)$ is the maximal solution of the ordinary differential equation: $y' = w(t, y)$, with the initial condition $y(0) = (0)$.

We shall show that Φ is a nonempty, compact and convex subset of the space C.
(i) Φ is nonempty, it contains the function

$$f(t) = \begin{cases} \xi(t) & \text{for } t \in [\beta, 0] \\ \xi(0) & \text{for } t \in [0,T] \end{cases}$$

(ii) That Φ is compact, follows from Arzela's Theorem: its elements are uniformly bounded and equicontinuous.
(iii) It is easy to establish that Φ is convex. Let us consider the map $L : \Phi \to C$ such that for $x \in \Phi$,

$$L(x) = \{y \in C : y(t) = \xi(t) \text{for } t \in [\beta, 0] \text{ and } (Pt\, y)(t) \subset F([x]_t) \text{ for } t \in [0,T]\}.$$

For each fixed function x in Φ, the set $L(x)$ is nonempty according by Lemma 3.9, convex by Lemma 3.8. and closed by Lemma 3.7.

Now we show that if for all $x \in \Phi$, $F([x]_t) \subset \overline{B}(0, w(t, \|x\|_t))$ for $t \in [0,T]$, then $L(x)$ is compact. Let $y \in L(x)$, i.e., $y(t) = \xi(t)$ for $t \in [\beta, 0]$ and $(Pt\, y)(t) \subset F([x]_t)$ for $t \in [0,T]$.

Let us show that $y \in \Phi$, i.e. that y verified the conditions (3.6), (3.7) and (3.8).
(i) Obviously we have $y(t) = \xi(t)$ for $t \in [\beta, 0]$.
(ii) From hypotheses $(Pt\, y)(t) \subset F([x]_t)$ for $t \in [0,T]$ and $F([x]_t) \subset \overline{B}(0, w(t, \|x\|_t))$ for $t \in [0,T]$, we obtain $(Pt\, y)(t) \subset \overline{B}(0, w(t, \|x\|_t))$ for $t \in [0,T]$. According to Lemma 3.5, we have $\|y\|_t \le M(t)$for $t \in [0,T]$.
(iii) Finally, in view of Lemma 3.6, we have $|y(t) - y(t')| \le \Omega_T |t - t'|$ for $t \in [0,T]$. Moreover, since $L(x) \subset \Phi$, all elements of $L(x)$ are uniformly bounded and equicontinuous; since $L(x)$ is closed, it is compact. Therefore, L maps Φ in the family of the nonempty, compact and convex subsets of Φ.

Let us show that the application L is upper semi-continuous. Let x_i, x, y_i , $i = 1, 2, \dots$, an elements of Φ such that $x_i \to x$ and $y_i \in L(x_i)$. Since Φ is compact, from sequence $\{y_i\}$ $i = 1, 2, \dots$, we can extract a subsequence $\{y_i\}$ which converges to a certain function y. According to Lemma 3.7, we have $(Pt\, y)(t) \subset F([x]_t)$ for $t \in [0,T]$ and $y(t) = \xi(t)$ for $t \in [\beta, 0]$. Therefore, $y \in L(x)$ and by applying Lemma 2.1, we show the upper semi-continuity of the map L.

Using the Glicksberg Ky Fan theorem on the fixed point for multimaps in locally convex spaces [4], the map L has a fixed point in Φ. Therefore, there exists a function $x_0 \in \Phi$ such that $x_0 \in L(x_0)$, i.e., we have

$$(Pt\, x_0)(t) \subset F([x_0]_t)$$

for $t \in [0,T]$, and $x_0(t) = \xi(t)$ for $t \in [\beta, 0]$. In other words, x_0 is a solution of the paratingent equation with delayed argument (2.1) with the initial condition (2.2). Moreover, we have an estimate of the solution x_0,

$$\|x_0\|_t \le M(t) \quad \text{for } t \in [0,T].$$

□

Remark. Kryzowa [15] assumed that $F([x]_t) \subset \overline{B}(0, M(t) + N(t)\|x\|_t)$ and Zygmunt [27] assumed that $F([x]_t) \subset \overline{B}(0, M(t) + N(t)\|x\|_t^{\alpha})$ with $M(t), N(t) \ge 0$ real-valued continuous functions and $0 < \alpha \le 1$ for $t \ge 0$. In our work we have assumed that F satisfies condition (3.1) which is more general than those of Kryszowa and Zygmunt.

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